Electrically charged black hole on AdS_3: Scale invariance and the Smarr formula

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The Einstein-Maxwell theory with negative cosmological constant in three spacetime dimensions is considered. It is shown that the Smarr relation for the electrically charged Bañados-Teitelboim-Zanelli (BTZ) black hole emerges from two different approaches based on the scaling symmetry of the asymptotic behavior of the fields at infinity. In the first approach, we prove that the conservation law associated to the scale invariance of the action for a class of stationary and circularly symmetric configurations, allows to obtain the Smarr formula as long as a special set of holographic boundary conditions is satisfied. This particular set is singled out making the integrability conditions for the energy compatible with the scale invariance of the reduced action. In the second approach, it is explicitly shown that the Smarr formula is recovered through the Euler theorem for homogeneous functions, provided the same set of holographic boundary conditions is fulfilled.

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I. INTRODUCTION

Since the early stage of the thermodynamical description of black holes, the Smarr formula [1] has been an intensive subject of study as an analogous of the Euler equation for black hole mechanics. This formula can be understood as an integrated form of the first law under certain homogeneity assumptions for the extensive variables. Specifically, this relation states the energy as a bilinear form of the global charges of the black hole along with their corresponding chemical potentials, as long as the entropy is a homogeneous function of a definite degree in the conserved charges. In the case of three-dimensional Einstein gravity on AdS_3, the Bañados-Teitelboim-Zanelli (BTZ) black hole [2,3] naturally satisfies this requirement. Indeed, the entropy of the BTZ black hole can be written in terms of the global charges through the Cardy formula [4];

$$S = 2\pi\sqrt{\frac{c}{12}(Ml + J)} + 2\pi\sqrt{\frac{c}{12}(Ml - J)}, \qquad (1)$$

where $c = \frac{3l}{2G}$ stands for the Brown-Henneaux central charge [5]. From (1) is evident that the entropy is a homogeneous function of degree $\frac{1}{2}$ in (M, J). Hence, by direct application of the Euler theorem, it is possible to write the energy as a Smarr relation [6,7];

$$M = \frac{1}{2}TS + \Omega J. \qquad (2)$$

In the Einstein-Maxwell theory with negative cosmological constant in three dimension the situation is rather different. As shown in [8,9], the energy spectrum of the electrically charged rotating BTZ black hole [10,11] is highly sensitive to the choice of boundary conditions,[1] what makes its thermodynamical description a very subtle problem. For instance, for the simplest choice of boundary conditions the energy spectrum turns out to be unbounded from below [11]. Indeed, by considering the same set of boundary conditions, the following formula of the energy holds

$$M = \frac{1}{2}TS + \Omega J + \frac{1}{2}\Phi_e Q_e + \frac{1}{8\pi}(1 - l^2\Omega^2)Q_e^2, \qquad (3)$$

which is certainly not a Smarr relation. As mentioned in [12], this is because of the logarithmic contributions of the gauge potential, which spoils the homogeneity property of the extensive quantities, in the sense that entropy is no

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[1]It is worthwhile to point out that we refer to boundary conditions to conditions that are held fixed at the boundary, while we mean by asymptotic conditions to the asymptotic behaviour of the fields at infinity.

longer a homogeneous function with a definite degree in the conserved charges. In what follows, we show that it possible to recover the aforementioned homogeneity property by considering the asymptotic conditions of the Einstein-Maxwell theory on AdS_3 introduced in [8,9] endowed with an appropriate set of boundary conditions.

The aim of this work is to show how the Smarr formula for the charged BTZ black hole emerges through two different approaches. Both of them are based on the preservation of the fall-off of the fields at infinity, given in [8], under a specific set of scale transformations that leaves the reduced action principle invariant for a wide class of configurations. In particular, we will use the method developed in [13] that recovers the Smarr formula for three-dimensional hairy black holes from a radial conservation law related to a scale invariance of the reduced action. However, the assumptions considered in this method are that the matter fields must be finite at the event horizon and vanish at infinity, where the latter is clearly not satisfied by the charged BTZ black hole because of the presence of the logarithmic terms. In spite of that, we will show herein that this method can still be applied in the case of the Einstein-Maxwell theory on AdS_3 by implementing the asymptotic conditions proposed in [8,9]. As consequence, it can be proved that the Smarr formula for the charged BTZ black hole holds as long as a special set of holographic boundary conditions is satisfied.

The next section is dedicated to a brief review of the main results found in [8], related to the global charges and their integrability conditions for stationary and circularly symmetric configurations in the Einstein-Maxwell theory on AdS_3. In Sec. III, we prove that the conservation law associated to the scale invariance of the action for the aforementioned class of configurations, allows us to obtain the Smarr formula as long as a special set of holographic boundary conditions is satisfied. This particular set is singled out by requiring compatibility of the integrability conditions for the energy with the scale invariance of the reduced action principle. In Sec. IV, it is shown that the same set of holographic boundary conditions ensures the right homogeneous transformation laws for the extensive quantities of the black hole under scaling transformations, allowing to recover the Smarr formula through the Euler theorem along the lines of its original derivation for the Kerr-Newman black hole in [1]. We conclude with some ending remarks in Sec. V.

