Y-junction intercommutations of current carrying strings

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Under certain conditions the collision and intercommutation of two cosmic strings can result in the formation of a third string, with the three strings then remaining connected at Y-junctions. The kinematics and dynamics of collisions of this type have been the subject of analytical and numerical analyses in the special case in which the strings are Nambu-Goto. Cosmic strings, however, may well carry currents, in which case their dynamics is not given by the Nambu-Goto action. Our aim is to extend the kinematic analysis to more general kinds of string model. We focus in particular on the collision of strings described by conservative elastic string models, characteristic of current carrying strings, and which are expected to form in a cosmological context. As opposed to Nambu-Goto strings collisions, we show that in this case the collision cannot lead to the formation of a third elastic string: if dynamically such a string forms then the joining string must be described by a more general equation of state. This process will be studied numerically in a forthcoming publication.

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I. INTRODUCTION

Strings with Y-shaped junctions occur in many models, and have been the subject of numerous studies, from QCD [1], to cosmic strings [2–9], and cosmic superstrings [10–13]. In a cosmological context, the presence of Y-junctions on a string network can give rise to many interesting effects: for instance, the lensing pattern of a distant light source background sources can pick up a distinctive triplified aspect [14,15]; when kinks (discontinuities on the tangent vector of a string) travel through a junction they multiply (in number) thus potentially sourcing more gravitational waves [16,17]; and also networks of cosmic superstrings lead to novel effects on the CMB temperature and polarization spectra [18,19].

Detailed analytic studies [3,4] of the collisions of strings and the subsequent formation of Y-junctions have, so far, focused on Nambu-Goto (NG) strings, which are idealized strings that are infinitely thin and carry no internal structure.[1] In this paper we consider the more general case of elastic string models, which provide a macroscopic description of different kinds of current-carrying and superconducting strings such as those originally proposed by Witten in [20]. Crucially, in a cosmological context, current carrying strings are expected to be formed in many supersymmetric models of inflation [21]. Thus while it seems probable that cosmic strings carry currents, the collision of such strings has not been fully investigated. That is the aim of this paper.

There are important differences between NG and current carrying strings, the latter of which carry internal degrees of freedom. From a technical point of view, in particular, the world-sheet gauge choices often made to study NG strings (the conformal and temporal gauges) do not apply to general elastic string models. Furthermore, the equations of motion of these strings are generally not integrable (the exception being the chiral case, in which the world sheet current is null [22]). The above-mentioned gauge choices were made in previous works on NG strings [3,4], thus rendering them inapplicable to the study of elastic string collisions. For this reason, here we develop a fully covariant formalism—with respect to both the string world sheet and the background spacetime [23–25]—to study the collision of current-carrying strings with Y-junctions. This formalism applies both to NG strings where it reduces to that of [3,4], and to different elastic-string models as discussed in Sec. VI.

The simplest field theory model in which Y-junctions form is the Abelian-Higgs model in the type I regime where higher winding number (labeled by the integer n)

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[1]See also [10] for an extension to strings governed by the Dirac-Born-Infeld action.

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strings are stable. It has been shown that when two $n = 1$ strings collide, under certain conditions—which depend on the relative velocity and angle of the two colliding strings [3–5]—a third $n = 2$ string is formed with two corresponding Y-junctions joining it to the original 2 strings. Elastic current-carrying strings are obtained by extending this model to a $U(1) \times U(1)$ theory, as first discussed by Witten [20]. Here the first $U(1)$ forms the strings while, if the coupling between the two $U(1)$ sectors is chosen appropriately, the second $U(1)$ can condensate on the strings and generate a current. This model has been studied in depth, both from a field theory (microscopic) [26] and effective action (macroscopic) point of view [27–29]. It is this second, effective action approach, that we follow in this paper. The questions we are interested in are: what happens to the currents when the two strings collide? Can junctions form? If so, what are the properties of the joining string? Initial numerical studies of collisions of this kind were presented in [30]; further studies will be presented in a companion paper [31].

This paper is set up as follows. In Sec. II we briefly review some of the main properties of elastic string models. In Sec. III we set up the necessary formalism to describe junctions between elastic strings, and derive the corresponding junction conditions. In Sec. IV we apply these junction conditions to V-junctions at which two strings join; and in Sec. V we apply them to Y-junctions. String collisions are studied in Sec. VI. Our conclusions are presented in Sec. VII.

II. PRELIMINARIES: ELASTIC STRINGS

Elastic strings contain internal degrees of freedom, and (for a string with no junctions) the models we consider are governed by an effective action of the form [23,24]

$$S = \int d^2\sigma \sqrt{-\det(\gamma_{ab})}\mathcal{L}(w), \tag{1}$$

where $\sigma^a = (\tau, \sigma)$ are the internal coordinates of the string world sheet. In terms of the string position $x^\mu(\sigma, \tau)$, the induced metric γ_{ab} is given by

$$\gamma_{ab} = \eta_{\mu\nu} x^\mu_{,a} x^\nu_{,b},$$

where a comma denotes partial differentiation, and $\eta_{\mu\nu}$ is the Minkowski metric (with mostly positive signature). For NG strings, the world-sheet Lagrangian $\mathcal{L}$ is constant—there are no internal world-sheet degrees of freedom—and hence the string is locally Lorentz invariant. Thus the tension T and energy per unit length U of NG strings are equal and constant.

