

Two-point function of a quantum scalar field in the interior region of a Reissner-Nordstrom black hole

Assaf Lanir, Adam Levi, Amos Ori, and Orr Sela

Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel

 (Received 23 October 2017; published 24 January 2018)

We derive explicit expressions for the two-point function of a massless scalar field in the interior region of a Reissner-Nordstrom black hole, in both the Unruh and the Hartle-Hawking quantum states. The two-point function is expressed in terms of the standard $lm\omega$ modes of the scalar field (those associated with a spherical harmonic Y_{lm} and a temporal mode $e^{-i\omega t}$), which can be conveniently obtained by solving an ordinary differential equation, the radial equation. These explicit expressions are the internal analogs of the well-known results in the external region (originally derived by Christensen and Fulling), in which the two-point function outside the black hole is written in terms of the external $lm\omega$ modes of the field. They allow the computation of $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ and the renormalized stress-energy tensor inside the black hole, after the radial equation has been solved (usually numerically). In the second part of the paper, we provide an explicit expression for the trace of the renormalized stress-energy tensor of a *minimally coupled* massless scalar field (which is nonconformal), relating it to the d'Alembertian of $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$. This expression proves itself useful in various calculations of the renormalized stress-energy tensor.

DOI: 10.1103/PhysRevD.97.024033

I. INTRODUCTION

In the framework of semiclassical general relativity, the gravitational field is treated classically as a curved spacetime while other fields are treated as quantum fields residing in this background spacetime. The relation between the spacetime geometry and the stress energy of the quantum fields is described by the semiclassical Einstein equation

$$G_{\mu\nu} = 8\pi \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}, \quad (1.1)$$

where $G_{\mu\nu}$ is the Einstein tensor of spacetime and $\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}$ is the renormalized stress-energy tensor (RSET), which is the renormalized expectation value of the stress-energy tensor operator \hat{T} , associated with the quantum fields. In Eq. (1.1) and throughout this paper we adopt standard geometric units $c = G = 1$, and signature $(-+++)$.

The main challenge in analyzing the semiclassical Einstein equation is the computation of the RSET. Even when the background geometry is fixed and the corresponding metric is given, performing a procedure of renormalization is not an easy task. The task becomes much more difficult upon trying to solve the full self-consistent problem represented by Eq. (1.1). One reason (besides the obvious numerical challenge) is that Eq. (1.1) admits runaway solutions [1].

An example of a quantum field that is often chosen for its simplicity is that of a scalar field, which satisfies the Klein-Gordon equation

$$(\square - m^2 - \xi R)\hat{\Phi} = 0, \quad (1.2)$$

where $\hat{\Phi}$ is the scalar field operator, m denotes the field's mass, and ξ is the coupling constant. As a first stage toward calculating the RSET, it is customary to begin by calculating $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$, as this quantity is endowed with many of the essential features of the RSET, but is simpler to compute. In other words, $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ serves as a simple toy model for the RSET.

The standard method to calculate quantities which are quadratic in the field and its derivatives is *point splitting* (or *covariant point separation*), developed by DeWitt for $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ and by Christensen for the RSET [2,3]. In this method, one splits the point x , in which $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ is evaluated, and writes it as a product of the field operators at two different points, namely the two-point function (TPF) $\langle \hat{\Phi}(x)\hat{\Phi}(x') \rangle$. One then subtracts from the TPF a known counterterm and takes the limit $x' \rightarrow x$, thereby obtaining $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$.

The numerical implementation of the limit described above turns out to be very difficult. To overcome this difficulty, practical methods to implement the point-splitting scheme were developed by Candelas and others [4–7]. These techniques, however, all relied on Wick rotation; namely, they required the background to admit a Euclidean sector (usually employing a high-order WKB approximation for the field modes on this sector).

Recently, a more versatile method to implement the point-splitting scheme was developed, the *pragmatic mode-sum regularization* (PMR) scheme. In this method, the background does not need to admit a Euclidean sector, and the WKB approximation is not used. Instead, the background

must only admit a single Killing field. The PMR method was tested and used to compute $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ and the RSET in the exterior part of a Schwarzschild black hole (BH) [8–11], a Reissner-Nordstrom (RN) BH [12], and a Kerr BH [13].

So far most of the calculations of $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ and the RSET were performed on the exterior part of BHs. The only exception we know of is Ref. [14], which calculated $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ for the interior part of Schwarzschild, in the Hartle-Hawking state. The main reason is probably that the internal calculation requires a longer analytical derivation for expressing the TPF in terms of the standard Eddington-Finkelstein modes. In addition, the internal calculation requires numerical computation of modes both inside and outside the BH.

Although it is not an easy task, there is a great interest in studying the semiclassical quantum effects in the interior of BHs. One obvious reason is the question whether semiclassical effects could resolve the spacetime singularity inside e.g. a Schwarzschild BH [15,16]. Yet another motivation is the quest to understand the fate of the inner horizon inside spinning or charged BHs, when realistic perturbations are taken into account. For both spinning and charged BHs, in the corresponding unperturbed classical solution (the Kerr or RN solution, respectively), the inner horizon is a perfectly smooth null hypersurface. Classical perturbations typically convert the smooth inner horizon into a curvature singularity, which is nevertheless null and weak (i.e. tidally nondestructive). One may expect, however, that semiclassical energy-momentum fluxes could have a stronger potential effect on the inner horizon (perhaps converting the latter into a strong, i.e. tidally destructive, spacelike singularity). A first and important step toward clarifying this issue is the RSET computation on the background of the classical RN or Kerr geometry—inside the BH (and particularly near the inner horizon).

Several works have previously been made in the attempt to address this issue analytically, for RN [17], Kerr [18], and even Kerr-Newmann [19] background spacetimes. Generally speaking, these works suggested that indeed the RSET is likely to diverge at the inner horizon. However, the results obtained so far [at least in four dimensions (4D)] are still not entirely conclusive. Thus, Ref. [17] analyzed a two-dimensional (2D) RN model and found RSET divergence at the Cauchy horizon (CH). In addition, Refs. [17–19] obtained several relations between various RSET components at the CH (at leading order) in 4D. They showed that certain nontrivial quantities have to vanish in order for the RSET to be regular there, which may suggest (but does not prove) divergence. The strongest CH result was derived by Hiscock in Ref. [19]: He explicitly showed that in the Unruh state in RN, the semiclassical fluxes must diverge at either the ingoing section (i.e. the CH) or the outgoing section of the inner horizon—or possibly at both. Still, this result leaves open the possibility of a perfectly regular RSET at the CH. A possible way to conclusively address this issue is to explicitly

compute the RSET inside a Kerr and/or RN black hole, and to obtain its asymptotic behavior on approaching the CH. This, however, requires the extension of the RSET computation infrastructure to the internal part of the BH. Here we undertake this goal in the case of RN background (deferring the more complicated Kerr case to future works).

More specifically, in this paper we address two different (and perhaps loosely related) issues. The first one is directly related to the main objective described above: We consider a massless quantum scalar field and construct explicit expressions for its TPF in the interior of a RN black hole, expressing it as a sum of standard Eddington-Finkelstein modes that are naturally defined in the interior region. Specifically, we focus on the symmetrized form of the TPF, which is also known as the Hadamard elementary function

$$G^{(1)}(x, x') = \langle \{ \hat{\Phi}(x), \hat{\Phi}(x') \} \rangle, \quad (1.3)$$

for both the Unruh and the Hartle-Hawking quantum states,¹ where $\{, \}$ denotes anticommutation. The final expressions can be found in Eqs. (3.13) and (4.3). Using these expressions one can calculate $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ (using e.g. the PMR method) after numerically solving the radial equation (an ordinary differential equation) needed for constructing the standard Eddington-Finkelstein modes.

The second issue is pointing out an identity for the RSET of a minimally coupled massless scalar field, which is analogous to the well-known trace-anomaly identity for conformal fields. The identity is easily derived using known expressions, and we found it very useful. Yet we could not find any explicit indications for it in the literature.

