

Stationary black holes with stringy hair

Jens Boos* and Valeri P. Frolov†

Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2G7, Canada

 (Received 16 November 2017; published 19 January 2018)

We discuss properties of black holes which are pierced by special configurations of cosmic strings. For static black holes, we consider radial strings in the limit when the number of strings grows to infinity while the tension of each single string tends to zero. In a properly taken limit, the stress-energy tensor of the string distribution is finite. We call such matter stringy matter. We present a solution of the Einstein equations for an electrically charged static black hole with the stringy matter, with and without a cosmological constant. This solution is a warped product of two metrics. One of them is a deformed 2-sphere, whose Gaussian curvature is determined by the energy density of the stringy matter. We discuss the embedding of a corresponding distorted sphere into a three-dimensional Euclidean space and formulate consistency conditions. We also found a relation between the square of the Weyl tensor invariant of the four-dimensional spacetime of the stringy black holes and the energy density of the stringy matter. In the second part of the paper, we discuss test stationary strings in the Kerr geometry and in its Kerr-NUT-(anti-)de Sitter generalizations. Explicit solutions for strings that are regular at the event horizon are obtained. Using these solutions, the stress-energy tensor of the stringy matter in these geometries is calculated. Extraction of the angular momentum from rotating black holes by such strings is also discussed.

DOI: [10.1103/PhysRevD.97.024024](https://doi.org/10.1103/PhysRevD.97.024024)

I. INTRODUCTION

Cosmic strings are one-dimensional topological defects which might be formed in the symmetry-breaking phase transitions in the early universe [1]. The string's tension μ_s and its width η_s are related to the characteristic energy scale of the corresponding phase transition m via

$$\mu_s \sim \frac{m_{\text{Pl}}}{l_{\text{Pl}}} \left(\frac{m}{m_{\text{Pl}}} \right)^3, \quad \eta_s \sim l_{\text{Pl}} \frac{m}{m_{\text{Pl}}}. \quad (1)$$

Cosmic strings formed in the early universe would distort the cosmological microwave background (CMB). However, the observations indicate that their contribution to CMB cannot be more than 10% [2].

The dynamics of a test string in an external gravitational field $g_{\mu\nu}$ is described by the Nambu-Goto action

$$I = -\mu_s \int d^2\zeta \sqrt{-\det(\gamma_{ab})}, \quad \gamma_{ab} = g_{\mu\nu} \partial_{\zeta^a} x^\mu \partial_{\zeta^b} x^\nu. \quad (2)$$

Here, μ_s is the string tension, and ζ^a ($a = 0, 1$) are coordinates on the string world-sheet. The functions $x^\mu(\zeta^a)$ determine string's embedding in the bulk spacetime. The stress-energy tensor of the string is localized on its surface and is of the form [1]

$$T^{\mu\nu} = \frac{\mu_s}{\sqrt{-g}} \int d^2\zeta \sqrt{-\gamma} \gamma^{ab} x^\mu_{,a} x^\nu_{,b} \delta^{(4)}(x^\sigma - x^\sigma(\zeta^a)). \quad (3)$$

For a straight string in z -direction in a flat spacetime with Cartesian coordinates (t, z, x, y) , it takes the form

$$T_\mu{}^\nu = \text{diag}(-\mu, -\mu, 0, 0), \quad \mu = \mu_s \delta(x) \delta(y). \quad (4)$$

The spacetime is locally flat outside the string, and it has the angle deficit $\nu = 8\pi\mu_s$. Such a space can be obtained by cutting out a wedge of angle ν along the z -axis and gluing together the edges [1,3].

Several interesting effects occur in a situation when a cosmic string, passing near a black hole, is caught by the latter (see, e.g., the book [4] and references therein). The simplest case corresponds to an infinitely long straight string piercing a black hole [5]. Properties of static black holes pierced by a polyhedral set of radial straight strings were discussed in [6,7]. In the paper [8], the authors introduce a notion of a “thorny sphere”, which is everywhere locally isometric to a round two-dimensional sphere except at a finite number of isolated points where it has conical singularities. Using thorny spheres, a general solution for a black hole pierced by an arbitrary number of radial strings was constructed. Such configurations can be used for quantum mining of energy from black holes [9,10]. Such a model was discussed in connection with the information loss paradox [11,12]. The interaction of classical strings with a rotating black hole can also be used

*boos@ualberta.ca

†frolov@ualberta.ca

for extraction of the energy from the latter [13–16]. In some aspects, this process is similar to the Blandford-Znajek mechanism [17,18].

In the present paper, we continue the study of interaction of cosmic strings with black holes. We generalize the results on the thorny black holes to the case when the number of strings attached to the black hole grows to infinity while their tension decreases. In the properly chosen limit (“smearing the string”), such a configuration describes a radial distribution of what is called the stringy matter. We discuss properties of static and stationary black holes with such stringy matter hair.

The paper is organized as follows. After the Introduction, we review geometrical properties of the spacetime for a straight smeared string (Sec. II). In Sec. III, we describe a solution of the Einstein equations for a static charged (anti-) de Sitter [(A)dS] black hole with the stringy matter. The corresponding metric is a warped product of a 2D metric of the (t, r) sector of the unperturbed solution and a 2D metric of the distorted sphere, the Gaussian curvature of which is determined by the stress-energy tensor of the stringy matter. The embedding of a distorted sphere into 3D Euclidean space, consistency conditions, and the relation between its 2D Gaussian curvature and 4D curvature invariants are also discussed in this section. Single strings piercing a Kerr black hole and the stress-energy tensor of corresponding stringy matter are discussed in Sec. IV. Section V contains the generalization of these results to the case of a Kerr-NUT-(A)dS black hole. Section VI contains a brief discussion of the obtained results.

II. GEOMETRY OF A STRAIGHT SMEARED STRING

Since the force between any two straight cosmic strings with arbitrary orientations vanishes, one can consider a static ensemble of such cosmic strings which are in passive equilibrium. In particular, one can choose a set of parallel cosmic strings and take a limit of their continuous distribution. In such a limit, the number of strings grows to infinity, while the tension of each individual string decreases. An action for such stringy matter was discussed in [19,20]. The gravitational field for such one-dimensional straight distribution of the stringy matter can be written as [1]

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + f^2(\rho, \phi)d\phi^2. \quad (5)$$

Substituting the metric (5) in (t, z, ρ, ϕ) coordinates into the Einstein equations

$$G_{\mu}{}^{\nu} \equiv R_{\mu}{}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R = 8\pi T_{\mu}{}^{\nu}, \quad (6)$$

where one finds

$$T_{\mu}{}^{\nu} = \text{diag}(\mu, \mu, 0, 0), \quad \mu = \mu(\rho, \phi) = -\frac{1}{8\pi}f^{-1}\partial_{\rho}^2 f. \quad (7)$$

This is exactly what one would expect for a one-dimensional stringy matter distribution.