For elastic strings [23,24], first introduced in the context of cosmic string theory for studying the mechanical effects of the currents that occur in "superconducting" strings [20,22], the Lagrangian is a function of the variable w (often referred to as the state parameter) defined by

$$w \equiv \kappa_0 \gamma^{ab} \varphi_{,a} \varphi_{,b} \tag{2}$$

where κ_0 is a freely adjustable positive dimensionless normalisation constant. The scalar field $\varphi(\sigma^i)$ can be viewed as a stream function associated with the conserved current,[2] which itself arises from the invariance of $\mathcal{L}(w)$ under $\varphi \to \varphi + \text{constant}$, see below. In many physical models, φ can be associated with a (dimensionless) phase: for instance, in a field theory with $U(1)_{\text{local}} \times U(1)_{\text{global}}$ symmetry, φ can be identified with the phase of the $U(1)_{\text{global}}$ field which condenses in the core of the string (which is itself formed by the breaking of the $U(1)_{\text{local}}$ symmetry group). The precise form of the world-sheet Lagrangian $\mathcal{L}(w)$ depends on the underlying field theory, and has been the subject of numerous studies, see for instance [28,29]. In terms of the mass m associated with the symmetry breaking scale and the mass M of the current carriers, *electric strings* (with timelike $w < 0$) in the $U(1)_{\text{local}} \times U(1)_{\text{global}}$ can be described by [29]

$$\mathcal{L}(w) = -m^2 - \frac{M^2}{2} \ln\left(1 + \frac{w}{M^2}\right). \tag{3}$$

The divergence at $w = -M^2$ corresponds to the threshold for current carrier particle creation. For *magnetic* strings for which w is spacelike, $w > 0$, a suitable Lagrangian is [28]

$$\mathcal{L}(w) = -m^2 - \frac{w}{2}\left(1 - \frac{w}{M^2}\right). \tag{4}$$

Lightlike null currents $w = 0$ must be treated separately, and in this case the equations of motion are integrable [22,32,33]. For most of this analysis we shall leave $\mathcal{L}(w)$ arbitrary, but we exclude chiral strings $w = 0$ which will be considered elsewhere.

The equations of motion obtained from (1) by varying with respect to φ and x^μ take the form

$$\nabla_a c^a = 0,$$
$$\nabla_a (T^{ab} x^\mu_{,b}) = 0,$$

where the conserved current c^a, and the stress energy tensor T^{ab}, are given by

$$c^a \equiv \frac{\sqrt{\kappa_0}}{\mathcal{K}(w)} \gamma^{ab} \varphi_{,b}, \tag{5}$$

$$T^{ab} \equiv \mathcal{L}(w)\gamma^{ab} + \mathcal{K}(w) c^a c^b, \tag{6}$$

with

$$\mathcal{K}^{-1} \equiv -2\frac{d\mathcal{L}}{dw} > 0. \tag{7}$$

[2]For simplicity we assume that the field is not charged under electromagnetism.

We define the norm of c_a as

$$\chi \equiv c^a c_a,$$

so that the state parameter defined in Eq. (2) is given by $w = \mathcal{K}^2\chi$. One of the eigenvectors of T^{ab} is c^a, whilst the other is $d^a \propto \epsilon^{ab}\varphi_{,b}$ (where ϵ^{ab} is the antisymmetric tensor with $\epsilon_{01} = 1$) with $c^a d_a = 0$. Indeed, from Eq. (6),

$$T^{ab}c_b = (\mathcal{L} + \mathcal{K}\chi)c^a,$$
$$T^{ab}d_b = (\mathcal{L})d^a.$$

The corresponding eigenvalues correspond to the energy density U or tension T depending on whether w is positive or negative:

$$U = -\mathcal{L}, \qquad T = -(\mathcal{L} + \mathcal{K}\chi) \quad \text{for } w > 0 \text{ (magnetic)}, \tag{8}$$

$$U = -(\mathcal{L} + \mathcal{K}\chi), \qquad T = -\mathcal{L} \quad \text{for } w < 0 \text{ (electric)}. \tag{9}$$

Note that $U = U(\chi)$ and $T = T(\chi)$ so that $U = U(T)$, and hence elastic strings are characterized by a barotropic equation of state. (This can be determined explicitly in the case of the two Lagrangians given in Eqs. (3) and (4), see e.g. [28].) It follows from Eqs. (8) and (9) that the relation

$$U - T = \mathcal{K}|\chi| \tag{10}$$

holds whatever the sign of $w = \mathcal{K}^2\chi$, and that the tension of the string is always of lower value than U (recall that $\mathcal{K} > 0$). As a result, the transverse (or "wiggle") perturbations on the string propagate at speeds [24] $c_{\mathrm{E}}^2 = T/U$ less than or equal to 1. There are also longitudinal (soundlike) perturbations, which travel at speed $c_{\mathrm{L}}^2 = -dT/dU$. For NG strings $c_{\mathrm{E}}^2 = 1 = c_{\mathrm{L}}^2$, whereas for elastic strings, all field theory models studied in detail so far give $c_{\mathrm{E}} > c_{\mathrm{L}}$.

