$\beta'_{\rm IR}$ at an infrared fixed point in chiral gauge theories

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We present scheme-independent calculations of the derivative of the beta function at an infrared (IR) fixed point, denoted $\beta'_{\rm IR}$, in several asymptotically free chiral gauge theories, namely SO$(4k+2)$ with $2 \le k \le 4$ with respective numbers N_f of fermions in the spinor representation, and E$_6$ with fermions in the fundamental representation. Some implications of these results are discussed.

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I. INTRODUCTION

A weakly coupled chiral gauge theory (χGT) associated with the $G_{\rm EW} = {\rm SU}(2)_L \otimes {\rm U}(1)_Y$ electroweak gauge symmetry plays a crucial role in nature, comprising the electroweak sector of the Standard Model (SM). However, the properties of strongly coupled chiral gauge theories are not well understood. Such strongly coupled chiral gauge theories have been of physical interest in the past for several reasons. In general, for a given gauge group G and set of matter fermion representations, one would like to be able to describe the behavior of the theory at both weak and strong coupling. We define a chiral gauge theory as irreducibly chiral if and only if the fermion content does not contain any vectorlike subsector. Such a theory forbids fermion mass terms in the fundamental Lagrangian.

One physical application of strongly coupled chiral gauge theories was in preon models [1,2]. These models addressed the still-unsolved puzzle of why there are three generations of quarks and leptons in nature, and attempted to offer a possible solution to this puzzle by hypothesizing that these SM fermions are actually composite bound states of more fundamental (spin-$1/2$) fermions, namely, the preons. This approach made use of an underlying asymptotically free, preonic chiral gauge theory with gauge group G_{pr}, which would become strongly coupled at some scale Λ_{pr} and confine the preons to massless G_{pr}-singlet spin-$1/2$ fermionic bound states of size $r_{pr} \sim 1/\Lambda_{pr}$. The 't Hooft anomaly matching conditions were a necessary, although not sufficient, condition for this scenario to occur [1]. Since SM fermions appear pointlike down to the smallest distances probed, the preonic chiral gauge symmetry with a sufficiently large Λ_{pr}, and hence a sufficiently small r_{pr}, could potentially account for this observed property of the quarks and leptons. It was anticipated that an appropriate ultraviolet (UV) completion of the preonic theory would then explain the actual nonzero masses of the SM fermions, and this UV completion, in conjunction with an understanding of the dynamics of the strongly coupled preonic gauge theory, would explain the observed three generations of SM fermions. However, there was only limited progress with this program, in part because of the lack of understanding of the nonperturbative properties of a chiral gauge theory.

A second application of strongly coupled chiral gauge theories has been in models of dynamical electroweak symmetry breaking (EWSB) [3–17]. Related general studies of strongly coupled chiral gauge theories include [16,18]. In dynamical EWSB models, this symmetry breaking is envisioned to occur as a result of an asymptotically free vectorial gauge interaction, with a set of associated fermions, which becomes strongly coupled and confining on the TeV scale, producing bilinear condensates of these fermions. To give adequate masses to SM fermions in such models, one extends the basic gauge symmetry to a larger gauge symmetry [4]. Reasonably ultraviolet-complete extensions, e.g., [7], make crucial use of an asymptotically free chiral gauge symmetry with an associated gauge interaction that becomes strong on the scale of $\sim 10^3$ TeV and self-breaks in a sequence of stages, thereby naturally producing the observed generational hierarchy of quark and charged lepton masses. A low-scale seesaw mechanism in these models could produce naturally small neutrino masses. A rough criterion to determine the minimal strength of the gauge coupling in the chiral gauge theory that can lead to this self-breaking was provided by the most-attractive-channel (MAC) criterion [5]. In order to be viable, the vectorial gauge interaction in these dynamical

EWSB models should exhibit quasiconformal behavior, which can occur naturally in the presence of an approximate infrared fixed point (IRFP) of the renormalization group (RG) at a value of the gauge coupling that is sufficiently strong to eventually cause the bilinear fermion condensate formation. Since this spontaneously breaks the approximate scale (dilatation) invariance, this can lead to a resultant light dilatonlike scalar, with Higgs-like properties (some papers on this include [10–15]). These models are strongly constrained by precision electroweak data, observed properties of the Higgs boson, and absence of definite manifestations of physics beyond the Standard Model in available data [17].

An asymptotically free (anomaly-free) chiral gauge theory with a gauge group G and a set of N_f chiral fermions in a representation R of G exhibits an IRFP for sufficiently large N_f. (The analogous phenomenon for vectorial gauge theories was discussed in [19].) Let us denote the running gauge coupling at the Euclidean energy/momentum scale μ as $g = g(\mu)$ and let $\alpha = g^2/(4\pi)$. The property of asymptotic freedom requires that $N_f < N_u$, where N_u is given below in Eq. (2.1). If N_f is only slightly less than N_u, then the theory is expected to evolve from the UV to a weakly coupled IRFP at a value $\alpha = \alpha_{\rm IR}$, at which it is in a chirally symmetric (deconfined) non-Abelian Coulomb phase (NACP). As N_f decreases below a value $N_{f,cr}$, the gauge interaction becomes strongly coupled and, depending on the fermion content, it might confine and produce massless composite fermions or produce bilinear fermion condensate(s), spontaneously breaking chiral global and gauge chiral symmetries. To construct a quasiconformal chiral gauge theory, it is therefore necessary to know the value of $N_{f,cr}$ for a given G and R. For vectorial gauge theories, an intensive program of lattice simulations has been underway for a number of years to investigate the properties of quasiconformal theories, including an estimate of $N_{f,cr}$ and measurements of anomalous dimensions and particle spectra [20–26]. Ideally, one would carry out a similar program of fully nonperturbative simulations of chiral gauge theories on the lattice to study their properties. However, it has proved much more difficult to try to simulate chiral, as contrasted with vectorial, gauge theories on the lattice, owing to fermion doubling [27,28]. Continuum studies of strongly coupled chiral gauge theories [5,7,8,29] have typically relied upon criteria such as the 't Hooft anomaly matching conditions [1], the most attractive channel criterion [5], and a conjectured inequality on field degrees of freedom [18] for guidance on likely nonperturbative behavior.