The general considerations described above motivate one to study the internal semiclassical effects primarily in quantum states that may be considered as “vacuum,” such as the Hartle-Hawking and Unruh states (and especially in the latter state, which characterizes the actual evaporating BHs). Indeed, the results of Secs. III and IV explicitly refer to the TPF in the Unruh and Hartle-Hawking states, respectively. Note, however, that the result derived in Sec. V (the trace of a minimally coupled scalar field) applies to *any* quantum state.

In forthcoming papers, the results of this paper will be used to compute $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ and the RSET in the interior region of Schwarzschild and RN spacetimes using two different approaches. One approach [20] uses analytical asymptotic approximations for the Eddington-Finkelstein modes, in order to analyze the leading-order divergence (if it occurs) of $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ and the RSET upon approaching the inner horizons. The other approach [21] uses a numerical computation of the Eddington-Finkelstein modes in the interior region of the BH. It then utilizes the aforementioned PMR method to calculate $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ in the interior. The

¹In the point-splitting procedure, it is common to use the Hadamard function $G^{(1)}(x, x')$ instead of $\langle \hat{\Phi}(x)\hat{\Phi}(x') \rangle$.

second approach was already used for the Schwarzschild case, and the results agree fairly well with those presented in [14]. The two approaches complement each other and allow for cross-checking the results.

The organization of this paper is as follows. In Sec. II we introduce all the necessary preliminaries needed for our derivation. Then, in Secs. III and IV, we develop expressions for the TPF calculated in Unruh and Hartle-Hawking states, respectively. Section V analyzes the trace of a massless, minimal-coupled, scalar field. In Sec. VI we discuss our results and possible extensions.

II. PRELIMINARIES

Before we begin the computation, let us start by defining the coordinate systems, sets of modes, and quantum states which we use in this paper, and the form of the Hadamard function outside the BH.

A. Coordinate systems

In this paper we consider the RN spacetime, which in the standard Schwarzschild coordinates has the following metric:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The event horizon ($r = r_+$) and the inner horizon ($r = r_-$) are located at the two roots of $g_{tt} = 0$, i.e.

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}.$$

Throughout this paper, only the region $r \geq r_-$ will be concerned. The surface gravity parameters at the two horizons, κ_{\pm} , are given by

$$\kappa_{\pm} = \frac{r_+ - r_-}{2r_{\pm}^2}.$$

We define the tortoise coordinate, r_* , in both the interior and the exterior regions, using the standard relation

$$\frac{dr}{dr_*} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

Specifically, we choose the integration constants in the interior and the exterior regions such that in both regions

$$r_* = r + \frac{1}{2\kappa_+} \ln\left(\frac{|r - r_+|}{r_+ - r_-}\right) - \frac{1}{2\kappa_-} \ln\left(\frac{|r - r_-|}{r_+ - r_-}\right).$$

Note that r_+ corresponds to $r_* \rightarrow -\infty$ (both for r_* defined in the exterior region and for that defined in the interior) and r_- to $r_* \rightarrow +\infty$.

The Eddington-Finkelstein coordinates are defined in the exterior region by

$$u_{\text{ext}} = t - r_*, \quad v = t + r_* \quad (\text{outside}),$$

while in the interior region they are

$$u_{\text{int}} = r_* - t, \quad v = r_* + t \quad (\text{inside}).$$

Note that the v coordinate is continuously defined in both regions I and II of Fig. 1.

The Kruskal coordinates (corresponding to the event horizon r_+) are defined in terms of the exterior and interior Eddington-Finkelstein coordinates by

$$U(u_{\text{ext}}) = -\frac{1}{\kappa_+} \exp(-\kappa_+ u_{\text{ext}}),$$

$$V(v) = \frac{1}{\kappa_+} \exp(\kappa_+ v) \quad (\text{outside}) \quad (2.1)$$

and

$$U(u_{\text{int}}) = \frac{1}{\kappa_+} \exp(\kappa_+ u_{\text{int}}),$$

$$V(v) = \frac{1}{\kappa_+} \exp(\kappa_+ v) \quad (\text{inside}). \quad (2.2)$$

We make the following notations: H_{past} denotes the past horizon [i.e. the hypersurface ($U < 0, V = 0$)], PNI denotes past null infinity [i.e. ($U = -\infty, V > 0$)]. H_L is the “left event horizon” ($U > 0, V = 0$), and H_R is the “right event horizon” ($U = 0, V > 0$). See Fig. 1.

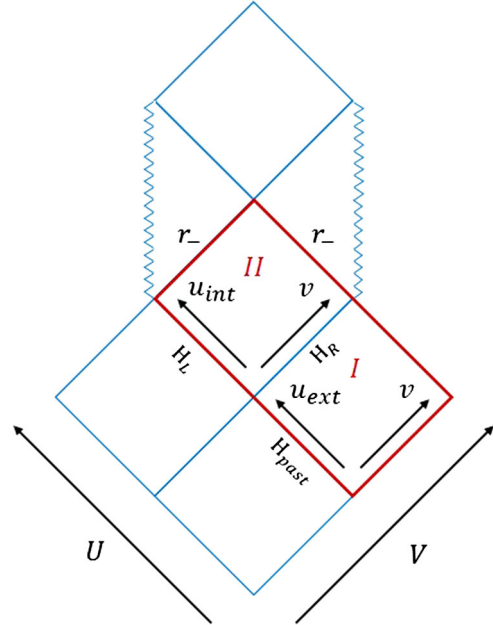


FIG. 1. Penrose diagram of Reissner-Nordstrom spacetime. In the exterior region (region I), we use the external Eddington-Finkelstein coordinates, while in the interior (region II), we use the internal ones. In addition, the Kruskal coordinate system is shown and is defined throughout both regions I and II. The red-framed area denotes the region in the eternal Reissner-Nordstrom spacetime which concerns this paper, i.e. regions I and II.

B. Modes

In this paper we are considering a massless quantum scalar field operator $\hat{\Phi}$ in RN, satisfying the Klein-Gordon equation

$$\square\hat{\Phi} = 0. \quad (2.3)$$

The quantum states considered in this paper are conveniently defined via a decomposition of this field into sets of modes satisfying Eq. (2.3) with certain initial conditions. Therefore it is useful to consider various complete sets of modes in different regions of spacetime.²

First we define the *Unruh modes*, $g_{\omega lm}^{\text{up}}$ and $g_{\omega lm}^{\text{in}}$ (for $\omega > 0$). Exploiting the spherical symmetry, we define these modes by decomposing them in the following standard way:

$$g_{\omega lm}^{\Lambda}(x) = \omega^{-1/2} C_{lm}(x) \tilde{g}_{\omega l}^{\Lambda}(x), \quad (2.4)$$

where

$$C_{lm}(x) = (4\pi)^{-1/2} \frac{1}{r} Y_{lm}(\theta, \varphi), \quad (2.5)$$

Λ denotes ‘‘in’’ and ‘‘up,’’ and $\tilde{g}_{\omega l}^{\Lambda}$ are solutions of the following two-dimensional wave equation, obtained by substituting Eq. (2.4) in Eq. (2.3):

$$\tilde{g}_{r_* r_*}^{\Lambda} - \tilde{g}_{tt}^{\Lambda} = V_l(r) \tilde{g}^{\Lambda}, \quad (2.6)$$

where

$$V_l(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}\right]. \quad (2.7)$$

The two sets of independent solutions, $\tilde{g}_{\omega l}^{\text{up}}$ and $\tilde{g}_{\omega l}^{\text{in}}$, are defined according to the following initial conditions:

$$\tilde{g}_{\omega l}^{\text{up}} = \begin{cases} e^{-i\omega U}, & H_{\text{past}} \\ e^{-i\omega U}, & H_L \\ 0, & \text{PNI} \end{cases}, \quad \tilde{g}_{\omega l}^{\text{in}} = \begin{cases} 0, & H_{\text{past}} \\ 0, & H_L \\ e^{-i\omega v}, & \text{PNI} \end{cases}. \quad (2.8)$$

We further introduce for later use the two parts of $\tilde{g}_{\omega l}^{\text{up}}$ which, too, satisfy Eq. (2.6) but with initial conditions

$$\tilde{g}_{\omega l}^{\text{past}} = \begin{cases} e^{-i\omega U}, & H_{\text{past}} \\ 0, & H_L \\ 0, & \text{PNI} \end{cases}, \quad \tilde{g}_{\omega l}^{\text{L}} = \begin{cases} 0, & H_{\text{past}} \\ e^{-i\omega U}, & H_L \\ 0, & \text{PNI} \end{cases}.$$

These modes are defined in both regions I and II, i.e. throughout the red-framed area of Fig. 1. Note that, by additivity, $\tilde{g}_{\omega l}^{\text{past}}(x) + \tilde{g}_{\omega l}^{\text{L}}(x) = \tilde{g}_{\omega l}^{\text{up}}(x)$.