We assume that the angle coordinate ϕ has the period 2π . Then the regularity of the metric (5) at $\rho = 0$ implies that $f \sim \rho$ near this point.

In the domain outside the matter, the Gaussian curvature vanishes, and the geometry can be embedded into three-dimensional Euclidean space as a cylinder [21]. The metric in this domain can be written in the form (5) with $f = b\rho$. The parameter b is connected to the angle deficit via $\nu = 2\pi(1 - b)$.

The tension μ of the stringy matter is directly related to a special geometric invariant. Namely, consider the two-dimensional metric

$$ds_{(2)}^2 = d\rho^2 + f^2(\rho, \phi)d\phi^2. \quad (8)$$

The Gaussian curvature K of this 2D metric, which is connected to the 2D Ricci scalar ${}^{(2)}R$ via $K = {}^{(2)}R/2$, is

$$K = \frac{1}{2}{}^{(2)}R = -f^{-1}\partial_{\rho}^2 f. \quad (9)$$

Thus, one has

$$K = 8\pi\mu. \quad (10)$$

III. STATIC BLACK HOLES WITH STRINGY HAIR

A. Geometry

Our starting point for constructing a solution for a static black hole with stringy hair is the following metric:

$$ds_0^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\omega_0^2, \quad (11)$$

$$d\omega_0^2 = d\theta^2 + \sin^2\theta d\phi^2. \quad (12)$$

For

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2, \quad (13)$$

this metric describes a static charged spherically symmetric black hole in an asymptotically A(d)S spacetime; see also [22]. It is a solution of the Einstein-Maxwell equations

$$G_{\mu}{}^{\nu} \equiv R_{\mu}{}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R = 8\pi T_{\mu}{}^{\nu}, \quad (14)$$

$$F^{\mu\nu}{}_{;\nu} = 0, \quad F_{\mu\nu} = 2A_{[\nu, \mu]}, \quad (15)$$

with the potential

$$A_{\mu} = (Q/r)\delta_{\mu}^t. \quad (16)$$

The stress-energy tensor which enters the Einstein equation is

$$T_{\mu\nu} = -\frac{1}{8\pi}\Lambda g_{\mu\nu} + T_{\mu\nu}^{(\text{em})}. \quad (17)$$

Here the first term in the right-hand side is just a cosmological constant while the second term is the stress-energy tensor of the static electric field

$$T_{\mu\nu}^{(\text{em})} = \frac{1}{4\pi} \left(F_{\mu\beta} F_{\nu}^{\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right),$$

$$g^{\rho\sigma} T_{\mu\rho}^{(\text{em})} = \text{diag}(-Y, -Y, Y, Y), \quad Y = \frac{Q^2}{8\pi r^4}. \quad (18)$$

The corresponding black hole solution with the stringy hair is obtained by the following deformation of the metric (11):

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\omega^2, \quad (19)$$

$$d\omega^2 = \exp(2\sigma) d\omega_0^2. \quad (20)$$

The metric is still a warped space where the 2D round sphere is distorted and possesses the metric $d\omega^2$. Let us note that the function $f(r)$ remains the same. The radius r_+ of the black hole horizon, $f(r_+) = 0$, and its surface gravity,

$$\kappa_H = \frac{1}{2} \left. \frac{df}{dr} \right|_H, \quad (21)$$

are the same as that for the undistorted black hole. Introducing the advanced time

$$dv = dt + \frac{dr}{f}, \quad (22)$$

one can check that the metric in these coordinates,

$$ds^2 = -f dv^2 - 2dvdr + r^2 d\omega^2, \quad (23)$$

is regular at the future event horizon.¹ The surface area of the horizon is

¹Let us notice that this metric is a special case of a general class of Robinson-Trautman metrics [23–26]. The metric (23) reproduces the Robinson-Trautman line element (see Chapter 28 of [26])

$$ds^2 = 2r^2 P^{-2} d\zeta d\bar{\zeta} - 2dudr - 2Hdu^2, \quad P_{,r} = 0. \quad (24)$$

after the evident changes $2H \rightarrow f$, $u \rightarrow v$. The angular line element $2P^{-2} d\zeta d\bar{\zeta}$ takes the form $d\omega^2$ after the transformation $\zeta = \sqrt{2} e^{i\phi} \tan \theta/2$. In order to obtain the metric for a stringy black hole, one can start with the ansatz (24) and obtain the metric functions P and H by solving the Einstein equations in presence of the stringy matter.

$$\mathcal{A} = r_+^2 \int d\theta d\phi \exp(2\sigma) \sin \theta. \quad (25)$$

It is easy to check that

- (i) The potential (16) is still a solution of the Maxwell equations in the distorted metric (19);
- (ii) The stress-energy tensor of this Maxwell field has the same form (18);
- (iii) The distorted metric (19) obeys the Einstein equation (14) where the stress-energy tensor (17) is modified by adding a term,

$$T_{\mu}{}^{\nu} = \text{diag} \left(\frac{-\Phi}{8\pi r^2}, \frac{-\Phi}{8\pi r^2}, 0, 0 \right), \quad (26)$$

$$\Phi = \exp(-2\sigma)[1 - \Delta\sigma] - 1. \quad (27)$$

Here, Δ is the Laplace operator on the unit sphere

$$\Delta = \frac{1}{\sin \theta} \partial_{\theta} (\sin \theta \partial_{\theta}) + \frac{1}{\sin^2 \theta} \partial_{\phi}^2. \quad (28)$$

We choose the notation for Φ so that for the matter with positive energy density the function Φ is also positive. We call the matter with the equation of state (26) stringy matter.

The Gaussian curvature of the 2D metric $d\omega^2$ is connected with the 2D Ricci scalar ${}^{(2)}R$ via $K = {}^{(2)}R/2$. Simple calculations give

$$K = \frac{1}{2} {}^{(2)}R = \exp(-2\sigma)(1 - \Delta\sigma). \quad (29)$$

For a unit round sphere, when $\sigma = 0$, one has $K = 1$. Substituting (29) in (27) gives

$$K = 1 + \Phi. \quad (30)$$

B. Embedding and consistency conditions

For positive energy density of the stringy matter, $\Phi \geq 0$, so that $K \geq 1$. If one applies the results to the case of $\Phi < 0$, we assume always that $\Phi > -1$. This means that K is positive, and because the sphere is compact, there exists such a positive number c that $K > c$. This essentially prevents the stringy matter from changing the topology of the distorted 2-sphere. In this case, the distorted sphere can be isometrically embedded into a flat three-dimensional space E^3 as a regular surface M^2 [27].