It the following it will be more transparent to work in terms of four-dimensional "extrinsic" quantities, namely

$$\bar{T}^{\mu\nu} = T^{ab}x^{\mu}_{,a}x^{\nu}_{,b}, \quad \text{and} \quad \bar{c}^{\mu} = c^a x^{\mu}_{,a}. \tag{11}$$

On using world-sheet reparametrization invariance to choose the conformal gauge, $\dot{x}\cdot x' = 0$, $x'^2 = -\dot{x}^2$ (here $\dot{x}^{\mu} = \frac{\partial x^{\mu}}{\partial \tau}$, $x'^{\mu} = \frac{\partial x^{\mu}}{\partial \sigma}$) so that the induced metric $\gamma_{ab} = x'^2\mathrm{diag}(-1, 1)$, it follows that the two vectors

$$u^{\mu} \equiv \frac{\dot{x}^{\mu}}{\sqrt{x'^2}}, \qquad v^{\mu} \equiv \frac{x'^{\mu}}{\sqrt{x'^2}} \tag{12}$$

with

$$u^{\mu}u_{\mu} = -1, \qquad v^{\mu}v_{\mu} = +1, \qquad u^{\mu}v_{\mu} = 0,$$

define a world-sheet orthogonal frame. Furthermore, it is important to notice that provided $w \neq 0$ (thus excluding NG and chiral strings), one can use the freedom of Lorentz rotation on the world sheet to choose $\varphi = \varphi(\tau)$ for electric strings, and $\varphi = \varphi(\sigma)$ for magnetic strings. In this *preferred rest frame*, which we shall use repeatedly below, it is straightforward to show using Eqs. (11) and (12), that the stress energy tensor is given by

$$\bar{T}^{\mu\nu} = U u^{\mu}u^{\nu} - T v^{\mu}v^{\nu}, \tag{13}$$

and that the components of the current c^a take a particularly simple form, see (5), from which $\bar{c}^{\mu} = \nu u^{\mu}$ for electric strings, and $\bar{c}^{\mu} = \nu v^{\mu}$ for magnetic ones, where

$$\nu = \frac{\chi}{\sqrt{|\chi|}} = (\mathrm{sign}(\chi))\sqrt{|\chi|}.$$

III. COVARIANT JUNCTION CONDITIONS FOR ELASTIC STRINGS

We now study the dynamics of elastic strings meeting at a junction (itself assumed massless). We label the strings by an index j, so that for a V-junction j runs from 1 to 2, and for a Y-junction j runs from 1 to 3. A V-junction is nothing other than a kink, or discontinuity in the tangent vector of the string. Since kinks are invariably formed when strings collide and create of Y-junctions, see [3], it is crucial to consider them in the following analysis.

Let string j have coordinates $x^{\mu}_j(\sigma, \tau)$, world-sheet scalar field φ_j, Lagrangian density $\mathcal{L}(w_j)$, and corresponding tension T_j and energy per unit length U_j. Furthermore the spatial world-sheet coordinate σ is taken to increase towards the junction where it takes the value $s_j(\tau)$, thus

$$-\infty < \sigma \leq s_j(\tau).$$

The action describing the dynamics of the system (strings and massless junction) is given by

$$\begin{aligned} S = {} & \sum_i \int d\tau d\sigma \theta(s_i(\tau) - \sigma)\sqrt{-\gamma_i}\mathcal{L}(w_i) \\ & + \sum_i \int d\tau \{\mathfrak{f}^{\mu}_i \cdot [x_{i,\mu}(s_i(\tau), \tau) - X_{\mu}(\tau)] \\ & + \mathfrak{g}_i \cdot [\varphi_i(s_i(\tau), \tau) - \Phi(\tau)]\}, \end{aligned}$$

where $\mathfrak{f}_i$ is a 4-vector Lagrange multiplier imposing that the strings meet at the same position, namely at the junction

$$X^{\mu}(\tau) = x^{\mu}_i(s_i(\tau), \tau), \tag{14}$$

and $\mathfrak{g}_i$ a scalar Lagrange multiplier imposing that the fields are continuous at the junction

$$\Phi(\tau) = \varphi_i(s_i(\tau), \tau).$$

Varying the action with respect to X^μ and Φ gives

$$\sum_i \mathfrak{f}_i^\mu = 0 = \sum_i \mathfrak{g}_i, \tag{15}$$

and with respect to x_i^μ and φ_i gives

$$\partial_a(\sqrt{-\gamma_i} T_i^{ab} x_{i,b}^\mu \theta(s_i(\tau) - \sigma)) = \mathfrak{f}_i^\mu \delta(s_i - \sigma),$$
$$\partial_a(\sqrt{-\gamma_i} z_i^a \theta(s_i(\tau) - \sigma)) = \mathfrak{g}_i \delta(s_i - \sigma),$$

