Determination of baryon-baryon elastic scattering phase shift from finite volume spectra in elongated boxes

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The relations between the baryon-baryon elastic scattering phase shifts and the two-particle energy spectrum in the elongated box are established. We studied the cases with both the periodic boundary condition and twisted boundary condition in the center of mass frame. The framework is also extended to the system of nonzero total momentum with periodic boundary condition in the moving frame. Moreover, we discussed the sensitivity functions $\sigma(q)$ that represent the sensitivity of higher scattering phases. Our analytical results will be helpful to extract the baryon-baryon elastic scattering phase shifts in the continuum from lattice QCD data by using elongated boxes.

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I. INTRODUCTION

The phase shift for strong elastic scattering of two hadrons is encoded in the basic knowledge on the strong interaction, one of the four elementary interactions in nature. The phase shift δ_l is related to the phase difference between the outgoing and the ingoing l wave outside the interaction range, and parametrizes the complicated form of this interaction. The scattering length can be extracted from its effective range expansion. The knowledge from the phase shift also serves to study the resonance, such as in Refs. [1–8]. In these references, the authors determined the masses, the widths of the resonances $K^*(892)$, ρ, and a_0 by calculating the scattering phase shifts of πK, ηK coupled channels, $\pi\pi$, $K\bar{K}$ coupled channels, and $\pi\eta$, $K\bar{K}$ coupled channels, etc. A phase shift crossing through $\pi/2$ is often a indication of a resonance, of which the sharpness of the rise allows to determine the property of resonance according to Breit-Wigner type functional form.

In the aspect of particle and nuclear physics, unraveling the origin of baryon forces based on the quantum chromodynamics (QCD) is one of the most challenging issues. The precise information on nuclear and hyperon forces serves as the key ingredients to calculate properties of nuclei, dense matter, and the structure of neutron stars [9–13]. According to the baryon-baryon scattering phase shifts measured in experiments, people established the relationships on realistic nuclear forces and the fundamental theories such as QCD. However, scattering experiments with hyperons are very difficult because of their short lives. Then, hyperon forces suffer from large uncertainties. Under these circumstances, it is most desirable to carry out the first-principles calculations of baryon forces by lattice QCD. By measuring appropriate correlation functions, energy eigenvalues of two-particle states in a finite box can be obtained. Lüscher found out a relation, now commonly known as Lüscher's formula, which relates the energy of two-particle state in a finite box of size L, i.e., $E(L)$, to the elastic scattering phase shift $\delta(E(L))$ of the two particles in the continuum [14–17].

Owing to the advent of Lüscher's formula, various lattice studies, both quenched [18–25] and unquenched [26–35], have been performed over the years to investigate the scattering of hadrons. The original Lüscher's formula was derived for systems of two identical spinless particles in center-of-mass (COM) frame with periodic boundary condition in a cubic box. It restrains the applicability of the formalism to the general hadron scattering. To overcome the difficulties, one can of course consider asymmetric volumes [36,37], or boosting the system to a frame that is different from COM frame [38–42]. Both will enhance the energy resolution of the problem. Another possible generalization is to use the so-called twisted boundary conditions advocated in Refs. [43–46]. Then, generalizations to particles with spin [47–49] are also possible. For example, in Refs. [50,51], Lüscher's formulas have been extended to elastic scattering of baryons. Generalized Lüscher's formula to the case of inelastic scattering commonly encountered in hadronic physics, is also important. Some attempts have been made over the years, see Refs. [50–54].

In a cubic box, the three momenta of a single particle are quantized according to $\mathbf{k} = (2\pi/L)\mathbf{n}$ with $\mathbf{n} \in \mathcal{Z}^3$, where $\mathcal{Z}^3$ is the set of 3-tuples of integers. In real simulations, large values of L are needed to control lattice artifacts because of

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these non-zero momentum modes. However, in the cubic lattice, many low momentum modes are energy degenerates such as modes (1,0,0), (0,1,0) and (0,0,1) since they relate to each other by the cubic symmetry. This means that the second lowest energy level of the particle with nonvanishing momentum corresponds to $\mathbf{n} = (1, 1, 0)$. If one would like to measure these states on the lattice, even larger values of L should be used. One way to remedy this is to use an elongated box of size $L \times L \times \eta L$, a specific asymmetric box, where η is the elongation factor in the z-direction. We would have two different low-lying one-particle energy with nonzero momenta corresponding to $\mathbf{n} = (1, 0, 0)$ and $\mathbf{n} = (0, 0, 1)$, respectively. This implies that the energy resolution can be enhanced by using the asymmetric box. The technique of asymmetric box has first been applied to the calculation of pion-pion elastic scattering [55] and later used in the study of $D^{*+} - D_1^0$ system [56]. In a recent comprehensive study, Frank X. Lee and Andrei Alexandru have derived Lüscher's phase shift formulas for mesons and baryons elastic scattering in elongated boxes [57]. In their work, various scenarios, such as moving and zero-momentum states in cubic and elongated boxes, are systematically studied and relations between them are also clarified. The derived formulas are applicable to a wide set of meson-meson and meson-baryon elastic scattering processes. Therefore they can be applied to investigate various resonances, such as a_1, Δ, Roper, and so on. In this paper, we would like to synthesize the above mentioned generalizations by trying to seek phase shift formulas those are applicable for baryon-baryon elastic scattering in the elongated box, which is helpful to study properties of the low energy baryon-baryon scattering by using lattice simulations.

Recently, in Ref. [47], nucleon-nucleon scattering lengths in the $^1S_0$ channel and $^3S_1$-$^3D_1$ coupled channels are determined. In Ref. [58], based on dispersion theory, the scattering length of $\Lambda - \Lambda$ and $\Xi - N$ are extracted. In Refs. [59–63], the authors investigated the scattering phase shifts of $N - \Omega$, $\Omega - \Omega$, $\Lambda_c - N$ and $\Sigma_c - N$ through the interaction potential calculated by the equal-time Nambu-Bethe-Salpeter (NBS) wave function measured on the lattice. In principle, these baryon-baryon scattering can also be studied by using lattice QCD simulations in elongated boxes.

The organization of the paper is as follows. In Sec. II, we start out by discussing scattering phase shift using a quantum-mechanical model in COM frame in an elongated box. Then, in Sec. III, we generalize the scattering phase shift formulas in the elongated box: in subsection III A, we provide scattering phase shift formulas for baryon-baryon scattering in MF, and in subsection III B, we show the scattering phase shift formulas for baryon-baryon scattering in COM frame with the twisted boundary condition. In Sec. IV, we obtain the results in a cubic box, and then perform the consistency checks and have confirmed the validation on the results in elongated boxes by comparing the two cases. In Sec. V, we discuss the low momentum expansion of scattering phase shift. The functions $\sigma(q)$ representing the sensitivity of higher scattering phases are obtained. In Sec. VI, we give a brief summary, and discuss the possible applications of our derived formulas in real lattice simulations.

II. PHASE SHIFT FORMULAS FOR BARYON-BARYON SCATTERING IN COM FRAME IN ELONGATED BOXES

In this section, we try to obtain phase shift formulas that are applicable for baryon-baryon scattering in COM frame in elongated boxes. Basically, we first derive the scattering phase shift formulas for baryon-baryon scattering in a cubic box in COM frame, and then generalize the formulas to the case in elongated boxes.

A. Brief review of the basic Lüscher's formula

In nonrelativistic quantum mechanics, after factoring out the center-of-mass motions, the asymptotic form of the wave function is written as

$$\psi_{s\nu}(\mathbf{r}) \xrightarrow{r\to\infty} \left(\chi_{s\nu} e^{i\mathbf{k}\cdot\mathbf{r}} + \sum_{s'\nu'} \chi_{s'\nu'} M_{s'\nu';s\nu} \frac{e^{ikr}}{r} \right). \quad (1)$$

In the remote past, the wave function reduces to an incident plane wave with prescribed quantum numbers. Here $\chi_{s\nu}$ designates spin-wave function, an eigenstate of spin angular momentum of s^2 and ν with the eigenvalues given by $s = 0$, $\nu = 0$ (singlet state), or $s = 1$, $\nu = 1, -1, 0$ (triplet state). $M_{s'\nu',s\nu}$ is the scattering amplitude. When one chooses the z-axis to coincide with $\mathbf{k}$, the scattering amplitudes introduced above is related to the S-matrix elements as [64]

$$\begin{aligned} M_{s'\nu';s\nu}(\mathbf{k}\cdot\mathbf{r}) = \frac{1}{2ik} \sum_{l'=0}^{\infty} \sum_{l=0}^{\infty} \sum_{J=l-1}^{l+1} \sqrt{4\pi(2l+1)}\, i^{(l-l')} \\ \times (S^J_{l's';ls} - \delta_{l'l}\delta_{s's}) \langle JM|l'm'; s'\nu'\rangle \\ \times \langle JM|l0; s\nu\rangle Y_{l'm'}(\mathbf{r}). \end{aligned} \quad (2)$$

From the Clebsch-Gordan coefficients $\langle JM|l'm'; s'\nu'\rangle$ and $\langle JM|l0; s\nu\rangle$, we find $M = \nu$ and $m = M - \nu'$. In the previous equation, we have ignored the sum over M and m for a given pair of ν and ν'. It is known that the phase shift enters via S-matrix. Lüscher obtained a relation, which relates the energy of two-particle state in a finite box of size L, i.e., $E(L)$, to the elastic scattering phase $\delta(E(L))$ of the two particles in the continuum. This inspires us that if one calculates two-particle scattering phase shifts, one should derive the corresponding Lüscher's formula firstly.

