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Comment on "Construction of regular black holes in general relativity"

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We claim that the paper by Zhong-Ying Fan and Xiaobao Wang on nonlinear electrodynamics coupled to general relativity [Phys. Rev. D **94**,124027 (2016)], although correct in general, in some respects repeats previously obtained results without giving proper references. There is also an important point missing in this paper, which is necessary for understanding the physics of the system: in solutions with an electric charge, a regular center requires a non-Maxwell behavior of Lagrangian function $L(f)$, ($f = F_{\mu\nu}F^{\mu\nu}$) at small f. Therefore, in all electric regular black hole solutions with a Reissner-Nordström asymptotic, the Lagrangian $L(f)$ is different in different parts of space, and the electromagnetic field behaves in a singular way at surfaces where $L(f)$ suffers branching.

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Nonlinear electrodynamics (NED) as a possible material source in general relativity (GR) and its extensions attracts much attention since, among other reasons, it leads to many space-time geometries of interest, in particular, regular black holes (BHs) and starlike, or solitonlike, configurations.

The paper by Fan and Wang [1] belongs to this trend but in some important points repeats already known results, and many relevant papers are absent in the list of references. There are also some well-known important physical properties of solutions with an electric charge, which are not mentioned but deserve mentioning as necessary information for readers (e.g., students) who are not experts in the field.

To begin with, the key inferences are based on the general static, spherically symmetric solution of GR coupled to NED in the case of an electric field obtained in 1969 by Pellicer and Torrence [2], not cited in the paper. This consideration was extended in [3] to systems containing both electric and magnetic charges. For a further discussion, let us briefly reproduce it here.

In GR coupled to NED one considers the action

$$S = \frac{1}{2}\int \sqrt{-g}d^4x[R - L(f)], \qquad f = F_{\mu\nu}F^{\mu\nu} \quad (1)$$

($F_{\mu\nu}$ is the Maxwell tensor, and units with $c = 8\pi G = 1$ are used) with an arbitrary function $L(f)$. Then, assuming static spherical symmetry, the stress-energy tensor (SET) satisfies the condition $T^t_t = T^r_r$; hence, due to the Einstein equations, the metric can be written as

$$ds^2 = A(r)dt^2 - dr^2/A(r) - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2)$$

The only nonzero components of $F_{\mu\nu}$ are $F_{tr} = -F_{rt}$ (a radial electric field) and $F_{\theta\phi} = -F_{\phi\theta}$ (a radial magnetic field). The Maxwell-like equations $\nabla_\mu(L_f F^{\mu\nu}) = 0$ and the Bianchi identities $\nabla_\mu {}^*F^{\mu\nu} = 0$ give

$$r^2 L_f F^{tr} = q_e, \qquad F_{\theta\phi} = q_m \sin\theta, \quad (3)$$

where q_e and q_m are the electric and magnetic charges, respectively, and $L_f \equiv dL/df$. Accordingly, the nonzero SET components are

$$T^t_t = T^r_r = \frac{1}{2}L + f_e L_f, \qquad T^\theta_\theta = T^\phi_\phi = \frac{1}{2}L - f_m L_f, \quad (4)$$

$$f_e = 2F_{tr}F^{rt} = \frac{2q_e^2}{L_f^2 r^4}, \qquad f_m = 2F_{\theta\phi}F^{\theta\phi} = \frac{2q_m^2}{r^4}, \quad (5)$$

so that the invariant f is $f = f_m - f_e$. The metric function $A(r)$ is found from the Einstein equations as

$$A(r) = 1 - \frac{2M(r)}{r}, \qquad M(r) = \frac{1}{2}\int T^t_t(r) r^2 dr, \quad (6)$$

where T^t_t is the energy density. It is a relation including both electric and magnetic fields, written in a general form [3].

Fan and Wang claim that they have presented a general procedure for constructing exact regular BH solutions with electric or magnetic charges in GR coupled to NED. However, this procedure was in fact already described in [3]. Indeed, for a magnetic solution ($q_e = 0$), given any $A(r)$, using Eq. (6), one easily calculates $T^t_t = L/2$ as a function of r, and $f = f_m(r)$ is known from (5); thus $L(f)$ is also determined. On the contrary, starting from $L(f)$ and using (5), we obtain $M(r)$ and $A(r)$ from (6). Moreover, a necessary condition for obtaining a regular center is that $L(f)$ should tend to a finite limit as $f \to \infty$ [3] (an observation absent in the paper). On the other hand, the

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description in the paper contains some additional relations, details, and explanations.

Electric solutions are obtained in a similar way using the "Hamiltonian" formulation of NED (see, e.g., [4]), produced from the original one by a Legendre transformation: one introduces the tensor $P_{\mu\nu} = L_f F_{\mu\nu}$ with its invariant $p = -P_{\mu\nu}P^{\mu\nu}$ and considers the Hamiltonian-like quantity $H = 2fL_f - L = 2T^t_t$ as a function of p; then $H(p)$ can be used to specify the whole theory. One has then

$$L = 2pH_p - H, \qquad L_f H_p = 1, \qquad f = pH_p^2, \qquad (7)$$

with $H_p \equiv dH/dp$. Then for electric solutions ($q = q_e \neq 0$, $q_m = 0$), specifying $H(p)$, we directly find $M(r)$ and $A(r)$ using (6) since we have simply $p = 2q^2/r^4$. If we specify $A(r)$, from (6) we easily find $H(p)$. All this was described in [3]; see there Eqs. (12) for magnetic solutions and (19) for electric ones.