Let us chose a point p on the deformed sphere M^2 and denote by \vec{n} an outward unit vector normal to M at this point. Let dA be the element of the surface area at p , then we denote

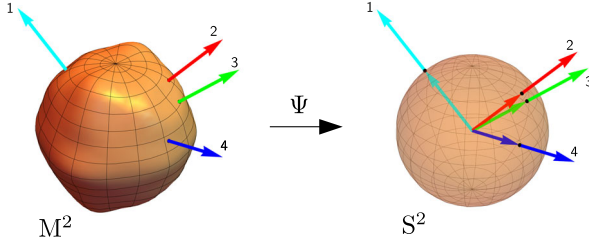


FIG. 1. Illustration of the Gauss map. This map $\Psi: M^2 \rightarrow S^2$ transforms points on the deformed 2-sphere M^2 to points on the unit sphere S^2 .

$$d\vec{A} = \vec{n}dA. \quad (31)$$

The Gauss map Ψ of the convex surface M^2 to the unit round sphere S^2 is defined as follows. It maps a point of M^2 with the normal vector \vec{n} to the point of S^2 with the same normal vector [28,29] (see Fig. 1). Let us denote by $d\alpha$ and $d\vec{\alpha} = \vec{n}d\alpha$ the scalar and vector surface area elements at $\Psi(p)$, respectively. Then one has

$$d\alpha = KdA, \quad d\vec{\alpha} = Kd\vec{A}. \quad (32)$$

The surface areas of M^2 and S^2 are

$$\mathcal{A} = \int_{M^2} dA, \quad \mathcal{S} = \int_{S^2} d\alpha = 4\pi. \quad (33)$$

The second of these relations can be rewritten as

$$\int_{M^2} dAK = 4\pi. \quad (34)$$

This is nothing but the Gauss-Bonnet theorem. Using the relations (30) and (33), one also obtains

$$\mathcal{A} + \int_{M^2} dA\Phi = 4\pi. \quad (35)$$

Hence, for non-negative energy density of the stringy matter, $\Phi \geq 0$, the surface area \mathcal{A} of the distorted sphere is less than or equal to 4π .

Another set of useful relations, called consistency conditions, can be obtained by using the following divergence theorem [30]:

$$\int_V d^3V \vec{\nabla} \varphi = \int_{\partial V} d\vec{A} \varphi. \quad (36)$$

Here, ∂V is the boundary surface restricting three-dimensional volume V , and $d\vec{A}$ is the vector surface area element on this surface. The application of (36) to M^2 and S^2 for a special choice $\varphi = 1$ gives

$$\int_{M^2} d\vec{A} = 0, \quad \int_{S^2} d\vec{\alpha} = 0. \quad (37)$$

The second of these relations can be written in the forms

$$\int_{M^2} dAK\vec{n} = \int_{M^2} dA\Phi\vec{n} = 0. \quad (38)$$

For the black hole with stringy matter, the second relation has a simple interpretation: The total force acting on the black hole, which is induced by stringy matter tension, must vanish [8]. It is this condition that secures our deformed black hole solution to be static.

C. Curvature invariants

In the warped geometry (19), the stringy matter directly affects the geometry of the distorted 2-sphere, and its stress-energy tensor contributes to the Gaussian curvature of the latter via the relation (30). Let us demonstrate that the square of the Weyl tensor, characterizing the four-dimensional spacetime curvature, is also simply related to the energy density of the stringy matter. We write the metric of the stringy black hole in the form

$$ds^2 = r^2 d\hat{s}^2, \quad d\hat{s}^2 = d\hat{\alpha}^2 + d\omega^2, \quad (39)$$

$$d\hat{\alpha}^2 = -fr^{-2}dt^2 + \frac{dr^2}{r^2f}. \quad (40)$$

We denote the quadratic invariants of the Weyl tensor for the metrics ds^2 and $d\hat{s}^2$ by

$$C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \quad \hat{C}^2 = \hat{C}_{\mu\nu\rho\sigma}\hat{C}^{\mu\nu\rho\sigma}, \quad (41)$$

respectively. Under the conformal transformation relating these metrics, $ds^2 = r^2 d\hat{s}^2$, they transform as

$$C = \frac{1}{r^2} \hat{C}. \quad (42)$$

Since the metric $d\hat{s}^2$ is a direct sum of two independent metrics $d\hat{\alpha}^2$ and $d\omega^2$, one has [31]

$$\hat{C} = \frac{2}{\sqrt{3}} ((\hat{\alpha})K + ({}^\omega)K), \quad (43)$$

where $(\hat{\alpha})K$ and $({}^\omega)K$ denote the Gaussian curvatures of the corresponding 2-metrics

$$(\hat{\alpha})K = -\frac{1}{2}r^2 f'' + rf' - f, \quad (44)$$

$$({}^\omega)K = e^{-2\sigma}(1 - \Delta\sigma). \quad (45)$$

Using (30), one obtains

$$C = \frac{2}{\sqrt{3}r^2} ((\hat{\alpha})K + 1 + \Phi). \quad (46)$$

At the event horizon $r = r_+$, this relation takes the form

$$C_H = \frac{2}{\sqrt{3}r_+^2}(B + \Phi), \quad B = 1 + r_+f'_+ - \frac{1}{2}r_+^2f''_+. \quad (47)$$

In the simplest case, when the charge and cosmological constant vanish, $B = 3$.

IV. ROTATING BLACK HOLE WITH STRINGY HAIR

A. Principal Killing string in the Kerr spacetime

Let us consider a stationary string in the Kerr geometry. This metric in the Boyer-Lindquist coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dt d\phi + \frac{A\sin^2\theta}{\Sigma}d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2. \quad (48)$$

Here,

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad (49)$$

$$A = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta.$$

This metric has two Killing vectors,

$$\xi = \partial_t, \quad \eta = \partial_\phi. \quad (50)$$

We denote, as usual, by $r_\pm = M \pm \sqrt{M^2 - a^2}$ the roots of $\Delta = 0$. Then the event horizon, which is a null surface, is located at $r = r_+$. The Killing vector

$$n = \xi + \Omega\eta, \quad \Omega = \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+}, \quad (51)$$

is tangent to null geodesics, which are generators of the horizon.