²Note that not all of these sets of modes are necessarily related to the definition of a quantum state. Specifically, the internal ‘‘right’’ and ‘‘left’’ modes (see below) are introduced merely for mathematical convenience.

Next, we turn to the definition of the *outer Eddington-Finkelstein modes*, $f_{\omega lm}^{\text{up}}$ and $f_{\omega lm}^{\text{in}}$. Similar to Eq. (2.4) above, we decompose the modes as

$$f_{\omega lm}^{\Lambda}(x) = |\omega|^{-1/2} C_{lm}(x) \tilde{f}_{\omega l}^{\Lambda}(x), \quad (2.9)$$

and as the functions $\tilde{g}_{\omega l}^{\Lambda}$, the functions $\tilde{f}_{\omega l}^{\Lambda}$ satisfy Eq. (2.6), but are defined only in region I (see Fig. 1). The two independent sets of modes that correspond to in and up are defined according to the following initial conditions:

$$\tilde{f}_{\omega l}^{\text{in}} = \begin{cases} 0, & H_{\text{past}} \\ e^{-i\omega v}, & \text{PNI} \end{cases}, \quad \tilde{f}_{\omega l}^{\text{up}} = \begin{cases} e^{-i\omega u_{\text{ext}}}, & H_{\text{past}} \\ 0, & \text{PNI} \end{cases}. \quad (2.10)$$

Note that we formally define the modes $f_{\omega lm}^{\Lambda}$ for negative values of ω as well [see Eq. (2.9)], although our final expressions will contain only modes of positive values of ω , i.e. positive frequency modes.

These outer Eddington-Finkelstein modes are especially useful, as they can be decomposed into radial functions which satisfy an *ordinary* differential equation and can therefore easily be computed numerically. The decomposition is as follows:

$$\tilde{f}_{\omega l}^{\text{in}}(r, t) = e^{-i\omega t} \Psi_{\omega l}^{\text{in}}(r), \quad \tilde{f}_{\omega l}^{\text{up}}(r, t) = e^{-i\omega t} \Psi_{\omega l}^{\text{up}}(r). \quad (2.11)$$

Substituting these decompositions into Eq. (2.6) yields the following radial equation for $\Psi_{\omega l}^{\Lambda}$:

$$\Psi_{r_* r_*}^{\Lambda} + [\omega^2 - V_l(r)] \Psi^{\Lambda} = 0, \quad (2.12)$$

where the effective potential V_l is given by Eq. (2.7). In terms of the radial functions $\Psi_{\omega l}^{\Lambda}$, the initial conditions given in Eq. (2.10) are translated to

$$\Psi_{\omega l}^{\text{in}}(r) \cong \begin{cases} \tau_{\omega l}^{\text{in}} e^{-i\omega r_*}, & r_* \rightarrow -\infty \\ e^{-i\omega r_*} + \rho_{\omega l}^{\text{in}} e^{i\omega r_*}, & r_* \rightarrow \infty \end{cases} \quad (2.13)$$

and

$$\Psi_{\omega l}^{\text{up}}(r) \cong \begin{cases} e^{i\omega r_*} + \rho_{\omega l}^{\text{up}} e^{-i\omega r_*}, & r_* \rightarrow -\infty \\ \tau_{\omega l}^{\text{up}} e^{i\omega r_*}, & r_* \rightarrow \infty \end{cases}, \quad (2.14)$$

where $\rho_{\omega l}^{\Lambda}$ and $\tau_{\omega l}^{\Lambda}$ are the reflection and transmission coefficients (corresponding to the mode $\tilde{f}_{\omega l}^{\Lambda}$), respectively. Solving numerically Eq. (2.12) together with the boundary conditions (2.13) and (2.14) yields $\Psi_{\omega l}^{\Lambda}(r)$, which then gives the modes $f_{\omega lm}^{\text{in}}$ and $f_{\omega lm}^{\text{up}}$ using Eqs. (2.11) and (2.9).

In a similar way, we define two sets of *inner Eddington-Finkelstein modes*, which are similarly decomposed according to Eq. (2.9), and the corresponding functions $\tilde{f}_{\omega l}^\Lambda$ again satisfy Eq. (2.6). Here, however, Λ denotes “right” (R) and “left” (L) corresponding to the following initial conditions on the left and right event horizons:

$$\tilde{f}_{\omega l}^L = \begin{cases} e^{-i\omega u_{\text{int}}}, & H_L \\ 0, & H_R \end{cases}, \quad \tilde{f}_{\omega l}^R = \begin{cases} 0, & H_L \\ e^{-i\omega v}, & H_R \end{cases}. \quad (2.15)$$

Note that these modes are defined only in region II (see Fig. 1).

As in the external region, the modes $f_{\omega l}^L$ and $f_{\omega l}^R$ are useful since they can be decomposed into a radial function, which can easily be computed numerically. (Note that in this case we have a single radial function instead of two.) The decomposition is as follows:

$$\tilde{f}_{\omega l}^L(r, t) = \psi_{\omega l}(r) e^{i\omega t}, \quad \tilde{f}_{\omega l}^R(r, t) = \psi_{\omega l}(r) e^{-i\omega t}. \quad (2.16)$$

As before, substituting these decompositions into Eq. (2.6) yields the radial equation (2.12) for $\psi_{\omega l}$. In terms of this radial function, the initial conditions given in Eq. (2.15) reduce to the single condition

$$\psi_{\omega l} \cong e^{-i\omega r_*}, \quad r_* \rightarrow -\infty. \quad (2.17)$$

Then, numerically solving Eq. (2.12) together with the initial condition (2.17) yields $\psi_{\omega l}(r)$, which in turn gives the modes $f_{\omega l}^L$ and $f_{\omega l}^R$ using Eqs. (2.16) and (2.9).

C. Quantum states

One can use the Unruh modes and the outer Eddington-Finkelstein modes to define the Unruh and Boulware states, respectively. In order to define the Unruh state, we decompose the scalar field operator in terms of the Unruh modes $g_{\omega l}^{\text{up}}$ and $g_{\omega l}^{\text{in}}$ as follows:

$$\hat{\Phi}(x) = \int_0^\infty d\omega \sum_{\Lambda, l, m} [g_{\omega l}^\Lambda(x) \hat{a}_{\omega l}^\Lambda + g_{\omega l}^{\Lambda*}(x) \hat{a}_{\omega l}^{\Lambda\dagger}]. \quad (2.18)$$

Then the Unruh state $|0\rangle_U$ is defined by [22]

$$\hat{a}_{\omega l}^\Lambda |0\rangle_U = 0,$$

where $\hat{a}_{\omega l}^\Lambda$ are the annihilation operators appearing in Eq. (2.18). This quantum state describes an evaporation of a BH; i.e. it involves an outgoing flux of radiation at infinity (with no incoming waves at PNI). The vacuum expectation value of the stress-energy tensor in this state is regular at H_R (but not at H_{past}) [23].

