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Complexity of AdS_5 black holes with a rotating string

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We consider computational complexity of AdS_5 black holes. Our system contains a particle moving on the boundary of AdS. This corresponds to the insertion of a fundamental string in AdS_5 bulk spacetime. Our results give a constraint for complexity. This gives us a hint for defining complexity in quantum field theories.

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I. INTRODUCTION

A. Background

Computational complexity is originally introduced in physics of quantum information [1] and its application to black hole physics is considered in many works [2–5]. It behaves in a similar way to the entropy as it is expected to satisfy the second law of thermodynamics [6]. In recent works, for example [7,8], the similarity or unsimilarity to the entanglement entropy is discussed.

The growth of complexity is related to problems of black holes, for example, information problem, the transparency of horizons or the existence of firewalls [9–13].

The complexity-action (CA) conjecture [14,15] predicts that complexity of the black hole is equal to the bulk action integrated in the region called the Wheeler-de Witt patch (WdW). This conjecture is checked by many works [16–20]. The time dependence is studied in [21–27]. This duality is studied also in a deformed case by adding probe branes [28].

Nevertheless, the strict and valid definition of complexity is still unclear. The most standard one so far is by the method of quantum operators which are called gates. Let us define a reference state as the simplest quantum state. Quantum gates operate a state and change its state. Computational complexity of a test state is defined as the minimum number of gates to prepare it from the reference state. But this definition has some unsatisfied points. One is that the definition of the reference state is unclear. Another is that the number of the gates depends on the choices of the primitive gates.

B. Our approach

To treat this problem, in this paper, we shed light on a new property of complexity and suggest a new constraint which complexity should possess. The above works so far treat stationary systems. A generalization to this conjecture to the system other than stationary states is an important work. As an example, we consider here a system moved by a drag force. Such a work is motivated by jet-quenching phenomena in heavy ion collisions. Energy loss of a charged particle in quark-gluon plasma (QGP) is calculated in several approaches [29–32]. Especially in [29] a particle moving in AdS_5-Schwarzschild spacetime was considered and its action was calculated. The particle is moved on the boundary of the AdS_5 space. Because of the effect of shear viscosity in QGP, this particle losses the energy. This system is a dissipative system.

While his work considered the action in the outside of the horizon, that is usual spacetime, we apply here this method also to the inside of the horizon. According to CA relation, we will obtain complexity of a time dependent system. In this work we consider a Wilson line operator located in AdS_5 spacetime by inserting a fundamental string. This Wilson line moves on a great circle in S^3 part in AdS_5. Such a nonlocal operator describes a test particle moving on the boundary gauge theory. This shows how complexity is deformed when adding a time-dependent operator, especially also nonlocal operator. Since inserting the Wilson loop is described by adding a Nambu-Goto (NG) term, the action is expected to consist of the Einstein-Hilbert term, the Nambu-Goto term and the boundary term. We study on the effect of the Wilson line operator focusing on the Nambu-Goto term and show the black-hole mass and particle's velocity dependences of the action. This will show the growth of complexity for a dissipative system.

This paper is constructed as follows. In Sec. II, we explain our setup which includes the AdS black hole with a kind of nonlocal operator, a Wilson line, and the action. In Sec. III, in order to find the effect of the test particle, we focus on the NG action which is a part of the action we introduced the former section. Section IV gives an interpretation of our calculation and some suggestions of the constraint for complexity.

II. SETUP

We consider a Wilson loop inserted on AdS_5 spacetime. The black hole geometry in AdS_5 spacetime is described by the metric:

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$$ds^2_{\rm AdS} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \tag{1}$$

$$f(r) = 1 - \frac{16\pi GM}{5\Omega_3 r^2} + \frac{r^2}{\ell^2_{\rm AdS}} = 1 - \frac{r_{\rm m}^2}{r^2} + \frac{r^2}{\ell^2_{\rm AdS}},$$
$$r_{\rm m}^2 := \frac{16\pi GM}{5\Omega_3}, \tag{2}$$

where $d\Omega_3^2$ is the square of the line element on three sphere, while $d\Omega_3$ is the volume form on the three sphere and Ω_3 is the volume obtained by integrating it.[1] We see in appendix A that $\ell_{\rm AdS}$ and other parameters are related to the gauge theory parameters as

$$\frac{r_{\rm m}^2}{\ell^2_{\rm AdS}} = \frac{16\pi GM}{5\Omega_3 \ell^2_{\rm AdS}} = \frac{8GM}{5\pi\ell^2_{\rm AdS}} = \frac{4M}{5}\frac{\sqrt{\alpha' g_{\rm YM}}}{N^{7/4}}. \tag{3}$$