With the help of Eqs. (1) and (2), we obtain the following asymptotic form of the wave function as

$$\psi_{s\nu}(\mathbf{r}) = \sum_{s'JMll'} \sqrt{4\pi(2l+1)}\, W^J_{l's';ls}\langle JM|l0;s\nu\rangle Y^{l's'}_{JM}(\mathbf{r}), \tag{3}$$

where $Y^{l's'}_{JM}(\mathbf{r})$ is the spin spherical harmonics whose explicit form is given by

$$Y^{ls}_{JM}(\mathbf{r}) = \sum_{m\nu} Y_{lm}(\mathbf{r})\chi_{s\nu}\langle JM|lm;s\nu\rangle. \tag{4}$$

In Eq. (3), $W^J_{l's';ls}(r)$ is the radial wave function of the two-particle scattering state. In the large r region, when the two-particle interaction is ignored and the potential vanishes, the wave function $W^J_{l's';ls}(r)$ has the following asymptotic form:

$$W^J_{l's';ls}(r) = \frac{1}{2ikr} i^{(l-l')}[S^J_{l's';ls}e^{ikr} + (-1)^{l+1}e^{-ikr}\delta_{ll'}\delta_{ss'}]. \tag{5}$$

For baryon-baryon scattering, we enclose the two-particle system in a cubic box of size L with periodic boundary condition. In the outer region where the potential vanishes, the wave function becomes

$$\psi(\mathbf{r}) = \sum_{s'JMll's} [F_{JMls}W^J_{l's';ls}(r)]Y^{l's'}_{JM}(\mathbf{r}). \tag{6}$$

Then, we can expand the wave function into a linear superposition of the singular periodic solutions, i.e., $G_{JMls}(\mathbf{r};k^2)$, of the Helmholtz equation in the outer region. Thus, one can get

$$\psi(\mathbf{r}) = \sum_{s=0}^{1}\sum_{l=0}^{\infty}\sum_{J=l-1}^{l+1}\sum_{M=-J}^{J} v_{JMls}G_{JMls}(\mathbf{r};k^2). \tag{7}$$

Here, $G_{JMls}(\mathbf{r};k^2)$ can be further expanded in terms of spherical harmonics:

$$G_{JMls} = \frac{(-1)^l k^{l+1}}{4\pi}\Bigg(Y^{ls}_{JM}n_l(kr) + \sum_{J'M'l'}\mathcal{M}^{\mathbf{c}(s)}_{JMl;J'M'l'}(k^2)Y^{l's}_{J'M'}j_{l'}(kr)\Bigg), \tag{8}$$

where the explicit form of $\mathcal{M}^{\mathbf{c}(s)}_{JMl;J'M'l'}(q^2)$ with $\mathbf{k} = (2\pi/L)\mathbf{q}$ is written as

$$\mathcal{M}^{\mathbf{c}(s)}_{JMl;J'M'l'}(q^2) = \sum_{mm'\nu}\langle JM|lm;s\nu\rangle\langle J'M'|l'm';s\nu\rangle \times \mathcal{M}^{\mathbf{c}}_{lm;l'm'}(q^2), \tag{9}$$

with

$$\mathcal{M}^{\mathbf{c}}_{lm;l'm'}(q^2) = \sum_{t=|l-l'|}^{l+l'}\sum_{t_z=-t}^{t}\frac{(-1)^l i^{l+l'}}{\pi^{3/2}q^{t+1}} \times Z^{\mathbf{c}}_{tt_z}(q^2,\eta)\langle l0t0|l'0\rangle\langle lmtt_z|l'm'\rangle \times\sqrt{\frac{(2l+1)(2t+1)}{(2l'+1)}}. \tag{10}$$

For convenience, we introduce the short-hand function $\omega^{\mathbf{c}}_{lm}(q^2)$ for the zeta function,

$$\omega^{\mathbf{c}}_{lm}(q^2) = \frac{Z^{\mathbf{c}}_{lm}(q^2)}{\pi^{3/2}q^{l+1}}. \tag{11}$$

Then the zeta function in cubic boxes is written as

$$Z^{\mathbf{c}}_{lm}(q^2) = \sum_{\mathbf{n}}\frac{\mathcal{Y}_{lm}(\mathbf{n})}{\mathbf{n}^2 - q^2}, \tag{12}$$

where $\mathcal{Y}_{lm}(\mathbf{n}) = \mathbf{n}^l Y_{lm}(\theta,\phi)$ with $\mathbf{n} = (n_x,n_y,n_z) \in \mathcal{Z}^3$, and $\mathcal{Z}^3$ is the set of 3-tuples of integers. One can get four linear equations of the coefficients by comparing Eq. (6) with Eq. (7). If there exist nontrivial solutions for them, the determinant of the corresponding matrix must vanish which leads to the basic form of Lüscher's formula, i.e.,

$$\Bigg|\sum_{l''}(S^J_{l''s';ls} - \delta_{ll''}\delta_{ss'})\mathcal{M}^{\mathbf{c}(s')}_{JMl'';J'M'l'}(k^2) - i\delta_{JJ'}\delta_{MM'}(S^J_{l's';ls} + \delta_{ll'}\delta_{ss'})\Bigg| = 0. \tag{13}$$

B. Extension in elongated boxes in COM frame

In the following, we generalize the basic form of Lüscher's formula in the cubic box to that in the elongated box. In this case, one can expand the wave function of the system in the outer region as series of the modified matrix elements $\mathcal{M}^{(s)}_{JMl;J'M'l'}(q^2,\eta)$, i.e.,

$$\mathcal{M}^{(s)}_{JMl;J'M'l'}(q^2,\eta) = \sum_{mm'\nu}\langle JM|lm;s\nu\rangle\langle J'M'|l'm';s\nu\rangle \times \mathcal{M}_{lm;l'm'}(q^2,\eta), \tag{14}$$

$$\mathcal{M}_{lm;l'm'}(q^2,\eta) = \sum_{t=|l-l'|}^{l+l'}\sum_{t_z=-t}^{t}\frac{(-1)^l i^{l+l'}}{\pi^{3/2}\eta q^{t+1}} \times Z_{tt_z}(1,q^2,\eta)\langle l0t0|l'0\rangle\langle lmtt_z|l'm'\rangle \times\sqrt{\frac{(2l+1)(2t+1)}{(2l'+1)}}. \tag{15}$$

Hereafter for convenience, we introduce the short-hand function $\omega_{lm}(q^2,\eta)$ with the zeta function as follows:

$$\omega_{lm}(q^2,\eta) = \frac{Z_{lm}(q^2,\eta)}{\pi^{3/2}\eta q^{l+1}} \tag{16}$$

Then the generalized zeta function in z-elongated boxes is given by

$$Z_{lm}(q^2,\eta) = \sum_{\mathbf{n}} \frac{\mathcal{Y}_{lm}(\tilde{\mathbf{n}})}{\tilde{\mathbf{n}}^2 - q^2}, \tag{17}$$

where $\mathcal{Y}_{lm}(\tilde{\mathbf{n}}) = \tilde{\mathbf{n}}^l Y_{lm}(\theta,\phi)$. Here, the modified index $\tilde{\mathbf{n}}$ is $\tilde{\mathbf{n}} = (n_x, n_y, n_z/\eta)$. Apart from the above-mentioned substitutions, an extra attention should also be paid to the difference in symmetry. In order to discuss Lüscher's formula in the elongated box, the symmetry of two-particle system is no longer O_h but reduces to D_{4h} group. This group contains 16 elements that can be divided into the following ten conjugate classes: $A_1^\pm$,$A_2^\pm$,$B_1^\pm$,$B_2^\pm$, and $E^\pm$. For instance, for $J = 0,\ 1,\ 2$ when the cutoff momentum $\Lambda = 2$, the decomposition into irreducible representation is given by $0^\pm = A_1^\pm,\ 1^\pm = A_2^\pm \oplus E^\pm,\ 2^\pm = A_1^\pm \oplus B_1^\pm \oplus B_2^\pm \oplus E^\pm$, respectively [57]. In a definite irreducible representation of the D_{4h} group, the basis vectors are labeled as $|\Gamma,\xi,J,l,s,n\rangle$, where Γ denotes the representation; ξ runs from 1 to the number of the dimensions, and n runs from 1 to the multiplicity of the representation. This basis can be expressed by linear combinations of $|JMls\rangle$. The corresponding matrix $\mathcal{M}$ is diagonal with respect to Γ and ξ by Schur's lemma [16]. If there is no multiplicity, the labels $n = 1$ and $n' = 1$ can be dropped. Therefore, in a definite symmetry sector Γ, the explicit form of Lüscher's formula is given by Eq. (18).

$$\left| \sum_{l''} (S^J_{l''s';ls} - \delta_{ll''}\delta_{ss'}) \mathcal{M}^{(s')}_{Jl'';J'l'}(\Gamma) - i\delta_{JJ'}(S^J_{l's';ls} + \delta_{ll'}\delta_{ss'}) \right| = 0. \tag{18}$$