The stationary string equations are completely integrable in this metric [32,33]. There exists a special interesting solution describing a stationary string in the Kerr geometry, the world-sheet of which is a principal Killing surface. This surface has two tangent vectors. One of them is ξ , whereas the other coincides with the null vector l tangent to a principal null geodesic. In [13], it was proven that the principal Killing surfaces are the only stationary timelike minimal 2-surfaces that (i) cross the static limit surface, where $\xi^2 = 0$, and (ii) are regular in its vicinity. Such a principal Killing surface represents what we call a principal Killing string. Such a string crosses the event horizon, and its representation in Boyer-Lindquist coordinates (for $a < M$) is

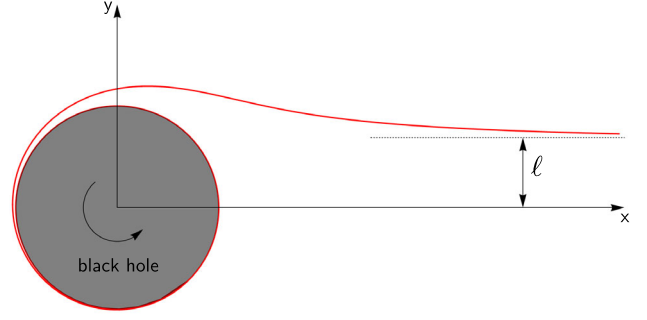


FIG. 2. Representing the string, a line (54) is shown in polar coordinates (ρ, ϕ) in a 2-plane.

$$\phi = \phi_0 + \frac{a}{r_+ - r_-} \ln\left(\frac{r - r_-}{r - r_+}\right), \quad \theta = \theta_0 = \text{const}. \quad (52)$$

The string makes an infinite number of rotations before it reaches the horizon. However, as we shall see in the next section, this is a coordinate effect connected with the choice of the angle variable ϕ in the Boyer-Lindquist coordinates. Let us emphasize that the dependence of the angle ϕ on the radius r is the same for any value θ_0 of the cone solution. Let us introduce dimensionless parameters

$$\rho = r/M, \quad \alpha = a/M, \quad \rho_\pm = 1 \pm \sqrt{1 - \alpha^2}. \quad (53)$$

Then Eq. (52) takes the form

$$\phi = \phi_0 + \frac{\alpha}{2\sqrt{1 - \alpha^2}} \ln\left(\frac{\rho - \rho_-}{\rho - \rho_+}\right). \quad (54)$$

Figure 2 schematically shows a line (54) in polar coordinates in a 2-plane.

B. Near horizon behavior

In order to discuss the properties of the principal Killing string near the horizon, we perform the following coordinate transformation:

$$dv = dt + (r^2 + a^2)\frac{dr}{\Delta}, \quad d\hat{\phi} = d\phi + a\frac{dr}{\Delta}. \quad (55)$$

In these ingoing Kerr coordinates $(v, r, \theta, \hat{\phi})$, which are regular at the future horizon, the Kerr metric takes the form

$$ds^2 = -\frac{\Delta}{\Sigma}(dv - a\sin^2\theta d\hat{\phi})^2 + \frac{\sin^2\theta}{\Sigma}[(r^2 + a^2)d\hat{\phi} - adv]^2 + \Sigma d\theta^2 + 2dr(dv - a\sin^2\theta d\hat{\phi}). \quad (56)$$

In these new coordinates,

$$\sqrt{-g} = \sin\theta\Sigma, \quad (57)$$

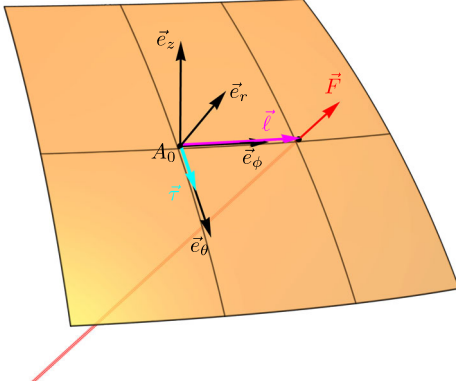


FIG. 3. Illustration of the calculation of the torque. The picture shows a small piece of the sphere of large radius r . At a point A_0 with spherical coordinates (θ_0, ϕ_0) , one introduces a triad of unit orthonormal vectors $\{\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi\}$. Two of these vectors, in the θ and ϕ directions, span a two-dimensional plane tangent to the sphere. Because the sphere's curvature is small, this plane practically coincides with the sphere at A_0 . The string enters the tangent plane orthogonally. Its direction coincides with \vec{e}_r , and its position is shifted from A_0 by the distance \vec{l} in the direction of \vec{e}_ϕ . The unit vector \vec{e}_z is in the direction of the angular momentum of the black hole.

and the Killing vectors (50) have the form

$$\xi = \partial_v, \quad \eta = \partial_{\hat{\phi}}. \quad (58)$$

The null generator of the horizon (51) is

$$n_\mu \stackrel{H}{=} \left(0, \frac{\Sigma_+}{r_+^2 + a^2}, 0, 0\right), \quad (59)$$

where $\Sigma_+ = \Sigma|_{r=r_+}$, and “ $\stackrel{H}{=}$ ” denotes an equality that is valid on the horizon.

Written in the incoming null coordinates $(v, r, \theta, \hat{\phi})$, the principal Killing string equation (52) takes the form

$$\hat{\phi} = \hat{\phi}_0 = \text{const}, \quad \theta = \theta_0 = \text{const}. \quad (60)$$

We use (v, r) as the coordinates on the string surface. Then the induced metric is

$$d\gamma^2 = \frac{\Xi}{\Sigma} dv^2 + 2dvdr, \quad \frac{\Xi}{\Sigma} \equiv \xi^2 = -(\Delta - a^2 \sin^2 \theta). \quad (61)$$

Inside the ergosphere, Ξ is positive, so that the Killing vector ξ , tangent to the string surface, is spacelike. One also has

$$\partial_\gamma^2 = \gamma^{ab} \partial_a \partial_b = 2\partial_v \partial_r - \frac{\Xi}{\Sigma} \partial_r^2, \quad \sqrt{-\gamma} = 1. \quad (62)$$