This geometry has some characteristic length; one is "the Schwarzschild radius" defined in Eq. (2)[2] and the other is r_h which is the radius of the black-hole horizon which is determined by the equation $f(r) = 0$:

$$r_h = \ell_{\rm AdS}\left(\frac{-1 + \sqrt{1 + 4r_{\rm m}^2/\ell^2_{\rm AdS}}}{2}\right)^{1/2}. \tag{4}$$

We parametrize S^3 part of the AdS spacetime (1) by

$$x^1 = \cos\theta\cos\phi, \qquad x^2 = \cos\theta\sin\phi,$$
$$x^3 = \sin\theta\cos\varphi, \qquad x^4 = \sin\theta\sin\varphi;$$
$$\theta \in [0, \pi/2), \phi, \varphi \in [0, 2\pi). \tag{5}$$

Let us take parameters τ and σ on the world sheet of the fundamental string:

$$t = \tau, \qquad r = \sigma, \qquad \phi = \omega\tau + \xi(\sigma), \tag{6}$$

where ω is the constant angular velocity and $\xi(\sigma)$ is a function which determines the shape of the string. Thus, the induced metric on the world sheet is

$$ds^2_{\rm ind} = -f(\sigma)d\tau^2 + \frac{d\sigma^2}{f(\sigma)}, \tag{7}$$

$$f(\sigma) = 1 - \frac{r_{\rm m}^2}{\sigma^2} + \frac{\sigma^2}{\ell^2_{\rm AdS}}. \tag{8}$$

[1]The volume of $(n-1)$-dim unit sphere S^{n-1} is $\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

[2]We use this definition in this paper but the term "Schwarzschild radius" is usually only defined for asymptotically flat Schwarzschild black holes, which are the same as the horizon radius.

The action consists of the Einstein-Hilbert term and the Nambu-Goto (NG) term:

$$S_{\rm EH} = \frac{1}{16\pi G}\int d^6x\sqrt{|g|}(\mathcal{R} - 2\Lambda), \qquad \Lambda = \frac{-6}{\ell^2_{\rm AdS}}, \tag{9}$$

$$S_{\rm NG} = -T_s \int d^2\sigma\sqrt{-\det g_{\rm ind}}, \tag{10}$$

where T_s is the tension of the fundamental string. We are afraid for the necessity of adding a boundary term like the York-Gibbos-Hawking (YGH) term. The development of the Einstein-Hilbert action is given in [15] as

$$\frac{dS_{\rm EH}}{dt} = -\frac{1}{2\pi G\ell^2_{\rm AdS}}\int_\Sigma r^3 dr d\Omega_3 = -\frac{1}{8\pi G}\frac{r_h^4\Omega_3}{\ell^2_{\rm AdS}}. \tag{11}$$

This is the leading contribution to complexity. In the next section we consider the effect of the addition of the Wilson loop.

III. EVALUATION OF THE NG ACTION

The addition of the Wilson loop corresponds to inserting a fundamental string whose world sheet has a boundary on the Wilson loop. We calculate here the NG action of this fundamental string. The time derivative of the NG action is obtained by integrating the square root of the determinant of the induced metric over the WdW (Fig. 1):

$$\frac{dS_{\rm NG}}{dt} = T_s\int_0^{r_h} d\sigma\sqrt{1 - \frac{\sigma^2\omega^2}{f(\sigma)} + \sigma^2\xi'(\sigma)^2 f(\sigma)}. \tag{12}$$