In this paper we apply a different approach to this problem of understanding the behavior of strongly coupled chiral gauge theories. We consider several asymptotically free (and anomaly-free) chiral gauge theories, namely theories with the gauge group $SO(4k+2)$, where $2 \le k \le 4$, containing N_f chiral fermions in the spinor representation, and a theory with the gauge group E_6 containing N_f chiral fermions in the fundamental representation. Without loss of generality, all fermions may be taken as left-handed. Our approach is to apply the renormalization group, starting in a perturbative regime, namely at a weakly coupled IRFP at a small value $\alpha_{\rm IR}$ in the non-Abelian Coulomb phase of the theory with N_f only slightly less than N_u. At this IRFP, the theory is scale-invariant and is inferred to be conformally invariant [30]. We then decrease N_f, thereby increasing $\alpha_{\rm IR}$ and moving toward stronger coupling. We analyze the derivative of the beta function at the IRFP,

$$\frac{d\beta}{d\alpha}\bigg|_{\alpha=\alpha_{\rm IR}} \equiv \beta'_{\rm IR}, \tag{1.1}$$

in the non-Abelian Coulomb phase of each chiral gauge theory. This is a physical quantity and is equivalent to the anomalous dimension of the operator ${\rm Tr}(F_{\mu\nu}F^{\mu\nu})$, where $F^a_{\mu\nu}$ is the field-strength tensor [31] (and a is a gauge group index). As a physical quantity, $\beta'_{\rm IR}$ must, of course, be independent of the scheme used for regularization and renormalization; a formal proof of its scheme independence was given in [32]. However, a conventional perturbative series expansion in powers of the coupling is scheme-dependent above the lowest loop orders and hence does not maintain this scheme independence of the exact $\beta'_{\rm IR}$. Here we achieve a significant advance in the study of $\beta'_{\rm IR}$ for chiral gauge theories by calculating it for the first time as a series expansion in the manifestly scheme-independent quantity

$$\Delta_f = N_u - N_f. \tag{1.2}$$

Our calculation extends to a high order, $O(\Delta_f^5)$. Our work makes use of the recently calculated five-loop beta function for a general group G and fermion representation R [33].

The trace of the energy-momentum tensor, T^μ_μ, satisfies the relation [34]

$$T^\mu_\mu = \frac{\beta}{4\alpha} F^a_{\mu\nu} F^{a\mu\nu}, \tag{1.3}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ is the gluon field strength tensor and the f^{abc} are the structure constants of the Lie algebra of G. For Euclidean scales μ such that $\alpha = \alpha(\mu)$ is close to the infrared zero of the beta function at $\alpha_{\rm IR}$, one can expand $\beta(\alpha)$ in a Taylor series around $\alpha_{\rm IR}$ and use the fact that the first term vanishes, since $\beta(\alpha_{\rm IR}) = 0$. Substituting this expansion in Eq. (1.3), one obtains, as an approximation that is applicable as $\alpha_{\rm IR} - \alpha \searrow 0$,

$$T^\mu_\mu \simeq -\frac{\beta'_{\rm IR}(\alpha_{\rm IR} - \alpha)}{4\alpha_{\rm IR}} F^a_{\mu\nu} F^{a\mu\nu}. \tag{1.4}$$

(Here, $\alpha_{\rm IR} - \alpha > 0$ since the approach to the IRFP is from smaller α, i.e., from the UV.) Thus, a second physical role of $\beta'_{\rm IR}$ is via its occurrence in Eq. (1.4).

The chiral gauge theories that we use for these calculations of $\beta'_{\rm IR}$ are particularly simple, in the sense that they contain chiral fermions transforming according to only one representation of the respective gauge groups $\mathrm{SO}(4k+2)$ and E_6. The chiral gauge theories that were used for phenomenological applications were typically more complicated, since they contained fermions transforming according to two (or more) different representations of the gauge group. For example, two widely studied preon models [2,5,18,29] used an $\mathrm{SU}(N)$ gauge group with fermions transforming according to (i) a symmetric rank-2 tensor representation and $N+4$ copies of fermions in the conjugate fundamental representation or (ii) an antisymmetric rank-2 tensor representation and $N-4$ copies of fermions in the conjugate fundamental representation of $\mathrm{SU}(N)$. These were irreducibly chiral; more complicated preon models [2,29] included also various vectorlike subsectors. Similarly, several reasonably ultraviolet-complete models with dynamical electroweak symmetry breaking studied in Ref. [7] made use of an SU(5) chiral gauge theory with several types of fermions in the fundamental and conjugate antisymmetric rank-2 tensor representation of SU(5). The renormalization-group flows and possible nonperturbative sequences of self-breakings of chiral gauge and global symmetries in these models depend in detail on the various different fermion representations. For our current first set of scheme-independent calculations of $\beta'_{\rm IR}$ in chiral gauge theories, there is thus a motivation to step back from these complicated phenomenological models and consider the simplest type of chiral gauge theories, namely those involving a single type of fermion representation. In future research, one could then move on to study more complicated chiral gauge theories with multiple different fermion representations.

Previously, we have presented scheme-independent series calculations of physical quantities in vectorial gauge theories [35–42]. Our present results for chiral gauge theories serve as useful inputs for both theories of strongly coupled chiral gauge theories for physics beyond the Standard Model, as discussed above, and to studies of conformal field theories [43].

This paper is organized as follows. In Sec. II we briefly review the overall theoretical context and methods of analysis. We present our results for $\mathrm{SO}(4k+2)$ theories in Sec. III and for the E_6 theory in Sec. IV. A discussion concerning the behavior of $\beta'_{\rm IR}$ in the vicinity of the lower end of the non-Abelian Coulomb phase is presented in Sec. V. We give our conclusions in Sec. VI and some relevant group-theoretic formulas in the Appendix.