II. A REVIEW ON THE EINSTEIN-MAXWELL THEORY ON AdS_3 AND GLOBAL CHARGES

This section is devoted to a brief review of the results found in [8]. It is shown the reduced action principle of the Einstein-Maxwell theory on AdS_3 in a canonical form for stationary and circularly symmetric configurations, the variation of the global charges and their appropriate integrability conditions.

A. Action principle for stationary and circularly symmetric configurations

The action of the Einstein-Maxwell theory with negative cosmological constant in three spacetime dimensions is given by

$$I_{\mathrm{EM}} = \int d^3x\sqrt{-g}\left[\frac{1}{2\kappa}(R-2\Lambda) - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right]. \tag{4}$$

Here the Newton constant G and the AdS radius l are defined through $\kappa = 8\pi G$ and $\Lambda = -l^{-2}$, respectively.

We consider stationary and circularly symmetric spacetimes, which describe a wide class of configurations already reported in the literature [2,10,11,14–29]. The line element for this family of solutions is given by

$$ds^2 = -\mathcal{N}(r)^2\mathcal{F}(r)^2dt^2 + \frac{dr^2}{\mathcal{F}(r)^2} + \mathcal{R}(r)^2(\mathcal{N}^\phi(r)dt + d\phi)^2, \tag{5}$$

where the gauge field is chosen as

$$A = \mathcal{A}_t(r)dt + \mathcal{A}_\phi(r)d\phi. \tag{6}$$

The reduced action principle in the canonical form can be obtained by replacing the class of configurations described by (5) and (6) in (4), which reads

$$I = -2\pi(t_2 - t_1)\int dr(\mathcal{N}\mathcal{H} + \mathcal{N}^\phi\mathcal{H}_\phi + \mathcal{A}_t\mathcal{G}) + B, \tag{7}$$

where the boundary term B must be added in order to have a well-defined variational principle. The surface deformation generators $\mathcal{H}$, $\mathcal{H}_\phi$ and the generator of gauge transformations $\mathcal{G}$, acquire the following form

$$\mathcal{H} = -\frac{\mathcal{R}}{\kappa l^2} + 4\kappa\mathcal{R}(\pi^{r\phi})^2 + \frac{(p^r)^2}{2\mathcal{R}} + \frac{\mathcal{F}^2(\mathcal{A}_\phi')^2}{2\mathcal{R}} + \frac{(\mathcal{F}^2)'\mathcal{R}'}{2\kappa} + \frac{\mathcal{F}^2\mathcal{R}''}{\kappa}, \tag{8}$$

$$\mathcal{H}_\phi = -p^r\mathcal{A}_\phi' - 2(\mathcal{R}^2\pi^{r\phi})', \tag{9}$$

$$\mathcal{G} = -\partial_r p^r, \tag{10}$$

where $\mathcal{N}$, $\mathcal{N}^\phi$ and $\mathcal{A}_t$ stand for their corresponding Lagrange multipliers. The only nonvanishing components of the momenta π^{ij} and p^i are explicitly given by

$$\pi^{r\phi} = -\frac{\mathcal{N}^{\phi\prime}\mathcal{R}}{4\kappa\mathcal{N}}; \qquad p^r = \frac{\mathcal{R}}{\mathcal{N}}(\mathcal{A}_\phi'\mathcal{N}^\phi - \mathcal{A}_t'). \tag{11}$$

Note that prime denotes derivative with respect to r.

B. Global charges and integrability conditions

Hereafter, it is considered that the class of configurations that we are dealing with are given by asymptotically AdS_3 spacetimes with the following behavior at infinity[2]

$$\begin{aligned}
\mathcal{R}^2 &= r^2 - \frac{\kappa l^2}{\pi}\left[h_{\mathcal{R}}\log\left(\frac{r}{l}\right) - \frac{f_{\mathcal{R}}}{2}\right] + \cdots \\
\mathcal{F}^2 &= \frac{r^2}{l^2} - \frac{\kappa}{\pi}\left[\left(2h_{\mathcal{R}} + \frac{1}{4\pi}(q_t^2 + q_\phi^2)\right)\log\left(\frac{r}{l}\right) + f_{\mathcal{F}}\right] \\
&\quad + \cdots \\
\mathcal{N}^\phi &= N_\infty^\phi + \frac{\kappa}{2\pi}N_\infty\left[\frac{l}{2\pi}q_t q_\phi \log\left(\frac{r}{l}\right) - j\right]\frac{1}{r^2} + \cdots \\
\mathcal{N} &= N_\infty + \cdots \\
\mathcal{A}_t &= -\frac{1}{2\pi}(q_t N_\infty + q_\phi l N_\infty^\phi)\log\left(\frac{r}{l}\right) + N_\infty^\phi \varphi_\phi + N_\infty \frac{\varphi_t}{l} \\
&\quad - \Phi + \cdots \\
\mathcal{A}_\phi &= -\frac{q_\phi l}{2\pi}\log\left(\frac{r}{l}\right) + \varphi_\phi + \cdots \qquad (12)
\end{aligned}$$

which was proposed in [8]. Here the coefficients $h_{\mathcal{R}}$, $f_{\mathcal{R}}$, $f_{\mathcal{F}}$, j, φ_t, φ_ϕ, q_t, q_ϕ are constants, which are assumed to vary in the phase space, while N_∞, N_∞^ϕ and Φ correspond to arbitrary constants without variation, which are kept fixed at the boundary.