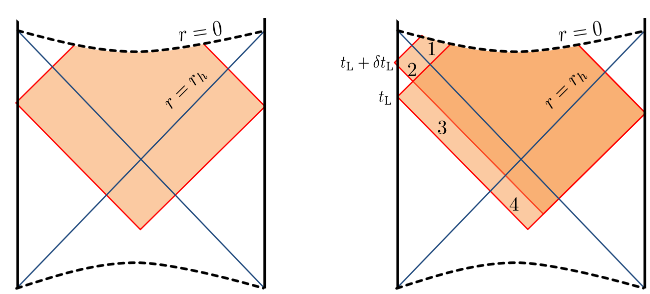

FIG. 1. Penrose diagram of the AdS black hole. The blue diagonal lines represent the black hole horizon. (left) The shaded rectangular region represents the Wheeler-de Witt patch (WdW). (right) Two WdW in different left CFT time t_L. The regions 1,2,3 and 4 represent the difference of these two patches. The contributions from regions 2 and 3 cancel. The contribution from region 4 becomes smaller as time develops. Then the integration over only the region 1 is relevant for our analysis.

The Lagrangian is

$$\mathcal{L} = T_s\sqrt{1 - \frac{\sigma^2\omega^2}{f(\sigma)} + \sigma^2\xi'(\sigma)^2 f(\sigma)}. \tag{13}$$

Then the equation of motion for $\xi(\sigma)$ is

$$0 = \frac{1}{T_s}\frac{d}{d\sigma}\frac{\partial\mathcal{L}}{\partial\xi'(\sigma)} = \frac{d}{d\sigma}\left(\frac{\sigma^2\xi'(\sigma)f(\sigma)}{\sqrt{1-\sigma^2\omega^2/f(\sigma)+\sigma^2\xi'(\sigma)^2 f(\sigma)}}\right), \tag{14}$$

$$c_\xi \coloneqq \frac{\sigma^2\xi'(\sigma)f(\sigma)}{\sqrt{1-\sigma^2\omega^2/f(\sigma)+\sigma^2\xi'(\sigma)^2 f(\sigma)}} = \frac{\sigma^2\xi'(\sigma)f(\sigma)}{\mathcal{L}/T_s}, \tag{15}$$

where by the second line we defined a conserved constant c_ξ. Solving it for $\xi(\sigma)$, this function is given by integrating the following:

$$\xi'(\sigma) = \frac{c_\xi}{\sigma f(\sigma)}\sqrt{\frac{f(\sigma)-\sigma^2\omega^2}{\sigma^2 f(\sigma) - c_\xi^2}}. \tag{16}$$

In order for this expression to give real values, the denominator in the square root must be negative when the numerator factor $f(\sigma) - \sigma^2\omega^2$ becomes negative. Thus they become zero coincidentally. From this condition, the constant c_ξ is determined to be

$$c_\xi = \omega\sigma_H^2 = \omega\frac{-1+\sqrt{1+4r_\mathrm{m}^2(1/\ell_\mathrm{AdS}^2 - \omega^2)}}{2(1/\ell_\mathrm{AdS}^2-\omega^2)}, \tag{17}$$

$$\sigma_H^2 = \frac{-1+\sqrt{1+4r_\mathrm{m}^2(1/\ell_\mathrm{AdS}^2 - \omega^2)}}{2(1/\ell_\mathrm{AdS}^2-\omega^2)}, \tag{18}$$

where $\sigma = \sigma_H$ is the solution for numerator of the square root in Eq. (16). We assumed that c_ξ is positive. Since the numerator and the denominator have the coincident solution, the cancellation gives

$$\xi'(\sigma) = \frac{\omega\sigma_H^2}{(\sigma^2 - r_h^2)(1/\ell_\mathrm{AdS}^2(\sigma^2+r_h^2)+1)} \times\sqrt{\frac{(1/\ell_\mathrm{AdS}^2-\omega^2)(\sigma^2+\sigma_H^2)+1}{1/\ell_\mathrm{AdS}^2(\sigma^2+\sigma_H^2)+1}}, \tag{19}$$

$$\mathcal{L} = T_s\sigma^2\xi'(\sigma)f(\sigma)/c_\xi = T_s\sqrt{\frac{(1/\ell_\mathrm{AdS}^2-\omega^2)(\sigma^2+\sigma_H^2)+1}{1/\ell_\mathrm{AdS}^2(\sigma^2+\sigma_H^2)+1}}. \tag{20}$$