Similarly, the Boulware state [24] is defined using the decomposition

$$\hat{\Phi}(x) = \int_0^\infty d\omega \sum_{\Lambda, l, m} [f_{\omega l}^\Lambda(x) \hat{b}_{\omega l}^\Lambda + f_{\omega l}^{\Lambda*}(x) \hat{b}_{\omega l}^{\Lambda\dagger}] \quad (2.19)$$

in terms of the outer Eddington-Finkelstein modes, $f_{\omega l}^{\text{up}}$ and $f_{\omega l}^{\text{in}}$, and the condition

$$\hat{b}_{\omega l}^\Lambda |0\rangle_B = 0,$$

where $\hat{b}_{\omega l}^\Lambda$ are the annihilation operators appearing in Eq. (2.19). The Boulware state matches the usual notion of a vacuum at infinity; i.e. the vacuum expectation value of the stress-energy tensor in this state vanishes at infinity. On the other hand, this vacuum expectation value, evaluated in a freely falling frame, is singular at the past and future event horizons [23]. Note that this quantum state is only defined in region I.

The Hartle-Hawking state [25] corresponds to a thermal bath of radiation at infinity. In this state the Hadamard function and the RSET are regular on both H_{past} and H_R [23]. Formally this quantum state may be defined by an analytic continuation to the Euclidean sector. Here we shall primarily be interested in the mode structure of the Hadamard function in this state, given in the next subsection.

D. The Hadamard function outside the black hole

As shown above in Eq. (2.18), the scalar field operator can be decomposed in terms of the Unruh modes. Substituting this expression into the definition of the Unruh state Hadamard function, using Eq. (1.3), readily yields a mode-sum expression for this function in terms of the Unruh modes. In the exterior region of the BH, these Unruh modes can be reexpressed in terms of the outer Eddington-Finkelstein modes. Then, using this relation, an expression for the Unruh state Hadamard function in terms of the latter modes can be obtained.

This procedure was carried out in [23,26], and we quote here the final result³:

$$G_U^{(1)}(x, x') = \int_0^\infty d\omega \sum_{l, m} \left[\coth\left(\frac{\pi\omega}{\kappa_+}\right) \{f_{\omega l}^{\text{up}}(x), f_{\omega l}^{\text{up}*}(x')\} + \{f_{\omega l}^{\text{in}}(x), f_{\omega l}^{\text{in}*}(x')\} \right], \quad (2.20)$$

where the subscript U stands for “Unruh state,” and the curly brackets denote symmetrization with respect to the arguments x and x' , i.e.

$$\{A(x), B(x')\} = A(x)B(x') + A(x')B(x).$$

A similar procedure can be applied to the Hartle-Hawking Hadamard function, yielding [23,26]

³In [23,26] the results were given specifically for the Schwarzschild case. Nevertheless, the translation of these results from Schwarzschild to RN is straightforward (one only needs to replace Schwarzschild’s surface gravity κ by the corresponding RN parameter κ_+).

$$G_H^{(1)}(x, x') = \int_0^\infty d\omega \sum_{l,m} \coth\left(\frac{\pi\omega}{\kappa_+}\right) [\{f_{\omega lm}^{\text{up}}(x), f_{\omega lm}^{\text{up}*}(x')\} + \{f_{\omega lm}^{\text{in}}(x), f_{\omega lm}^{\text{in}*}(x')\}]. \quad (2.21)$$

This expression may be obtained by replacing the above Unruh modes with corresponding ‘‘Hartle-Hawking modes,’’ which are Kruskal-based at both H_{past} and PNI [more specifically, by changing $e^{-i\omega v} \rightarrow e^{-i\omega V}$ at PNI in Eq. (2.8)].

III. THE UNRUH STATE HADAMARD FUNCTION INSIDE THE BLACK HOLE

As discussed in Sec. IID, outside the BH the Unruh modes, and thereby the corresponding Hadamard function, can be expressed in terms of the outer Eddington-Finkelstein modes. These modes are naturally defined in the exterior region of the BH, and they can be computed numerically by solving the ordinary differential equation (2.12) together with the boundary conditions given by Eqs. (2.13) and (2.14). This way, using the relation between the former and the latter modes, the Unruh state Hadamard function can easily be computed numerically.

In the interior region, the Unruh state Hadamard function has the same expression in terms of the Unruh modes as the one in the exterior, since the Unruh modes are continuously defined throughout both regions I and II of Fig. 1 (as discussed in Sec. IIB). Using the same method as in the exterior region, we can similarly obtain a relation between the Unruh modes and the *inner* Eddington-Finkelstein modes, which serve as the internal analogues of the outer Eddington-Finkelstein modes. These modes can be computed numerically by solving the ordinary differential equation (2.12) in the interior region together with the initial condition given by Eq. (2.17). Again, this will facilitate the computation of the Unruh state Hadamard function in the BH interior.

We begin with the mode-sum expression for the Unruh state Hadamard function in terms of the Unruh modes $g_{\omega lm}^{\text{up}}$ and $g_{\omega lm}^{\text{in}}$,

$$G_U^{(1)}(x, x') = \langle \{\hat{\Phi}(x), \hat{\Phi}(x')\} \rangle_U = \int_0^\infty d\omega \sum_{\Lambda, l, m} \{g_{\omega lm}^\Lambda(x), g_{\omega lm}^{\Lambda*}(x')\}. \quad (3.1)$$

Now, in order to express this function in the interior region in terms of the inner Eddington-Finkelstein modes, we need to express the Unruh modes using the latter. We first find the relation between these two sets of modes on H_R and H_L . For this purpose, we write $\tilde{g}_{\omega l}^{\text{past}}$ in terms of $\tilde{f}_{\omega l}^{\text{up}}$ on H_{past} , using Fourier transform, in the following way:

$$\tilde{g}_{\omega l}^{\text{past}}|_{H_{\text{past}}}(u_{\text{ext}}) = e^{-i\omega U(u_{\text{ext}})} = \frac{1}{2\pi} \int_{-\infty}^\infty \alpha_{\omega \tilde{\omega}}^{\text{past}} e^{-i\tilde{\omega} u_{\text{ext}}} d\tilde{\omega}, \quad (3.2)$$

where $\alpha_{\omega \tilde{\omega}}^{\text{past}}$ are the Fourier coefficients given by the inverse Fourier transform

$$\alpha_{\omega \tilde{\omega}}^{\text{past}} = \int_{-\infty}^\infty e^{-i\omega U(u_{\text{ext}})} e^{i\tilde{\omega} u_{\text{ext}}} du_{\text{ext}}.$$

Substituting Eq. (2.1), defining the variable $s = u_{\text{ext}} - \frac{1}{\kappa_+} \log(\omega/\kappa_+)$ and using the identity [27]

$$\int_{-\infty}^\infty e^{ie^{-\kappa_+ s}} e^{i\tilde{\omega} s} ds = \frac{1}{\kappa_+} e^{\tilde{\omega}\pi/2\kappa_+} \Gamma\left(-i\frac{\tilde{\omega}}{\kappa_+}\right), \quad \text{Im}(\tilde{\omega}) > 0,$$

we get (after assigning $\tilde{\omega}$ a small imaginary part and taking it to vanish in the end)

$$\begin{aligned} \alpha_{\omega \tilde{\omega}}^{\text{past}} &= \int_{-\infty}^\infty e^{i(\omega/\kappa_+)e^{-\kappa_+ u_{\text{ext}}}} e^{i\tilde{\omega} u_{\text{ext}}} du_{\text{ext}} \\ &= \left(\frac{\omega}{\kappa_+}\right)^{i\tilde{\omega}/\kappa_+} \int_{-\infty}^\infty e^{ie^{-\kappa_+ s}} e^{i\tilde{\omega} s} ds \\ &= \frac{1}{\kappa_+} \left(\frac{\omega}{\kappa_+}\right)^{i\tilde{\omega}/\kappa_+} e^{\tilde{\omega}\pi/2\kappa_+} \Gamma\left(-i\frac{\tilde{\omega}}{\kappa_+}\right). \end{aligned} \quad (3.3)$$