where $z_j^a = \sqrt{\kappa_{0,j}} c_j^a$. Thus *at* the junction [where $\sigma = s_i(\tau)$], on using Eq. (15), one deduces the conservation equations

$$\sum_i \sqrt{-\gamma_i}\,(T_i^{0b}\dot{s}_i - T_i^{1b}) x_{i,b}^\mu = 0, \tag{16}$$

$$\sum_i \sqrt{-\gamma_i}\,(z_i^0 \dot{s}_i - z_i^1) = 0. \tag{17}$$

In terms of 4-dimensional quantities defined in Eq. (11) these conservation equations can be written in a simpler and more physically obvious form [25], namely

$$\sum_j \lambda_{j\nu} \bar{T}_j^{\nu\mu} = 0, \tag{18}$$

$$\sum_j \lambda_{j\nu} \bar{c}_j^\nu = 0, \tag{19}$$

where the λ_j^μ, which we construct below, are outward directed unit normal vectors at the junction:

$$\lambda_j^\mu \lambda_{\mu,j} = +1, \qquad \lambda_j^\mu \dot{X}_\mu = 0. \tag{20}$$

Here the 4-velocity of the junction $\dot{X}^\mu(\tau)$ is obtained from Eq. (14) as

$$\dot{X}^\mu = \dot{x}_j^\mu + x'^\mu_j \dot{s}_j,$$

where all quantities are evaluated at the junction and $\dot{s}_j = ds_j/d\tau$. The corresponding unit timelike vector $\hat{\dot{X}}^\mu(\tau)$ is thus

$$\hat{\dot{X}}^\mu(\tau) = \frac{\dot{X}^\mu(\tau)}{N(\tau)}, \quad \hat{\dot{X}}^2 = -1, \quad N^2(\tau) = \dot{X}^2 = x'^2_j(1 - \dot{s}_j^2) \tag{21}$$

which, in terms of the orthonormal unit frame vectors defined in Eq. (12), becomes

$$\hat{\dot{X}}^\mu = \Gamma_j(u_j^\mu + \dot{s}_j v_j^\mu) \tag{22}$$

where $\Gamma_j \equiv (1 - \dot{s}_j^2)^{-1/2}$. It finally follows from Eq. (20) that

$$\lambda_j^\mu = \Gamma_j(\dot{s}_j u_j^\mu + v_j^\mu). \tag{23}$$

Using this expression, together with the definitions of $\bar{T}^{\mu\nu}$ and $\bar{c}^\mu$ in (11) it is now straightforward to show the equivalence of "intrinsic" expressions for current and energy conservation at the junction Eqs. (16), (17), with their extrinsic form in Eqs. (18), (19).

Before proceeding, we note that Eqs. (22) and (23) can be inverted to determine u_j^μ and v_j^μ at the junction, namely

$$u_j^\mu = \Gamma_j(\hat{\dot{X}}^\mu - \dot{s}_j \lambda_j^\mu),$$
$$v_j^\mu = \Gamma_j(\lambda_j^\mu - \hat{\dot{X}}^\mu \dot{s}_j).$$

When the vectors $\hat{\dot{X}}^\mu$ and λ_j^μ, as well as $\dot{s}_j$, are constants—a particular case we will meet repeatedly below where we consider straight segments of string—then it is straightforward to integrate these equations to determine the string position $x^\mu(\sigma, \tau)$. Indeed

$$x_j^\mu(\sigma, \tau) = \Gamma_j[\hat{\dot{X}}^\mu(\tau - \dot{s}_j\sigma) + \lambda_j^\mu(\sigma - \dot{s}_j\tau)], \tag{24}$$

where we have chosen the integration constant to be such that at the junction $x'^2_j = 1$ (that is $N = 1/\Gamma_j$) so that the induced metric is exactly Minkowski.

Now consider the junction condition Eq. (18). Working in the preferred rest frame of each string where $\bar{T}_j^{\mu\nu}$ takes the form given in Eq. (13), it becomes

$$\sum_j \bar{T}_j^{\mu\nu} \lambda_{j\nu} = \sum_j [f_j \lambda_j^\mu + g_j \hat{\dot{X}}^\mu] = 0, \tag{25}$$

where the 4D-Lorentz scalars f_j and g_j are given by

$$f_j = \Gamma_j^2(U_j \dot{s}_j^2 - T_j) \tag{26}$$

$$g_j = -\Gamma_j^2 \dot{s}_j(U_j - T_j). \tag{27}$$

Assuming that strings meeting at the junction are either all electric, $\bar{c}_j^\mu = \nu_j u_j^\mu$, or all magnetic $\bar{c}_j^\mu = \nu_j v_j^\mu$, the current conservation equation Eq. (19) reduces, on using Eq. (23), to

$$\sum_j \Gamma_j \nu_j \dot{s}_j = 0 \quad \text{(electric)}, \tag{28}$$

$$\sum_j \Gamma_j \nu_j = 0 \quad \text{(magnetic)}. \tag{29}$$

We now apply this formalism to study the dynamics of V and Y-shaped junctions, as well as to string collisions.

