Grassmannian sigma model on a finite interval

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We discuss the two-dimensional Grassmannian sigma model $\mathbb{G}_{N,M}$ on a finite interval L. The different boundary conditions which allow one to obtain analytical solutions by the saddle-point method in the large N limit are considered. The nontrivial phase structure of the model on the interval similar to the $\mathbb{C}P(N)$ model is found.

I. INTRODUCTION

Two-dimensional sigma models are of great interest because of their similarity with the four-dimensional Yang-Mills theory and serve as a good platform for studying nonperturbative effects [1,2]. For the first time they [$\mathbb{O}(N)$ and $\mathbb{C}P(N)$] were solved in [1,3,4] in the large N limit [5]. The $\mathbb{C}P(N)$ and Grassmannian sigma models also emerge as the worldsheet theories on the single and multiple non-Abelian strings in QCD-like nonsupersymmetric theories [6,7] (see [8,9] for review). Therefore, the corresponding models on the interval correspond to the non-Abelian strings stretched between the boundary branes representing the domain walls. The $\mathbb{C}P(N)$ model on the interval has been considered in [10–12]. It was found that the solution of the model strongly depends on the boundary conditions and for the particular choice of the boundary conditions can undergo the phase transition at some value of the length of the interval.

In this paper we consider field theory on the complex Grassmannian manifold [13]

$$\mathbb{G}_{N,M} = \frac{SU(N)}{SU(N-M) \times U(M)}, \qquad M < N.$$

The fields $\varphi(x_0, x_1)$ are a complex matrix $N \times M$ satisfying the additional conditions $\varphi^\dagger \varphi = I_M$ (unit matrix $M \times M$); hence it is a set of N-dimensional vectors orthogonal to each other. We will get effective action and will study the large N solution of the saddle-point equation with various types of boundary conditions: Dirichlet-Dirichlet (D-D), Neumann-Neumann (N-N), and mixed Dirichlet-Neumann (D-N). Similar studies were carried out in [11,12] for $\mathbb{C}P(N)$ model $(\mathbb{C}P(N) = \mathbb{G}_{N,1})$. In [12] boundary conditions admitting an analytical solution have been found and which yields the existence of two phases: Higgs phase with broken $U(1)$ gauge symmetry and Coulomb phase with unbroken $U(1)$ symmetry. We will show that the similar result is true for $\mathbb{G}_{N,M}$ as well.

II. EFFECTIVE ACTION

Let us remind the reader of the Lagrangian of the model. The unique $SU(N)$-invariant metric on the $\mathbb{G}_{N,M}$ is $ds^2 = \text{Tr}(d\varphi d\varphi)$; hence the Lagrangian reads as

$$L = \text{Tr}[N/g^2 (D_\mu \varphi)^\dagger D_\mu \varphi + \lambda(I_M - \varphi^\dagger \varphi)], \qquad (1)$$

where λ is Lagrange multiplier, g is the coupling constant, $D_\mu = \partial_\mu + iA_\mu$ and $x \in [0, L], t \in (-\infty, +\infty)$.

We consider the partition function in the Euclidean space:

$$Z = \int DAD\lambda D\varphi^\dagger D\varphi$$

$$\times \exp\left[-\int d^2 x \text{Tr}(N/g^2 (D_\mu \varphi)^\dagger D_\mu \varphi + \lambda(\varphi^\dagger \varphi - I_M))\right]$$

$$= \int DAD\lambda D\varphi_{ij} D\varphi_{ij}^*$$

$$\times \exp\left[-\int d^2 x (N/g^2 (\partial_\mu - iA_\mu)\varphi_{ij}^*(\partial_\mu + iA_\mu)\varphi_{ij} \right.$$

$$\left. + \lambda(\varphi_{ij}^* \varphi_{ij} - M))\right]. \qquad (2)$$

Integrating by parts and taking into account the relation $\varphi^\dagger \varphi = I_M$, we obtain

$$\int d^2 x (D_\mu \varphi_{ij})^* D_\mu \varphi_{ij} = -\int d^2 x \varphi_{ij}^* D_\mu^2 \varphi_{ij}$$

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and

$$Z = \int DAD\lambda DRe\varphi_{ij}DIm\varphi_{ij}$$

$$\times \exp\left[-\int d^2x(Re\varphi_{ij}(-N/g^2(\partial_\mu + iA_\mu)^2 + \lambda)Re\varphi_{ij}\right.$$

$$\left. + Im\varphi_{ij}(-N/g^2(\partial_\mu + iA_\mu)^2 + \lambda)Im\varphi_{ij} - \lambda M)\right]. \quad (3)$$

In more detail, the quantity in the exponent is

$$\sum_{i,j}\left(-\int d^2x Re\varphi_{ij}\left(-\frac{N}{g^2}D_\mu^2 + \lambda\right)Re\varphi_{ij}\right)$$

$$+ \sum_{i,j}\left(-\int d^2x Im\varphi_{ij}\left(-\frac{N}{g^2}D_\mu^2 + \lambda\right)Im\varphi_{ij}\right)$$

$$+ \int d^2x\lambda M.$$

This integral is Gaussian, so we can separate $\varphi_{11} = \sigma$ and easily integrate out the remaining $NM - 1$ fields. Then we obtain an action depending only on λ and σ.