Following the canonical approach given in [31], as shown in [8], the variation of the energy for the class of configurations (5), (6) endowed with the fall-off for the fields (12), is given by

$$\delta M = \delta\left[f_{\mathcal{R}} + f_{\mathcal{F}} + h_{\mathcal{R}} + \frac{1}{l}q_\phi\varphi_\phi\right] - \frac{1}{l}\varphi_\mu \delta q^\mu, \qquad (13)$$

where $\varphi_\mu = (l^{-1}\varphi_t, \varphi_\phi)$ and $q_\mu = (l^{-1}q_t, q_\phi)$ are assumed to be Lorentz covariant vectors, whose indices are raised and lowered by the flat (conformal) boundary metric $\eta_{\mu\nu} = \mathrm{diag}(-l^{-2}, 1)$. The rest of the global charges; angular momentum J and electric charge Q_e, can be directly integrated, and they read

$$J = j + \frac{l}{4\pi}q_t q_\phi - q_t\varphi_\phi, \qquad (14)$$

$$Q_e = q_t. \qquad (15)$$

As explained in [8], (13) yields a nontrivial integrability condition for φ_μ and q_μ. The integrability for the energy is ensured by the condition $\delta^2 M = 0$, which is satisfied provided

$$\varphi_\mu = -\frac{\delta\mathcal{V}}{\delta q^\mu}, \qquad (16)$$

where $\mathcal{V} = \mathcal{V}(q^\mu)$ is an arbitrary function of q_t and q_ϕ. In consequence, the energy and the angular momentum are then given by[3]

$$M = f_{\mathcal{R}} + f_{\mathcal{F}} + h_{\mathcal{R}} + \frac{1}{l}\left(\mathcal{V} - q_\phi\frac{\delta\mathcal{V}}{\delta q_\phi}\right), \qquad (17)$$

$$J = j + \frac{l}{4\pi}q_t q_\phi + q_t\frac{\delta\mathcal{V}}{\delta q_\phi}. \qquad (18)$$

In sum, the global charges are determined by the function $\mathcal{V}$ that describes the set of boundary conditions compatible with the integrability of the energy.

III. SCALE INVARIANCE AND RADIAL CONSERVATION LAW

In this section, we will make use of the approach given in [13], where the Smarr formula for three-dimensional hairy black holes is recovered from a radial conservation law associated to a scale invariance of the reduced action. The assumptions considered in this approach are that the matter fields must be finite at the event horizon and vanish at infinity, which due to the presence of the logarithmic terms is clearly not satisfied by the charged BTZ black hole. Nonetheless, we will show that this method can still be applied in the case of the Einstein-Maxwell theory on AdS_3 by implementing the asymptotic conditions proposed in [8,9].

In this case, it is possible to prove that the reduced action principle given in (7) is invariant under the following set of transformations

$$\begin{aligned}
\bar{\mathcal{R}}(\bar{r}) &= \lambda\mathcal{R}(r), & \bar{\mathcal{N}}(\bar{r}) &= \lambda^{-2}\mathcal{N}(r), \\
\bar{\mathcal{F}}(\bar{r})^2 &= \lambda^2\mathcal{F}(r)^2, & \bar{\mathcal{N}}^\phi(\bar{r}) &= \lambda^{-2}\mathcal{N}^\phi(r), \\
\bar{\mathcal{A}}_\phi(\bar{r}) &= \lambda\mathcal{A}_\phi(r), & \bar{\mathcal{A}}_t(\bar{r}) &= \lambda^{-1}\mathcal{A}_t(r), \\
\bar{p}^r(\bar{r}) &= \lambda p^r(r), & \bar{\pi}^{r\phi}(\bar{r}) &= \pi^{r\phi}(r), \qquad (19)
\end{aligned}$$

spanned by the scalings $\bar{r} = \lambda r$, $\bar{t} = t$ and $\bar{\phi} = \phi$, where λ is a positive constant. Note that similar scaling symmetries

[2]The dots "…" stand for subleading terms that can be consistently gauged away because they do not appear either in the global charges or in the gauge transformations of the dynamical fields [30].

[3]Note that as shown in [9], the integrability condition (16) endowed with a very precise set of boundary conditions ensures that the canonical generators fulfill a Poisson bracket algebra, otherwise the Jacobi identity would not be satisfied, spoiling the whole canonical structure. As a consequence, the reduced phase space (5), (6), with the asymptotic conditions (12), does possess a well-defined symplectic structure since it is already contained in the fall-off of the fields in [9].

were firstly observed in the matter-free case [32] and in the context of three-dimensional hairy black holes in [13].