IV. V-SHAPED JUNCTIONS

For a V-shaped junction formed by two strings, $j = 1, 2$, energy-momentum conservation at the junction [Eq. (25)] imposes[3]

$$f_1 = f_2 = 0, \qquad g_1 = -g_2.$$

The first equality implies, from Eq. (26), that

$$\dot{s}_j^2 = \frac{T_j}{U_j} = c_{\mathrm{E},j}^2, \tag{30}$$

so that junction propagates along the strings with extrinsic velocity c_{E}. Hence, using Eq. (10),

$$\Gamma_j^2 = \frac{U_j}{U_j - T_j} = \frac{U_j}{\mathcal{K}_j|\chi_j|},$$

and condition $g_1 = -g_2$ becomes [on using Eq. (27)]

$$\sqrt{U_1 T_1} = \sqrt{U_2 T_2}.$$

For NG strings ($U_i = T_i$) it follows that $U_1 = U_2$, and the two strings meeting at the junction must be identical. For a current-carrying string, the current conservation condition leads—for both electric (28) and magnetic (29) strings—to the same condition, namely

$$\frac{\mathcal{L}_1(w_1)}{\mathcal{K}_1(w_1)} = \frac{\mathcal{L}_2(w_2)}{\mathcal{K}_2(w_2)},$$

where $\mathcal{K}(w)$ is defined in Eq. (7). Assuming the strings meeting at the kink are described by the same underlying field theory and hence the same underlying Lagrangian, current conservation therefore imposes that

$$w_1 = w_2$$

as expected—namely the strings carry the same current.

For illustrative purposes, choose the first string to lie along the x-axis and the other to be the (x, y)-plane, so that in the V-junction rest frame (see Fig. 1)

$$\begin{aligned} \dot{\hat{X}}(\tau) &= (1, 0, 0, 0), \\ \lambda_1^\mu &= (0, 1, 0, 0), \\ \lambda_2^\mu &= (0, -\cos\theta, \sin\theta, 0). \end{aligned}$$

Since $w_1 = w_2 \equiv w$ are constants, $\dot{s}_1 = -\dot{s}_2 = c_{\mathrm{E}}$ are also constant and given by Eq. (30). Then from Eq. (24),

$$x_1^\mu(\sigma, \tau) = \Gamma_{\mathrm{E}}(\tau - c_{\mathrm{E}}\sigma, \sigma - c_{\mathrm{E}}\tau, 0, 0).$$

Notice that the point $x_1^\mu(0, \tau)$ is not fixed but moves along string 1 in the negative x-direction with speed c_E. Physically, therefore, it is more transparent to work in a frame in which points of fixed σ do not have a component of velocity along the string, namely in a frame in which $\dot{\vec{x}} \cdot \vec{x}' = 0$. Hence, we now boost to a frame moving with velocity $-c_{\mathrm{E}}$ along the x-axis, and with a transverse velocity v_z in the z-direction: this will correspond precisely to the situation at hand when considering string collisions in Sec. VI. In this case

$$\begin{aligned} \dot{\hat{X}}^\mu &= \Gamma_{\mathrm{E}}(\gamma_z, c_{\mathrm{E}}, 0, \gamma_z v_z), \\ \lambda_1^\mu &= \Gamma_{\mathrm{E}}(\gamma_z c_{\mathrm{E}}, 1, 0, \gamma_z v_z c_{\mathrm{E}}), \end{aligned} \tag{31}$$

so that from Eq. (24),

$$x_1^\mu(\sigma, \tau) = (\gamma_z \tau, \sigma, 0, \gamma_z v_z \tau). \tag{32}$$

When $v_z = 0$ this reduces to $x_1^\mu(\sigma, \tau) = (\tau, \sigma, 0, 0)$ as required.

V. Y-SHAPED JUNCTIONS

In this section we consider a Y junction where, as shown in Fig. 1, for simplicity we assume that two of the strings deviate by the same angle θ from the direction of the first. Working in the junction rest frame, we thus have

$$\begin{aligned} \dot{\hat{X}}^\mu &= (1, 0, 0, 0) \\ \lambda_1^\mu &= (0, 1, 0, 0) \\ \lambda_2^\mu &= (0, -\cos\theta, \sin\theta, 0) \\ \lambda_3^\mu &= (0, -\cos\theta, -\sin\theta, 0) \end{aligned}$$

so that $\lambda_2^\mu + \lambda_3^\mu = -2\lambda_1^\mu \cos\theta$. The y component of the energy-momentum conservation Eq. (25) gives