II. THEORETICAL CONTEXT AND METHODS OF ANALYSIS

A. Theoretical context

Here we briefly review some background and methods relevant for our work. As noted above, we consider several asymptotically free chiral gauge theories, namely theories with the gauge group $\mathrm{SO}(N)$, where $N=4k+2$ with $k\ge 2$, containing N_f chiral fermions in the spinor representation, and a theory with the gauge group E_6, containing N_f chiral fermions in the fundamental representation. These theories have complex representations [44] and vanishing gauge anomaly [45]. They also have vanishing global π_4 anomaly [46]. The requirement of asymptotic freedom limits our consideration of $\mathrm{SO}(4k+2)$ theories to those with $k=2$, 3, 4, i.e., SO(10), SO(14), and SO(18). Specifically, this requirement of asymptotic freedom implies that N_f must be less than an upper (u) bound N_u, where

$$N_u = \frac{11C_A}{2T_f} \tag{2.1}$$

(see Appendix for definitions of the group invariants C_A and T_f). For the $\mathrm{SO}(4k+2)$ theories, this imposes the following upper limits on N_f: $N_f\le 21$ for SO(10), $N_f\le 8$ for SO(14), and $N_f\le 2$ for SO(18). There are no asymptotically free $\mathrm{SO}(4k+2)$ chiral gauge theories with fermions in the spinor representation if $k\ge 5$, i.e., for SO(22) and higher-lying members of this family. Similarly, the asymptotic freedom constraint imposes the upper limit $N_f\le 21$ in the E_6 theory.

The renormalization-group flow from the UV, where the gauge coupling approaches zero, to the IR, is described by the beta function, $\beta = d\alpha/d\ln\mu$. The maximal loop order at which the beta function is scheme-independent is two loops [32]. The two-loop (2ℓ) beta function has an IR zero if N_f lies in the interval I defined by $N_\ell < N_f < N_u$, where N_u was given in Eq. (2.1) and [47]

$$N_\ell = \frac{17C_A^2}{T_f(5C_A+3C_f)}. \tag{2.2}$$

This IR zero occurs at

$$\alpha_{{\rm IR},2\ell} = \frac{2\pi(11C_A-2T_fN_f)}{T_f(5C_A+3C_f)N_f-17C_A^2}. \tag{2.3}$$

Formally generalizing N_f from positive integers $\mathbb{N}_+$ to positive real numbers, $\mathbb{R}_+$, one can let N_f approach N_u from below, thereby making $\alpha_{{\rm IR},2\ell}$ arbitrarily small. Thus, for the UV to IR evolution in this regime of N_f, one infers that the theory evolves from weak coupling in the UV to an IRFP in a non-Abelian Coulomb phase (NACP).

Physical quantities at this IRFP can be expressed perturbatively as series expansions in powers of $\alpha_{\rm IR}$ (e.g., [48–52]). However, beyond respective low loop orders, the coefficients in these expansions depend on the scheme used for regularization and renormalization of the theory. Since $\alpha_{\rm IR}$ becomes small as N_f approaches N_u from below, one can reexpress physical quantities as series expansions in the manifestly scheme-independent variable Δ_f. This has the advantage, relative to conventional calculations of $\beta'_{\rm IR}$ as power series in the coupling [50], that the coefficients in the expansion are scheme-independent.

We denote the lowest value of N_f in the NACP as $N_{f,cr}$. In general, the value of $N_{f,cr}$ is not known precisely for the theories under consideration here. A method for obtaining a rough estimate of $N_{f,cr}$ will be reviewed and applied below in Sec. V. Our calculations assume that the IRFP is exact, as is the case in the non-Abelian Coulomb phase $N_{f,cr} < N_f < N_u$. In the analytic expressions and plots given below, this restriction will be understood implicitly.

B. Interval I for SO$(4k+2)$ theories

For our SO(N) theories with $N = 4k+2$, $k = 2, 3, 4$, and chiral fermions in the spinor representation S, one has

$$T_S = 2^{(N/2)-4} \tag{2.4}$$

and

$$C_2(S) = \frac{N(N-1)}{16}, \tag{2.5}$$

so

$$N_u = \frac{11(N-2)}{2^{(N/2)-3}}. \tag{2.6}$$

Here N_u takes on the values (i) 22 for $k = 2$, i.e., SO(10); (ii) $33/4 = 8.25$ for $k = 3$, i.e., SO(14); (iii) $11/4 = 2.75$ for $k = 4$, i.e., SO(18); and (iv) $55/64 = 0.859375$ for $k = 5$, i.e., SO(22), decreasing monotonically toward zero for larger k. Hence, the only asymptotically free SO$(4k+2)$ chiral gauge theories with chiral fermions in the spinor representation are as follows, for physical integral N_f:

(1) SO(10) with $1 \le N_f \le 21$,
(2) SO(14) with $1 \le N_f \le 8$, and
(3) SO(18) with $1 \le N_f \le 2$.

(The theories with $N_f = 0$ are pure gluonic theories and hence are not of interest here.)

For the SO(N) gauge theories with $N = 4k+2$, containing N_f chiral fermions in the spinor representation, N_ℓ is given by

$$N_\ell = \frac{17(N-2)^2}{2^{(N/2)-8}(3N^2 + 77N - 160)}. \tag{2.7}$$

N_ℓ takes on the value (i) $4352/455 = 9.564835$ for SO(10); (ii) $816/251 = 3.250996$ for SO(14); and (iii) $1088/1099 = 0.9899909$ for SO(18). In Table I we list the resultant intervals I in N_f for which the asymptotically free chiral gauge theories of SO$(4k+2)$ type have a two-loop beta function with an IR zero. For each case, we give two ranges, namely one for N_f formally generalized to $\mathbb{R}_+$, and the second for physical, integral $N_f \in \mathbb{N}_+$.

TABLE I. Interval I in terms of N_f, for N_f formally generalized to real numbers, $\mathbb{R}_+$ and for physical, integral values of $N_f \in \mathbb{N}_+$, for the $G = \mathrm{SO}(4k+2)$ chiral gauge theories with $k = 2, 3, 4$, i.e., SO(10), SO(14), and SO(18) and chiral fermions in the spinor representation.