Recalling that $\tilde{g}_{\omega l}^{\text{past}}$ vanishes at PNI, each mode $e^{-i\tilde{\omega} u_{\text{ext}}}$ at H_{past} in Eq. (3.2) evolves in time into a mode $\rho_{\omega l}^{\text{up}} e^{-i\tilde{\omega} v}$ on H_R (and an outgoing field $\tau_{\omega l}^{\text{up}} e^{-i\tilde{\omega} u_{\text{ext}}}$ at future null infinity, which will not concern us however). Thus from linearity we get on H_R

$$\begin{aligned} \tilde{g}_{\omega l}^{\text{past}}|_{H_R} &= \frac{1}{2\pi} \int_{-\infty}^\infty \alpha_{\omega \tilde{\omega}}^{\text{past}} \rho_{\omega l}^{\text{up}} e^{-i\tilde{\omega} v} d\tilde{\omega} \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \alpha_{\omega \tilde{\omega}}^{\text{past}} \rho_{\omega l}^{\text{up}} \tilde{f}_{\omega l}^{\text{R}}|_{H_R} d\tilde{\omega}. \end{aligned}$$

Also, recalling that $\tilde{g}_{\omega l}^{\text{past}}$ vanishes on H_L and so does $\tilde{f}_{\omega l}^{\text{R}}$, we get

$$\tilde{g}_{\omega l}^{\text{past}}|_{H_L} = 0 = \frac{1}{2\pi} \int_{-\infty}^\infty \alpha_{\omega \tilde{\omega}}^{\text{past}} \rho_{\omega l}^{\text{up}} \tilde{f}_{\omega l}^{\text{R}}|_{H_L} d\tilde{\omega}.$$

Since the wave equation (2.6) is linear, these relations between $\tilde{g}_{\omega l}^{\text{past}}$ and $\tilde{f}_{\omega l}^{\text{R}}$ will hold not only for points x on H_L and H_R but also for any point x in the interior region of the BH. Therefore, throughout region II we can write

$$\tilde{g}_{\omega l}^{\text{past}}(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \alpha_{\omega \tilde{\omega}}^{\text{past}} \rho_{\omega l}^{\text{up}} \tilde{f}_{\omega l}^{\text{R}}(x) d\tilde{\omega}.$$

In a similar fashion we now express $\tilde{g}_{\omega l}^{\text{L}}$ in terms of $\tilde{f}_{\omega l}^{\text{L}}$ on H_L using Fourier transform in the following way:

$$\begin{aligned}
\tilde{g}_{\omega l}^L|_{H_L}(u_{\text{int}}) &= e^{-i\omega U(u_{\text{int}})} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_{\omega\tilde{\omega}}^L e^{-i\tilde{\omega}u_{\text{int}}} d\tilde{\omega} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_{\omega\tilde{\omega}}^L \tilde{f}_{\omega l}^L|_{H_L} d\tilde{\omega}, \quad (3.4)
\end{aligned}$$

where again $\alpha_{\omega\tilde{\omega}}^L$ are the Fourier coefficients given by the inverse Fourier transform

$$\alpha_{\omega\tilde{\omega}}^L = \int_{-\infty}^{\infty} e^{-i\omega U(u_{\text{in}})} e^{i\tilde{\omega}u_{\text{in}}} du_{\text{in}}.$$

A similar computation to that shown in Eq. (3.3) yields

$$\alpha_{\omega\tilde{\omega}}^L = \frac{1}{\kappa_+} \left(\frac{\omega}{\kappa_+}\right)^{-i\tilde{\omega}/\kappa_+} e^{\tilde{\omega}\pi/2\kappa_+} \Gamma\left(i\frac{\tilde{\omega}}{\kappa_+}\right) = \alpha_{\omega\tilde{\omega}}^{\text{past}*}. \quad (3.5)$$

Using the same reasoning as above (recalling that $\tilde{g}_{\omega l}^L$ and $\tilde{f}_{\omega l}^L$ both vanish at H_R), Eq. (3.4) applies for a general point x in the interior region; therefore

$$\tilde{g}_{\omega l}^L(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha_{\omega\tilde{\omega}}^L \tilde{f}_{\omega l}^L(x) d\tilde{\omega}.$$

We can now write the total up Unruh modes $\tilde{g}_{\omega l}^{\text{up}}$ in the interior of the BH in terms of the inner Eddington-Finkelstein modes as

$$\begin{aligned}
\tilde{g}_{\omega l}^{\text{up}}(x) &= \tilde{g}_{\omega l}^{\text{past}}(x) + \tilde{g}_{\omega l}^L(x) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} [\alpha_{\omega\tilde{\omega}}^{\text{past}} \rho_{\omega l}^{\text{up}} \tilde{f}_{\omega l}^{\text{R}}(x) + \alpha_{\omega\tilde{\omega}}^L \tilde{f}_{\omega l}^L(x)] d\tilde{\omega}. \quad (3.6)
\end{aligned}$$

Using Eqs. (2.9) and (2.4), Eq. (3.6) can readily be written using the full modes as

$$\begin{aligned}
g_{\omega l m}^{\text{up}}(x) &= \frac{|\omega|^{-1/2}}{2\pi} \int_{-\infty}^{\infty} [\alpha_{\omega\tilde{\omega}}^{\text{past}} \rho_{\omega l}^{\text{up}} f_{\omega l m}^{\text{R}}(x) \\
&\quad + \alpha_{\omega\tilde{\omega}}^L f_{\omega l m}^L(x)] |\tilde{\omega}|^{1/2} d\tilde{\omega}. \quad (3.7)
\end{aligned}$$

We are now left with the in Unruh modes $\tilde{g}_{\omega l}^{\text{in}}$. Since the initial condition of these modes at PNI is given by $e^{-i\omega v}$, the values of these modes on H_R are $\tau_{\omega l}^{\text{in}} e^{-i\omega v}$ (and they vanish on H_L). Recalling the initial conditions for $\tilde{f}_{\omega l}^{\text{R}}$, we find that for a general point x in the interior region

$$\tilde{g}_{\omega l}^{\text{in}}(x) = \tau_{\omega l}^{\text{in}} \tilde{f}_{\omega l}^{\text{R}}(x). \quad (3.8)$$

Here as well, we can use Eqs. (2.9) and (2.4) and rewrite Eq. (3.8) as

$$g_{\omega l m}^{\text{in}}(x) = \tau_{\omega l}^{\text{in}} f_{\omega l m}^{\text{R}}(x). \quad (3.9)$$

We are now ready to compute Hadamard's function [Eq. (1.3)] in the interior region of the BH in the Unruh state, using Eqs. (3.1), (3.7), and (3.9). It will be convenient to split the Unruh state Hadamard function into a sum of two terms, one resulting from the contribution of the up modes and the other from that of the in modes,

$$G_U^{(1)}(x, x') = G_U^{(1)\text{up}}(x, x') + G_U^{(1)\text{in}}(x, x').$$

Let us first consider $G_U^{(1)\text{up}}(x, x')$. Writing it as a mode sum, as in Eq. (3.1), we have

$$G_U^{(1)\text{up}}(x, x') = \int_0^{\infty} d\omega \sum_{l,m} \{g_{\omega l m}^{\text{up}}(x), g_{\omega l m}^{\text{up}*}(x')\}.$$