Before writing out the explicit form of the phase shift formula, one should exploit the symmetry properties to simplify the $\mathcal{M}$ matrix and the parity of the two-particle system. First, the matrix is hermitian which constrains half of the off-diagonal elements. Furthermore, a lot of short-hand functions $\omega_{lm}(q^2,\eta)$ vanish to satisfy certain constraints, which can be traced back to how the zeta function behaves under the symmetry operations in the elongated box. The properties follow Ref. [57]. Second, the total angular momentum for two scattering particles with spin s_1 and s_2 is $J = s_1 + s_2 + l$ with l the relative orbital angular momentum dubbed "partial waves." For the asymptotic states, the two particles are far away from each other so that they are not interacting. Then, we can label the states with s and s_i, where $s = s_1 + s_2$ is the total spin. The total angular momentum J is conserved during the scattering process, but both l and s may change. For the baryon-baryon elastic scattering considered in this paper, the total spin s can take 0 (singlet state) or 1 (triplet states). Moreover, due to that the orbital angular momentum also remains fixed in the case with $s = 0$, we have $l = J$ which is conserved, and the parity is simply $(-1)^J$. When $s = 1$ for a given J, l takes three different values, i.e., $l = J+1, J-1, J$. The first two have the same parity $(-1)^{J+1}$, but the parity for the third one is $(-1)^J$. The total parity of the two-particle state is equal to $P_{\rm tol} = P_1 P_2 (-1)^l$, where P_1 and P_2 are the intrinsic parities of the two scattering particles. For simplicity we assume that the intrinsic parity $P_1 P_2$ is positive, then the total parity is $(-1)^l$. For parity-conserving theories like QCD, there is no scattering between states with opposite parity. Then we divide Lüscher's formulas into the *Case a* and *Case b* corresponding to the states with parity $(-1)^{J+1}$ and parity $(-1)^J$ respectively.

C. Application to *Case a*

In this case ($s = s' = 1,\ l = J \pm 1$), Lüscher's formula becomes

$$\left| \sum_{l''} (S^J_{l''1;l1} - \delta_{ll''}) \mathcal{M}^{(1)}_{Jl'';J'l'} - i\delta_{JJ'}(S^J_{l'1;l1} + \delta_{ll'}) \right| = 0. \tag{19}$$

According to matrix elements $\mathcal{M}^{(1)}_{Jl'';J'l'}$ given in Table I, one can obtain the phase shift in the definite symmetry. Then we list the phase shift with different irreducible representations of the D_{4h} group.

If we consider the explicit parity and also suppose that the cutoff angular momentum is $\Lambda = 2$, the decomposition in this case becomes $0^- = A_1^-,\quad 1^+ = A_2^+ \oplus E^+$, $2^- = A_1^- \oplus B_1^- \oplus B_2^- \oplus E^-$. For instance, in the A_1^- representation, the phase shift formula is Eq. (20) if we ignore the mixing with $J = 2,\ l = 3$.

$$\begin{vmatrix} \cot\delta_{01} - \omega_{00} & -\frac{\sqrt{10}}{5}\omega_{20} \\ -\frac{\sqrt{10}}{5}\omega_{20} & \cot\delta_{21} - (\omega_{00} + \frac{\sqrt{5}}{5}\omega_{20}) \end{vmatrix} = 0 \tag{20}$$

If we ignore the mixing with $J = 2$, one can extract phase shift δ_{01} from

$$\cot\delta_{01} = \omega_{00}. \tag{21}$$

Next, to calculate phase shift δ_{21}, we can consider representation B_1^-, B_2^-, and E^-. If we ignore the mixing with $J = 2,\ l = 3$, the phase shift formulas in these three representations are written as

$$\begin{cases} \cot\delta_{21} = \omega_{00} - \frac{\sqrt{5}}{5}\omega_{20} \\ \cot\delta_{21} = \omega_{00} - \frac{\sqrt{5}}{5}\omega_{20} \\ \cot\delta_{21} = \omega_{00} + \frac{\sqrt{5}}{10}\omega_{20} \end{cases}. \tag{22}$$

Finally, let us discuss the quantum number $J^P = 1^+$ of the two-particle state in A_2^+ and E^+ representations, i.e., the

TABLE I. D_{4h} symmetry group for angular momentum up to $J=2$ and $l=3$. For the parity $(-1)^{J\pm1}$, the matrix elements $\mathcal{M}^{(1)}_{Jl;J'l'}(\Gamma)$ are presented in each irreducible representations sector.

Γ	J	l	J'	l'	$\mathcal{M}^{(1)}_{Jl;J'l'}(\Gamma)$
A_1^-	0	1	0	1	ω_{00}
	0	1	2	1	$-\frac{\sqrt{10}}{5}\omega_{20}$
	0	1	2	3	$-\frac{\sqrt{15}}{5}\omega_{20}$
	2	1	2	1	$\omega_{00}+\frac{\sqrt{5}}{5}\omega_{20}$
	2	1	2	3	$\frac{\sqrt{30}}{35}\omega_{20}+\frac{2\sqrt{6}}{7}\omega_{40}$
	2	3	2	3	$\omega_{00}+\frac{8\sqrt{5}}{35}\omega_{20}+\frac{2}{7}\omega_{40}$
A_2^+	1	0	1	0	ω_{00}
	1	0	1	2	$\frac{\sqrt{10}}{5}\omega_{20}$
	1	2	1	2	$\omega_{00}+\frac{\sqrt{5}}{5}\omega_{20}$
B_1^-	2	1	2	1	$\omega_{00}-\frac{\sqrt{5}}{5}\omega_{20}$
	2	1	2	3	$-\frac{\sqrt{30}}{35}\omega_{20}+\frac{\sqrt{6}}{21}\omega_{40}+\frac{2\sqrt{105}}{21}\omega_{44}$
	2	3	2	3	$\omega_{00}+\frac{1}{21}\omega_{40}-\frac{8\sqrt{5}}{35}\omega_{20}+\frac{\sqrt{70}}{21}\omega_{44}$
B_2^-	2	1	2	1	$\omega_{00}-\frac{\sqrt{5}}{5}\omega_{20}$
	2	1	2	3	$-\frac{\sqrt{30}}{35}\omega_{20}+\frac{\sqrt{6}}{21}\omega_{40}-\frac{2\sqrt{105}}{21}\omega_{44}$
	2	3	2	3	$\omega_{00}+\frac{1}{21}\omega_{40}-\frac{8\sqrt{5}}{35}\omega_{20}-\frac{\sqrt{70}}{21}\omega_{44}$
E^+	1	0	1	0	ω_{00}
	1	0	1	2	$-\frac{\sqrt{10}}{10}\omega_{20}$
	1	2	1	2	$\omega_{00}-\frac{\sqrt{5}}{10}\omega_{20}$
E^-	2	1	2	1	$\omega_{00}+\frac{\sqrt{5}}{10}\omega_{20}$
	2	1	2	3	$\frac{\sqrt{30}}{70}\omega_{20}-\frac{4\sqrt{6}}{21}\omega_{40}$
	2	3	2	3	$\omega_{00}+\frac{4\sqrt{5}}{35}\omega_{20}-\frac{4}{21}\omega_{40}$

total angular momentum J is fixed for 1. There is a mixing between $l=J-1=0$ (S-wave) and $l=J+1=2$ (D-wave). To proceed, we need to parametrize S-matrix. First we note that S must be unitary (conservation of probability) and symmetric (reciprocity) due to the T-invariance of the strong interactions. Thus the 2×2 S-matrix is determined by three real parameters. We use the "eigenphase convention" of Blatt and Biedenharn [65] in Eq. (23),

$$S_{2\times2}=\begin{pmatrix}\cos\epsilon & -\sin\epsilon\\ \sin\epsilon & \cos\epsilon\end{pmatrix}\begin{pmatrix}e^{2i\delta_\alpha} & 0\\ 0 & e^{2i\delta_\beta}\end{pmatrix}\begin{pmatrix}\cos\epsilon & \sin\epsilon\\ -\sin\epsilon & \cos\epsilon\end{pmatrix} \tag{23}$$

where δ_α and δ_β are the scattering phase shifts corresponding to two eigenstates of the S-matrix called "α" and "β" waves respectively. At low energies, the α-wave is predominantly S-wave with a small admixture of the D-wave, while the β-wave is predominantly D-wave with a small admixture of the S-wave. Therefore, it hardly needs to be emphasized that the eigenphaseshifts δ_α and δ_β are not to be thought of as phase shifts for the states $l=0$ and $l=2$, respectively. There are no such phase shifts due to that neither of these two states is an eigenstate of the scattering matrix. The "mixture parameter" ϵ, which determines the "correct" mixtures of these two states, is an essential parameter in the scattering matrix and enters explicitly into the differential cross section. Thus, in A_2^+ representation, the phase shift formula is shown in Eq. (24),

$$\left|(S_{2\times2}-I_{2\times2})\begin{pmatrix}\omega_{00} & \frac{\sqrt{10}}{5}\omega_{20}\\ \frac{\sqrt{10}}{5}\omega_{20} & \omega_{00}+\frac{\sqrt{5}}{5}\omega_{20}\end{pmatrix} - i(S_{2\times2}+I_{2\times2})\right|=0, \tag{24}$$

where $I_{2\times2}$ is a 2×2 unit matrix. Similarly, for E^+ representation, we can obtain the phase shift formula by substituting corresponding $\mathcal{M}^{(1)}_{Jl'';J'l'}$ according to Table I with Eq. (24).