G	I, $N_f \in \mathbb{R}_+$	I, $N_f \in \mathbb{N}_+$
SO(10)	$9.565 < N_f < 22$	$10 \le N_f \le 21$
SO(14)	$3.251 < N_f < 8.25$	$4 \le N_f \le 8$
SO(18)	$0.990 < N_f < 2.75$	$1 \le N_f \le 2$

C. Interval I for E_6 theory

For the E_6 chiral gauge theory with N_f fermions in the fundamental (27-dimensional) representation, F, $C_A \equiv C_2(G) = 12$, $T_F = 3$, and $C_2(F) = 26/3$, so $N_u = 22$. Hence, to maintain asymptotic freedom in this E_6 theory, we require that $N_f < 22$. Furthermore, we calculate that $N_\ell = 408/43 = 9.488372$. Therefore, the interval I for this E_6 theory is

$$\begin{aligned} E_6 : I : 9.488 < N_f < 22 \quad &\text{for } N_f \in \mathbb{R}_+, \\ I : 10 \le N_f \le 21 \quad &\text{for } N_f \in \mathbb{N}_+. \end{aligned} \tag{2.8}$$

In passing, we note that the interval of physical, integral N_f for this E_6 theory is the same as that for the SO(10) theory with chiral fermions in the spinor representation, given in Table I.

D. Scheme-independent expansion for $\beta'_{\rm IR}$

Given the property of asymptotic freedom, β is negative in the region $0 < \alpha < \alpha_{\rm IR}$, and since β is continuous, it follows that this function passes through zero at $\alpha = \alpha_{\rm IR}$ with positive slope, i.e., $\beta'_{\rm IR} > 0$. This derivative $\beta'_{\rm IR}$ has the scheme-invariant expansion

$$\beta'_{\rm IR} = \sum_{j=2}^{\infty} d_j \Delta_f^j. \tag{2.9}$$

As indicated, $\beta'_{\rm IR}$ has no term linear in Δ_f. In general, the calculation of the scheme-independent coefficient d_j requires, as inputs, the ℓ-loop coefficients in the beta function, b_ℓ, for $1 \le \ell \le j$. For our calculation of $\beta'_{\rm IR}$ to $O(\Delta_f^5)$ for vectorial gauge theories in [39], we thus made use of the five-loop beta function from [33]. In the literature, the beta function coefficients have usually been given for a vectorial gauge theory with N_f Dirac fermions in a representation R of the gauge group G. In the case of a chiral gauge theory with fermions in a single representation of the gauge group, one can take over these results with the replacement $N_f \to N_f/2$, reflecting the replacement of

Dirac with chiral fermions. In particular, we can use our previous calculations of the d_j with $2 \le j \le 4$ in [37] and d_5 in [39] in a VGT for the χGTs under consideration, with the correspondence, for a given G and representation R,

$$(d_j)_{\chi\rm GT} = 2^{-j}(d_j)_{\rm VGT}. \tag{2.10}$$

Let us denote the full scaling dimension of an operator $\mathcal{O}$ as $D_{\mathcal{O}}$ and its free-field value as $D_{\mathcal{O},{\rm free}}$. We define the anomalous dimension of $\mathcal{O}$, denoted $\gamma_{\mathcal{O}}$, by $D_{\mathcal{O}} = D_{\mathcal{O},{\rm free}} - \gamma_{\mathcal{O}}$ [53]. Let the full scaling dimension of ${\rm Tr}(F_{\mu\nu}F^{\mu\nu})$ be denoted D_{F^2} (with free-field value 4). At an IRFP, $D_{F^2,{\rm IR}} = 4 + \beta'_{\rm IR}$ [31], so $\beta'_{\rm IR} = -\gamma_{F^2,{\rm IR}}$. Given that the theory at an IRFP in the non-Abelian phase is conformally invariant, there is a conformality bound from unitarity, namely $D_{F^2} \ge 1$ [54]. Since $\beta'_{\rm IR} > 0$, this bound is obviously satisfied.

Discussions of the accuracy of finite-order series expansions of physical quantities in powers of Δ_f were given for vectorial gauge theories in [35–41], and similar comments apply here. Quantitatively, in each of the figures below, for the range of N_f where the $O(\Delta_f^4)$ and $O(\Delta_f^5)$ curves are close to each other, these finite-order calculations are expected to be most accurate. As is evident, this accuracy is greatest at the upper end of the NACP and decreases toward the lower end of the NACP.

III. CALCULATION OF $\beta'_{\rm IR}$ TO $O(\Delta_f^5)$ ORDER FOR SO$(4k+2)$ THEORIES

For the SO(N) theories with $N = 4k+2$ considered here, namely SO(10), SO(14), and SO(18) with N_f fermions in the spinor representation, and N_f in the respective intervals in Table I, we calculate

$$d_2 = \frac{2^{N-1}}{3^2(N-2)(11N^2+101N-224)}, \tag{3.1}$$

$$d_3 = \frac{2^{(3N/2)-4}(3N^2+77N-160)}{3^3(N-2)^2(11N^2+101N-224)^2}, \tag{3.2}$$

$$d_4 = \frac{2^{2N-9}}{3^5(N-2)^3(11N^2+101N-224)^5}[(-3993N^8+967780N^7-3621142N^6+40922980N^5 + 385439463N^4 - 5018429440N^3 + 18335731200N^2 - 28558381056N + 16524705792) + 2^8\cdot 33(11N^2+101N-224)(11N^5-108N^4-1913N^3+17210N^2-50720N+53376)\zeta_3], \tag{3.3}$$

