PHYSICAL REVIEW D **96**, 126004 (2017)

Entanglement entropy and Fisher information metric for closed bosonic strings in homogeneous plane wave background

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(Received 23 May 2017; published 7 December 2017)

We derive the extended renormalized entanglement entropy and the Fisher information metric in the case of closed bosonic strings in a homogeneous plane wave background. Our investigations are conducted within the framework of thermo field dynamics. The formalism is also illustrated in the example of some particular models in condensed matter physics and the nonequilibrium case for a system with dissipations.

DOI: 10.1103/PhysRevD.96.126004

I. INTRODUCTION

Entanglement entropy (EE) is a measure of how much information is stored in a quantum system. One expects that EE is directly related to the degrees of freedom. In this sense it can be used to gain insight into the quantum dynamics of diverse and complex phenomena. In recent years it also became a powerful bridge between string/gravity and condensed matter physics. For example, in holographic systems, the entanglement entropy is encoded in the geometric features of different string backgrounds [1,2]. This is closely related to the concept of emergent spacetime in such models [3–5]. The progress so far suggests that one can also study relevant aspects of string theory on the microscopic level by making use of thermodynamic and information-theoretic quantities.

Our interest is focused specifically on the study of quantum entanglement entropy and the Fisher information for a particular string model, namely, closed bosonic strings on homogeneous plane wave backgrounds. In general, it is difficult to calculate the entanglement entropy, especially in quantum field theory on curved spacetime. However, the recent progress in thermo field dynamics (TFD) [6–10] offers a relatively easy and straightforward way of treating quantum states, which facilitates the derivation of the EE and the Fisher matrix for the relevant models considered in this paper.

Thermo field dynamics requires a "statistical" state defined in a double Hilbert space, which is a direct product of the original space and an isomorphic copy of it. If one chooses to work in the energy basis $\{|n\rangle\}$, where $\hat{H}|n\rangle = E_n|n\rangle$, $n = 0, 1, \ldots$, then the bases in the double Hilbert space are labeled as $\{|n\rangle \otimes |\tilde{n}\rangle\} = \{|n\rangle|\tilde{n}\rangle\} = \{|n, \tilde{n}\rangle\}$. The extended states were defined originally by [6,8,9]

$$|\Psi\rangle = \frac{1}{Z} e^{-\beta H/2} |I\rangle, \qquad |I\rangle = \sum_n |n, \tilde{n}\rangle, \qquad (1.1)$$

where $Z = Z(\beta)$ is the partition function. It was shown in [11] that the extended state $|I\rangle$ is invariant for any orthogonal complete set $\{|\alpha\rangle\}$, $|I\rangle = \sum_n |n, \tilde{n}\rangle = \sum_\alpha |\alpha, \tilde{\alpha}\rangle$. Thus, the statistical state $|\Psi\rangle$ is independent of the chosen representation. This result is known as "the general representation theorem" in TFD. It allows one to use TFD techniques even in the nonequilibrium case. The notion of double Hilbert space is very useful in treating quantum states directly and facilitates the calculation of entanglement entropy of the quantum systems. Although TFD works for arbitrary nondiagonal Hamiltonians the calculations simplify if one is allowed to work only with diagonal Hamiltonians, which is the case we prefer in this study.

Let the Hamiltonian be a bilinear function in creation and annihilation operators. One can diagonalize it by an appropriate procedure, commonly known as the Bogoliubov transformation [12,13]. It mixes the creation and annihilation operators, but leaves the form of the commutation relations unchanged. In this case, operator eigenvalues, calculated with the diagonalized Hamiltonian on the transformed state functions, remain unchanged. Many such examples exist with important applications in condensed matter physics and string theory.

This paper is structured as follows. In Sec. II we consider a rather generic case of a system in equilibrium, where one applies TFD techniques to calculate the extended entanglement entropy and the Fisher information metric. In Sec. III we show that our result is applicable for certain bosonic and fermionic systems, naturally found in condensed matter systems such as superfluidity, superconductivity, and spin chains. In Sec. IV we calculate the extended renormalized

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entanglement entropy (EREE) and the Fisher metric for a nontrivial example of closed bosonic string theory in a class of curved plane wave backgrounds. In Sec. V we consider a nonequilibrium case with dissipation and generalize the formula for EREE found in [14]. Finally, in Sec. VI we make a short summary of our results.