D. Application to *Case b*

For this case ($l=l'=l''=J$, $s=0,1$), the Lüscher's formula can be expressed as Eq. (25).

$$\left|\sum_{l''}(S^J_{l''s';ls}-\delta_{ll''}\delta_{ss'})\mathcal{M}^{(s')}_{Jl'';J'l'}-i\delta_{JJ'}(S^J_{ls';ls}+\delta_{ss'})\right|=0. \tag{25}$$

If we consider the explicit parity and also suppose that the cutoff angular momentum is $\Lambda=2$, the decomposition in this case becomes $0^+=A_1^+$, $1^-=A_2^-\oplus E^-$, $2^+=A_1^+\oplus B_1^+\oplus B_2^+\oplus E^+$. Then, we parameterize the S-matrix again using the "eigenphase convention" of Blatt and Biedenharn [65], whose explicit form is similar to Eq. (23). Thus, δ_α and δ_β are the scattering phase shifts corresponding to two eigenstates of the S-matrix named "α" and "β" waves respectively. At low energy, however, the α-wave is predominantly $s=0$ with a small admixture of the $s=1$, while the β-wave is predominantly $s=1$ with a small admixture of the $s=0$. According to the nonzero matrix elements, one can obtain phase shift formula in a definite symmetry. For example, if we focus on the A_1^+ representation with positive parity that corresponds $J=0$ and ignore the index l and l', both of which are unity, the phase shift formula is given by Eq. (26).

TABLE II. For the parity $(-1)^J$ with angular momentum up to $J=2$ and $l=2$, the matrix elements $\mathcal{M}^{(1)}_{Jl;J'l'}(\Gamma)$ presented in each irreducible representations sector of D_{4h} symmetry group. The total spin $s=s'=1$.

Γ	J	l	J'	l'	$\mathcal{M}^{(1)}_{Jl;J'l'}(\Gamma)$
A_1^+	2	2	2	2	$\omega_{00}-\frac{4}{7}\omega_{40}+\frac{\sqrt{5}}{7}\omega_{20}$
A_2^-	1	1	1	1	$\omega_{00}-\frac{\sqrt{5}}{5}\omega_{20}$
B_1^+	2	2	2	2	$\omega_{00}-\frac{2}{21}\omega_{40}-\frac{\sqrt{5}}{7}\omega_{20}-\frac{2\sqrt{70}}{21}\omega_{44}$
B_2^+	2	2	2	2	$\omega_{00}-\frac{2}{21}\omega_{40}-\frac{\sqrt{5}}{7}\omega_{20}+\frac{2\sqrt{70}}{21}\omega_{44}$
E^-	1	1	1	1	$\omega_{00}+\frac{\sqrt{5}}{10}\omega_{20}$
E^+	2	2	2	2	$\omega_{00}+\frac{8}{21}\omega_{40}+\frac{\sqrt{5}}{14}\omega_{20}$

$$\left|(S_{2\times2}-I_{2\times2})\begin{pmatrix}\omega_{00}+\frac{6}{7}\omega_{40}+\frac{2\sqrt{5}}{7}\omega_{20} & 0\\ 0 & \omega_{00}-\frac{4}{7}\omega_{40}+\frac{\sqrt{5}}{7}\omega_{20}\end{pmatrix} - i(S_{2\times2}+I_{2\times2})\right|=0 \tag{26}$$

In addition, phase shifts in B_1^+, B_2^+, and E^+ are similar to Eq. (26) expect that the matrix elements $\mathcal{M}^{(s')}_{Jl'';J'l'}$ in the equation are replaced by the corresponding ones according to Table II and Table III. Next, taking A_2^- representation with negative parity as an example, the formula is written as Eq. (27).

$$\left|(S_{2\times2}-I_{2\times2})\begin{pmatrix}\omega_{00}+\frac{2\sqrt{5}}{5}\omega_{20} & 0\\ 0 & \omega_{00}-\frac{\sqrt{5}}{5}\omega_{20}\end{pmatrix} - i(S_{2\times2}+I_{2\times2})\right|=0 \tag{27}$$

The formula in E^- representation is similar to Eq. (27) expect that the matrix elements $\mathcal{M}^{(s')}_{Jl'';J'l'}$ in the equation are replaced by the corresponding ones according to Table II and Table III.

TABLE III. For the parity $(-1)^J$ with angular momentum up to $J=2$ and $l=2$, the matrix elements $\mathcal{M}^{(0)}_{Jl;J'l'}(\Gamma)$ are presented in each irreducible representations sector of D_{4h} symmetry group. The total spin $s=s'=0$.

Γ	J	l	J'	l'	$\mathcal{M}^{(0)}_{Jl;J'l'}(\Gamma)$
A_1^+	2	2	2	2	$\omega_{00}+\frac{6}{7}\omega_{40}+\frac{2\sqrt{5}}{7}\omega_{20}$
A_2^-	1	1	1	1	$\omega_{00}+\frac{2\sqrt{5}}{5}\omega_{20}$
B_1^+	2	2	2	2	$\omega_{00}+\frac{1}{7}\omega_{40}-\frac{2\sqrt{5}}{7}\omega_{20}+\frac{\sqrt{70}}{7}\omega_{44}$
B_2^+	2	2	2	2	$\omega_{00}+\frac{1}{7}\omega_{40}-\frac{2\sqrt{5}}{7}\omega_{20}-\frac{\sqrt{70}}{7}\omega_{44}$
E^-	1	1	1	1	$\omega_{00}-\frac{\sqrt{5}}{5}\omega_{20}$
E^+	2	2	2	2	$\omega_{00}+\frac{\sqrt{5}}{7}\omega_{20}-\frac{4}{7}\omega_{40}$

In the above discussion, we have not taken into account the possibility for the identical nature of the two scattering particles, and thus the singlet-triplet transition within the same parity is allowed. However, for the two identical particles, the singlet-triplet transition is forbidden, since the singlet state has an antisymmetric spin wave function which then requires a symmetric spatial one that necessarily has positive parity while the triplet state has the opposite parity. Below, we list Lüscher's formulas for $s=s'=0$ and $s=s'=1$ cases respectively:

$$\left|\sum_{l''}(S^J_{l''0;l0}-\delta_{ll''})\mathcal{M}^{(0)}_{Jl'';J'l'}-i\delta_{JJ'}(S^J_{l0;l0}+1)\right|=0. \tag{28}$$

$$\left|\sum_{l''}(S^J_{l''1;l1}-\delta_{ll''})\mathcal{M}^{(1)}_{Jl'';J'l'}-i\delta_{JJ'}(S^J_{l1;l1}+1)\right|=0. \tag{29}$$

From these explicit expressions, we find they are quite similar to those in the case of meson-meson scattering. In particular, Lüscher's formula for $s=s'=0$ is the same as that in the meson-meson scattering case. Then, we only discuss the case for $s=s'=1$. In this case, we denote the phase shift as δ_J (we have ignored the index l due to $J=l$ in this case) in the definite representation. According to the nonzero matrix elements listed in Table II, one can obtain phase shift δ_J in the definite representation.

If we focus on the phase shift δ_2, one can consider A_1^+, B_1^+, B_2^+, and E^+ representations. The phase shift formulas in these representations are similar to each other expect that matrix elements $\mathcal{M}^{(1)}_{Jl;J'l'}(\Gamma)$ are different. We list the explicit form of these matrix elements in Table II. For instance, we take A_1^+ representation as an example, where the phase shift formula is

$$\cot\delta_2=\omega_{00}-\frac{4}{7}\omega_{40}+\frac{\sqrt{5}}{7}\omega_{20}. \tag{30}$$

Then, if one wants to obtain δ_1, one should consider A_2^- and E^- representations. Here, we take A_2^- representation as an example, the phase shift δ_1 is contained in the flowing equation:

$$\cot\delta_1=\omega_{00}-\frac{\sqrt{5}}{5}\omega_{20}. \tag{31}$$

III. SOME GENERALIZATIONS OF PHASE SHIFT FORMULAS FOR BARYON-BARYON SCATTERING IN ELONGATED BOXES

A. Phase shift formulas for baryon-baryon scattering in MF

In this subsection, we extend the phase shift formula that has been obtained in the previous subsection for baryon-baryon scattering to moving frames (MF) in an elongated box. We will follow the notations in Ref. [38] below. We denote the four momenta of the two particles with periodic boundary conditions in the lab frame by

$$k = (E_1, \mathbf{k}), \qquad P - k = (E_2, \mathbf{P} - \mathbf{k}), \tag{32}$$

where $E_1 = \sqrt{\mathbf{k}^2 + m_1^2}$ and $E_2 = \sqrt{(\mathbf{P}-\mathbf{k})^2 + m_2^2}$ are energy values of the two particles respectively, m_1 and m_2 are the mass values of the two baryons respectively. The total three momentum $\mathbf{P} \neq 0$ of the two-particle system is quantized by the condition $\mathbf{P} = (2\pi/L)\mathbf{d}$ with $\mathbf{d} \in \mathbb{Z}^3$. The COM frame is then moving relative to the lab frame with a velocity

$$\mathbf{v} = \mathbf{P}/(E_1 + E_2). \tag{33}$$

Then, we denote the momenta of the two particles with $\mathbf{k}^*$ and $(-\mathbf{k}^*)$ for the two scattering particles. Thus, $\mathbf{k}^*$ is related to $\mathbf{k}$ by conventional Lorentz boost:

$$\mathbf{k}^{*\parallel} = \gamma(\mathbf{k}^{\parallel} - \mathbf{v}E_1), \qquad \mathbf{k}^{*\perp} = \mathbf{k}^{\perp}, \tag{34}$$

where the symbol $\perp$ and $\parallel$ designate the components of the corresponding vector perpendicular and parallel to $\mathbf{v}$, respectively. For simplicity, the above relation is also denoted by the shorthand notation: $\mathbf{k}^* = \vec{\gamma}\mathbf{k}$. A similar transformation relation holds for the other particles.