The integral of (20) over the WdW is

$$\frac{dS_\mathrm{NG}}{dt} = \int_\mathrm{WdW}\mathcal{L} = T_s\int_0^{r_h} d\sigma\sqrt{\frac{(1/\ell_\mathrm{AdS}^2-\omega^2)(\sigma^2+\sigma_H^2)+1}{1/\ell_\mathrm{AdS}^2(\sigma^2+\sigma_H^2)+1}}. \tag{21}$$

Using the integral formula (C2) shown in the Appendix,

$$\frac{dS_\mathrm{NG}}{dt} = -iT_s\sqrt{\sigma_H^2(1-\ell_\mathrm{AdS}^2\omega^2)+\ell_\mathrm{AdS}^2}E \times\left[\arcsin\left(i\frac{r_h}{\sqrt{\sigma_H^2+\ell_\mathrm{AdS}^2}}\right), \sqrt{\frac{(\sigma_H^2+\ell_\mathrm{AdS}^2)(1-\ell_\mathrm{AdS}^2\omega^2)}{\sigma_H^2(1-\ell_\mathrm{AdS}^2\omega^2)+\ell_\mathrm{AdS}^2}}\right]. \tag{22}$$

Substituting Eq. (18) and the horizon radius r_h, (4), we can express the growth of the action as

$$\frac{dS_\mathrm{NG}}{dt} = -iT_s\ell_\mathrm{AdS}\left(\frac{1+\sqrt{1+4s^2(1-v^2)}}{2}\right)^{1/2} \times E\left[\arcsin\left(i\left(\frac{(-1+\sqrt{4s^2+1})(1-v^2)}{(1-2v^2)+\sqrt{1+4s^2(1-v^2)}}\right)^{1/2}\right), \left(\frac{(1-2v^2)+\sqrt{1+4s^2(1-v^2)}}{1+\sqrt{1+4s^2(1-v^2)}}\right)^{1/2}\right], \tag{23}$$

where for simplicity we defined the following dimensionless constants:

$$s \coloneqq r_\mathrm{m}/\ell_\mathrm{AdS}, \qquad v \coloneqq \ell_\mathrm{AdS}\omega. \tag{24}$$

A. Limit behavior

The behavior in the limits $s\to 0$ and $s\to\infty$ are, respectively, as follows:

$$\left.\frac{dS_\mathrm{NG}}{dt}\right|_{s\to 0} = -i4T_s\ell_\mathrm{AdS}E\left[0, \sqrt{1-v^2}\right] = 0, \tag{25}$$

$$\left.\frac{dS_\mathrm{NG}}{dt}\right|_{s\to \infty} = 4T_s\ell_\mathrm{AdS}\sqrt{s}(1-v^2)^{1/2}. \tag{26}$$

The behavior in the limits $v\to 0$ and $v\to 1$ are

$$\left.\frac{dS_\mathrm{NG}}{dt}\right|_{v=0} = \sqrt{2}T_s\ell_\mathrm{AdS}\left(\frac{-1+\sqrt{4s^2+1}}{2}\right)^{1/2} \sim \sqrt{s} \times (s\gg 1), \tag{27}$$

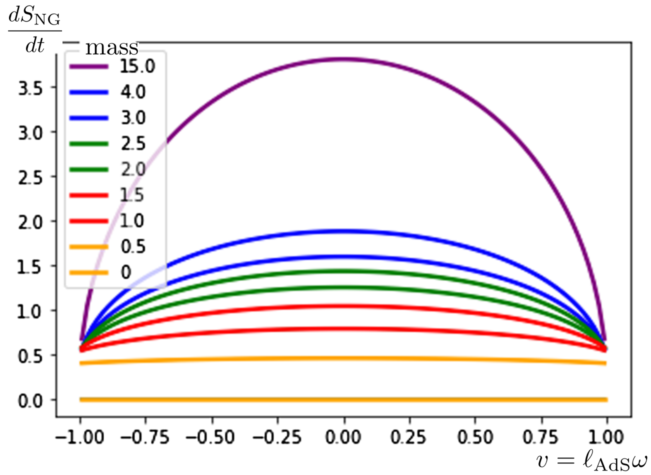

FIG. 2. Angular velocity dependence: Different colored lines represent curves for different masses.