$$\Gamma_2^2(T_2 - U_2\dot{s}_2) = \Gamma_3^2(T_3 - U_3\dot{s}_3)$$

where $\Gamma_i = \Gamma(\dot{s}_i)$. This will be automatically satisfied if strings 2 and 3 are identical, which we will suppose in the following (and drop the subscripts 2 and 3). Then the x-component yields

$$2\cos\theta\,\Gamma^2(T - U\dot{s}^2) = -\Gamma_1^2(T_1 - U_1\dot{s}_1^2), \tag{33}$$

which, in the case of NG strings, yields the condition on θ found in [3,4], namely $\cos\theta = U_1/2U$ so that $U_1 \leq 2U$ as expected energetically. Contracting Eq. (25) with $\dot{\hat{X}}^\mu$ gives

$$2\Gamma^2\dot{s}(T - U) = -\Gamma_1^2\dot{s}_1(T_1 - U_1), \tag{34}$$

[3] on contracting Eq. (25) with $\lambda_{1,2}^\mu$ and $\dot{\hat{X}}^\mu$.

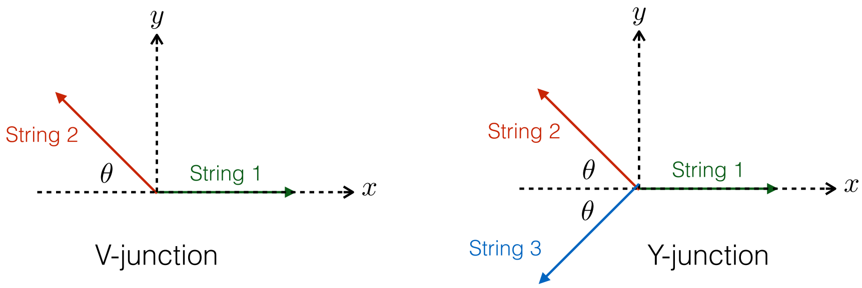

FIG. 1. Static V and Y junctions. The arrows indicate the direction of the outward pointing tangent vectors.

a condition which vanishes for NG strings. Finally, current conservation Eq. (28) gives

$$2\Gamma\dot{s}\nu = -\Gamma_1\dot{s}_1\nu_1 \quad \text{(electric)}, \tag{35}$$

$$2\Gamma\nu = -\Gamma_1\nu_1 \quad \text{(magnetic)}. \tag{36}$$

Let us assume that the Lagrangian of each string is known, see for instance Eq. (3), as well as the current on that string, namely the values of (w, w_1). (That is, the tension and energy density of each string is known.) Then the unknown variables describing the junction are $(\dot{s}, \dot{s}_1, \cos\theta)$. The first two are determined from Eqs. (34) and (35), and $\cos\theta$ from Eq. (33). For instance, for electric strings it follows from Eqs. (34) and (35) that

$$\dot{s} = -\frac{\mathcal{K}(w)}{2\mathcal{K}(w_1)}\dot{s}_1, \qquad \dot{s}^2 = \frac{|w_1| - |w|}{|w_1| - |w|\frac{4\mathcal{K}^2(w_1)}{\mathcal{K}^2(w)}} \quad \text{(electric)},$$

which implies that $|w_1| > |w|$ and $2\mathcal{K}(w_1) \leq \mathcal{K}(w)$ where, for the explicit electric string model of (3), $\mathcal{K}(w) = 1 - |w|/M^2$. The condition $\cos\theta \leq 1$ further restricts the parameter space.

Finally, the position of each string can be obtained from (24) as before.

VI. STRING COLLISIONS

We now consider a string collision between two incoming and identical strings at angles $\pm\alpha$ in the (x, y) plane with velocities $\pm v_z$ in the z-direction, as shown in Fig. 2. The values of α and v_z, as well as the type of incoming