Using (3), one obtains

$$T_s^{\mu\nu} = q\tau^{\mu\nu}, \quad \tau^{\mu\nu} = \frac{1}{\Sigma} \left(-2\delta_v^{(\mu} \delta_r^{\nu)} + \frac{\Xi}{\Sigma} \delta_r^\mu \delta_r^\nu \right), \quad (63)$$

$$q = q(\theta, \hat{\phi} | \theta_0, \hat{\phi}_0) = \mu_s \frac{\delta(\theta - \theta_0) \delta(\hat{\phi} - \hat{\phi}_0)}{\sin \theta}. \quad (64)$$

We include the subscript “s” in order to indicate that this is the expression valid for a single string. The horizon surface element is

$$d\sigma_\mu \stackrel{H}{=} -\sin \theta \Sigma_+ \delta_\mu^r dv d\theta d\hat{\phi} \stackrel{H}{=} -(r_+^2 + a^2) n_\mu \sin \theta dv d\theta d\hat{\phi}. \quad (65)$$

Thus,

$$T_s^{\mu\nu} d\sigma_\nu \stackrel{H}{=} j^\mu \delta(\theta - \theta_0) \delta(\hat{\phi} - \hat{\phi}_0) dv d\theta d\hat{\phi}, \quad (66)$$

$$j^\mu \stackrel{H}{=} \mu_s \left(-\delta_v^\mu + \frac{a^2 \sin^2 \theta_0}{\Sigma_H} \delta_r^\mu \right). \quad (67)$$

It is easy to show that

$$j^\mu \xi_\mu \stackrel{H}{=} 0, \quad j^\mu \eta_\mu \stackrel{H}{=} -\mu_s a \sin^2 \theta_0. \quad (68)$$

To obtain the fluxes of some observable through the horizon, one has to project $T^{\mu\nu} d\sigma_\nu$ on the corresponding generator and integrate the obtained scalar over v , θ , and $\hat{\phi}$. Since the integrand does not depend on v , the integral over this variable gives the constant Δv , which is the duration of time for which the flux is calculated. To obtain the flux per a unit of time v , one hence needs to divide the flux by Δv , resulting in the following flux rates of energy, \dot{E} , and the angular momentum, \dot{J} :

$$\dot{E} \stackrel{H}{=} j^\mu \xi_\mu = 0, \quad \dot{J} \stackrel{H}{=} j^\mu \eta_\mu = -\mu_s a \sin^2 \theta_0. \quad (69)$$

Here, \dot{J} takes its maximal (negative) value when $\theta_0 = \pi/2$, that is, when the string lies in the equatorial plane. Since $a = J/M$, one obtains the following equation for the dynamics of the angular momentum of the black hole:

$$\dot{J} = -\frac{\mu_s}{M} J. \quad (70)$$

Its solution is

$$J = J_0 \exp(-t/t_\mu), \quad (71)$$

where $t_\mu = M/\mu_s$ is the characteristic time of the deceleration of black hole rotation. We can rewrite this as

$$t_\mu(M) \sim \left(\frac{M}{m_{\text{Pl}}}\right) \left(\frac{m_{\text{Pl}}}{m}\right)^3 t_{\text{Pl}}. \quad (72)$$

Clearly, for the electroweak scale, this time scale is much larger than the age of the Universe (for both stellar mass black holes and supermassive black holes). For the Planck scale, one has $t_\mu(M_\odot) \sim \mu\text{s}$ and $t_\mu(10^6 M_\odot) \sim \text{s}$.

C. Asymptotics at spatial infinity

Let us now discuss the string properties at far distances from the black hole. In this asymptotically flat domain, the Boyer-Lindquist coordinates reduce to the standard spherical coordinates. Consider a sphere with radius r and a point A_0 with spherical coordinates (θ_0, ϕ_0) on this sphere. These coordinates characterize the chosen string, which at large r has the following asymptotic form:

$$\theta = \theta_0, \quad \phi = \phi_0 + \frac{a}{r} + \dots \quad (73)$$

To leading order, the stress-energy tensor is

$$T_s^{\mu\nu} = -\frac{q}{r^2} \left(\delta_t^\mu \delta_t^\nu + \delta_r^\mu \delta_r^\nu - \frac{2a}{r^2} \delta_r^\mu \delta_\phi^\nu + \dots \right). \quad (74)$$

The radially inward pointing surface element is $d\sigma_\mu = -\delta_\mu^r r^2 \sin\theta d\theta d\phi$, such that one obtains for the flux vector in the limit $r \rightarrow \infty$ the following expression:

$$j^\mu = \int_{S^2} T_s^{\mu\nu} d\sigma_\nu = \mu_s \left(\delta_r^\mu - \frac{a}{r^2} \delta_\phi^\mu \right). \quad (75)$$

This results in a change of energy E and angular momentum J according to

$$\dot{E} = j^\mu \xi_\mu = 0, \quad \dot{J} = j^\mu \eta_\mu = -\mu_s a \sin^2 \theta_0. \quad (76)$$

Let us now interpret these results. We introduce three mutually orthonormal vectors $\{\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi\}$ directed along r , θ , and ϕ coordinate lines, respectively. For this choice, the triad of the vectors has the right-hand orientation. The displacement $\vec{\ell}$ of the string position from the origin of the frame is $a \sin \theta_0$ in the positive ϕ direction,

$$\vec{\ell} = +a \sin \theta_0 \vec{e}_\phi. \quad (77)$$

In order to keep the string in equilibrium, one needs to apply a force along the string's spatial tangent vector, which asymptotically takes the form

$$\vec{F} = \mu_s \vec{e}_r. \quad (78)$$

In the frame at A_0 , this force would provide the torque

$$\vec{\tau} = \vec{\ell} \times \vec{F} = \mu_s a \sin \theta_0 \vec{e}_\theta. \quad (79)$$

A unit vector \vec{e}_z , which is parallel to the direction of the angular momentum of the black hole, is

$$\vec{e}_z = \cos \theta_0 \vec{e}_r - \sin \theta_0 \vec{e}_\theta. \quad (80)$$

Thus, the projection of the torque on the direction of the angular momentum of the black hole is

$$\tau_z = -\mu_s a \sin^2 \theta_0. \quad (81)$$

This torque decreases the angular momentum of the black hole, and the rate of this process is in agreement with the result (69).² For a visualization of this calculation, see Fig. 3.