Substituting Eq. (3.7) in the above equation yields

$$G_U^{(1)\text{up}}(x, x') = I_{RR} + I_{LL} + I_{RL} + I_{LR},$$

where

$$\begin{aligned}
I_{RR} &= \frac{1}{4\pi^2} \sum_{l,m} \int_{-\infty}^{\infty} |\tilde{\omega}|^{1/2} d\tilde{\omega} \int_{-\infty}^{\infty} |\tilde{\omega}|^{1/2} d\tilde{\omega} \rho_{\omega l}^{\text{up}} \rho_{\tilde{\omega} l}^{\text{up}*} \{f_{\omega l m}^{\text{R}}(x), f_{\tilde{\omega} l m}^{\text{R}*}(x')\} \int_0^{\infty} \frac{d\omega}{\omega} \alpha_{\omega\tilde{\omega}}^{\text{past}} \alpha_{\omega\tilde{\omega}}^{\text{past}*}, \\
I_{LL} &= \frac{1}{4\pi^2} \sum_{l,m} \int_{-\infty}^{\infty} |\tilde{\omega}|^{1/2} d\tilde{\omega} \int_{-\infty}^{\infty} |\tilde{\omega}|^{1/2} d\tilde{\omega} \{f_{\omega l m}^L(x), f_{\tilde{\omega} l m}^{L*}(x')\} \int_0^{\infty} \frac{d\omega}{\omega} \alpha_{\omega\tilde{\omega}}^L \alpha_{\omega\tilde{\omega}}^{L*}, \\
I_{RL} &= \frac{1}{4\pi^2} \sum_{l,m} \int_{-\infty}^{\infty} |\tilde{\omega}|^{1/2} d\tilde{\omega} \int_{-\infty}^{\infty} |\tilde{\omega}|^{1/2} d\tilde{\omega} \rho_{\omega l}^{\text{up}} \{f_{\omega l m}^{\text{R}}(x), f_{\tilde{\omega} l m}^{L*}(x')\} \int_0^{\infty} \frac{d\omega}{\omega} \alpha_{\omega\tilde{\omega}}^{\text{past}} \alpha_{\omega\tilde{\omega}}^{L*}, \\
I_{LR} &= \frac{1}{4\pi^2} \sum_{l,m} \int_{-\infty}^{\infty} |\tilde{\omega}|^{1/2} d\tilde{\omega} \int_{-\infty}^{\infty} |\tilde{\omega}|^{1/2} d\tilde{\omega} \rho_{\tilde{\omega} l}^{\text{up}*} \{f_{\omega l m}^L(x), f_{\tilde{\omega} l m}^{\text{R}*}(x')\} \int_0^{\infty} \frac{d\omega}{\omega} \alpha_{\omega\tilde{\omega}}^L \alpha_{\omega\tilde{\omega}}^{\text{past}*} = I_{RL}^*.
\end{aligned}$$

Substituting Eqs. (3.3) and (3.5) in the above expressions, and using the results

$$\int_0^{\infty} \frac{d\omega}{\omega} \alpha_{\omega\tilde{\omega}}^{\text{past}} \alpha_{\omega\tilde{\omega}}^{\text{past}*} = \int_0^{\infty} \frac{d\omega}{\omega} \alpha_{\omega\tilde{\omega}}^L \alpha_{\omega\tilde{\omega}}^{L*} = \frac{4\pi^2}{\tilde{\omega}} \frac{1}{1 - e^{-2\pi\tilde{\omega}/\kappa_+}} \delta(\tilde{\omega} - \tilde{\omega})$$

and

$$\begin{aligned} \int_0^\infty \frac{d\omega}{\omega} \alpha_{\omega\tilde{\omega}}^{\text{past}} \alpha_{\omega\tilde{\omega}}^{L*} &= \int_0^\infty \frac{d\omega}{\omega} \alpha_{\omega\tilde{\omega}}^L \alpha_{\omega\tilde{\omega}}^{\text{past}*} \\ &= \frac{2\pi^2}{\tilde{\omega}} \sinh^{-1}\left(\frac{\pi\omega}{\kappa_+}\right) \delta(\tilde{\omega} + \tilde{\omega}), \end{aligned}$$

we get (renaming the integration variable)

$$\begin{aligned} I_{RR} &= \sum_{l,m} \int_{-\infty}^{\infty} d\omega \text{sgn}(\omega) \\ &\quad \times \frac{1}{1 - e^{-2\pi\omega/\kappa_+}} |\rho_{\omega l}^{\text{up}}|^2 \{f_{\omega l m}^R(x), f_{\omega l m}^{R*}(x')\}, \\ I_{LL} &= \sum_{l,m} \int_{-\infty}^{\infty} d\omega \text{sgn}(\omega) \\ &\quad \times \frac{1}{1 - e^{-2\pi\omega/\kappa_+}} \{f_{\omega l m}^L(x), f_{\omega l m}^{L*}(x')\}, \\ I_{RL} &= \frac{1}{2} \sum_{l,m} \int_{-\infty}^{\infty} d\omega \text{sgn}(\omega) \sinh^{-1}\left(\frac{\pi\omega}{\kappa_+}\right) \\ &\quad \times \rho_{\omega l}^{\text{up}} \{f_{\omega l m}^R(x), f_{\omega l m}^{L*}(x')\}, \\ I_{LR} &= I_{RL}^*. \end{aligned}$$

We now split each of these integrals into two integrals, one over the positive values of ω and the other on the negative values. We then use the identity

$$C_{lm}(x) C_{lm}^*(x') = C_{l(-m)}^*(x) C_{l(-m)}(x'),$$

which further implies

$$\sum_{m=-l}^l C_{lm}(x) C_{lm}^*(x') = \sum_{m=-l}^l C_{lm}^*(x) C_{lm}(x'), \quad (3.10)$$

to obtain relations between the different modes, such as

$$\sum_{m=-l}^l f_{(-\omega)lm}^{R*}(x') f_{\omega lm}^L(x) = \sum_{m=-l}^l f_{(-\omega)lm}^{L*}(x) f_{\omega lm}^R(x').$$

Using these relations and (3.10) after summing the four expressions for I_{RR} , I_{LL} , I_{RL} , and I_{LR} , we finally get for the up part of Hadamard's function

$$\begin{aligned} G_U^{(1)\text{up}}(x, x') &= \int_0^\infty d\omega \sum_{l,m} \left[\coth\left(\frac{\pi\omega}{\kappa_+}\right) \{f_{\omega l m}^L(x), f_{\omega l m}^{L*}(x')\} \right. \\ &\quad + |\rho_{\omega l}^{\text{up}}|^2 \{f_{\omega l m}^R(x), f_{\omega l m}^{R*}(x')\} \\ &\quad + 2 \sinh^{-1}\left(\frac{\pi\omega}{\kappa_+}\right) \\ &\quad \left. \times \text{Re}(\rho_{\omega l}^{\text{up}} \{f_{\omega l m}^R(x), f_{(-\omega)lm}^{L*}(x')\}) \right]. \quad (3.11) \end{aligned}$$

Next we consider the much simpler term $G_U^{(1)\text{in}}(x, x')$. Writing it as a mode sum, we have

$$G_U^{(1)\text{in}}(x, x') = \int_0^\infty d\omega \sum_{l,m} \{g_{\omega l m}^{\text{in}}(x), g_{\omega l m}^{\text{in}*}(x')\}.$$

Substituting Eq. (3.9) in the above expression yields

$$G_U^{(1)\text{in}}(x, x') = \int_0^\infty d\omega \sum_{l,m} |\tau_{\omega l}^{\text{in}}|^2 \{f_{\omega l m}^R(x), f_{\omega l m}^{R*}(x')\}. \quad (3.12)$$