Taking into account that

$$\int DRe\varphi_{ij}\exp\left[\sum_{i,j}\left(-\int d^2x Re\varphi_{ij}\left(-\frac{N}{g^2}D_\mu^2 + \lambda\right)Re\varphi_{ij}\right)\right]$$

$$\sim [\det(-N/g^2 D_\mu^2 + \lambda)]^{-(NM-1)/2}$$

we get

$$\tilde{Z} = \int DAD\lambda D\sigma D\sigma^* \exp\left[-(NM-1)Tr\ln\left(-D_\mu^2 + \frac{g^2}{N}\lambda\right)\right.$$

$$\left. - \int d^2x\left((D_\mu\sigma)^2 + \lambda\frac{g^2}{N}\sigma\sigma^* - \lambda M\right)\right].$$

In the last equation the fields were rescaled. Redesignating λ and introducing the value $r = MN/g^2$, we obtain an expression for the effective action:

$$S_{\text{eff}} = (NM-1)Tr\ln(-D_\mu^2 + \lambda)$$

$$+ \int d^2x((D_\mu\sigma)^2 + \lambda(|\sigma|^2 - r)). \quad (4)$$

III. SADDLE-POINT EQUATIONS

We will calculate the path integral by the saddle-point method in the large N limit. For this we are looking for saddle points with respect to λ and σ of the action function. With the chosen boundary conditions, translational symmetry with respect to x is broken, but it is conserved with respect to t. Therefore we consider $\lambda = \lambda(x)$, $\sigma = \sigma(x)$. Let us set the gauge $A_t = 0$ and consider the eigenvalues and eigenfunctions of the operator:

$$(-D_x^2 + \lambda(x))\psi_n(x) = \mu_n\psi_n(x). \quad (5)$$

For our boundary conditions, $\psi_n(x) \sim \sin(E_n x + b_n)$ and $\mu_n = E_n^2$. Thus, using the relation Tr ln = ln det we can represent effective action in the form:

$$S_{\text{eff}} = (NM-1)\sum_n E_n + \int d^2x((D_\mu\sigma)^2 + \lambda(|\sigma|^2 - r)). \quad (6)$$

Let us vary effective action with respect to λ and use the first-order correction in perturbation theory, then we get the first saddle-point equation:

$$\frac{NM-1}{g^2}\sum_n\frac{\psi_n^2(x)}{E_n} + |\sigma(x)|^2 - r = 0. \quad (7)$$

The second equation is obtained by varying action with respect to σ which yields the equation of motion:

$$D_x^2\sigma^* - \lambda(x)\sigma^* = 0. \quad (8)$$

Further assume that $A_x = 0$ and σ is real. In this case the saddle-point equation with respect to A_x will be identically satisfied.

IV. D-N BOUNDARY CONDITIONS

Now we are considering mixed Dirichlet-Neumann boundary conditions. Let $NM = 2Z + 1$, $Z \in \mathbb{N}$, and following [12] suppose that on φ_{11} the boundary conditions N-N are imposed:

$$D_x\sigma(0) = D_x\sigma(L) = 0, \quad (9)$$

Z fields have N-D:

$$D_x\varphi_{ij}(0) = \varphi_{ij}(L) = 0, \quad (10)$$

and the remaining Z fields have D-N:

$$\varphi_{ij}(0) = D_x\varphi_{ij}(L) = 0. \quad (11)$$

The eigenfunctions and eigenvalues corresponding to the boundary conditions: for D-N:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left[\frac{\pi x(n-1/2)}{L}\right], \quad E_n^2 = \left[\frac{\pi(n-1/2)}{L}\right]^2 + \lambda,$$

$$n = 1, 2, \ldots,$$

for N-D:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\cos\left[\frac{\pi x(n-1/2)}{L}\right], \quad E_n^2 = \left[\frac{\pi(n-1/2)}{L}\right]^2 + \lambda,$$

$$n = 1, 2, \ldots$$

Substituting them into the Eq. (7), we can see that dependence on x disappears due to the trigonometric identity, hence $\sigma = $ const. Then it follows from (8) that $\sigma\lambda = 0$.