II. ENTANGLEMENT ENTROPY FOR TIME-INDEPENDENT QUADRATIC HAMILTONIANS

A. Extended entanglement entropy for systems in equilibrium

Let $|\phi_n\rangle$ be a complete basis of eigenfunctions of the operator F,

$$F|\phi_n\rangle = F_n|\phi_n\rangle, \qquad \langle\phi_m|\phi_n\rangle = \delta_{mn}. \tag{2.1}$$

In general, the Hamiltonian in such a basis is nondiagonal,

$$H = \sum_{mn} H_{mn} a_m^\dagger a_n, \qquad H_{mn} = \langle\phi_m|H|\phi_n\rangle, \tag{2.2}$$

where the creation and annihilation operators satisfy standard commutation relations,

$$[a_m, a_n^\dagger] = \delta_{mn}, \quad [a_m, a_n] = 0, \quad [a_m^\dagger, a_n^\dagger] = 0. \tag{2.3}$$

If the Hamiltonian is diagonalizable one can write it in the following form [12,13,15][1]:

$$H = \sum_{i=1}^{N} E_i b_i^\dagger b_i + E_0, \tag{2.4}$$

where the energy coefficients E_i and the energy E_0 of the ground state depend on the matrix elements H_{mn} of the original Hamiltonian (2.2). The new creation and annihilation operators $b_i^\dagger$ and b_i satisfy the same commutation relations as the previous operators $a_n^\dagger$ and a_n:

$$\begin{aligned} &[b_i, b_j^\dagger] = \delta_{ij}, \qquad [b_i, b_j] = 0, \\ &[b_i^\dagger, b_j^\dagger] = 0, \qquad i, j = 1, \ldots, N. \end{aligned} \tag{2.5}$$

Following [14,17] we can apply TFD techniques to find the EREE for the new system of quasiparticles, described by the Hamiltonian (2.4). Consider the excited states $|n_1, \ldots, n_N\rangle$, which satisfy the orthonormal relation

$$\langle m_1, \ldots, m_N|n_1, \ldots, n_N\rangle = \prod_{i=1}^{N} \delta_{m_i, n_i}. \tag{2.6}$$

[1]For more general discussion on quantum quadratic Hamiltonians see the lecture notes [16].

One can write the Hamiltonian in matrix form such as[2]

$$\hat{H} = \sum_{\{n_i\}=0}^{\infty} \left(\sum_{i=1}^{N} E_i n_i + E_0\right) |n_1, \ldots, n_N\rangle\langle n_1, \ldots, n_N|, \tag{2.7}$$

where $n_i = b_i^\dagger b_i$ are the number operators, $\{n_i\} = \{n_i\}_{i=1}^N = n_1, \ldots, n_N$. Once the Hamiltonian assumes diagonal form it is straightforward to compute the relevant statistical quantities. The first one is the partition function Z,

$$\begin{aligned} Z &= \mathrm{Tr}_{\{i\}}(e^{-\beta\hat{H}}) \\ &= \sum_{\{\ell_i\}=0}^{\infty} \langle\{\ell_i\}|e^{-\beta\hat{H}}|\{\ell_i\}\rangle \\ &= \prod_{i=1}^{N} \frac{e^{-\beta E_0}}{1 - e^{-\beta E_i}} = \prod_{i=1}^{N} \frac{e^{-K_0}}{1 - e^{-K_i}}, \end{aligned} \tag{2.8}$$

where $\langle\{\ell_i\}| = \langle\ell_1, \ell_2, \ldots, \ell_N| = \langle\ell_1|\langle\ell_2|\ldots\langle\ell_N|$ and $\beta = 1/T$, $(k_B = 1)$. We also introduce the notations $K_0 = \beta E_0$ and $K_i = \beta E_i$, $i = 1, \ldots, N$, usually called inverse scaled temperatures. The ordinary density matrix in equilibrium is given by

$$\hat{\rho}_{eq} = \frac{e^{-\beta\hat{H}}}{Z} = \frac{1}{Z} \sum_{\{n_i\}=0}^{\infty} e^{-\sum_{i=1}^{N} K_i n_i - K_0} |\{n_i\}\rangle\langle\{n_i\}|. \tag{2.9}$$

In order to define the entanglement entropy the whole system is divided into two subsystems A and B, traditionally called "Alice" and "Bob." Then, the standard EE Σ_A for the first system is found as[3]

$$\Sigma_A = -k_B \mathrm{Tr}_A \rho_A \log \rho_A, \qquad \rho_A = \mathrm{Tr}_B \hat{\rho}_{eq}. \tag{2.10}$$