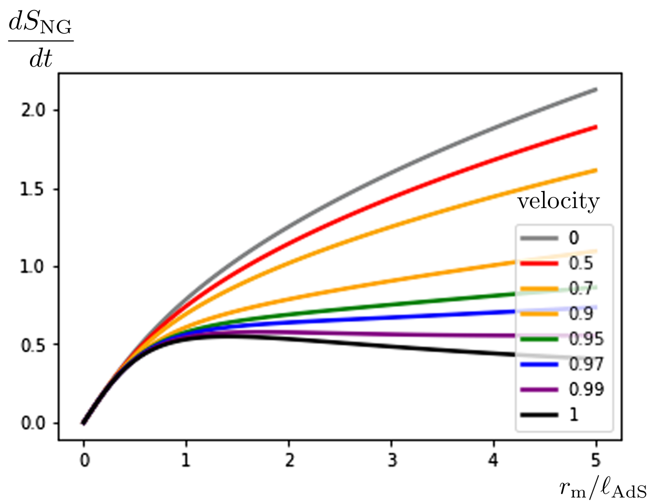

FIG. 3. Mass dependence: Different colored lines represent curves for different velocities.

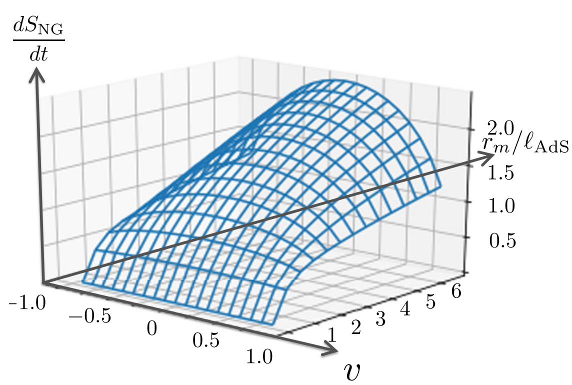

FIG. 4. 3d plot: velocity and mass dependence.

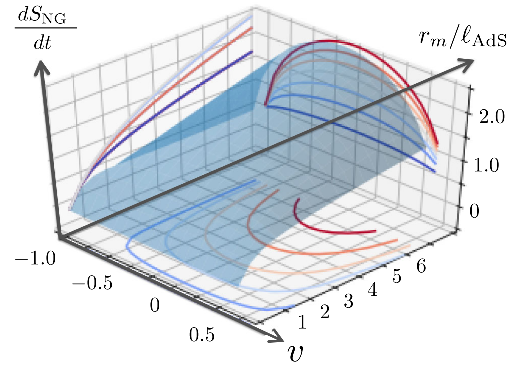

FIG. 5. Same graph with projections: the same curves (Figs. 2 and 3) can be seen in the individual planes.

$$\left.\frac{dS_{\rm NG}}{dt}\right|_{v\to 1} = -iT_s\ell_{\rm AdS}\arcsin i\left(\frac{-1+\sqrt{4s^2+1}}{2(1+s^2)}\right)^{1/2}$$
$$= T_s\ell_{\rm AdS}\mathrm{arcsinh}\left(\frac{-1+\sqrt{4s^2+1}}{2(1+s^2)}\right)^{1/2}. \qquad (28)$$

The Figs. 2 and 3 show these behavior and also the curves of all parameter ranges. The function (28) has an extremum at $s = \sqrt{2}$.

And the next two figures (Figs. 4 and 5) are the 3d plots which represent the velocity and mass dependences simultaneously.

IV. DISCUSSION

Figure 2 shows the relation between the growth of the action and the angular velocity ($v = \ell_{\rm AdS}\omega$). We can see that the effect of the drag force to the growth of the action is larger as the mass increases when the string moves relatively slower. As the string moves faster the effect become smaller. For any mass, the effect vanishes in the limit $v \to 1$. This behavior is similar to the one that of a rotating black holes—the complexity growth bound changes from $2M$ to $2\sqrt{M^2 - J^2/\ell_{\rm AdS}^2}$, where J is the angular momentum of the black holes.