Following similar steps listed in Sec. II, we can readily obtain the Lüscher's formula in this case. Lüscher's formula takes exactly the same form as Eq. (13) except that all the labels for $\mathcal{M}^{\mathbf{c}(s)}_{JMl;J'M'l'}$ are replaced with the modified matrix elements $\mathcal{M}^{\mathbf{d}(s)}_{JMl;J'M'l'}$. The explicit form for the matrix element is written as

$$\mathcal{M}^{\mathbf{d}(s)}_{JMl;J'M'l'}(\boldsymbol{\kappa}^2, \eta) = \sum_{mm'\nu} \langle JM|lm; s\nu\rangle \langle J'M'|l'm'; s\nu\rangle \times \mathcal{M}^{\mathbf{d}}_{lm;l'm'}(\boldsymbol{\kappa}^2, \eta) \tag{35}$$

with $\boldsymbol{\kappa} = \mathbf{k}L/(2\pi)$. The explicit form of the reduction matrix element in moving frames is given by [38,41]

$$\mathcal{M}^{\mathbf{d}}_{lm;l'm'}(\boldsymbol{\kappa}^2, \eta) = \sum_{t=|l-l'|}^{l+l'} \sum_{t'=-t}^{t} \frac{(-1)^l i^{l+l'}}{\gamma \pi^{3/2} \eta \boldsymbol{\kappa}^{t+1}} \times Z^{\mathbf{d}}_{tt'}(\boldsymbol{\kappa}^2, \eta) \langle l0t0|l'0\rangle \langle lmtt'|l'm'\rangle \times \sqrt{\frac{(2l+1)(2t+1)}{(2l'+1)}}, \tag{36}$$

where γ is the Lorentz factor of the boost, and the modified zeta function $Z^{\mathbf{d}}_{tt'}(\boldsymbol{\kappa}^2, \eta)$ is defined via

$$Z^{\mathbf{d}}_{tt'}(\boldsymbol{\kappa}^2, \eta) = \sum_{\tilde{\mathbf{n}}_{\mathbf{d}} \in \mathcal{P}_{\mathbf{d}}} \frac{\mathcal{Y}_{tt'}(\tilde{\mathbf{n}}_{\mathbf{d}})}{\tilde{\mathbf{n}}_{\mathbf{d}}^2 - \boldsymbol{\kappa}^2}. \tag{37}$$

In the above formulas, $\mathcal{P}_{\mathbf{d}}$ is the following set,

$$\mathcal{P}_{\mathbf{d}} = \left\{ \tilde{\mathbf{n}}_{\mathbf{d}} \in \mathbb{R}^3 | \tilde{\mathbf{n}}_{\mathbf{d}} = \gamma\left(\mathbf{n} + \frac{1}{2}\alpha\mathbf{d}\right), \mathbf{n} \in \mathbb{Z}^3 \right\}, \tag{38}$$

where $\mathbf{d} = (L/2\pi)\mathbf{P}$, $\alpha = 1 + \frac{w^{*2}_{1\mathbf{k}} - w^{*2}_{2\mathbf{k}}}{E^{*2}}$, and $w^{*2}_{i\mathbf{k}} = m_i^2 + \mathbf{k}^{*2}$ with $i = 1$, 2 being the square of the energy for the two scattering particles. We will also use E^* to denote the total energy in the center of mass frame and introduce the short-hand function for the zeta function,

$$\omega^{\mathbf{d}}_{lm}(\boldsymbol{\kappa}^2, \eta) = \frac{Z^{\mathbf{d}}_{lm}(\boldsymbol{\kappa}^2, \eta)}{\gamma \pi^{3/2} \eta \boldsymbol{\kappa}^{l+1}}. \tag{39}$$

In addition, an extra attention should also be paid to the difference in symmetries. In order to discuss phase shift formula in MF, we should introduce the space group $\mathcal{G}$, which is a semidirect product of lattice translation group $\mathcal{T}$ and the cubic group O. The representations are characterized by the little group Γ and the corresponding total momentum $\mathbf{P}$. For example, for the case with $\mathbf{P} = \frac{2\pi}{L}\mathbf{e}_3$, the corresponding little group is C_{4v} [42]. Then, following similar steps as in the COM frame, one can readily obtain the explicit formulas for the little group C_{4v}. Before giving the explicit form of phase shift, we should calculate nonvanished matrix elements $\mathcal{M}^{\mathbf{d}(1)}_{Jl;J'l'}(\Gamma)$ listed in Table IV and Table V for the C_{4v} group. Then, following similar steps as in the Sec. II, one can also readily obtain the phase shift formulas in the moving frame. Here, we take phase shift formula in *Case a* ($s = s' = 1$, $l = J \pm 1$) as an example. For instance, in the A_1 representation, the phase shift formula is given by Eq. (40) if we ignore the mixing with $l = 2$.

$$\begin{vmatrix} \cot\delta_{01} - \omega^{\mathbf{d}}_{00} & \frac{\sqrt{3}i}{3}\omega^{\mathbf{d}}_{10} & -\frac{\sqrt{10}}{5}\omega^{\mathbf{d}}_{20} \\ -\frac{\sqrt{3}i}{3}\omega^{\mathbf{d}}_{10} & \cot\delta_{10} - \omega^{\mathbf{d}}_{00} & \frac{\sqrt{6}i}{3}\omega^{\mathbf{d}}_{10} \\ -\frac{\sqrt{10}}{5}\omega^{\mathbf{d}}_{20} & -\frac{\sqrt{6}i}{3}\omega^{\mathbf{d}}_{10} & \cot\delta_{21} - (\omega^{\mathbf{d}}_{00} + \frac{\sqrt{5}}{5}\omega^{\mathbf{d}}_{20}) \end{vmatrix} = 0 \tag{40}$$

We note that there is mixing between odd and even J due to the lack of the parity in the boosted two-particle state.

B. Phase shift formulas for baryon-baryon scattering in COM frame with twisted boundary condition

In this subsection, we impose the twisted boundary condition on the two-particle system in the COM frame. The quantized momentum is $\mathbf{k} = \frac{2\pi}{L}(\mathbf{n} + \frac{\boldsymbol{\theta}}{2\pi})$ with $\mathbf{n}$ the three-dimensional vector and $\boldsymbol{\theta}$ the twist angle. By adjusting the twist angle, one can gain more momenta for two-particle energy levels. Lüscher's formula takes exactly the same form as shown in Eq. (13) except that all the labels of $\mathcal{M}^{\mathbf{c}(s)}_{JMl;J'M'l'}$ are replaced by $\mathcal{M}^{\phi(s)}_{JMl;J'M'l'}$. The explicit form of $\mathcal{M}^{\phi(s)}_{JMl;J'M'l'}$ is given by

TABLE IV. For the parity $(-1)^{J\pm 1}$ with angular momentum up to $J=2$ and $l=3$, the matrix elements $\mathcal{M}^{\mathbf{d}(1)}_{Jl;J'l'}(\Gamma)$ are presented in each irreducible representations sector of C_{4v} symmetry group. The total spin $s=s'=1$.