First we set λ not equal to zero, hence $\sigma = 0$. In this solution the vacuum expectation valus $\langle\varphi_{ij}\rangle = 0$ and the $U(1)$ symmetry is not broken. Therefore it is called "Coulomb" phase. Equation (7) takes the following form:

$$\frac{Z}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{(n-1/2)^2 + (\lambda L/\pi)^2}}\right) - r = 0. \quad (12)$$

In order to eliminate the divergence of the sum we represent it in the form

$$\frac{Z}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{(n-1/2)^2 + (\lambda L/\pi)^2}} - \frac{1}{n}\right) + \frac{Z}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} - r = 0, \quad (13)$$

and use UV cutoff:

$$\frac{Z}{\pi}\sum_{n=1}^{\infty}\frac{\exp(-\frac{n\pi}{L\Lambda_{uv}})}{n} = -\ln\left(1 - \exp\left(-\frac{\pi}{L\Lambda_{uv}}\right)\right)$$
$$\approx -\ln(\pi/L\Lambda_{uv}). \quad (14)$$

Note that the sigma model has dynamical mass generation via dimensional transmutation: $\Lambda^2 = \Lambda_{uv}^2 \exp(-4\pi/g^2)$. This allows us to express r in terms of dynamical scale Λ and Λ_{uv}: $r = \frac{Z}{\pi}\ln(\Lambda_{uv}/\Lambda)$. Thus,

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{(n-1/2)^2 + (\lambda L/\pi)^2}} - \frac{1}{n}\right) = \ln\left(\frac{\pi}{L\Lambda}\right). \quad (15)$$

Now let us show that this phase does not exist on the entire interval. Indeed, the left-hand side in the power series in λL has the form

$$\frac{1}{\sqrt{(n-1/2)^2 + (\lambda L/\pi)^2}} = \frac{1}{n-\frac{1}{2}} - 4\left(\frac{\lambda L}{\pi}\right)\frac{1}{2n-1} + \cdots. \quad (16)$$

Using the representation of the Riemann zeta function

$$\sum_{n=1}^{\infty}\frac{4}{(2n-1)^3} = \frac{7}{2}\zeta(3), \quad (17)$$

we obtain the equation

$$\frac{7}{2}\zeta(3)\left(\frac{\lambda L}{\pi}\right) = \ln\left(\frac{4\lambda L}{\pi}\right). \quad (18)$$

Therefore, the Coulomb phase exists only for $L > \pi/4\Lambda$.

Now let $\sigma = $ const, $\lambda = 0$. This is corresponds to the "Higgs" phase, because nonzero σ breaks $U(1)$ symmetry. In this case, from the first saddle-point equation, we obtain that

$$\frac{Z}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n}\right) + \sigma^2 = \frac{Z}{\pi}\ln\left(\frac{\pi}{L\Lambda}\right); \quad (19)$$

the second Eq. (8) is satisfied. Again using (18) we get

$$\sigma^2 = \frac{N}{\pi}\ln\left(\frac{\pi}{4L\Lambda}\right). \quad (20)$$

It can be seen that there is a solution only for $L \leq \frac{\pi}{4\Lambda}$.

V. D-D AND N-N BOUNDARY CONDITIONS

Suppose now that φ_{11} still has N-N boundary conditions, Z fields have D-D boundary conditions and other Z field have N-N boundary conditions. In this case, we have for D-D:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left[\frac{\pi xn}{L}\right], \qquad E_n^2 = \left[\frac{\pi n}{L}\right]^2 + \lambda,$$
$$n = 1, 2, \ldots,$$

and for N-N:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\cos\left[\frac{\pi xn}{L}\right], \qquad E_n^2 = \left[\frac{\pi n}{L}\right]^2 + \lambda,$$
$$n = 0, 1, \ldots.$$

There is a zero mode with $n = 0$ and the first saddle-point Eq. (7) has singularity at $\lambda = 0$. Hence, for given boundary conditions the solution has only Coulomb phase with $\lambda = $ const $\neq 0$. Substituting this eigenfunctions and eigenvalues into the Eq. (7) and using the UV cutoff, we have

$$\frac{Z}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n^2 - \lambda L/\pi} - \frac{1}{n}\right) + \frac{Z}{\lambda L} = \frac{Z}{\pi}\ln\left(\frac{\pi}{\Lambda L}\right). \quad (21)$$

It is easy to see that for fixed Λ and L, it is always possible to find λ that satisfies the equation, since the left-hand side, as a function of λ, takes a value on the whole $\mathbb{R}$. Thus, the only Coulomb phase exists for all L.

VI. CONCLUSION

In this paper we have presented a two-dimensional Grassmannian sigma model $\mathbb{G}_{N,M}$ on a finite interval in the large N limit. For special boundary conditions we have found an analytical solution to the saddle-points equation. Under the D-D and N-N boundary conditions solution has only one Coulomb phase ($\sigma = 0$) for all values of the length of the interval L; under the mixed Dirichlet-Neumann boundary conditions solution undergo the phase transition:

there is Coulomb phase for $L > \frac{\pi}{4\Lambda}$ and Higgs phase ($\sigma \neq 0$) for $L < \frac{\pi}{4\Lambda}$. We note that it is also of interest to consider a phase structure of sigma model on a flag target space.

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