A direct application of the Noether theorem, by considering the infinitesimal transformation laws derived from (19) on the reduced action principle (7), yields the following conserved quantity[4]

$$
\begin{aligned}
C(r) = {} & 2\pi p^r(\mathcal{A}_t + \mathcal{N}^\phi \mathcal{A}_\phi) + 8\pi \mathcal{N}^\phi \mathcal{R}^2 \pi^{r\phi} \\
& - \frac{2\pi \mathcal{F}^2 \mathcal{N} \mathcal{A}_\phi \mathcal{A}'_\phi}{\mathcal{R}} + \frac{\pi \mathcal{N} \mathcal{R} (\mathcal{F}^2)'}{\kappa} \\
& + \frac{2\pi \mathcal{F}^2 \mathcal{R} \mathcal{N}'}{\kappa} - \frac{2\pi \mathcal{F}^2 \mathcal{N} \mathcal{R}'}{\kappa} \\
& + \mathcal{N}\mathcal{R}\left(\frac{2\pi}{\kappa l^2} - \frac{\pi (p^r)^2}{\mathcal{R}^2} - 8\pi\kappa (\pi^{r\phi})^2 \right. \\
& \left. + \frac{\pi \mathcal{F}^2 (\mathcal{A}'_\phi)^2}{\mathcal{R}^2} - \frac{\pi (\mathcal{F}^2)' \mathcal{R}'}{\kappa \mathcal{R}} - \frac{2\pi \mathcal{F}^2 \mathcal{N}' \mathcal{R}'}{\kappa \mathcal{N} \mathcal{R}} \right) r, \qquad (20)
\end{aligned}
$$

along the radial direction, i.e. $C' = 0$, by virtue of the field equations. We will explore whether it is possible to find a Smarr formula for the charged rotating black hole [10,11] from the conserved quantity (20). Thus, in the particular case of the black hole solution with event horizon located at r_+, $C(\infty) = C(r_+)$.

In what follows, we proceed to compute $C(r)$ at infinity by considering the asymptotic behaviour of the fields given in Sec. II B, and then at the event horizon by imposing appropriate regularity conditions.

A. Conserved charge at infinity: Holographic boundary conditions

By considering the fall-off of the fields (12), the radial conserved charge (20) at the asymptotic region becomes

$$
\begin{aligned}
C(\infty) = {} & 2N_\infty \left[f_\mathcal{F} + f_\mathcal{R} + h_\mathcal{R} + \frac{l}{8\pi}(q_\phi^2 - q_t^2) \right. \\
& \left. + \frac{1}{l}(q_t \varphi_t + q_\phi \varphi_\phi) \right] \\
& - 2N_\infty^\phi \left(j + \frac{l}{4\pi} q_t q_\phi - q_t \varphi_\phi \right) - \Phi q_t, \qquad (21)
\end{aligned}
$$

recalling that $\varphi_\mu = -\frac{\delta \mathcal{V}}{\delta q^\mu}$.

In order to determine the functional form of $\mathcal{V}$ some physically reasonable criteria must be used. In this case, we will require compatibility of the asymptotic conditions given in (12) and the scale invariance of the reduced action under transformations (19), which allows to find the explicit form of this function. In particular, considering the scale transformations $\bar{A}_t(\bar{r}) = \lambda^{-1} A_t(r)$ and $\bar{A}_\phi(\bar{r}) = \lambda A_\phi(r)$ implies the following transformation rules for φ_μ and q_μ

$$
\bar{\varphi}_\mu = \lambda \left(\varphi_\mu + \frac{l q_\mu}{2\pi} \log(\lambda) \right), \qquad \bar{q}_\mu = \lambda q_\mu. \qquad (22)
$$

It must be highlighted that the transformations rules (22) precisely coincide with the ones found in [8], where it was made use of a scaling symmetry that leaves the configuration invariant and rescales the reduced action as $\bar{I} = \lambda^2 I$.

Compatibility of the transformation rules (22) under scalings with the integrability condition for the energy (16) requires, up to an arbitrary integration constant without variation, that the function $\mathcal{V}$ must obey the following differential equation

$$
q_t \frac{\partial \mathcal{V}}{\partial q_t} + q_\phi \frac{\partial \mathcal{V}}{\partial q_\phi} = 2\mathcal{V} + \frac{l}{4\pi}(q_t^2 - q_\phi^2). \qquad (23)
$$

The general solution of equation (23) is given by

$$
\mathcal{V} = q_t^2 F\left(\frac{q_\phi}{q_t}\right) + \frac{l}{8\pi}[q_t^2(\log[q_t^2] - 1) - q_\phi^2(\log[q_\phi^2] - 1)], \qquad (24)
$$

where F is an arbitrary function that describes a special set of boundary conditions compatible with the scale invariance. Hereafter, following [8], we will refer them as “holographic boundary conditions.”[5]

By using the integrability condition (16) and holographic boundary condition determined by (23), it is found that $C(\infty)$ in terms of the global charges (13), (14) and (15) is given by

$$
C(\infty) = 2N_\infty M - 2N_\infty^\phi J - \Phi Q_e. \qquad (25)
$$

In the following subsection we focus in the value of the Noether quantity (20) in the case of the configurations that possess an event horizon, that is the case of the charged BTZ black hole, where it is mandatory to impose some regularity conditions in order to ensure a well-defined Euclidean action principle (see e.g. [33]).