Γ	J	l	J'	l'	$\mathcal{M}^{\mathbf{d}(1)}_{Jl;J'l'}(\Gamma)$
A_1	0	1	0	1	$\omega^{\mathbf{d}}_{00}$
	0	1	1	0	$\frac{\sqrt{3}i}{3}\omega^{\mathbf{d}}_{10}$
	0	1	1	2	$\frac{\sqrt{150}i}{15}\omega^{\mathbf{d}}_{10}$
	0	1	2	1	$-\frac{\sqrt{10}}{5}\omega^{\mathbf{d}}_{20}$
	0	1	2	3	$-\frac{\sqrt{15}}{5}\omega^{\mathbf{d}}_{20}$
	1	0	1	0	$\omega^{\mathbf{d}}_{00}$
	1	0	1	2	$\frac{\sqrt{10}}{5}\omega^{\mathbf{d}}_{20}$
	1	0	2	1	$\frac{\sqrt{6}i}{3}\omega^{\mathbf{d}}_{10}$
	1	0	2	3	$\frac{\sqrt{21}i}{7}\omega^{\mathbf{d}}_{30}$
	1	2	1	2	$\omega^{\mathbf{d}}_{00}+\frac{\sqrt{5}}{5}\omega^{\mathbf{d}}_{20}$
	1	2	2	1	$\frac{\sqrt{3}i}{15}\omega^{\mathbf{d}}_{10}+\frac{9\sqrt{7}i}{35}\omega^{\mathbf{d}}_{30}$
	1	2	2	3	$\frac{3\sqrt{2}i}{7}\omega^{\mathbf{d}}_{10}+\frac{2\sqrt{42}i}{35}\omega^{\mathbf{d}}_{30}$
	2	1	2	1	$\omega^{\mathbf{d}}_{00}+\frac{\sqrt{5}}{5}\omega^{\mathbf{d}}_{20}$
	2	1	2	3	$\frac{\sqrt{30}}{35}\omega^{\mathbf{d}}_{20}+\frac{2\sqrt{6}}{7}\omega^{\mathbf{d}}_{40}$
	2	3	2	3	$\omega^{\mathbf{d}}_{00}+\frac{8\sqrt{5}}{35}\omega^{\mathbf{d}}_{20}+\frac{2}{7}\omega^{\mathbf{d}}_{40}$
B_1	2	1	2	1	$\omega^{\mathbf{d}}_{00}-\frac{\sqrt{5}}{5}\omega^{\mathbf{d}}_{20}$
	2	1	2	3	$-\frac{\sqrt{30}}{35}\omega^{\mathbf{d}}_{20}+\frac{\sqrt{6}}{21}\omega^{\mathbf{d}}_{40}+\frac{2\sqrt{105}}{21}\omega^{\mathbf{d}}_{44}$
	2	3	2	3	$\omega^{\mathbf{d}}_{00}+\frac{1}{21}\omega^{\mathbf{d}}_{40}-\frac{8\sqrt{5}}{35}\omega^{\mathbf{d}}_{20}+\frac{\sqrt{70}}{21}\omega^{\mathbf{d}}_{44}$
B_2	2	1	2	1	$\omega^{\mathbf{d}}_{00}-\frac{\sqrt{5}}{5}\omega^{\mathbf{d}}_{20}$
	2	1	2	3	$-\frac{\sqrt{30}}{35}\omega^{\mathbf{d}}_{20}+\frac{\sqrt{6}}{21}\omega^{\mathbf{d}}_{40}-\frac{2\sqrt{105}}{21}\omega^{\mathbf{d}}_{44}$
	2	3	2	3	$\omega^{\mathbf{d}}_{00}+\frac{1}{21}\omega^{\mathbf{d}}_{40}-\frac{8\sqrt{5}}{35}\omega^{\mathbf{d}}_{20}-\frac{\sqrt{70}}{21}\omega^{\mathbf{d}}_{44}$
E	1	0	1	0	$\omega^{\mathbf{d}}_{00}$
	1	0	1	2	$-\frac{\sqrt{10}}{10}\omega^{\mathbf{d}}_{20}$
	1	0	2	1	$\frac{\sqrt{2}i}{2}\omega^{\mathbf{d}}_{10}$
	1	0	2	3	$-\frac{\sqrt{7}i}{7}\omega^{\mathbf{d}}_{30}$
	1	2	1	2	$\omega^{\mathbf{d}}_{00}-\frac{\sqrt{5}}{10}\omega^{\mathbf{d}}_{20}$
	1	2	2	1	$\frac{i}{10}\omega^{\mathbf{d}}_{10}-\frac{3\sqrt{21}i}{35}\omega^{\mathbf{d}}_{30}$
	1	2	2	3	$\frac{3\sqrt{6}i}{10}\omega^{\mathbf{d}}_{10}-\frac{2\sqrt{14}i}{35}\omega^{\mathbf{d}}_{30}$
	2	1	2	1	$\omega^{\mathbf{d}}_{00}+\frac{\sqrt{5}}{10}\omega^{\mathbf{d}}_{20}$
	2	1	2	3	$\frac{\sqrt{30}}{70}\omega^{\mathbf{d}}_{20}-\frac{4\sqrt{6}}{21}\omega^{\mathbf{d}}_{40}$
	2	3	2	3	$\omega^{\mathbf{d}}_{00}+\frac{4\sqrt{5}}{35}\omega^{\mathbf{d}}_{20}-\frac{4}{21}\omega^{\mathbf{d}}_{40}$

$$\mathcal{M}^{\phi(s)}_{JMl;J'M'l'}(q^2,\eta)=\sum_{mm'\nu}\langle JM|lm;s\nu\rangle\langle J'M'|l'm';s\nu\rangle \times \mathcal{M}^{\phi}_{lm;l'm'}(q^2,\eta), \quad (41)$$

TABLE V. For the parity $(-1)^{J}$ with angular momentum up to $J=2$ and $l=2$, the matrix elements $\mathcal{M}^{\mathbf{d}(1)}_{Jl;J'l'}(\Gamma)$ are presented in each irreducible representations sector of C_{4v} symmetry group. The total spin $s=s'=1$.

Γ	J	l	J'	l'	$\mathcal{M}^{\mathbf{d}(1)}_{Jl;J'l'}(\Gamma)$
A_1	1	1	1	1	$\omega^{\mathbf{d}}_{00}-\frac{\sqrt{5}}{5}\omega^{\mathbf{d}}_{20}$
	1	1	2	2	$\frac{\sqrt{15}i}{5}\omega^{\mathbf{d}}_{10}+\frac{3\sqrt{35}i}{35}\omega^{\mathbf{d}}_{30}$
	2	2	2	2	$\omega^{\mathbf{d}}_{00}-\frac{4}{7}\omega^{\mathbf{d}}_{40}+\frac{\sqrt{5}}{7}\omega^{\mathbf{d}}_{20}$
B_1	2	2	2	2	$\omega^{\mathbf{d}}_{00}-\frac{2}{21}\omega^{\mathbf{d}}_{40}-\frac{\sqrt{5}}{7}\omega^{\mathbf{d}}_{20}-\frac{2\sqrt{70}}{21}\omega^{\mathbf{d}}_{44}$
B_2	2	2	2	2	$\omega^{\mathbf{d}}_{00}-\frac{2}{21}\omega^{\mathbf{d}}_{40}-\frac{\sqrt{5}}{7}\omega^{\mathbf{d}}_{20}+\frac{2\sqrt{70}}{21}\omega^{\mathbf{d}}_{44}$
E	1	1	1	1	$\omega^{\mathbf{d}}_{00}+\frac{\sqrt{5}}{10}\omega^{\mathbf{d}}_{20}$
	1	1	2	2	$\frac{3\sqrt{5}i}{10}\omega^{\mathbf{d}}_{10}+\frac{\sqrt{105}i}{35}\omega^{\mathbf{d}}_{30}$
	2	2	2	2	$\omega^{\mathbf{d}}_{00}+\frac{8}{21}\omega^{\mathbf{d}}_{44}+\frac{\sqrt{5}}{14}\omega^{\mathbf{d}}_{20}$

where

$$\mathcal{M}^{\phi}_{lm;l'm'}(q^2,\eta)=\sum_{t=|l-l'|}^{l+l'}\sum_{t'=-t}^{t}\frac{(-1)^l i^{l+l'}}{\pi^{3/2}\eta q^{t+1}} \times Z^{\phi}_{tt'}(q^2,\eta)\langle l0t0|l'0\rangle\langle lmtt'|l'm'\rangle \times\sqrt{\frac{(2l+1)(2t+1)}{(2l'+1)}}, \quad (42)$$

with

$$Z^{\phi}_{tt'}(q^2,\eta)=\sum_{\mathbf{r}\in\Gamma^{\phi}}\frac{\mathcal{Y}_{tt'}(\tilde{\mathbf{n}})}{\tilde{\mathbf{n}}^2-q^2}. \quad (43)$$

Here, $\Gamma^{\phi}=\{\tilde{\mathbf{n}}\in\mathbb{R}^3|\tilde{\mathbf{n}}=\mathbf{n}+(2\pi)^{-1}\phi,\mathbf{n}\in\mathbb{Z}^3\}$, and $\mathbb{R}^3$ is the set of real 3-tuples. The short-hand function for the zeta function is

$$\omega^{\phi}_{lm}(q^2,\eta)=\frac{Z^{\phi}_{lm}(q^2,\eta)}{\pi^{3/2}\eta q^{l+1}}. \quad (44)$$

For the case with $\phi=(0,0,\phi)$, the corresponding little group is written as C_{4v}. Taking the phase shift formula in the A_1 representation in *Case a* ($s=s'=1$, $l=J\pm1$) in Sec. II as an example, the phase shift formula is same as Eq. (40) except that all the labels of $\omega^{\mathbf{d}}_{lm}$ are replaced by ω^{ϕ}_{lm}. In particular, for $\phi=(0,0,\pi)$, the corresponding little group is D_{4h}. The phase shift formula is same as Eq. (20) except that all the labels of ω_{lm} are replaced by ω^{ϕ}_{lm}.

IV. PHASE SHIFT FORMULAS FOR BARYON-BARYON SCATTERING IN COM FRAME IN CUBIC BOX

In this section, we briefly discuss phase shift formulas for baryon-baryon scattering in COM frame in a cubic box. First,

TABLE VI. For the parity $(-1)^{J\pm 1}$ with the angular momentum up to $J=2$ and $l=3$, the matrix elements $\mathcal{M}^{\mathbf{c}(1)}_{Jl;J'l'}(\Gamma)$ are presented in each irreducible representations sector of O_h symmetry group.