and

$$d_5 = \frac{2^{(5N/2)-9}}{3^6(N-2)^4(11N^2+101N-224)^7}[(464519N^{12} - 18008914N^{11} + 359281505N^{10} - 6749294188N^9 - 41411922215N^8 + 459185530094N^7 - 1073251892065N^6 + 3394219370864N^5 - 32099048433664N^4 + 142779222543872N^3 - 306826058932224N^2 + 326234208075776N - 138794015653888) + 2^5(11N^2+101N-224)(363N^{10} + 38181N^9 + 1922118N^8 - 35102518N^7 - 149165913N^6 + 3972049185N^5 - 27149012488N^4 + 105670102816N^3 - 249943359104N^2 + 325769932800N - 176231645184)\zeta_3 - 2^7\cdot 55(N-2)(11N^2+101N-224)^2(33N^6 - 27N^5 - 9221N^4 + 1879N^3 + 440008N^2 - 2031648N + 2755584)\zeta_5, \tag{3.4}$$

where $\zeta_s = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

Evaluating these general results for the SO(N) theories under consideration, we obtain the following results for $\beta'_{\rm IR}$ calculated up to $O(\Delta_f^5)$ order (in floating-point format):

$$\text{SO}(10):\ \beta'_{{\rm IR},\Delta_f^5} = (3.7704725\times10^{-3})\Delta_f^2 + (3.032105\times10^{-4})\Delta_f^3 - (1.2664165\times10^{-6})\Delta_f^4 - (5.4744784\times10^{-7})\Delta_f^5, \quad \text{where } \Delta_f = 22 - N_f \tag{3.5}$$

$$\text{SO}(14):\ \beta'_{{\rm IR},\Delta_f^5} = (2.266941\times10^{-2})\Delta_f^2 + (4.534786\times10^{-3})\Delta_f^3 + (2.0571128\times10^{-4})\Delta_f^4 - (1.5915337\times10^{-5})\Delta_f^5, \quad \text{where } \Delta_f = 8.25 - N_f \tag{3.6}$$

$$\text{SO}(18):\ \beta'_{{\rm IR},\Delta_f^5} = 0.176468\Delta_f^2 + 0.100265\Delta_f^3 + (2.499877\times10^{-2})\Delta_f^4 + (2.156910\times10^{-3})\Delta_f^5, \quad \text{where } \Delta_f = 2.75 - N_f. \tag{3.7}$$

In Figs. 1–3 we plot the resultant values of $\beta'_{{\rm IR},\Delta_f^p}$ with $2 \le p \le 5$ for these theories.

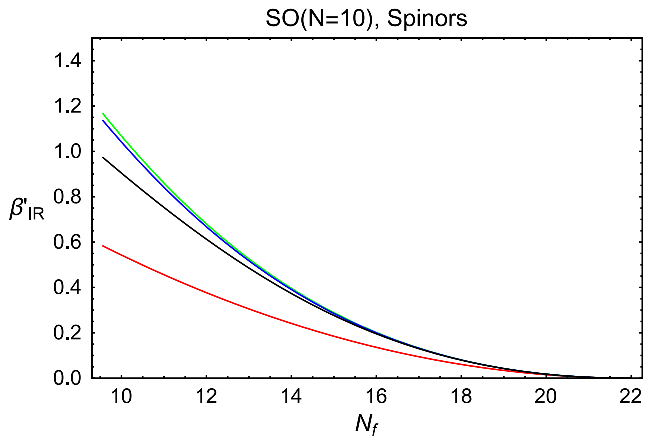

FIG. 1. Plot of $\beta'_{\mathrm{IR},\Delta_f^p}$ (labeled as β'_{IR} on the vertical axis) for an SO(10) chiral gauge theory with fermions in the spinor representation, with $2 \le p \le 5$, as a function of $N_f \in I$. At a given N_f, from bottom to top, the curves (with colors online) refer to $\beta'_{\mathrm{IR},F,\Delta_f^2}$ (red), $\beta'_{\mathrm{IR},\Delta_f^5}$ (black), $\beta'_{\mathrm{IR},\Delta_f^4}$ (blue), and $\beta'_{\mathrm{IR},\Delta_f^3}$ (green).

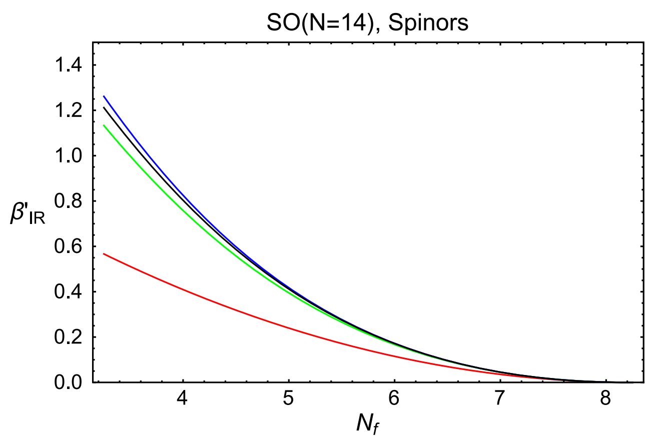

FIG. 2. Plot of $\beta'_{\mathrm{IR},\Delta_f^p}$ (labeled as β'_{IR} on the vertical axis) for an SO(14) chiral gauge theory with fermions in the spinor representation, with $2 \le p \le 5$, as a function of $N_f \in I$. At a given N_f, from bottom to top, the curves (with colors online) refer to $\beta'_{\mathrm{IR},F,\Delta_f^2}$ (red), $\beta'_{\mathrm{IR},\Delta_f^3}$ (green), $\beta'_{\mathrm{IR},\Delta_f^5}$ (black), and $\beta'_{\mathrm{IR},\Delta_f^4}$ (blue).