Figure 3 shows mass dependence of the growth of the action. When the angular velocity is small, it is a monotonically increasing function of the mass. As the angular velocity becomes larger, it ceases to increase rapidly and the extremum appears at about $v \gtrsim 0.97$. That is a notable phenomena—complexity has different velocity dependence in the relativistic region.

According to CA duality these represent the growth of the black-hole complexity. For a large mass black hole, complexity changes rapidly when the particle moves slowly and this effect becomes small for a relativistic particle. As the black hole becomes smaller, the complexity does not change whether the particle moves quickly or not. Complexity is expected to increase as the entropy increases by the second law of thermodynamics [6]. So the results shown in Figs. 2 and 3 are consistent with this expectation.

The velocity dependence of energy loss is changed by particle's mass. It is due to collisions between quarks and gluons, and gluon bremsstrahlung. Which of them is dominant depends on its velocity [30]. The behavior of the action for large mass, $(\sim\sqrt{s})$, is consistent with the expected behavior from CA relation [15], that is, the growth of complexity is bounded by two times mass. These behavior characterize the growth of complexity and gives a hint to define complexity in black hole spacetime.

In our analysis the boundary term, which was discussed in detail in [18], is not taken into consideration. An example of the boundary term calculation is given in [33]. They considered AdS boundaries. It is different from our case since in our calculation we need boundary terms which come from the boundary of region 1 (see Fig. 1). There is a cancellation and the contributions only from $r=0$ (black-hole singularity) and $r=r_h$ (horizon) survive. The detailed analysis will be considered as a future work. However, there is a reason that I think this term is not needed. In the case of Einstein-Hilbert action these boundary terms were needed because this action includes the second derivative of the field, especially the metric in this case. When the Dirichlet boundary condition (derivation of the field is zero at the boundary) is imposed, the boundary terms which include the first derivative is needed to give the correct equation of motion. In contrast of it, my calculation treated Nambu-Goto action which includes up to the first derivative of the fields. So my calculation does not need the boundary terms. Then we can expect that the boundary term is not need in this situation.

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APPENDIX A: GAUGE THEORY PARAMETER

The Newton's gravitational constant we used is related to the gauge theory parameter. Since 5-dimensional gravitational constant and AdS radius are as follows [34]:

$$G_5 = \frac{\pi\ell_{\rm AdS}^3}{2N^2},$$
$$\ell_{\rm AdS}^2 = \alpha'\sqrt{g_{\rm YM}^2 N} \Rightarrow G_5 = \frac{\pi}{2N^2}\alpha'^{3/2}(g_{\rm YM}^2 N)^{3/4} = \frac{\pi}{2}\frac{(\alpha' g_{\rm YM})^{3/2}}{N^{5/4}}, \tag{A1}$$

we obtain

$$G_5/\ell_{\rm AdS}^2 = \frac{\pi}{2N^2}\sqrt{\alpha'}(g_{\rm YM}^2 N)^{1/4} = \frac{\pi\sqrt{\alpha' g_{\rm YM}}}{2N^{7/4}}. \tag{A2}$$

Note that some people may use the different convention where $\ell_{\rm AdS}^2$ is different by the factor $\sqrt{2}$, for example [35], but we follow the above definition in this paper.

APPENDIX B: BLACK HOLE TEMPERATURE

Here we derive the temperature of AdS_5 black hole. This result can be seen in [36]. We consider the Euclidean version of the metric (1) and take the coordinates:

$$r = r_h(1+\rho^2). \tag{B1}$$

Near the horizon the metric is approximated as

$$\begin{aligned} ds_{\rm AdS}^2 &\approx 2\rho^2\left(\frac{r_{\rm m}^2}{r_h^2}+\frac{r_h^2}{\ell_{\rm AdS}^2}\right)dt_{\rm E}^2 + 2r_h^2\left(\frac{r_{\rm m}^2}{r_h^2}+\frac{r_h^2}{\ell_{\rm AdS}^2}\right)^{-1}d\rho^2 + r_h^2 d\Omega_3^2 \\ &= 2r_h^2\left(\frac{r_{\rm m}^2}{r_h^2}+\frac{r_h^2}{\ell_{\rm AdS}^2}\right)^{-1}\left(d\rho^2 + \frac{\rho^2}{r_h^2}\left(\frac{r_{\rm m}^2}{r_h^2}+\frac{r_h^2}{\ell_{\rm AdS}^2}\right)^2 dt_{\rm E}^2\right) + r_h^2 d\Omega_3^2. \end{aligned} \tag{B2}$$