Γ	J	l	J'	l'	$\mathcal{M}^{\mathbf{c}(1)}_{Jl;J'l'}(\Gamma)$
A_1^-	0	1	0	1	$\omega^{\mathbf{c}}_{00}$
E^-	2	1	2	1	$\omega^{\mathbf{c}}_{00}$
	2	1	2	3	$\frac{\sqrt{6}}{6}\omega^{\mathbf{c}}_{40}+\frac{\sqrt{105}}{21}\omega^{\mathbf{c}}_{44}$
	2	3	2	3	$\omega^{\mathbf{c}}_{00}+\frac{1}{6}\omega^{\mathbf{c}}_{40}+\frac{\sqrt{70}}{42}\omega^{\mathbf{c}}_{44}$
T_1^+	1	0	1	0	$\omega^{\mathbf{c}}_{00}$
	1	0	1	2	0
	1	2	1	2	$\omega^{\mathbf{c}}_{00}$
T_2^-	2	1	2	1	$\omega^{\mathbf{c}}_{00}$
	2	1	2	3	$-\frac{\sqrt{6}}{9}\omega^{\mathbf{c}}_{40}-\frac{2\sqrt{105}}{63}\omega^{\mathbf{c}}_{44}$
	2	3	2	3	$\omega^{\mathbf{c}}_{00}-\frac{1}{9}\omega^{\mathbf{c}}_{40}-\frac{\sqrt{70}}{63}\omega^{\mathbf{c}}_{44}$

we can perform the consistency checks and validation on the elongated results by comparing the two cases. In addition, this can serve as a basis for exploring the relationship between the two cases, providing valuable insights into how the results transform from one to the other.

Lüscher's formula takes the form given by Eq. (13). To write out a more explicit formula, we should consider the definite cubic symmetries. For the case of integer total momentum J, we need to consider the group of O_h, which contains 48 elements those can be divided into 10 conjugate classes: $A_1^\pm$, $A_2^\pm$, $E^\pm$, $T_1^\pm$ and $T_2^\pm$. For instance, for $J=0$, 1, 2, $\Lambda=2$, the decomposition into irreducible representation is given by $0^\pm=A_1^\pm$, $1^\pm=T_1^\pm$, $2^\pm=T_2^\pm\oplus E^\pm$ respectively [57]. Then, we take Lüscher's formula in *Case a* ($s=s'=1$, $l=J\pm 1$) as an example. Before writing out the explicit phase formula in this case, we list matrix elements $\mathcal{M}^{\mathbf{c}(s)}_{Jl;J'l'}(\Gamma)$ in Table VI. According to the nonzero matrix elements, one can obtain phase shift formula in the definite symmetry. If we focus in A_1^- representation, the phase shift formula is written as Eq. (45).

$$\cot\delta_{01}=\omega^{\mathbf{c}}_{00}. \tag{45}$$

If we consider D-wave resonance, we can give the phase shift formulas in E^-, i.e.,

$$\left|(S_{2\times 2}-I_{2\times 2})\begin{pmatrix}\omega^{\mathbf{c}}_{00} & \frac{\sqrt{6}}{6}\omega^{\mathbf{c}}_{40}+\frac{\sqrt{105}}{21}\omega^{\mathbf{c}}_{44}\\ \frac{\sqrt{6}}{6}\omega^{\mathbf{c}}_{40}+\frac{\sqrt{105}}{21}\omega^{\mathbf{c}}_{44} & \omega^{\mathbf{c}}_{00}+\frac{1}{6}\omega^{\mathbf{c}}_{40}+\frac{\sqrt{70}}{42}\omega^{\mathbf{c}}_{44}\end{pmatrix} - i(S_{2\times 2}+I_{2\times 2})\right|=0 \tag{46}$$

Then, if we ignore the mixing with $l=3$ the phase shift formulas in T_2^- representations is Eq. (47).

$$\cot\delta_{21}=\omega^{\mathbf{c}}_{00}. \tag{47}$$

Finally, for the T_1^+ representation with O_h group, the phase shift formula is similar to Eq. (46) expect that the matrix elements $\mathcal{M}^{\mathbf{c}(1)}_{Jl;J'l'}(\Gamma)$ in the equation are replaced by the corresponding ones according to Table VI.

Then, let us discuss the relation of two-particle scattering Lüscher's formulas between the cubic case and the elongated case. In Table VII, we listed the symmetry relations between O_h and D_{4h} [57]. By using the relations in Table VII, we can readily obtain the following relationships. For ($J=J'=0$), the A_1^- has one-to-one correspondence:

$$\mathcal{M}^{\mathbf{c}(1)}_{01;01}(A_1^-)=\mathcal{M}^{(1)}_{01;01}(A_1^-). \tag{48}$$

Then, for ($J=J'=2$), the E^- splits into A_1^- and B_1^-, i.e.,

$$\mathcal{M}^{\mathbf{c}(1)}_{21;21}(E^-)=\frac{1}{2}\mathcal{M}^{(1)}_{21;21}(A_1^-)+\frac{1}{2}\mathcal{M}^{(1)}_{21;21}(B_1^-). \tag{49}$$

Next, for ($J=J'=1$), the T_1^+ is divided into A_2^+ and E^+ so that

$$\begin{cases}\mathcal{M}^{\mathbf{c}(1)}_{10;10}(T_1^+)=\frac{1}{3}\mathcal{M}^{(1)}_{10;10}(A_2^+)+\frac{2}{3}\mathcal{M}^{(1)}_{10;10}(E^+)\\ \mathcal{M}^{\mathbf{c}(1)}_{10;12}(T_1^+)=\frac{1}{3}\mathcal{M}^{(1)}_{10;12}(A_2^+)+\frac{2}{3}\mathcal{M}^{(1)}_{10;12}(E^+)\\ \mathcal{M}^{\mathbf{c}(1)}_{12;12}(T_1^+)=\frac{1}{3}\mathcal{M}^{(1)}_{12;12}(A_2^+)+\frac{2}{3}\mathcal{M}^{(1)}_{12;12}(E^+)\end{cases}. \tag{50}$$

Then, for ($J=J'=2$), the T_2^- splits into B_2^- and E^-, and we have

$$\mathcal{M}^{\mathbf{c}(1)}_{21;21}(T_2^-)=\frac{1}{3}\mathcal{M}^{(1)}_{21;21}(B_2^-)+\frac{2}{3}\mathcal{M}^{(1)}_{21;21}(E^-). \tag{51}$$

Finally, we discuss the relations of phase shift formulas between the cubic boxes and the elongated boxes. Here, we take T_2^- representation in cubic boxes as an example. From the elongated box to the cubic symmetry, the matrix elements $\mathcal{M}^{(1)}_{21;21}(B_2^-)$ and $\mathcal{M}^{(1)}_{21;21}(E^-)$ will individually approach $\mathcal{M}^{\mathbf{c}(1)}_{21;21}(T_2^-)$ in the limit $\eta=1$. From Eq. (51), one can see how to follow this limit by the subduction rule in this particular channel. The relation translates directly into one for the phase shift as follows:

TABLE VII. Subduction rules in the descent in symmetry in the group from cubic box(O_h) to the elongated box(D_{4h}).

O_h	A_1^+	A_2^+	E^+	T_1^+	T_2^+	A_1^-	A_2^-	E^-	T_1^-	T_2^-
D_{4h}	A_1^+	B_1^+	$A_1^+\oplus B_1^+$	$A_2^+\oplus E^+$	$B_2^+\oplus E^+$	A_1^-	B_1^-	$A_1^-\oplus B_1^-$	$A_2^-\oplus E^-$	$B_2^-\oplus E^-$

$$\cot\delta_{21}(T_2^-) = \frac{1}{3}\cot\delta_{21}(B_2^-) + \frac{2}{3}\cot\delta_{21}(E^-). \quad (52)$$

V. LOW MOMENTUM EXPANSION OF SCATTERING PHASE SHIFT

It is known that the scattering phases behave as $\delta_J = n_J\pi + a_J k^{2J+1} + \mathcal{O}(k^{2J+3})$ with small relative momentum k. Therefore, it is reasonable to assume that the phase shift is dominated by the lowest angular momentum scattering channel, and the higher angular momentum channels are relatively small in the process of low energy elastic scattering [37,66]. Taking phase shift formula Eq. (20) in the elongated box for A_1^- representation in *Case a* as an example, we find that Eq. (20) can be simplified to

$$\cot\delta_{01} - \mathcal{M}^{(1)}_{01;01} = \frac{\mathcal{M}^{(1)2}_{01;21}}{\cot\delta_{21} - \mathcal{M}^{(1)}_{21;21}}, \quad (53)$$

where $\mathcal{M}^{(1)}_{01;01}$,$\mathcal{M}^{(1)2}_{01;21}$, and $\mathcal{M}^{(1)}_{21;21}$ are listed in Table I; If the D-wave scattering phase shift vanishes, namely $\delta_{21} = 0$ as we expected, it is easy to check

$$\cot\delta_{01} = \mathcal{M}^{(1)}_{01;01} = \omega_{00}. \quad (54)$$

In general, Eq. (53) offers the desired relation between the energy eigenvalues and the scattering phases for the cases in the elongated box in the A_1^- representation in *Case a*. In Eq. (53), it is easy to verify that contributions those appear in the right-hand side of the equations are smaller by a factor of q^2 compared with $\mathcal{M}^{(1)}_{01;01}$ on the left-hand side. The right-hand side of the equations are negligible as long as the relative momentum q is small enough. Thus, the S-wave scattering length will be determined by the zero momentum limit of Eq. (54). Therefore, in the low-energy limit, it is a good approximation to treat the D-wave scattering phase as a small perturbation, and it is also possible to work out the corrections because of higher scattering phases to the S-wave scattering phase:

$$n\pi - \delta_{01} = \phi(q) + \sigma(q)\tan\delta_{21}, \quad (55)$$

where the angle $\phi(q)$ is defined as $\tan\phi(q) = -1/\mathcal{M}^{(1)}_{01;01}$. The sensitivity functions $\sigma(q)$ represents the sensitivity of D-wave scattering phases. For the elongated box, the sensitivity function is found to be

$$\sigma(q) = \mathcal{M}^{(1)2}_{01;21}/(1 + \mathcal{M}^{(1)2}_{01;01}), \quad (56)$$