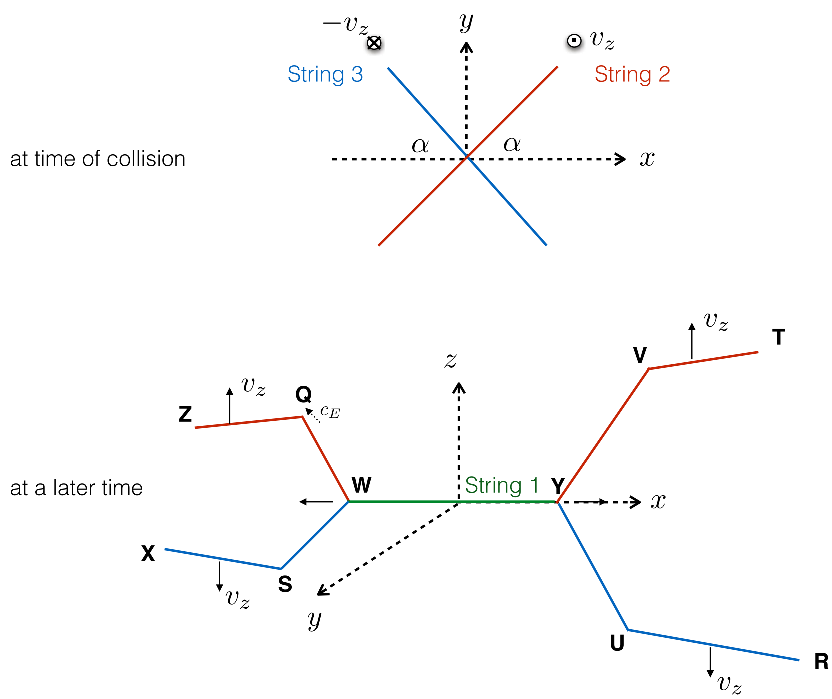

FIG. 2. String collision. Before the collision two identical are parallel to the (x, y) plane forming an angle $\pm\alpha$ with the x-axis and moving in the z-direction with velocities $\pm v_z$. After the collision, two Y-junctions (labeled W and Y) moves along the x-axis while four V-junctions (labeled Q, S, U, V) travel along the strings at velocity c_{E} in the string's preferred frame.

string, namely its tension T from which $U(T)$ is determined (or equivalently $\mathcal{L}(w)$ and w) are thus given initial conditions for this problem. Note that this means that $c_{\rm E}$, the speed of any V-junction formed, is also therefore known from the initial conditions.

There are at least 3 different possible outcomes of such a collision: (i) the strings could cross without interaction, (ii) they could intercommute, or (iii) the two initial strings could become joined by the formation of connecting segment (string 1 in the figure) which, by symmetry, is at rest in the x direction. It is scenario (iii) that is envisaged here.

For NG strings (when $U = T$), it is has been show in [3,4] that the formation of a connecting segment is only possible in a well defined region of the (α, v_z) plane. This kinematic constraint comes from the physical requirement that the length of the connecting segment should increase in time. Our aim in this section is to see if a similar constraint exists for current carrying string collisions.

After the collision, two Y-junctions (labeled W and Y in Fig. 2) move along string 1 parallel to the x-axis, while four V-junctions (labeled Q, S ,U, V) travel along the strings with velocity $c_{\rm E}$. By causality, the segments ZQ and VT of the string 2 remain unperturbed (and analogously for string 3), and are given by an expression similar to that of Eq. (32), namely

$$x_2^\mu(\sigma,t) = \gamma_z(e_t^\mu + v_z e_z^\mu)t + (\cos\alpha e_x^\mu + \sin\alpha e_y^\mu)\sigma,$$

where the unit vectors $e_t^\mu = (1,0,0,0)$, $e_x^\mu = (0,1,0,0)$ and analogously for e_y^μ and e_z^μ. The unit-velocity 4-vector of the V-junction V is then given by, see Eq. (31),

$$\dot{\hat{X}}_V^\mu(t) = \Gamma_{\rm E}\gamma_z(e_t^\mu + v_z e_z^\mu) + \Gamma_{\rm E} c_{\rm E}(\cos\alpha e_x^\mu + \sin\alpha e_y^\mu).$$

The corresponding vector $\dot{\hat{X}}_U^\mu(t)$ for the junction U is simply obained from the above by flipping $v_z \to -v_z$ and $\alpha \to -\alpha$.

On the other hand, string 1 is given by

$$x_1^\mu(\sigma,t) = t e_t^\mu + \sigma e_x^\mu,$$

and at the Y-junction Y, $\sigma = s_1(t)$. Hence

$$\dot{\hat{X}}_Y^\mu(t) = \Gamma_1(e_t^\mu + \dot{s}_1 e_x^\mu),$$
$$\lambda_1^\mu = \Gamma_1(\dot{s}_1 e_t^\mu + e_x^\mu).$$

Junctions V and Y therefore have a relative speed v_+ with $\gamma_+ = (1 - v_+^2)^{-1/2}$ given by

$$\gamma_+ = -\dot{\hat{X}}_Y^\mu(t)\dot{\hat{X}}_\mu^V(t) = \Gamma_1\Gamma_{\rm E}(\gamma_z - \dot{s}_1 c_{\rm E}\cos\alpha). \quad (37)$$

In order to apply the junction conditions at the Y-junction Y, we need to construct outward pointing tangent vectors on strings 2 and 3 at Y. On string 2, λ_2^μ is a linear combination of $\dot{\hat{X}}_Y^\mu$ and $\dot{\hat{X}}_V^\mu$, with coefficients determined by the normalization of λ_2^μ and its orthogonality with $\dot{\hat{X}}_Y^\mu$:

$$\lambda_2^\mu = -(v_+\gamma_+)^{-1}(\dot{\hat{X}}_V^\mu - \gamma_+\dot{\hat{X}}_Y^\mu). \quad (38)$$