Concerning the signs of these coefficients, d_2 and d_3 are manifestly positive (for a general G and R) [37], while the signs of d_4 and d_5 depend on the theory. For our SO(N) theories with $N = 4k + 2$, we find that the signs of d_4 and d_5 depend on N; they are both negative for SO(10); they are mixed for SO(14); and they are both positive for SO(18). We summarize these results in Table II. It is interesting to compare these findings with the corresponding signs that we found for the d_j in vectorial gauge theories. For example, we may recall the signs for the d_j with j up to 5, as summarized in Table VII of [39] for vectorial gauge theories with gauge group SU(N_c) and various fermion representations R. As is evident from that table, for the fundamental representation (F) d_4 and d_5 are both negative

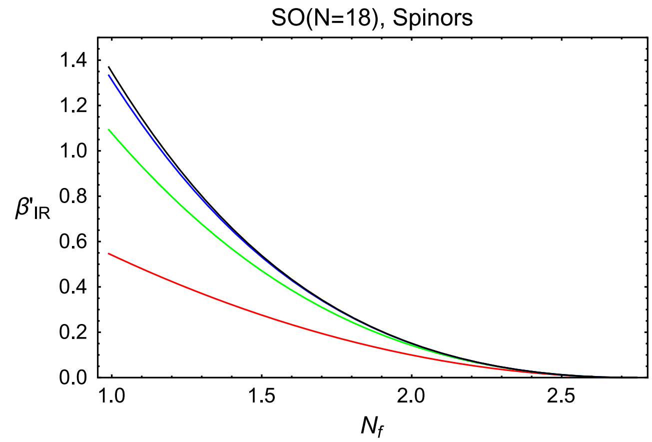

FIG. 3. Plot of $\beta'_{\mathrm{IR},\Delta_f^p}$ (labeled as β'_{IR} on the vertical axis) for an SO(18) chiral gauge theory with fermions in the spinor representation, with $2 \le p \le 5$, as a function of $N_f \in I$. At a given N_f, from bottom to top, the curves (with colors online) refer to $\beta'_{\mathrm{IR},F,\Delta_f^2}$ (red), $\beta'_{\mathrm{IR},\Delta_f^3}$ (green), $\beta'_{\mathrm{IR},\Delta_f^4}$ (blue), and $\beta'_{\mathrm{IR},\Delta_f^5}$ (black).

for all N_c, while $d_4 > 0$ and $d_5 < 0$ for the adjoint (A) and symmetric rank-2 tensor, S_2. For the antisymmetric rank-2 tensor representation, A_2, we found that the sign of d_4 depends on N_c, while d_5 is negative for all N_c. In [41] we carried out corresponding scheme-independent calculations of the d_j coefficients for vectorial gauge theories based on the gauge groups SO(N) with $N \ge 3$ [the SO(2) $\approx$ U(1) gauge theory being excluded by the requirement of asymptotic freedom] and Sp(N) with even $N \ge 2$, containing these fermion representations, F, A, S_2, and A_2. For example, we found that for the fundamental representation, d_4 is positive for $N = 3$ and negative for $N \ge 4$, while d_5 is negative for $N \ge 3$. Our present results may also be compared with the properties of a vectorial SU(N_c) gauge theory with $\mathcal{N} = 1$ supersymmetry; for this theory, the lower end of the NACP is known exactly, and, although there is no exact expression for β'_{IR}, it has been established that β'_{IR} vanishes (quadratically) at the lower end of the NACP [55]. In the supersymmetric case, this vanishing of β'_{IR}, and hence also the vanishing of the anomalous dimension of $F^a_{\mu\nu}F^{a\mu\nu}$, can be understood, via duality arguments [56], as reflecting the fact that, although the IRFP in the original ("electric") theory is quite strongly

TABLE II. Signs of the d_j coefficients for $2 \le j \le 5$ for the chiral gauge theories considered here. (Recall the general results that $d_1 = 0$, and $d_2 > 0$ and $d_3 > 0$ for all G and R.)

j	SO(10)	SO(14)	SO(18)	E_6
2	+	+	+	+
3	+	+	+	+
4	−	+	+	−
5	−	−	+	−

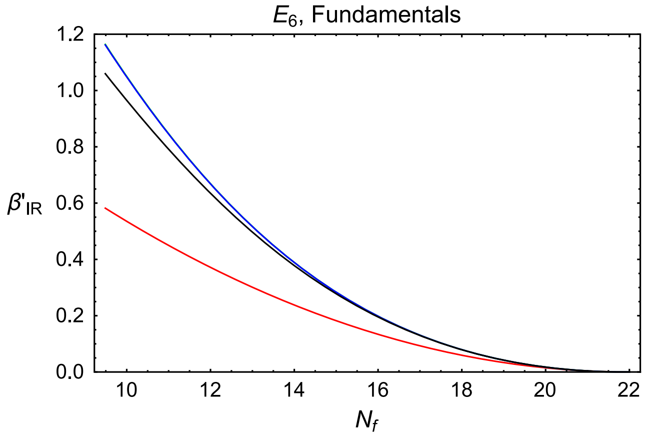

FIG. 4. Plot of $\beta'_{{\rm IR},\Delta_f^p}$ (labeled as $\beta'_{\rm IR}$ on the vertical axis) for an E_6 chiral gauge theory with fermions in the fundamental representation, with $2 \le p \le 5$, as a function of $N_f \in I$. At a given N_f, from bottom to top, the curves (with colors online) refer to $\beta'_{{\rm IR},F,\Delta_f^2}$ (red), $\beta'_{{\rm IR},\Delta_f^5}$ (black), $\beta'_{{\rm IR},\Delta_f^4}$ (blue), and $\beta'_{{\rm IR},\Delta_f^3}$ (green). Note that the curves for $p = 3$ and $p = 4$ are too close to each other to be distinguished in the plot.

coupled, the IRFP in the dual ("magnetic") theory goes to zero, i.e., this theory becomes free, at this lower end of the NACP. We have calculated the d_j coefficients up to $j = 3$ in this vectorial supersymmetric gauge theory and have found that (in addition to the positive d_1 and d_2), d_3 is negative for all N_c [40].