B. Conserved charge at the event horizon: Regularity conditions

In order to evaluate the conserved charge (20) at the event horizon, that for the class of configurations (5) is determined by $\mathcal{F}^2(r_+) = 0$, we have to consider smooth

[4] Note that the conserved charge (20) could also be obtained from the surface integral defining the canonical generators. Hence, it would be interesting to study its role in the Poisson bracket algebra of the global charges.

[5] It is worth to note that requiring invariance of the holographic boundary conditions under Lorentz symmetry singles out a very special set of boundary conditions, which remarkably, has shown to be compatible with the full conformal symmetry at infinity [9].

configurations which must satisfy regularity of the Euclidean geometry around the event horizon. These regularity conditions are generically given by

$$\mathcal{A}_\tau(r_+) = 0, \qquad \mathcal{N}^\phi(r_+) = 0,$$
$$\mathcal{N}(r_+)(\mathcal{F}^2)'(r_+) = 4\pi. \tag{26}$$

Hence, making use of the constraint $\mathcal{H} = 0$, (20) at the event horizon becomes

$$C(r_+) = \frac{4\pi^2}{\kappa}\mathcal{R}(r_+) = S, \tag{27}$$

which turns out to be the well-known Bekenstein-Hawking entropy of the charged BTZ black hole, i.e., $S = \frac{A}{4G}$ recalling that $\kappa = 8\pi G$.

Finally, the Smarr formula naturally emerges as consequence of the equality $C(r_+) = C(\infty)$, and it is given by[6]

$$S = 2N_\infty M - 2N_\infty^\phi J - \Phi Q_e. \tag{28}$$

Once the regularity conditions are taking into account it is possible to identify $N_\infty \equiv \beta$, $N_\infty^\phi \equiv \beta\Omega$ and $\Phi \equiv \beta\Phi_e$, where β is the inverse of the Hawking temperature, while Ω and Φ_e are the chemical potentials thermodynamically conjugated to the angular momentum J and the electric charge Q_e, respectively. Thus, the Smarr formula reads

$$M = \frac{1}{2}TS + \Omega J + \frac{1}{2}\Phi_e Q_e. \tag{29}$$

It is reassuring to verify that the charged BTZ black hole does satisfy the Smarr formula (29), as long as the special set of holographic boundary conditions, determined by (23), is satisfied. In order to carry out this computation one has to consider the explicit form of the global charges for the black hole, obtained in [8], which are given by

$$M = \frac{\pi r_+^2}{\kappa l^2}\left(\frac{1+\omega^2}{1-\omega^2}\right) - \frac{q_t^2}{4\pi}\left(\omega^2 + (1+\omega^2)\log\left(\frac{r_+}{l}\right)\right) + \frac{1}{l}\left(\mathcal{V} - q_\phi\frac{\delta\mathcal{V}}{\delta q_\phi}\right), \tag{30}$$

$$J = \frac{2\pi r_+^2\omega}{\kappa l(1-\omega^2)} - \frac{q_t^2\omega l}{4\pi}\left(1 + \log\left(\frac{r_+^2}{l^2}\right)\right) + q_t\frac{\delta\mathcal{V}}{\delta q_\phi}, \tag{31}$$

$$Q_e = q_t, \tag{32}$$

with $\omega = -q_\phi/q_t$. The Hawking temperature and the chemical potentials are

$$T = \frac{\sqrt{1-\omega^2}}{2\pi l^2 r_+}\left(r_+^2 - \frac{\kappa l^2}{8\pi^2}q_t^2(1-\omega^2)\right), \tag{33}$$

$$\Omega = \frac{\omega}{l}, \tag{34}$$

$$\Phi_e = -\frac{q_t}{2\pi}(1-\omega^2)\log\left(\frac{r_+}{l}\right) - \frac{\omega}{l}\frac{\delta\mathcal{V}}{\delta q_\phi} + \frac{1}{l}\frac{\delta\mathcal{V}}{\delta q_t}, \tag{35}$$

where the entropy is explicitly given by $S = \frac{4\pi^2 r_+}{\kappa\sqrt{1-\omega^2}}$.

IV. SMARR FORMULA FROM THE EULER THEOREM

In this section we will show that by virtue of the special set of holographic boundary conditions determined by (23), it is possible to obtain a homogeneous transformation law for the extensive quantities, allowing to use the Euler theorem in order to recover the Smarr formula (29) along the lines of its original derivation given in [1] for the Kerr-Newman black hole.