The expression for λ_3^μ is the same, but with $\dot{\hat{X}}_V^\mu$ replaced by $\dot{\hat{X}}_U^\mu$. Finally (up to an irrelevant overall additive constant) from Eq. (24), the string on segment YV is given by[4]

$$x_{\rm YV}^\mu(\sigma,t) = \Gamma_2[\dot{\hat{X}}_Y^\mu(t - \dot{s}_2\sigma) + \lambda_2^\mu(\sigma - \dot{s}_2 t)], \quad (39)$$

where $\sigma = \dot{s}_2 t$ at the Y-junction. This string had better connect the V and Y-junctions. Or, in other words, the value of $\dot{s}_2$ must such that at the V-junction where $\sigma = c_{\rm E} t$, the equality $\frac{d}{dt}(x_{\rm YV}^\mu(c_{\rm E}t,t)) = \dot{X}_V^\mu$ holds. Hence, since $\dot{\hat{X}}_V^\mu = \Gamma_{\rm E}\dot{X}_V^\mu$, it follows from Eq. (39) that

$$\dot{\hat{X}}_V^\mu = \Gamma_{\rm E}\Gamma_2[\dot{\hat{X}}_Y^\mu(1 - c_{\rm E}\dot{s}_2) + \lambda_2^\mu(c_{\rm E} - \dot{s}_2)]. \quad (40)$$

Combining Eqs. (38) and (40) therefore gives

$$\dot{s}_2 = \frac{c_{\rm E} + v_+}{1 + c_{\rm E} v_+} = \dot{s}_3 \equiv \dot{s}, \quad (41)$$

where $v_+ = -(1 - \gamma_+^{-2})^{1/2}$. Finally, the angle θ between strings 1 and 2 is given by

$$\cos\theta = -\lambda_1^\mu \lambda_{2,\mu} = \frac{\Gamma_1\Gamma_{\rm E}}{v_+\gamma_+}(\dot{s}_1\gamma_z - c_{\rm E}\cos\alpha)$$
$$= \frac{1}{v_+}\frac{(\dot{s}_1\gamma_z - c_{\rm E}\cos\alpha)}{(\gamma_z - \dot{s}_1 c_{\rm E}\cos\alpha)} \quad (42)$$

where v_+ itself obtained from Eq. (37).

We are now finally in a position to apply the Y-junction conditions Eqs. (33)–(35) to Y. First, however, let us understand the NG limit. Here there is obviously no current conservation condition, and furthermore one of the other stress-energy conservation equations [namely Eq. (34)] is trivially satisfied since $U = T$ for NG strings. How does the resulting condition reduce to the kinematic constraint of [3]? The NG limit is obtained when $c_{\rm E} \to 1$, and from Eq. (37) that obviously implies $v_+ \to -1$ as expected. Then from Eq. (42),

$$\cos\theta \to \frac{\dot{s}_1\gamma_z - \cos\alpha}{\gamma_z - \dot{s}_1\cos\alpha} \quad \text{(NG limit)}.$$

The only remaining junction condition is Eq. (33), which for NG strings reads

[4]Assuming that $\dot{\hat{X}}_Y^\mu$, λ_2^μ, and $\dot{s}_2$ are constants, as expected for straight strings.

$$2\cos\theta T = -T_1.$$

On substituting for $\cos\theta$ one arrives directly at the result

$$\dot{s}_1 = \frac{2T\gamma_z^{-1}\cos\alpha - T_1}{2T - T_1\gamma_z^{-1}\cos\alpha}.$$

Since the length of the joining string must increase, $\cos\alpha > \gamma_z T_1/(2T)$. Hence for a given T_1 and T, the formation of Y-junctions in such a collision can only occur in a certain region of the (α, v_z) plane (namely small α and v_z), as discussed in depth in [3].

Now let us turn to current carrying string collisions, for which the three junction conditions (33) to (35) must be satisfied. Relative to the static Y-junction considered in Sec. V, it is important to notice two crucial differences. The first is that $\dot{s}$ is not free variable: from Eq. (41), given α, v_z and c_{E} (known from the initial conditions), it is a function of $\dot{s}_1$. Second, the angle $\cos\theta$ is also no longer a free variable but determined by $\dot{s}_1$. Therefore, contrary to the case considered in Sec. V where there were 3 unknown variables, there are only *two* unknown variables in this problem involving colliding strings namely $(\dot{s}_1, w_1)$. However, there are 3 junction conditions. The system is therefore overdetermined. Indeed, we arrive at the conclusion that in such a collision between two elastic current carrying strings, if a joining string forms, then this joining string *cannot* be described by an elastic string model (that is, by a barotropic equation of state). We expect that there must be time dependence and dissipative processes at the Y-junctions, which will generically require the use of a nonconservative model. It is also of great interest to investigate numerically, as will be reported elsewhere [31].

VII. CONCLUSIONS

In this paper we have extended the analysis of string collisions, with the subsequent formation of Y-junctions, to *elastic* string models which characterise current-carrying strings. To do so, we have developed a fully covariant formalism. We have shown that when Y-junctions form, the resulting system of equations is overdetermined: the number of the unknown variables is smaller than the number of junction conditions. Therefore our main result is that if the collision of two elastic current-carrying strings leads to the formation of a joining string, then this joining string cannot be described by the elastic string model: a nonconservative model must be used. This will be the subject of future work, together with a numerical study of the collision of field theory current-carrying strings in the $U(1) \times U(1)$ model.

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