IV. CALCULATION OF $\beta'_{\rm IR}$ TO $O(\Delta_f^5)$ ORDER FOR E_6 THEORY

For the E_6 theory with N_f fermions in the fundamental (27-dimensional) representation, we calculate

$$d_2 = \frac{1}{269} = 3.717472 \times 10^{-3}, \tag{4.1}$$

$$d_3 = \frac{43}{2\cdot(269)^2} = 2.971214 \times 10^{-4}, \tag{4.2}$$

$$d_4 = \frac{660297341}{2^3\cdot 3^2\cdot (269)^5} - \frac{2\cdot 14355}{(269)^4}\zeta_3 = -(0.7999706\times 10^{-7}), \tag{4.3}$$

and

$$d_5 = -\frac{328284821696663}{2^5\cdot 3^4\cdot (269)^7} - \frac{2^8\cdot 18928393}{3^3\cdot (269)^6}\zeta_3 + \frac{2^3\cdot 251075}{(269)^5}\zeta_5 = -(3.3333007\times 10^{-7}). \tag{4.4}$$

Hence, to $O(\Delta_f^5)$, with $\Delta_f = 22 - N_f$ here (in floating-point format),

$$\beta'_{{\rm IR},\Delta_f^5} = (3.717472\times 10^{-3})\Delta_f^2 + (2.971214\times 10^{-4})\Delta_f^3 - (0.7999706\times 10^{-7})\Delta_f^4 - (3.3333007\times 10^{-7})\Delta_f^5. \tag{4.5}$$

Thus, as was the case with SO(10), we find that both d_4 and d_5 are negative. These results are summarized in Table II. In Fig. 4 we plot the resultant values of $\beta'_{{\rm IR},\Delta_f^p}$ with $2 \le p \le 5$ for this E_6 theory. Because $|d_4| \ll d_3$, the curve for $\beta'_{{\rm IR},\Delta_f^4}$ is too close to the curve for $\beta'_{{\rm IR},\Delta_f^3}$ to be distinguished from it in the plot.

V. BEHAVIOR OF $\beta'_{\rm IR}$ NEAR THE LOWER END OF THE NON-ABELIAN COULOMB PHASE

In this section we comment on the behavior of $\beta'_{\rm IR}$ near the lower end of the non-Abelian Coulomb phase, as one is moving into the region of strong coupling. For this purpose, we first review a method of estimating the value of $N_{f,cr}$ at this lower end of the NACP that was first used in vectorial gauge theories and later applied to chiral gauge theories.

In a vectorial gauge theory with a gauge group G and massless fermions in a representation R of G, the most attractive channel for bilinear fermion condensation is $R \times \bar{R} \to 1$, where here 1 denotes the singlet representation. An approximate solution of the Schwinger-Dyson equation for the fermion propagator in the iterated one-gluon exchange approximation yields a rough estimate of the minimum strength of the gauge coupling, denoted α_{cr}, that leads to spontaneous chiral symmetry breaking via the formation of a bilinear fermion condensate. This is given by the condition (see [6] and references therein)

$$\frac{3\alpha_{cr}C_2(R)}{\pi} \simeq 1. \tag{5.1}$$

In an asymptotically free (anomaly-free) chiral gauge theory with gauge group G and (massless) fermions ψ_{iL}, $i = 1,\ldots,N_f$, in a representation R of G, let us consider the decomposition of the direct product $R\times R$ into (irreducible) representations of G that occur in a fermion bilinear condensate of the form $\langle \psi_{iL}^T C \psi_{jL}\rangle$, namely

$$R\times R = R_1 + \cdots R_p, \tag{5.2}$$

where here p denotes the number of representations that occur in the direct product. For example, in SO(10), one has [57]

$$16\times 16 = 10_s + 120_a + 126_s, \tag{5.3}$$

while in E_6, one has [57]

$$27\times 27 = \overline{27}_s + \overline{351}_a + \overline{351}'_a, \tag{5.4}$$

where the subscripts s and a denote symmetric and antisymmetric combinations in the direct products. The MAC is defined as the channel that yields a bilinear fermion condensate whose quadratic Casimir invariant is minimal [5]. That is, if one defines

$$\Delta C_2 = 2C_2(R) - C_2(R_{\rm cond}), \tag{5.5}$$

where $R_{\rm cond}$ denotes the representation of the condensate, then the MAC is defined as the channel such that $C_2(R_{\rm cond})$ is minimal, i.e., ΔC_2 is maximal. The analog of Eq. (5.1) for a chiral gauge theory is then

$$\frac{3\alpha_{cr}\Delta C_2}{2\pi} \simeq 1. \tag{5.6}$$

[Note that $\Delta C_2 = 2C_2(R)$ for a vectorial gauge theory.] This rough criterion was used in a number of papers studying self-breaking of strongly coupled chiral gauge theories [5,7]. In this approach, one equates the value of $\alpha_{\rm IR}$ calculated to the maximal scheme-independent order, i.e., two-loop order, denoted $\alpha_{{\rm IR},2\ell}$, with the value of α_{cr} from Eq. (5.6) and then solves for $N_{f,cr}$. One of the most extensive comparisons of the results from this method was for a vectorial SU(N_c) theory with fermions in the fundamental representation. In a vectorial gauge theory, the number of Dirac (D) fermions is 1/2 the number of chiral components of fermions, so to discuss this vectorial theory, we define $N_{f,D} = N_f/2$ and thus $N_{f,D,cr} = N_{f,cr}/2$. The above approach for the vectorial SU(N_c) theory yielded the result [6]

$$N_{f,D,cr} = \frac{2N_c(50N_c^2 - 33)}{5(5N_c^2 - 3)}, \tag{5.7}$$

i.e., $N_{f,D,cr} \simeq 12$ for SU(3). We have obtained estimates of $N_{f,D,cr}$ in this theory and others by calculating scheme-independent series expansions for the anomalous dimension of the (gauge-invariant) fermion bilinear, estimating results of an all-order summation of this series, and equating the result to the upper bound from conformal invariance in the NACP [36,37,39]. For SU(3), we obtained $N_{f,D,cr} \simeq 8$–9 [36], in agreement with the estimates from lattice simulations in [20–23] (see also [24]). Hence, at least in this case of an SU(3) vectorial gauge theory with fermions in the fundamental representation, the estimate (5.7) of $N_{f,D,cr}$ obtained from equating $\alpha_{{\rm IR},2\ell}$ with α_{cr} appears to be somewhat larger than the actual value of $N_{f,D,cr}$ as inferred from lattice measurements (although there is not a complete consensus among lattice groups on the value of $N_{f,D,cr}$ for this theory [20–24,26]).