Let us consider the scale transformations for the coefficients of the fall-off (12) appearing in the energy (17) and angular momentum (18), which are given by

$$\bar{f}_\mathcal{F} = \lambda^2\left[f_\mathcal{F} - \left(2h_R + \frac{1}{4\pi}(q_t^2 + q_\phi^2)\right)\log(\lambda)\right],$$
$$\bar{f}_\mathcal{R} = \lambda^2[f_\mathcal{R} + 2h_\mathcal{R}\log(\lambda)],$$
$$\bar{j} = \lambda^2\left[j + \frac{l}{2\pi}q_t q_\phi\log(\lambda)\right],$$
$$\bar{h}_\mathcal{R} = \lambda^2 h_\mathcal{R}. \tag{36}$$

In the case of a generic $\mathcal{V}$, the mass (17), the angular momentum (18) and the electric charge (15) transform as

$$\bar{M} = \lambda^2(f_\mathcal{R} + f_\mathcal{F} + h_\mathcal{R}) - \frac{1}{4\pi}(q_t^2 + q_\phi^2)\lambda^2\log(\lambda) + \frac{1}{l}\left(\bar{\mathcal{V}} - q_\phi\frac{\delta\bar{\mathcal{V}}}{\delta q_\phi}\right), \tag{37}$$

$$\bar{J} = \lambda^2\left(j + \frac{l}{4\pi}q_t q_\phi\right) + \frac{l}{2\pi}q_t q_\phi\lambda^2\log(\lambda) + q_t\frac{\delta\bar{\mathcal{V}}}{\delta q_\phi}, \tag{38}$$

$$\bar{Q}_e = \lambda Q_e. \tag{39}$$

Note that the electric charge already transforms as homogeneous function of degree one, while the energy and the

[6] Similar relations for the entropy of three-dimensional black holes and cosmological configurations as a bilinear combination of the global charges along with their corresponding chemical potentials have been found in the context of higher spin gravity [33,34,35], hypergravity [36,37] and extended supergravity [38], where the coefficients in front of each term in the entropy formula turn out to be the spin (conformal weight) of the corresponding generator.

angular momentum possess anomalous scale transformation laws.

By implementing the holographic boundary conditions, one gets that the scale transformation of the function $\mathcal{V}$ inherited from (22) is given by

$$\bar{\mathcal{V}}(\lambda q_\mu) = \lambda^2 \left[\mathcal{V} + \frac{l}{4\pi}(q_t^2 - q_\phi^2)\log(\lambda)\right]. \qquad (40)$$

Remarkably, by replacing (40) into (37) and (38), the additional anomalous logarithmic terms precisely cancel out, such that the energy and the angular momentum now transform as homogeneous functions of degree two, i.e.,

$$\bar{M} = \lambda^2 M, \qquad \bar{J} = \lambda^2 J, \qquad (41)$$

where M and J are given in (17) and (18), respectively.

At this point, we focus in the charged BTZ black hole, whose entropy, given by $S = \frac{4\pi^2}{\kappa}\mathcal{R}(r_+)$, transforms as a homogeneous function of degree one, i.e., $\bar{S} = \lambda S$. In consequence, the entropy is a homogeneous function of degree $\frac{1}{2}$ in (M, J, Q_e^2), i.e.,

$$S(\sigma M, \sigma J, \sigma Q_e^2) = \sigma^{1/2} S(M, J, Q_e^2). \qquad (42)$$

Hence, by direct application of the Euler theorem for homogeneous functions, the above equation yields

$$\frac{1}{2}S = M\frac{\partial S}{\partial M}\bigg|_{J,Q_e} + J\frac{\partial S}{\partial J}\bigg|_{M,Q_e} + \frac{1}{2}Q_e\frac{\partial S}{\partial Q_e}\bigg|_{M,J}, \qquad (43)$$

and by virtue of the first law $\delta S = N_\infty \delta M - N_\infty^\phi \delta J - \Phi \delta Q_e$;

$$N_\infty = \frac{\partial S}{\partial M}\bigg|_{J,Q_e}, \qquad N_\infty^\phi = -\frac{\partial S}{\partial J}\bigg|_{M,Q_e}, \qquad \Phi = -\frac{\partial S}{\partial Q_e}\bigg|_{M,J}, \qquad (44)$$

the Eq. (43) can be cast directly to the expected Smarr formula

$$S = 2N_\infty M - 2N_\infty^\phi J - \Phi Q_e, \qquad (45)$$

provided $N_\infty = \beta$, $N_\infty^\phi = \beta\Omega$ and $\Phi = \beta\Phi_e$.

Note that for the simplest choice of boundary conditions, $\mathcal{V} = 0$, the global charges given by (17), (18) correspond to the ones found in [11]. In this case, the logarithmic terms in (36) lead to non-homogeneous contributions in the scale transformations of the global charges (17), (18), as it can be seen explicitly

$$\bar{M} = \lambda^2 M - \frac{1}{4\pi}(q_t^2 + q_\phi^2)\lambda^2 \log(\lambda), \qquad (46)$$

$$\bar{J} = \lambda^2 J + \frac{l}{2\pi} q_t q_\phi \lambda^2 \log(\lambda), \qquad (47)$$

yielding, by means of the Euler theorem, the relation (3)

$$M = \frac{1}{2}TS + \Omega J + \frac{1}{2}\Phi_e Q_e + \frac{1}{8\pi}(1 - l^2\Omega^2)Q_e^2. \qquad (48)$$

Therefore, the logarithmic terms in (46) and (47) preclude the possibility to obtain the Smarr formula (29), since the assumption of the homogeneity scaling property of the global charges are no longer satisfied.