In both cases, selection of special families of such solutions governed by a few parameters (as is done in [1]) is quite an easy task since the function $A(r)$, specifying the solutions, is arbitrary; hence the number of free parameters can also be arbitrary. One should only take care of the boundary condition $A(r) = 1 + O(r^2)$ as $r \to 0$ if a regular center is required, and provide $A(r) \approx 1–2M/r$ as $r \to \infty$ ($M = \text{const}$) to have a Schwarzschild asymptotic. If this $A(r)$ has zeros, corresponding to horizons, it is a BH solution, while if everywhere $A > 0$, it is a particle-like or soliton solution (the latter opportunity is not mentioned in the paper).

An important point concerning electric solutions is the existence of a no-go theorem [5] (which was probably unknown to the authors of the paper), saying that there is no such Lagrangian function $L(f)$ having a Maxwell weak-field limit ($L \sim f$ as $f \to 0$) that the electric solution (2), (3), and (6) has a regular center. The reason is that at such a center the electric field should be zero but the field equations then imply $fL_f^2 \to \infty$; hence $L_f \to \infty$ as $r \to 0$. It was further shown in [3] that a regular center is also impossible in dyonic configurations, with both $q_e \neq 0$ and $q_m \neq 0$, if $L(f)$ has a Maxwell weak-field limit.

An alternative (but equivalent) formulation of this no-go theorem is that if a static, spherically symmetric solution to the theory (1) with $q_e \neq 0$ contains a regular center, then $L(f)$ is non-Maxwell at small f.

A natural question is the following: how does this no-go theorem combine with the existing examples of regular electric solutions, e.g., the one given in [4] and others, mentioned or cited in [1]? An answer was given in [6]: in all such cases, in a regular solution there are different Lagrangian functions $L(f)$ at large and small r. At large r, where $f \to 0$, we have $L \sim f$ whereas at small r the theory is strongly non-Maxwell ($f \to 0$ but $L_f \to \infty$), in agreement with the no-go theorem. An inspection showed that it is indeed the case in all examples.

According to [3,6], in the Hamiltonian framework, at a regular center we have $p \to \infty$ but a finite limit of H, and the integral in (6) gives the mass function and $A(r)$. However, in all regular solutions where $f = 0$ at both $r = 0$ and $r = \infty$, the function f inevitably has at least one maximum at some $p = p^*$, violating the monotonicity of $f(p)$, which is necessary for equivalence of the f and p frameworks. It has been shown [3] that at an extremum of $f(p)$ the Lagrangian function $L(f)$ suffers branching, its plot forming a cusp, and different functions $L(f)$ correspond to $p < p^*$ and $p > p^*$. Another kind of branching occurs at extrema of $H(p)$, if any, and the number of Lagrangians $L(f)$ on the way from infinity to the center equals the number of monotonicity ranges of $f(p)$.

It was mentioned in [1] that "the original $L(f)$ formalism may not be appropriate any longer in this case because one will end with a multivalued $L(f)$, which has different branches for a well-defined single one $H(p)$." It should be stressed, however, that this branching is an *inevitable* property of all regular electric solutions with a Reissner-Nordström asymptotic behavior.

It might seem that the Hamiltonian framework is not worse than the Lagrangian one, even though the latter is directly related to the least action principle. However, as shown in [3], at $p = p^*$ the electromagnetic field exhibits a singular behavior, well revealed using the effective metric [7,8] in which NED photons move along null geodesics. This metric is singular at extrema of $f(p)$, and the effective potential for geodesics exhibits infinitely deep wells where NED photons are infinitely blueshifted [3,8] and can after all create a curvature singularity due to backreaction on the metric. Thus any such solution not only fails to correspond to a fixed Lagrangian $L(f)$ but has other important undesired features. In my opinion, it is a necessary addition to the description of electric solutions in the paper.

Is it possible to circumvent the above no-go theorem for electric solutions? The answer is yes [9]: one can consider a kind of phase transition on a certain sphere, outside of which there is a purely electric field $F_{\mu\nu}$ but inside of which the field is purely magnetic. An external observer then sees an electrically charged BH or soliton.

Fan and Wang also describe a straightforward extension of static, spherically symmetric NED solutions to GR with a nonzero negative cosmological constant Λ, leading to their anti–de Sitter asymptotic behavior; however, this extension (with both positive and negative Λ) has been already considered, e.g., in [10–12]. Actually, if we add -2Λ to R in the action (1), the only change in the expression (6) for the metric is that the term $-\Lambda r^2/3$ is added to $A(r)$. With or without Λ, if $A(r)$ is known (or chosen by hand), the form of the theory is easily restored from (6): $dM(r)/dr$ directly gives $H(p)$ for electric configurations or $L(f)$ for magnetic ones since $p(r)$ or $f(r)$, respectively, are known in these cases. On the contrary, knowing $L(f)$ or $H(p)$, it is easy to find $A(r)$ in magnetic or electric configurations, respectively.

To summarize, there is a substantial gap in [1], connected with the fact that the "regular black hole construction procedure" was already described earlier. An important point missing in the paper is the inevitable undesired property of regular electric solutions if one requires a Maxwell weak-field limit of NED at large radii [multivaluedness of the Lagrangian function $L(f)$ and troubles with the electromagnetic field at its branching points]. Somewhat less important is a missing mention of possible solitonic and asymptotically de Sitter solutions. An evident shortcoming is the absence of necessary references, directly related to the subject, such as [2,3,9–12], and maybe some others.

Does all that mean that there are no new results of interest in [1]? Certainly not. New examples of regular BH solutions both with and without a cosmological constant are obtained and discussed, some useful general relations have been obtained for the GR-NED set of equations, and the whole Sect. V entitled "The first law of thermodynamics" is quite interesting and is not restricted to the first law only: there are a generalization of Smarr's formula and new expressions for the entropy products. So, despite the above criticism, this paper seems to be quite a useful contribution to the studies of regular black holes.

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