Bearing this in mind, we may proceed to use this method to obtain a rough estimate of $N_{f,cr}$ for our present chiral gauge theories and evaluate $\beta'_{\rm IR}$ at this value of N_f, to the order $O(\Delta_f^5)$ to which we have calculated it. We begin with our SO(10) theory, where the spinor representation has dimension 16. Bilinear fermion condensates in this theory involve the direct product (5.3). The MAC is $16 \times 16 \to 10_s$. Calculating $\alpha_{{\rm IR},2\ell}$ and α_{cr} via the above method for this condensation channel, setting $\alpha_{{\rm IR},2\ell} = \alpha_{cr}$, and solving for $N_{f,cr}$, we obtain the estimate $N_{f,cr} \simeq 14.7$. We next proceed to combine this rough estimate of $N_{f,cr}$ at the lower end of the NACP with our scheme-independent calculation of $\beta'_{\rm IR}$ for this theory. From our calculation to the highest order, namely $O(\Delta_f^5)$, as presented in Fig. 1, we infer that $\beta'_{\rm IR} \sim 0.3$ as N_f decreases toward the neighborhood of this value of N_f (while still in the NACP). Corresponding estimates may be made in a similar way for SO(14) and SO(18). For E_6, we use the fact that the MAC for bilinear condensation is $27 \times 27 \to \overline{27}_s$. With the above method, we obtain $N_{f,cr} = 14.2$, and observe that $\beta'_{\rm IR} \sim 0.4$ as N_f decreases toward this value of N_f from within the NACP. We emphasize that higher-order terms $d_j\Delta_f^j$ with $j \geq 6$ may significantly change these values of $\beta'_{\rm IR}$ and, separately, that the estimate of $N_{f,cr}$ calculated by this method is only a rough estimate. In future work, it will be of interest to investigate the behavior of $\beta'_{\rm IR}$ further in the vicinity of the lower end of the NACP.

VI. CONCLUSIONS

In conclusion, in this paper we have presented scheme-independent calculations, up to order $O(\Delta_f^5)$ inclusive, of $\beta'_{\rm IR}$ at an IR fixed point in the non-Abelian Coulomb phase of several asymptotically free (and anomaly-free) chiral gauge theories, namely theories with the gauge groups SO($4k+2$), $k = 2, 3, 4$, containing various numbers N_f of chiral fermions in the spinor representation, and a theory with the gauge group E_6, containing N_f chiral fermions in the fundamental representation. These scheme-independent expansions have an advantage, relative to conventional expansions in powers of the gauge coupling at the IRFP, that at each order they maintain the property of scheme independence of the exact $\beta'_{\rm IR}$. The derivative $\beta'_{\rm IR}$ is of physical interest, since it is equivalent to the anomalous dimension of the operator ${\rm Tr}(F_{\mu\nu}F^{\mu\nu})$ and, related to this, since it appears in an expansion of the trace of the energy-momentum tensor of the theory near the IR fixed point. We have combined our series calculations of $\beta'_{\rm IR}$ with estimates of the value of N_f at the lower end of the non-Abelian Coulomb phase to obtain an estimate of $\beta'_{\rm IR}$ in this vicinity. Our results contribute to the knowledge of conformal field theories. Quasiconformal gauge theories have also been of interest as possible ultraviolet extensions of the Standard Model, and these have led to the study of the properties of the theories for fermion numbers slightly below the lower end of the NACP. Our methods provide a different and complementary way to get information about the properties of the theory in this region by approaching this lower end of the NACP phase from within this phase.

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APPENDIX: SOME GROUP-THEORETIC QUANTITIES

In this appendix we discuss some relevant group-theoretic quantities. The generators of the Lie algebra of G in the representation R are denoted T_R^a, where a is a group index. These satisfy $[T_R^a, T_R^b] = i f^{abc} T_R^c$. We denote the dimension of a given representation R as $d_R = \dim(R)$, and denote A as the adjoint representation. The trace invariant is defined by $\mathrm{Tr}_R(T_R^a T_R^b) = T(R)\delta^{ab}$ and the quadratic Casimir invariant $C_2(R)$ is given by $T_R^a T_R^a = C_2(R) I$, where I is the $d_R \times d_R$ identity matrix. For a fermion f in R, a compact notation is $T_f \equiv T(R)$, $C_f \equiv C_2(R)$, and $C_A \equiv C_2(A)$. As discussed in [41], although these group invariants depend on a convention for the normalization of the structure constants f^{abc}, the d_j are independent of this convention.

The general expressions for the coefficients d_4 and d_5 [37,39] involve certain quartic group invariants [58]. For SO(N) with $N = 4k + 2$, we calculate these to be

$$\mathrm{SO}(N), \qquad R = \text{spinor}:$$
$$\frac{d_R^{abcd} d_A^{abcd}}{d_A} = -\frac{2^{(N/2)-8}(N-2)(N^2-22N+52)}{3},$$
$$\frac{d_R^{abcd} d_R^{abcd}}{d_A} = \frac{2^{N-15}(13N^2-61N+76)}{3}. \tag{A1}$$

We gave the quartic invariant $d_A^{abcd} d_A^{abcd}/d_A$ for SO(N) previously in [41]; for reference, it is

$$\frac{d_A^{abcd} d_A^{abcd}}{d_A} = \frac{(N-2)(N^3-15N^2+138N-296)}{24}. \tag{A2}$$

For E_6 with $R = F$, the fundamental representation, we calculate

$$E_6: \quad \frac{d_A^{abcd} d_A^{abcd}}{d_A} = 540, \qquad \frac{d_F^{abcd} d_A^{abcd}}{d_A} = 90,$$
$$\frac{d_F^{abcd} d_F^{abcd}}{d_A} = 15. \tag{A3}$$

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