V. ENDING REMARKS

In this work it has been shown that the Smarr fomula for the charged BTZ black hole emerges from two different approaches. Both of them are based on the preservation of the fall-off of the fields under scale transformations which leave the reduced action principle invariant. In the first approach, we have proved that the scale invariance of the theory for stationary and circularly symmetric configurations is associated to a radially conserved charge. This conservation law leads to the Smarr formula as long as a special set of holographic boundary conditions is fulfilled. In the second approach, it was found that the same set of holographic boundary conditions confers the homogeneity scaling property to the global charges, allowing to derive the Smarr formula of the black hole through the Euler theorem.

Throughout this work, we have considered that the cosmological constant is a coupling constant fixed without variation. Nonetheless, the problem related to the spoil of the homogeneity property in the extensive variables of the electrically charged BTZ black hole has been addressed at some extent under the rescaling of the cosmological constant, which leads to the introduction of additional thermodynamical terms both in the first law and in the energy formula (3) (see e.g. [39,40,41,42]). This treatment might be consistently carried out promoting the cosmological constant to a canonical variable through the mechanism described in [43]. However, recent results has shown the existence of a superselection rule that forbids a superposition of quantum states with different values of the cosmological constant in three dimensions [44], so that its value would be definite, and in consequence, it cannot be rescaled.

Finally, it is noteworthy that apart of obtaining the Smarr formula for the charged BTZ black hole, the set of holographic boundary conditions endowed with the additional requirement of Lorentz symmetry makes the energy spectrum of the black hole nonnegative, and the electric charge bounded from above, for a fixed value of the energy [8]. This strongly suggests that the solution might be stable, so it would be interesting to carry out a thermodynamical analysis of the stability of the charged BTZ black hole by considering a generic set of boundary conditions.

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[1] L. Smarr, Mass Formula for Kerr Black Holes, Phys. Rev. Lett. **30**, 71 (1973); Erratum, Phys. Rev. Lett. **30**, 521(E) (1973).
[2] M. Banados, C. Teitelboim, and J. Zanelli, The Black Hole in Three-Dimensional Space-time, Phys. Rev. Lett. **69**, 1849 (1992).
[3] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Geometry of the $(2+1)$ black hole, Phys. Rev. D **48**, 1506 (1993); Erratum, Phys. Rev. D **88**, 069902(E) (2013).
[4] A. Strominger, Black hole entropy from near horizon microstates, J. High Energy Phys. 02 (1998) 009.
[5] J. D. Brown and M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity, Commun. Math. Phys. **104**, 207 (1986).
[6] R. G. Cai, Z. J. Lu, and Y. Z. Zhang, Critical behavior in $(2+1)$-dimensional black holes, Phys. Rev. D **55**, 853 (1997).
[7] G. Clement, Black hole mass and angular momentum in $2+1$ gravity, Phys. Rev. D **68**, 024032 (2003).
[8] A. Pérez, M. Riquelme, D. Tempo, and R. Troncoso, Conserved charges and black holes in the Einstein-Maxwell theory on AdS_3 reconsidered, J. High Energy Phys. 10 (2015) 161.
[9] A. Pérez, M. Riquelme, D. Tempo, and R. Troncoso, Asymptotic structure of the Einstein-Maxwell theory on AdS_3, J. High Energy Phys. 02 (2016) 015.
[10] G. Clement, Classical solutions in three-dimensional Einstein-Maxwell cosmological gravity, Classical Quantum Gravity **10**, L49 (1993).
[11] C. Martinez, C. Teitelboim, and J. Zanelli, Charged rotating black hole in three space-time dimensions, Phys. Rev. D **61**, 104013 (2000).
[12] M. Bravo-Gaete, S. Gomez, and M. Hassaine, Cardy formula for charged black holes with anisotropic scaling, Phys. Rev. D **92**, 124002 (2015).
[13] M. Banados and S. Theisen, Scale invariant hairy black holes, Phys. Rev. D **72**, 064019 (2005).
[14] S. Deser and P. O. Mazur, Static solutions in $D=3$ Einstein-Maxwell theory, Classical Quantum Gravity **2**, L51 (1985).
[15] J. R. Gott, J. Z. Simon, and M. Alpert, General relativity in a $(2+1)$-dimensional space-time: An electrically charged solution, Gen. Relativ. Gravit. **18**, 1019 (1986).
[16] P. Peldan, Unification of gravity and Yang-Mills theory in $(2+1)$-dimensions, Nucl. Phys. **B395**, 239 (1993).
[17] M. Kamata and T. Koikawa, The electrically charged BTZ black hole with self (antiself) dual Maxwell field, Phys. Lett. B **353**, 196 (1995).
[18] K. C. K. Chan, Comment on the calculation of the angular momentum for the (anti)self dual charged spinning BTZ black hole, Phys. Lett. B **373**, 296 (1996).
[19] G. Clement, Spinning charged BTZ black holes and selfdual particle-like solutions, Phys. Lett. B **367**, 70 (1996).
[20] E. W. Hirschmann and D. L. Welch, Magnetic solutions to $(2+1)$ gravity, Phys. Rev. D **53**, 5579 (1996).
[21] M. Cataldo and P. Salgado, Static Einstein-Maxwell solutions in $(2+1)$-dimensions, Phys. Rev. D **54**, 2971 (1996).
[22] M. Kamata and T. Koikawa, $(2+1)$-dimensional charged black hole with (anti-)selfdual Maxwell fields, Phys. Lett. B **391**, 87 (1997).
[23] M. Cataldo and P. Salgado, Three dimensional extreme black hole with self (anti-self) dual Maxwell field, Phys. Lett. B **448**, 20 (1999).
[24] M. Cataldo, Azimuthal electric field in a static rotationally symmetric $(2+1)$-dimensional space-time, Phys. Lett. B **529**, 143 (2002).
[25] O. J. C. Dias and J. P. S. Lemos, Rotating magnetic solution in three-dimensional Einstein gravity, J. High Energy Phys. 01 (2002) 006.
[26] M. Cataldo, J. Crisostomo, S. del Campo, and P. Salgado, On magnetic solution to $(2+1)$ Einstein-Maxwell gravity, Phys. Lett. B **584**, 123 (2004).
[27] J. Matyjasek and O. B. Zaslavskii, Extremal limit for charged and rotating $(2+1)$-dimensional black holes and Bertotti-Robinson geometry, Classical Quantum Gravity **21**, 4283 (2004).
[28] A. A. Garcia-Diaz, Three dimensional stationary cyclic symmetric Einstein-Maxwell solutions; black holes, Ann. Phys. (Berlin) **324**, 2004 (2009).
[29] E. Ayon-Beato, M. Cataldo, and A. A. Garcia, Electromagnetic fields in stationary cyclic symmetric 2+1 gravity, *Proceedings of the 10th International Symposium Northeastern University* (World scientific, Singapore, 2004).
[30] R. Benguria, P. Cordero, and C. Teitelboim, Aspects of the Hamiltonian dynamics of interacting gravitational gauge and Higgs fields with applications to spherical symmetry, Nucl. Phys. **B122**, 61 (1977).
[31] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) **88**, 286 (1974).
[32] G. T. Horowitz and V. E. Hubeny, Quasinormal modes of AdS black holes and the approach to thermal equilibrium, Phys. Rev. D **62**, 024027 (2000).