Summing Eqs. (3.11) and (3.12) gives the final expression for Hadamard's function in the Unruh state in terms of the inner Eddington-Finkelstein modes in the interior region of the BH. It is given by

$$\begin{aligned} G_U^{(1)}(x, x') &= G_U^{(1)\text{up}}(x, x') + G_U^{(1)\text{in}}(x, x') \\ &= \int_0^\infty d\omega \sum_{l,m} \left[\coth\left(\frac{\pi\omega}{\kappa_+}\right) \{f_{\omega l m}^L(x), f_{\omega l m}^{L*}(x')\} \right. \\ &\quad + \left(\coth\left(\frac{\pi\omega}{\kappa_+}\right) |\rho_{\omega l}^{\text{up}}|^2 + |\tau_{\omega l}^{\text{up}}|^2 \right) \\ &\quad \times \{f_{\omega l m}^R(x), f_{\omega l m}^{R*}(x')\} \\ &\quad \left. + 2 \text{csch}\left(\frac{\pi\omega}{\kappa_+}\right) \text{Re}(\rho_{\omega l}^{\text{up}} \{f_{\omega l m}^R(x), f_{(-\omega)lm}^{L*}(x')\}) \right], \quad (3.13) \end{aligned}$$

where we used the relation

$$|\tau_{\omega l}^{\text{up}}| = |\tau_{\omega l}^{\text{in}}|. \quad (3.14)$$

As discussed above in Sec. II B, the modes $f_{\omega l m}^L$ and $f_{\omega l m}^R$ can be obtained from Eqs. (2.9) and (2.16) by numerically solving the radial equation (2.12) for $\psi_{\omega l}$, and can then be used to construct $G_U^{(1)}$.

IV. THE HARTLE-HAWKING STATE HADAMARD FUNCTION INSIDE THE BLACK HOLE

The Hartle-Hawking state, like the Unruh state, is regular across the event horizon (H_R). In particular, quantities like the Hadamard function and the renormalized stress-energy tensor should be analytic [14] across r_+ . The expression for $G_H^{(1)}(x, x')$ outside the BH is known [see Eq. (2.21)], and in principle all we need is to analytically extend it from $r > r_+$ to $r < r_+$. The main complication is that the functions $f_{\omega l m}^{\text{up}}(x)$ are irregular at the event horizon (their asymptotic behavior is $\propto e^{-i\omega u_{\text{ext}}}$, which oscillates infinite times on approaching the event horizon), making their analytic extension tricky.

To circumvent this difficulty we recall that since $G_U^{(1)}(x, x')$ is regular at the event horizon, too, the difference $G_H^{(1)}(x, x') - G_U^{(1)}(x, x')$ is also analytic at the event horizon. From Eqs. (2.20) and (2.21) we obtain in the exterior region

$$\begin{aligned}
& G_H^{(1)}(x, x') - G_U^{(1)}(x, x') \\
&= \int_0^\infty d\omega \sum_{l,m} \left[\coth\left(\frac{\pi\omega}{\kappa_+}\right) - 1 \right] \{f_{\omega lm}^{\text{in}}(x), f_{\omega lm}^{\text{in}*}(x')\}, \\
& r > r_+.
\end{aligned} \tag{4.1}$$

It only involves the function $f_{\omega lm}^{\text{in}}(x)$, which is regular across the event horizon. This function is the solution of the wave equation (2.3) with boundary conditions $\propto e^{-i\omega v}$ at PNI and zero along $V = 0$ (namely the union of H_{past} and H_L). This in itself guarantees the regularity of this function at $r = r_+$, and also uniquely determines its extension to $r < r_+$. It is convenient to describe this extension in terms of the associated function $\tilde{f}_{\omega lm}^{\text{in}}(x)$ (the two functions are related by the trivial factor $|\omega|^{-1/2} C_{lm}$). The asymptotic behavior of $\tilde{f}_{\omega lm}^{\text{in}}$ is $e^{-i\omega v}$ at PNI and $\tau_{\omega l}^{\text{in}} e^{-i\omega v}$ at H_R (and zero at $V = 0$). From Eq. (2.15) it immediately follows that its extension to $r < r_+$ is just $\tau_{\omega l}^{\text{in}} \tilde{f}_{\omega lm}^{\text{R}}$. Likewise, the extension of $f_{\omega lm}^{\text{in}}$ to $r < r_+$ is simply

$$f_{\omega lm}^{\text{in}}(x) \rightarrow \tau_{\omega l}^{\text{in}} f_{\omega lm}^{\text{R}}(x).$$

Implementing this extension to Eq. (4.1) yields the $H - U$ difference inside the BH,

$$\begin{aligned}
& G_H^{(1)}(x, x') - G_U^{(1)}(x, x') \\
&= \int_0^\infty d\omega \sum_{l,m} |\tau_{\omega l}^{\text{up}}|^2 \left[\coth\left(\frac{\pi\omega}{\kappa_+}\right) - 1 \right] \{f_{\omega lm}^{\text{R}}(x), f_{\omega lm}^{\text{R}*}(x')\}, \\
& r < r_+,
\end{aligned} \tag{4.2}$$

where again we have used Eq. (3.14). Adding it to the Unruh state expression (3.13), we finally obtain the expression for the Hartle-Hawking state Hadamard function inside the BH,

$$\begin{aligned}
G_H^{(1)}(x, x') &= \int_0^\infty d\omega \sum_{l,m} \left[\coth\left(\frac{\pi\omega}{\kappa_+}\right) (\{f_{\omega lm}^{\text{L}}(x), f_{\omega lm}^{\text{L}*}(x')\}) \right. \\
&\quad \left. + \{f_{\omega lm}^{\text{R}}(x), f_{\omega lm}^{\text{R}*}(x')\} \right] \\
&\quad + 2\text{csch}\left(\frac{\pi\omega}{\kappa_+}\right) \text{Re}(\rho_{\omega l}^{\text{up}} \{f_{\omega lm}^{\text{R}}(x), f_{(-\omega)lm}^{\text{L}*}(x')\}), \tag{4.3}
\end{aligned}$$

where we have used the relation

$$|\rho_{\omega l}^{\text{up}}|^2 + |\tau_{\omega l}^{\text{up}}|^2 = 1.$$

Again, as discussed in the previous section for the Unruh state, $G_H^{(1)}$ can be expressed in terms of the radial function $\psi_{\omega l}$, which can be computed numerically from the radial equation (2.12) together with the boundary condition (2.17).

V. TRACE OF THE RSET FOR A MINIMALLY COUPLED SCALAR FIELD

In this section we derive a simple expression for the trace $\langle T_\alpha^\alpha \rangle_{\text{ren}}$ of a minimally coupled massless scalar field. Such an explicit expression turns out to be useful for RSET analysis in curved spacetime (particularly inside BHs).

For a conformally coupled massless scalar field the trace T_α^α of the classical energy-momentum tensor strictly vanishes. As a consequence, at the quantum level the renormalized expectation value $\langle T_\alpha^\alpha \rangle_{\text{ren}}$ becomes a purely local quantity (independent of the quantum state). This is the well-known trace anomaly [28],

$$\langle T_\alpha^\alpha \rangle_{\text{ren}} = T_{\text{anomaly}} \quad (\text{conformal field}), \tag{5.1}$$

where

$$T_{\text{anomaly}} \equiv \frac{1}{2880\pi^2} \left(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - R_{\alpha\beta} R^{\alpha\beta} + \frac{5}{2} R^2 + 6\Box R \right).$$

For a minimally coupled scalar field ϕ (which is not conformally coupled in 4D), this expression no longer holds, because the classical trace T_μ^μ does not vanish. Nevertheless, a simple generalization of Eq. (5.1) still holds as we now show.