D. Rotating black holes with stringy hair

It is rather straightforward to smear the string and consider a continuous distribution of the stringy matter around a rotating black hole. It is sufficient to use the approach developed in [19,20]. In our case, the description of a continuous ensemble of strings is greatly simplified because, in the incoming coordinates $(v, r, \theta, \hat{\phi})$, the string looks like a straight object with θ and $\hat{\phi}$ constant on its world-sheet. Using this property, we define the stress-energy tensor of the string distribution as follows:

$$T^{\mu\nu} = \int d\theta_0 d\hat{\phi}_0 \sin \theta_0 \mu_s^{-1} \mu(\theta_0, \hat{\phi}_0) T_s^{\mu\nu}, \quad (82)$$

where the function $\mu(\theta, \hat{\phi})$ is the density distribution of the stringy matter. The result of the averaging (82) is

$$T^{\mu\nu} = \mu \tau^{\mu\nu}. \quad (83)$$

The flux of the energy through the horizon vanishes, while the flux of angular momentum per unit time is

$$J = -a \int d\hat{\phi} d\theta \sin^2 \theta \mu(\theta, \hat{\phi}). \quad (84)$$

V. PRINCIPAL KILLING STRINGS IN KERR-NUT-(A)DS SPACETIME

A. Principal Killing strings

We consider now a generalization of a stationary single string solution, which we discussed in the previous section. Namely, instead of the Kerr metric, we consider its generalization. For this purpose, we first rewrite the Kerr

²The force of the string on the black hole would also result in the motion of the black hole as a whole in the external space. In order to compensate this effect, it is sufficient to attach an additional ("dual") string with parameters $\theta'_0 = \pi - \theta_0$ and $\phi'_0 = \pi + \phi_0$. The action of such a dual string compensates the force of the original one, whereas the loss of angular momentum would be doubled.

metric (48) in new coordinates

$$\tau = t - a\phi, \quad y = a \cos \theta, \quad \psi = \phi/a. \quad (85)$$

Its form is

$$ds^2 = -\frac{\Delta_r}{\Sigma}(d\tau + y^2 d\psi)^2 + \frac{\Delta_y}{\Sigma}(d\tau - r^2 d\psi)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_y} dy^2. \quad (86)$$

In the above, $\Sigma = r^2 + y^2$. For the Kerr metric, one has $\Delta_r = r^2 - 2Mr + a^2$ and $\Delta_y = a^2 - y^2$.

In what follows, we consider what is called off-shell version of the canonical metric (86). Namely, we assume that Δ_r and Δ_y are arbitrary functions of their arguments, r and y , respectively. In particular, this means that in the general case, the metric does not obey the vacuum Einstein equations. The properties of such metrics are discussed in [34,35]. For the special choice,

$$\Delta_r = (r^2 + a^2)(1 - \Lambda r^2/3) - 2Mr, \quad (87)$$

$$\Delta_y = (a^2 - y^2)(1 + \Lambda y^2/3) + 2Ny, \quad (88)$$

the metric (86) describes the Kerr-NUT-(A)dS black hole, with N being the NUT parameter.

In what follows, we shall use the following results:

- (i) The metric (86) possesses the principal tensor \mathbf{h} , which is generated by the potential \mathbf{b} :

$$\mathbf{h} = d\mathbf{b}, \quad \mathbf{b} = -\frac{1}{2}[(r^2 - y^2)d\tau + r^2 y^2 d\psi]; \quad (89)$$

- (ii) The Killing vector $\xi = \partial_\tau$ is related to \mathbf{h} as follows:

$$\xi^\mu = \frac{1}{3}\nabla_\nu h^{\nu\mu}; \quad (90)$$

- (iii) The principal tensor \mathbf{h} has four eigenvectors:

$$h^\mu{}_\nu l^\nu_\pm = \mp r l^\mu_\pm, \quad (91)$$

$$h^\mu{}_\nu m^\nu_\pm = \pm i y m^\mu_\pm. \quad (92)$$

These eigenvectors can be written as follows:

$$l^\mu_\pm = \left(\frac{r^2}{\Delta_r}, \pm 1, 0, \frac{1}{\Delta_r} \right), \quad (93)$$

$$m^\mu_\pm = \left(\frac{i y^2}{\Delta_y}, 0, \pm 1, -\frac{i}{\Delta_y} \right). \quad (94)$$

- (iv) The null vectors l_\pm are generators of principal null geodesics in the affine parametrization,

$$l^\nu_\pm l^\mu_{\pm;\nu} = 0. \quad (95)$$

- (v) The principal null vectors l_\pm are also eigenvectors of the 2-form $F_{\mu\nu} = \xi_{\mu;\nu}$ constructed from the primary Killing vector ξ :

$$F^\mu{}_\nu l^\nu_\pm = \pm \kappa l^\mu_\pm, \quad \kappa = \partial_r \left(\frac{\Delta_r - \Delta_y}{2\Sigma} \right). \quad (96)$$

- (vi) Let us denote the Lie derivative along ξ by \mathcal{L}_ξ . Then one has

$$\mathcal{L}_\xi l_\pm = [\xi, l_\pm] = 0, \quad \mathcal{L}_\xi \mathbf{h} = 0. \quad (97)$$

These relations can be easily checked by using computer programs, e.g., GRTENSOR.

The stationary string equations in four- and higher-dimensional Kerr-NUT-(A)dS spacetime allow a complete separation of variables [33,36]. Here, we consider a special case of a stationary string which regularly crosses the event horizon. We call this solution of the string equation a principal Killing string.

To construct this solution, we choose one of two null principal vector fields and denote it by l (without a subscript \pm)³ Since two vectors ξ and l commute, according to the Frobenius theorem, the spacetime is foliated by two-dimensional surfaces Σ , such that both of these vectors are tangent to it. There also exist coordinates $x^\mu = (v, \lambda, y^i)$, ($i = 2, 3$), such that for each Σ one has $y^i = \text{const}$ and $z^a = (v, \lambda)$, ($a = 0, 1$), which are coordinates on Σ such that

$$\xi = \partial_v, \quad l = \partial_\lambda. \quad (98)$$

In the coordinates $x^\mu = (z^a, y^i)$, one has

$$ds^2 = d\gamma^2 + b_{ij} dy^i dy^j, \quad (99)$$

where $d\gamma^2$ is the induced geometry on Σ ,

$$d\gamma^2 = \gamma_{ab} dz^a dz^b = \xi^2 dv^2 + 2(\xi, l) dv d\lambda. \quad (100)$$

Denote by $n^\mu_{(i)}$ two mutually orthogonal unit normal vectors to Σ . An extrinsic curvature of Σ is

$$\Omega_{(i)ab} = g_{\mu\nu} n^\mu_{(i)} x^{\rho;a} \nabla_\rho x^\nu_{;b}. \quad (101)$$

³In what follows, in our construction of the stationary string solution, we choose $l = l_-$, which is regular at the future event horizon.