[33] C. Bunster, M. Henneaux, A. Pérez, D. Tempo, and R. Troncoso, Generalized black holes in three-dimensional spacetime, J. High Energy Phys. 05 (2014) 031.
[34] M. Gary, D. Grumiller, M. Riegler, and J. Rosseel, Flat space (higher spin) gravity with chemical potentials, J. High Energy Phys. 01 (2015) 152.
[35] J. Matulich, A. Perez, D. Tempo, and R. Troncoso, Higher spin extension of cosmological spacetimes in 3D: Asymptotically flat behaviour with chemical potentials and thermodynamics, J. High Energy Phys. 05 (2015) 025.
[36] M. Henneaux, A. Perez, D. Tempo, and R. Troncoso, Hypersymmetry bounds and three-dimensional higher-spin black holes, J. High Energy Phys. 08 (2015) 021.
[37] M. Henneaux, A. Pérez, D. Tempo, and R. Troncoso, Extended anti-de Sitter hypergravity in $2+1$ dimensions and hypersymmetry bounds, arXiv:1512.08603.
[38] O. Fuentealba, J. Matulich, and R. Troncoso, Asymptotic structure of $\mathcal{N}=2$ supergravity in 3D: extended super-BMS_3 and nonlinear energy bounds, J. High Energy Phys. 09 (2017) 030.
[39] E. A. Larranaga Rubio, On the first law of thermodynamics for $(2+1)$ dimensional charged BTZ black hole and charged de Sitter space, Turk. J. Phys. **32**, 1 (2008).
[40] S. Gunasekaran, R. B. Mann, and D. Kubiznak, Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, J. High Energy Phys. 11 (2012) 110.
[41] A. M. Frassino, R. B. Mann, and J. R. Mureika, Lower-dimensional black hole chemistry, Phys. Rev. D **92**, 124069 (2015).
[42] M. Dehghani, Thermodynamics of $(2+1)$-dimensional charged black holes with power-law Maxwell field, Phys. Rev. D **94**, 104071 (2016).
[43] M. Henneaux and C. Teitelboim, The cosmological constant as a canonical variable, Phys. Lett. B **143B**, 415 (1984).
[44] C. Bunster and A. Perez, Superselection rule for the cosmological constant in three-dimensional spacetime, Phys. Rev. D **91**, 024029 (2015).