In the minimally coupled massless case the classical stress-energy tensor is

$$T_{\mu\nu} = \phi_{;\mu} \phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi^{;\alpha} \phi_{;\alpha}; \tag{5.2}$$

hence

$$T_\alpha^\alpha = -\phi^{;\alpha} \phi_{;\alpha}, \tag{5.3}$$

which is nonvanishing. To this expression we now add the quantity $\Box(\phi^2) \cdot \text{const}$. The field equation $\Box\phi = 0$ implies

$$\Box(\phi^2) = 2\phi^{;\alpha} \phi_{;\alpha}.$$

Correspondingly, we set $\text{const} = 1/2$ and obtain the classical relation

$$T \equiv T_\alpha^\alpha + \frac{1}{2} \Box(\phi^2) = 0.$$

It then follows that in quantum field theory $\langle T \rangle_{\text{ren}}$ must be purely local. Namely,

$$\langle T_\alpha^\alpha \rangle_{\text{ren}} + \frac{1}{2} \langle \hat{\Phi}^2 \rangle_{\text{ren}} = T_{\text{loc}}, \tag{5.4}$$

where T_{loc} is some local geometric quantity.

The explicit form of T_{loc} may be obtained from the local (Hadamard-based) short-distance asymptotic behavior of T_α^α . Such a local analysis was carried out by Brown and

Ottewill [29]; see in particular Eq. (2.24) therein⁴ (setting $m = \xi = 0$). One readily finds that T_{loc} is just the usual trace-anomaly term. Our final result is thus

$$\langle T_{\alpha}^{\alpha} \rangle_{\text{ren}} = T_{\text{anomaly}} - \frac{1}{2} \square \langle \hat{\Phi}^2 \rangle_{\text{ren}} \quad (\text{minimally coupled massless field}). \quad (5.5)$$

The procedure of adding a surface term to the stress-energy tensor (to form a new tensor with a vanishing trace) is basically well known.⁵ Here we used a similar idea to obtain the simple explicit relation (5.5) between the two quantities $\langle T_{\alpha}^{\alpha} \rangle_{\text{ren}}$ and $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ that are routinely computed in semiclassical calculations.

VI. DISCUSSION

In this paper we developed explicit expressions for the Hadamard function on the interior part of a RN black hole, using the internal Eddington-Finkelstein modes, in both the Unruh and the Hartle-Hawking states. Although we do not present numerical results here, this scheme was recently

⁴In this equation, $w_A/(8\pi^2)$ should coincide with $\langle \phi^2 \rangle_{\text{ren}}$. We are grateful to A. Ottewill for pointing this out to us.

⁵An example of such operations with surface terms is given in [30].

used [21] to reproduce the results by Candelas and Jensen [14] for $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ in the interior of a Schwarzschild BH in the Hartle-Hawking state.

This infrastructure is part of an ongoing effort to study the RSET inside a RN black hole, both analytically and numerically, with special emphasis on the asymptotic behavior on approaching the Cauchy horizon.

As a by-product of the above research we encountered the relation (5.5) between the trace of the RSET and the d'Alembertian of $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ for a minimally coupled massless scalar field. The identity reveals another piece of the puzzle and also allows us to check the results obtained for $\langle \hat{\Phi}^2 \rangle_{\text{ren}}$ against those obtained for the RSET.

The foundations laid here, together with the PMR method, allow for studying the quantum effects in the interior of charged BHs. We view this as a first step toward the more important and more ambitious goal of studying quantum effects inside rotating BHs, which are of course much more realistic.

ACKNOWLEDGMENTS

We are grateful to Robert Wald, Paul Anderson, and Adrian Ottewill for helpful discussions. This research was supported by the Asher Fund for Space Research at the Technion.

-
- [1] E. Flanagan and R. Wald, Does backreaction enforce the averaged null energy condition in semiclassical gravity?, *Phys. Rev. D* **54**, 6233 (1996).
 - [2] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
 - [3] S.M. Christensen, Vacuum expectation value of the stress tensor in an arbitrary curved background: The covariant point-separation method, *Phys. Rev. D* **14**, 2490 (1976).
 - [4] P. Candelas and K. W. Howard, Vacuum $\langle \phi^2 \rangle$ in Schwarzschild spacetime, *Phys. Rev. D* **29**, 1618 (1984).
 - [5] K. W. Howard, Vacuum $\langle T_{\mu}^{\nu} \rangle$ in Schwarzschild spacetime, *Phys. Rev. D* **30**, 2532 (1984).
 - [6] P.R. Anderson, A method to compute $\langle \phi^2 \rangle$ in asymptotically flat, static, spherically symmetric spacetimes, *Phys. Rev. D* **41**, 1152 (1990).
 - [7] P.R. Anderson, W. A. Hiscock, and D. A. Samuel, Stress-energy tensor of quantized scalar fields in static spherically symmetric spacetimes, *Phys. Rev. D* **51**, 4337 (1995).
 - [8] A. Levi and A. Ori, Pragmatic mode-sum regularization method for semiclassical black-hole spacetimes, *Phys. Rev. D* **91**, 104028 (2015).
 - [9] A. Levi and A. Ori, Mode-sum regularization of $\langle \phi^2 \rangle$ in the angular-splitting method, *Phys. Rev. D* **94**, 044054 (2016).
 - [10] A. Levi and A. Ori, Versatile method for renormalized stress-energy computation in black-hole spacetimes, *Phys. Rev. Lett.* **117**, 231101 (2016).
 - [11] A. Levi, Renormalized stress-energy tensor for stationary black holes, *Phys. Rev. D* **95**, 025007 (2017).
 - [12] A. Levi (unpublished).
 - [13] A. Levi, E. Eilon, A. Ori, and M. van de Meent, Renormalized Stress-Energy Tensor of an Evaporating Spinning Black Hole, *Phys. Rev. Lett.* **118**, 141102 (2017).
 - [14] P. Candelas and B. P. Jensen, Feynman Green function inside a Schwarzschild black hole, *Phys. Rev. D* **33**, 1596 (1986).
 - [15] W. A. Hiscock, S. L. Larson, and P. R. Anderson, Semiclassical effects in black hole interiors, *Phys. Rev. D* **56**, 3571 (1997).
 - [16] For earlier works related to this question see V. P. Frolov and G. A. Vilkovisky, Spherically symmetric collapse in quantum gravity, *Phys. Lett.* **106B**, 307 (1981); and I. L. Shapiro, Effective action of vacuum: Semiclassical approach, *Classical Quantum Gravity* **25**, 103001 (2008).
 - [17] N. D. Birrell and P. C. W. Davies, On falling through a black hole into another universe, *Nature (London)* **272**, 35 (1978).
 - [18] A. C. Ottewill and E. Winstanley, Renormalized stress tensor in Kerr space-time: General results, *Phys. Rev. D* **62**, 084018 (2000).

- [19] W. A. Hiscock, Quantum-mechanical instability of the Kerr-Newman black-hole interior, *Phys. Rev. D* **21**, 2057 (1980).
- [20] O. Sela and A. Ori (to be published).
- [21] A. Lanir, A. Levi, and A. Ori (to be published).
- [22] W. G. Unruh, Notes on black-hole evaporation, *Phys. Rev. D* **14**, 870 (1976).
- [23] P. Candelas, Vacuum polarization in Schwarzschild space-time, *Phys. Rev. D* **21**, 2185 (1980).
- [24] D. G. Boulware, Quantum field theory in Schwarzschild and Rindler spaces, *Phys. Rev. D* **11**, 1404 (1975).
- [25] J. B. Hartle and S. W. Hawking, Path-integral derivation of black-hole radiance, *Phys. Rev. D* **13**, 2188 (1976).
- [26] S. M. Christensen and S. A. Fulling, Trace anomalies and the Hawking effect, *Phys. Rev. D* **15**, 2088 (1977).
- [27] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 2014), Formula 3.331(1).
- [28] See e.g. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, UK, 1982).
- [29] M. R. Brown and A. C. Ottewill, Photon propagators and the definition and approximation of renormalized stress tensors in curved space-time, *Phys. Rev. D* **34**, 1776 (1986).
- [30] M. Asorey, G. de Berredo-Peixoto, and I. L. Shapiro, Renormalization ambiguities and conformal anomaly in metric-scalar backgrounds, *Phys. Rev. D* **74**, 124011 (2006).