The functions $\sigma(q)$ can be calculated using the matrix elements given in Table I. The generalized zeta function for z-elongated boxes is given by

$$Z_{lm}(q^2,\eta) = \sum_{\mathbf{n}} \frac{\mathcal{Y}_{lm}(\tilde{\mathbf{n}})}{\tilde{\mathbf{n}}^2 - q^2}, \quad (57)$$

where $\mathcal{Y}_{lm}(\tilde{\mathbf{n}}) = \tilde{\mathbf{n}}^l Y_{lm}(\theta,\phi)$. Here, the modified index is $\tilde{\mathbf{n}} = (n_x, n_y, n_z/\eta)$. It goes back to the case of cubic box with $\eta = 1$. For the analytic continuation [37], the explicit form of zeta function in elongated boxes is found to be

$$\begin{aligned} Z_{lm}(q^2,\eta) = {} & e^{q^2}\sum_{\mathbf{n}}\left[\frac{\mathcal{Y}_{lm}(\tilde{\mathbf{n}})}{\tilde{\mathbf{n}}^2 - q^2}\right]e^{-\tilde{\mathbf{n}}^2} \\ & - \pi\eta\delta_{l0}\delta_{m0} + \frac{\pi\eta}{2}\delta_{l0}\delta_{m0}\int_0^1 dt t^{-3/2}(e^{tq^2} - 1) \\ & + \pi\eta\int_0^1 dt t^{-3/2}\left[\sum_{\mathbf{n}\neq 0}\mathcal{Y}_{lm}\left(-i\frac{\pi}{t}\hat{\mathbf{n}}\right)e^{tq^2}e^{-\frac{\pi^2}{t}\hat{\mathbf{n}}^2}\right] \end{aligned} \quad (58)$$

where $\hat{\mathbf{n}} = (n_x, n_y, n_z\eta)$.

In this study, we calculate the sensitivity function $\sigma(q)$ with $\eta = 1.2$, $\eta = 1.4$. We find that it mostly varies in the range 0–100. In Figs. 1 and 2, the sensitivity functions are plotted versus q^2 for D_{4h} symmetry with $\eta = 1.2$, and $\eta = 1.4$, respectively. It is seen that the sensitivity function $\sigma(q)$ remains finite for all $q^2 > 0$. For some particular values of q^2, however, the sensitivity functions become quite large in magnitude. This is because of the almost coincidence of singularities of the numerator in matrix

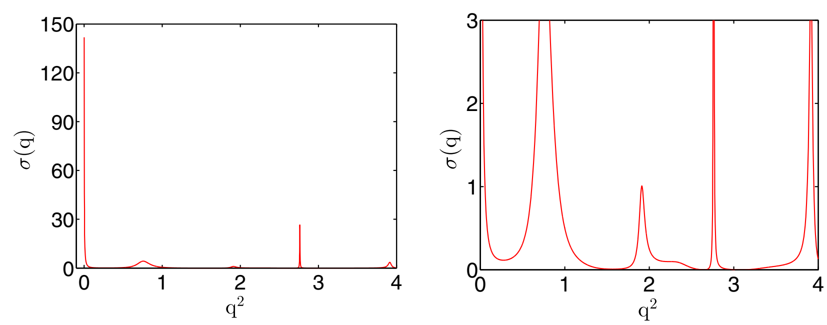

FIG. 1. The function $\sigma(q)$ is plotted versus q^2 for D_{4h} symmetry with $\eta = 1.2$. It can be calculated using the matrix elements given in Tab. I. The right panel in the plot are simply the same function as in the left panel with the scale of the vertical axis being magnified, in order to show the detailed variation of the functions.

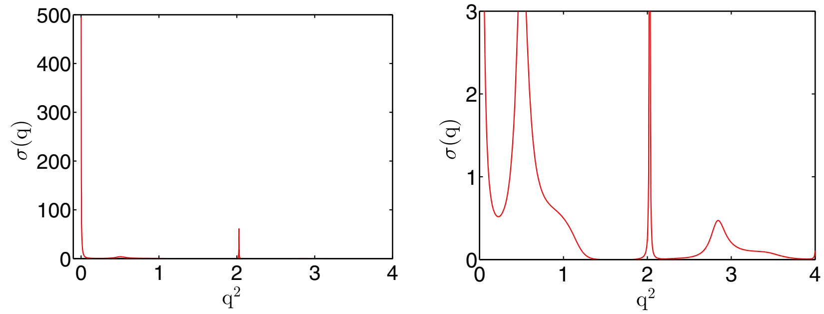

FIG. 2. The function $\sigma(q)$ is plotted versus q^2 for D_{4h} symmetry with $\eta = 1.4$. It can be calculated using the matrix elements given in Table I. The right panel in the plot are simply the same function as in the left panel with the scale of the vertical axis being magnified, in order to show the detailed variation of the functions.

elements $\mathcal{M}^{(1)2}_{01;21}$ and denominator in matrix elements $\mathcal{M}^{(1)2}_{01;01}$ which happens for some choices of η. For all other values of q^2 away from these values, the functional values of $\sigma(q)$ remain moderate. The Eq. (54) is considered to be a good approximation to extract the S-wave scattering phase shift.

However, it is very difficult to extract the phase shift from the energy spectrum in the A_1^- representation if the D-wave scattering phase shift, i.e., δ_{21} is not small. This is because there are two unknown functions δ_{01} and δ_{21} but we have only Eq. (53). In principle, we still can extract the S-wave scattering phase shift δ_{01} from Eq. (53) through dividing the D-wave phase shift by lattice simulations at various energy values. Then, due to scattering phases with higher angular momentum the corrections can be estimated from other representations for the group of D_{4h}. For example, from Table I, it is obviously seen that, for lattices with D_{4h} symmetry, by inspecting energy eigenstate with E^-, B_1^-, and B_2^- symmetries on the lattice, one can obtain a rough estimate for the D-wave scattering phase δ_{21} which dominates this symmetry sector. It seems to be difficult, but naturally, it is still possible to compute the energy spectrum and extract the phase shift functions.

VI. DISCUSSIONS AND CONCLUSIONS

In this paper, we have derived baryon-baryon scattering phase shift from finite volume spectra in elongated boxes. We show the cases where the baryon-baryon states are in COM frame with periodic boundary condition and twisted boundary condition, or moving frame with periodic boundary condition along the elongated direction. As a consistency check and validation on the elongated results, we have also derived the results in the cubic box by using the same approach. There are two differences between these two cases. One is the symmetry of the two-particle system and the other is the matrix $\mathcal{M}$. We take the two-baryon scattering in COM frame as an example. In the cubic case, the symmetry group is O_h, but in the elongated case, the symmetry group becomes D_{4h}. In addition, the case with factor $\eta = 1$ in matrix $\mathcal{M}$ in the elongated box is equivalent to the cubic case. Our interest in elongated boxes stems from the fact that they allow us to change the geometry of the box, and consequently the kinematics, with minimal amount of computer resources. In addition, elongated boxes have a different symmetry group compared with the cubic case. This has to be taken into account when designing interpolators and connecting the infinite volume phase shifts with the two-body energy values.

Finally, let us discuss some possible applications of phase shift formulas derived in this paper. Some typical examples for scattering with two spin-$1/2$ particles are listed here, and all examples are highly relevant in the study of the composition of dense nuclear matter which forms the neutron stars. The first class of typical examples includes the $\Lambda-\Lambda$, $N-\Xi$, $\Sigma-\Sigma$ scattering, where the existence of H dibaryon state has been discussed as the remaining of bound state in flavor singlet channel. Another class of typical example includes $\Lambda-N$ and $\Sigma-N$ scattering which is useful to study properties of hyperonic matters inside the neutron stars. $\Xi-\Xi$, $N-N$, $N\Sigma$, and $N\Lambda$ scattering follow. These examples have been studied by using HAL QCD method in Refs. [67–70]. In principle, these issues can also be studied with Lüscher's formulas in detail by using lattice QCD simulations in elongated boxes.

To summarize, in this paper we have generalized two-particle elastic scattering phase shift formulas to the case of particles with spin $1/2$ in the COM frame and MF in the elongated box, and COM frame in cubic box, respectively. By using a quantum mechanical model, we established a relation between the energy of the two-particle system and phase shift. It is verified that the phase shift formulas in elongated box in the limit $\eta = 1$ and cubic box are consistent. Although we focus on the scattering between two particles with the spin of both being $1/2$, there are not essential difficulties in generalizing phase shift formulas to cases with arbitrary spin in any number of channels. We expect that these relations will be helpful for the study of baryon-baryon elastic scattering in lattice QCD simulations.

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