The surface Σ is minimal if the following conditions are valid:

$$\Omega_{(i)} \equiv \gamma^{ab} \Omega_{(i)ab} = g_{\mu\nu} n_{(i)}^\mu Z^\nu = 0, \quad (102)$$

$$Z^\nu = \gamma^{ab} x_{,a}^\rho \nabla_\rho x_{,b}^\nu. \quad (103)$$

Using the relations

$$\partial_\gamma^2 = \frac{2}{(\xi, l)} \partial_v \partial_\lambda - \frac{\xi^2}{(\xi, l)^2} \partial_\lambda^2, \quad x_{,v}^\mu = \xi^\mu, \quad x_{,\lambda}^\mu = l^\mu, \quad (104)$$

one gets

$$Z^\nu = \frac{1}{(\xi, l)} (\xi^\rho \nabla_\rho l^\nu + l^\rho \nabla_\rho \xi^\nu) - \frac{\xi^2}{(\xi, l)^2} l^\rho \nabla_\rho l^\nu. \quad (105)$$

The properties of l and ξ , mentioned above imply that $Z^\nu \sim l^\nu$. Hence, the 2-surface generated by these vectors is minimal. We call it principal Killing surface. Since one of the tangent vectors, l , is null, the minimal surface Σ is timelike and represents a special time-independent solution of the Nambu-Goto equations. We call such strings principal Killing strings.

B. Principal Killing strings in the incoming null coordinates

Let us change coordinates in (86) to

$$d\tau = dv - \frac{r^2}{\Delta_r} dr - ad\hat{\phi}, \quad d\psi = a^{-1} d\hat{\phi} - \frac{dr}{\Delta_r}. \quad (106)$$

The off-shell canonical metric in these new coordinates $(v, r, y, \hat{\phi})$ is then given by

$$ds^2 = -\frac{\Delta_r}{\Sigma} \left(dv - \frac{a^2 - y^2}{a} d\hat{\phi} \right)^2 + \frac{\Delta_y}{\Sigma} \left(dv - \frac{r^2 + a^2}{a} d\hat{\phi} \right)^2 + 2 \left(dv - \frac{a^2 - y^2}{a} d\hat{\phi} \right) dr + \frac{\Sigma}{\Delta_y} dy^2, \quad (107)$$

such that $\sqrt{-g} = \Sigma/a$. The incoming principal null vector l , see Eq. (93), takes the form $l = -\partial_r$. Hence, one can identify the affine parameter λ with the coordinate r . One also has

$$\xi^2 = \frac{\Xi}{\Sigma}, \quad (\xi, l) = -1, \quad (108)$$

$$\Xi = \Delta_y - \Delta_r, \quad (109)$$

so that the induced metric on the surface of the principal Killing string in the metric (107) is

$$d\gamma^2 = \frac{\Xi}{\Sigma} dv^2 + 2dvdr. \quad (110)$$

The string in the incoming coordinates is ‘‘straightened’’, so that θ and $\hat{\phi}$ are constant on its surface. For this reason, the calculation of the stress-energy tensor for such a string is straightforward and can be simply obtained by repeating the calculations for the Kerr metric:

$$T_s^{\mu\nu} = q\tau^{\mu\nu}, \quad \tau^{\mu\nu} = \frac{1}{\Sigma} \left(2\delta_v^{(\mu} \delta_r^{\nu)} - \frac{\Xi}{\Sigma} \delta_r^\mu \delta_r^\nu \right), \quad (111)$$

$$q = q(y, \hat{\phi}|_{y_0, \hat{\phi}_0}) = \mu_s a \delta(y - y_0) \delta(\hat{\phi} - \hat{\phi}_0). \quad (112)$$

Due to $dy = -\sin\theta d\theta$, the horizon surface element is

$$d\sigma_\mu^H = \frac{\Sigma_\pm}{a} \delta_r^\mu dv dy d\hat{\phi}. \quad (113)$$

The flux vector is hence given by

$$j^\mu = \mu_s \left(-\delta_v^\mu + \frac{\Delta_y}{\Sigma_H} \delta_r^\mu \right). \quad (114)$$

The fluxes of the energy and angular momentum through the horizon are

$$\dot{E} \stackrel{H}{=} \xi_\mu j^\mu = 0, \quad J \stackrel{H}{=} \eta_\mu j^\mu = -\frac{\mu_s}{a} \Delta_{y_0}. \quad (115)$$

In general, the off-shell metric (107) is not asymptotically flat, which is why a discussion of asymptotic properties is not well-defined. We omit this here.

However, by repeating the arguments presented above, we can define smeared principal Killing string matter:

$$T^{\mu\nu} = \int dy_0 d\hat{\phi}_0 \mu_s^{-1} \mu(y_0, \hat{\phi}_0) T_s^{\mu\nu}, \quad (116)$$

such that the flux of angular momentum turns out to be

$$j = -\frac{1}{a} \int dy d\hat{\phi} \Delta_y \mu(y, \psi). \quad (117)$$

VI. DISCUSSION

Strings that pierce a black hole can be used for effective quantum energy mining from them. When the number of strings becomes large, one can approximate their distribution by stringy matter. One might say that such black holes have stringy hair. For static black holes, the stringy matter deforms the black hole geometry, described by a warped metric. This metric is a direct sum of the (t, r) sector of the original solution of the corresponding Einstein equations, whereas the warped part is a deformed unit 2-sphere. The Gaussian curvature $K = 1 + \Phi$ differs from the Gaussian

curvature of a round unit sphere by a term Φ proportional to the energy density of the stringy matter.

The case when the black hole is rotating is much more complicated. The reason is that if a single string is attached to the black hole, it produces a torque that permanently decreases the angular momentum of the black hole. Correspondingly, the stringy matter has a similar effect, and we calculated the rate of the loss of the angular momentum of the black hole for both cases. Moreover, we obtained a new solution for a stationary string in the spacetime of the Kerr-NUT-(A)dS black hole which we call a principal Killing string. It has the property that both the primary Killing vector and the tangent vector to the principal null rays are tangent to it. Also, this string configuration is regular in the vicinity of the event horizon. We calculated the flux of the angular momentum from such a black hole in the cases of a single string and stringy

matter. Unfortunately, the backreaction problem is rather difficult in this case: this is caused by the time dependency of the geometry, including the backreaction effects. The study of the evolution of rotating black holes interacting with cosmic strings is a quite interesting problem.