Seeing the period of Euclidean time $t_{\rm E}$, the black hole temperature $(1/\beta)$ is

$$\beta = 2\pi r_h\left(\frac{r_{\rm m}^2}{r_h^2}+\frac{r_h^2}{\ell_{\rm AdS}^2}\right)^{-1}. \tag{B3}$$

Expressed it by using mass, this leads

$$\begin{aligned} T &= \frac{1}{2\pi r_h}\left(\frac{r_{\rm m}^2/\ell_{\rm AdS}^2}{r_h^2/\ell_{\rm AdS}^2}+\frac{r_h^2}{\ell_{\rm AdS}^2}\right) = \frac{1}{2\pi\ell_{\rm AdS}}\left(\frac{-1+\sqrt{1+4x}}{2}\right)^{-1/2}\left(\frac{2x}{-1+\sqrt{1+4x}}+\frac{-1+\sqrt{1+4x}}{2}\right), \\ x &\coloneqq \frac{8GM}{5\pi\ell_{\rm AdS}^2}, \qquad T\ell_{\rm AdS} = \frac{1}{2\pi}\left(x\left(\frac{-1+\sqrt{1+4x}}{2}\right)^{-3/2}+\left(\frac{-1+\sqrt{1+4x}}{2}\right)^{1/2}\right). \end{aligned} \tag{B4}$$

where we used Eqs. (4) and (3) and in the last line we used dimensionless quantities $T\ell_{\rm AdS}$ and x. This function is plotted in Fig. 6.

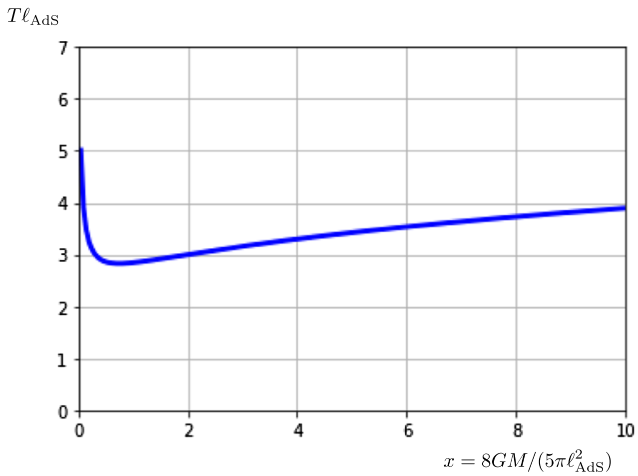

FIG. 6. Temperature-mass.

APPENDIX C: ELLIPTIC INTEGRAL

For calculating the action in the Wheeler-de Witt patch (WdW) we use the elliptic integral of second kind. The elliptic integral of the second kind is defined as

$$E[\varphi, k] \coloneqq \int_0^{\sin\varphi} \sqrt{\frac{1-k^2t^2}{1-t^2}}dt. \tag{C1}$$

We need the following integral formula:

$$\int_0^c \sqrt{\frac{x^2+a}{x^2+b}}dx = -i\sqrt{a}E\left[\arcsin\left(i\frac{c}{\sqrt{b}}\right), \sqrt{\frac{b}{a}}\right]. \tag{C2}$$

It can be proved as

$$\begin{aligned}
\int_0^c \sqrt{\frac{x^2+a}{x^2+b}}dx &= \sqrt{\frac{a}{b}}\int_0^c \sqrt{\frac{1+x^2/a}{1+x^2/b}}dx \\
&= \sqrt{\frac{a}{b}}\int_0^{ic/\sqrt{b}} \sqrt{\frac{1-(b/a)t^2}{1-t^2}}(-i\sqrt{b}dt), \\
ix/\sqrt{b} = t, \\
&= -i\sqrt{a}E(\arcsin(ic/\sqrt{b}, \sqrt{b/a})). \qquad (C3)
\end{aligned}$$

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