In the TFD formulation of the double Hilbert space the statistical state, $|\Psi\rangle$, is defined as

[2]Let us clarify the notations to avoid unnecessary confusion. If we define $I = \{n_1, \ldots, n_N\}$, then a nondiagonal Hamiltonian can be written in the form

$$\begin{aligned} \hat{H} &= \sum_{\substack{n_1,\ldots,n_N, \\ m_1,\ldots,m_N}} H_{n_1,\ldots,n_N,m_1,\ldots,m_N} |n_1, \ldots, n_N\rangle\langle m_1, \ldots, m_N| \\ &= \sum_{IJ} H_{IJ} |I\rangle\langle J|. \end{aligned}$$

The last expression allows one to write $\hat{H}$ as a matrix, where I and J run over all possible states, defined by the quantum numbers n_i (for explicit examples see [17,18]).

[3]We prefer to work in units $k_B = 1$, where k_B is the Boltzmann constant.

$$|\Psi\rangle = \sum_{\{n_i\}=0}^{\infty} \sqrt{\hat{\rho}_{eq}} |\{n_i\}\rangle |\{\tilde{n}_i\}\rangle = \frac{1}{\sqrt{Z}} \sum_{\{n_i\}=0}^{\infty} e^{-\frac{1}{2}(\sum_{i=1}^{N} K_i n_i + K_0)} |\{n_i\}\rangle |\{\tilde{n}_i\}\rangle. \quad (2.11)$$

Thus, the extended density operator assumes the form

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{1}{Z} \sum_{\{n_i\}=0}^{\infty} \sum_{\{m_i\}=0}^{\infty} e^{-\frac{1}{2}(\sum_{i=1}^{N} K_i (n_i+m_i) + 2K_0)} \times |\{n_i\}\rangle\langle\{m_i\}| |\{\tilde{n}_i\}\rangle\langle\{\tilde{m}_i\}|. \quad (2.12)$$

One can choose a bipartite system, namely,

$$\{n_i\}_{i=1}^{N} = \{n_\mu\}_{\mu=1}^{p} \bigcup \{n_k\}_{k=p+1}^{N}, \quad p \le N-1, \quad N \ge 2. \quad (2.13)$$

The extended density matrix $\hat{\rho}_A$ for Alice is obtained as a trace over the parameters of the second system B,

$$\hat{\rho}_A = \mathrm{Tr}_{\{B\}} \hat{\rho} = \sum_{\{\ell_k\}=0}^{\infty} \sum_{\{\tilde{\ell}_k\}=0}^{\infty} \langle\{\ell_k\}|\langle\{\tilde{\ell}_k\}|\hat{\rho}|\{\ell_k\}\rangle|\{\tilde{\ell}_k\}\rangle, \quad (2.14)$$

which leads to

$$\hat{\rho}_A = \sum_{\{n_\mu\}=0}^{\infty} \sum_{\{m_\mu\}=0}^{\infty} e^{-\frac{1}{2}\sum_{\mu=1}^{p} K_\mu (2+n_\mu+m_\mu)} \times |\{n_\mu\}\rangle\langle\{m_\mu\}| |\{\tilde{n}_\mu\}\rangle\langle\{\tilde{m}_\mu\}| \prod_{\alpha=1}^{p} (e^{K_\alpha} - 1). \quad (2.15)$$

Finally, the extended renormalized entanglement entropy, $S_A = -\mathrm{Tr}_{\{A\}}(\hat{\rho}_A \ln \hat{\rho}_A)$, follows as

$$S_A(K_\mu) = -\sum_{\mu=1}^{p} \left\{ \ln(e^{K_\mu} - 1) - K_\mu - \frac{K_\mu \prod_{\gamma\neq\mu}(e^{K_\gamma/2} - 1)}{\prod_{\alpha=1}^{p}(e^{K_\alpha/2} - 1)} \right\} \prod_{\alpha=1}^{p} \coth\frac{K_\alpha}{4}. \quad (2.16)$$

The result simplifies in terms of hyperbolic functions:

$$S_A(K_\mu) = \frac{1}{2} \left(\prod_{\mu=1}^{p} \coth\frac{K_\mu}{4} \right) \sum_{\mu=1}^{p} \left\{ K_\mu \left(1 + \coth\frac{K_\mu}{4} \right) - 2\ln(e^{K_\mu} - 1) \right\}. \quad (2.17)$$