ACKNOWLEDGMENTS

We thank our anonymous referee for pointing out the relation of Eq. (23) to the class of Robinson-Trautman spacetimes. J. B. is grateful for a Vanier Canada Graduate Scholarship administered by the Natural Sciences and Engineering Research Council of Canada as well as for the Golden Bell Jar Graduate Scholarship in Physics by the University of Alberta. V. F. thanks the Natural Sciences and Engineering Research Council of Canada and the Killam Trust for financial support.

-
- [1] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2000).
 - [2] P. A. R. Ade *et al.* (Planck Collaboration), Planck 2013 results. XXV. Searches for cosmic strings and other topological defects, *Astron. Astrophys.* **571**, A25 (2014).
 - [3] R. A. Puntigam and H. H. Soleng, Volterra distortions, spinning strings, and cosmic defects, *Classical Quantum Gravity* **14**, 1129 (1997).
 - [4] M. R. Anderson, *The Mathematical Theory of Cosmic Strings: Cosmic Strings in the Wire Approximation*, Series in High Energy Physics, Cosmology and Gravitation (CRC Press, Boca Raton, FL, 2015).
 - [5] M. Aryal, L. H. Ford, and A. Vilenkin, Cosmic strings and black holes, *Phys. Rev. D* **34**, 2263 (1986).
 - [6] J. S. Dowker and P. Chang, Polyhedral cosmic strings, *Phys. Rev. D* **46**, 3458 (1992).
 - [7] V. P. Frolov and D. V. Fursaev, Black holes with polyhedral multistring configurations, *Classical Quantum Gravity* **18**, 1535 (2001).
 - [8] V. P. Frolov, D. V. Fursaev, and D. N. Page, Thorny spheres and black holes with strings, *Phys. Rev. D* **65**, 104029 (2002).
 - [9] A. E. Lawrence and E. J. Martinec, Black hole evaporation along macroscopic strings, *Phys. Rev. D* **50**, 2680 (1994).
 - [10] V. P. Frolov and D. V. Fursaev, Mining energy from a black hole by strings, *Phys. Rev. D* **63**, 124010 (2001).
 - [11] S. B. Giddings, Possible observational windows for quantum effects from black holes, *Phys. Rev. D* **90**, 124033 (2014).
 - [12] S. B. Giddings, Nonviolent unitarization: Basic postulates to soft quantum structure of black holes, *J. High Energy Phys.* **12** (2017) 047.
 - [13] V. P. Frolov, S. Hendy, and A. L. Larsen, How to create a two-dimensional black hole, *Phys. Rev. D* **54**, 5093 (1996).
 - [14] V. P. Frolov, D. Hendy, and J. P. De Villiers, Rigidly rotating strings in stationary axisymmetric spacetimes, *Classical Quantum Gravity* **14**, 1099 (1997).
 - [15] V. S. Semenov, String mechanism for energy extraction from a Kerr black hole, *Phys. Scr.* **62**, 123 (2000).
 - [16] S. Kinoshita, T. Igata, and K. Tanabe, Energy extraction from Kerr black holes by rigidly rotating strings, *Phys. Rev. D* **94**, 124039 (2016).
 - [17] V. S. Semenov, S. A. Dyadchkin, I. B. Ivanov, and H. K. Biernat, Energy confinement for a relativistic magnetic flux tube in the ergosphere of a Kerr black hole, *Phys. Scr.* **65**, 13 (2002).
 - [18] S. Kinoshita and T. Igata, The essence of the Blandford-Znajek process, [arXiv:1710.09152](https://arxiv.org/abs/1710.09152).
 - [19] M. G. Ivanov, Black holes with complex multistring configurations, *Gravitation Cosmol.* **8**, 171 (2002).
 - [20] M. G. Ivanov, Models of delocalized membranes, *Gravitation Cosmol.* **9**, 45 (2003).
 - [21] W. S. Massey, Surfaces of Gaussian curvature zero in Euclidean 3-space, *Tohoku Math. J.* **14**, 73 (1962).
 - [22] T. Moskalets and A. Nurmagambetov, Liouville mode in gauge/gravity duality, *Eur. Phys. J. C* **75**, 551 (2015).
 - [23] I. Robinson and A. Trautman, Spherical Gravitational Waves, *Phys. Rev. Lett.* **4**, 431 (1960).
 - [24] I. Robinson and A. Trautman, Some spherical gravitational waves in general relativity, *Proc. R. Soc. A* **265**, 463 (1962).
 - [25] J. Bicak and J. Podolsky, Global structure of Robinson-Trautman radiative space-times with a cosmological constant, *Phys. Rev. D* **55**, 1985 (1997).
 - [26] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's*

- Field Equations* (Cambridge University Press, Cambridge, England, 2003), 2nd edition.
- [27] A. V. Pogorelov, *Extrinsic Geometry of Convex Surfaces*, Translations of mathematical monographs (American Mathematical Society, Providence, 1973).
- [28] R. G. Burns, B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry—Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields*, Graduate Texts in Mathematics (Springer, New York, 1991).
- [29] T. Banchoff, T. Gaffney, and C. McCrory, *Cusps of Gauss mappings*, Research Notes in Mathematics (Pitman, New York, 1982).
- [30] K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering: A Comprehensive Guide* (Cambridge University Press, Cambridge, England, 2006).
- [31] F. A. Ficken, The Riemannian and affine differential geometry of product-spaces, *Ann. Math.* **40**, 892 (1939).
- [32] V. P. Frolov, V. Skarzhinsky, A. Zelnikov, and O. Heinrich, Equilibrium configurations of a cosmic string near a rotating black hole, *Phys. Lett. B* **224**, 255 (1989).
- [33] B. Carter and V. P. Frolov, Separability of string equilibrium equations in a generalized Kerr-de Sitter background, *Classical Quantum Gravity* **6**, 569 (1989).
- [34] V. P. Frolov and A. Zelnikov, *Introduction to Black Hole Physics* (OUP, Oxford, 2011).
- [35] V. P. Frolov, P. Krtous, and D. Kubiznak, Black holes, hidden symmetries, and complete integrability, *Living Rev. Relativity* **20**, 6 (2017).
- [36] D. Kubiznak and V. P. Frolov, Stationary strings and branes in the higher-dimensional Kerr-NUT-(A)dS spacetimes, *J. High Energy Phys.* **02** (2008) 007.