This is the desired expression for the EREE. If $p = 1$, the formula reduces to (3.14). If $p = 2$, it reproduces the result for the EE of the Pais-Uhlenbeck oscillator, found in [19]. For comparison the standard entanglement entropy from (2.10) is written by

$$\Sigma_A(K_\mu) = \sum_{\mu=1}^{p} \left[\frac{K_\mu}{4} \prod_{\gamma\neq\mu} (e^{K_\gamma} - 1) \prod_{\alpha=1}^{p} \left\{ (1 - e^{-K_\alpha}) \mathrm{csch}^2\left(\frac{K_\alpha}{2}\right) \right\} - \ln(1 - e^{-K_\mu}) \right]. \quad (2.18)$$

B. Fisher information metric

Equation (2.17) allows one to calculate the Fisher information metric. It can be expressed as a second derivative of the entanglement entropy [20,21]:

$$g_{\mu\nu} = \partial_\mu \partial_\nu S_A = -\frac{1}{8} F (A_\mu B_\nu + A_\nu B_\mu + C_{\mu\nu} + E D_{\mu\nu}), \quad (2.19)$$

where $\partial_\mu = \partial/\partial K_\mu$ and

$$A_\mu = 2\mathrm{csch}\frac{K_\mu}{2}, \quad (2.20)$$

$$B_\mu = 1 + \coth\frac{K_\mu}{4} - \frac{K_\mu}{4}\mathrm{csch}^2\frac{K_\mu}{4} - \frac{2}{1 - e^{-K_\mu}}, \quad (2.21)$$

$$C_{\mu\nu} = \delta_{\mu\nu} \left[\left(2 - \frac{K_\mu}{2}\coth\frac{K_\mu}{4} \right) \mathrm{csch}^2\frac{K_\mu}{4} + \frac{4}{1 - \cosh K_\mu} \right], \quad (2.22)$$

$$D_{\mu\nu} = 2\mathrm{csch}^2\frac{K_\nu}{4} \left(\delta_{\mu\nu} + \tanh\frac{K_\nu}{4} \sum_{\tau\neq\nu} \left\{ \delta_{\mu\tau}\mathrm{csch}\frac{K_\tau}{2} \right\} \right), \quad (2.23)$$

$$E = -\frac{1}{4} \sum_{\alpha=1}^{p} \left[K_\alpha \left(1 + \coth\frac{K_\alpha}{4} \right) - 2\ln(e^{K_\alpha} - 1) \right], \quad (2.24)$$

$$F = \prod_{\sigma=1}^{p} \coth\frac{K_\sigma}{4}. \quad (2.25)$$

Formula (2.19) differs by a sign from the standard definition of the metric due to the requirement that the metric components be positive defined, which is a necessary condition for thermodynamic stability (for extended discussion see [22] and references therein). The case of $p = 2$ corresponds to the Fisher metric obtained by [19,23].

On the level of the space of probability distributions the Fisher metric represents a continuous setting even if the underlying features of the system (for example, the state space) are discrete. This allows one to take advantage of the powerful framework of differential geometry to treat statistical structures as geometrical ones. As it turns out the expressions for the EE and Fisher metric are applicable for variety of systems as shown below.

III. EXAMPLES FROM CONDENSED MATTER PHYSICS

Diagonalizable or approximately diagonalizable bosonic and fermionic Hamiltonians naturally arise in condensed matter physics such as spin wave theory, Heisenberg ferro- and antiferromagnets, spin chains, spin liquids, and BCS theory of superconductivity [24], but also in quantum field theory and string theory. In this section we give explicit examples of EREE for bosonic and fermionic systems, correspondingly.

A. Entanglement entropy for bosonic system

For simplicity let us consider the following BCS type bosonic Hamiltonian:

$$H = Aa_1^\dagger a_1 + Ba_2^\dagger a_2 + C(a_1^\dagger a_2^\dagger + a_1 a_2), \tag{3.1}$$

where A, B, and C are some energy coefficients and

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = 0, \qquad i, j = 1, 2. \tag{3.2}$$

Following [12], we want to transform the given Hamiltonian by introducing a new set of operators $b_i^\dagger$ and b_i, such that (3.1) takes the following diagonal form:

$$H = E_0 + E_1 b_1^\dagger b_1 + E_2 b_2^\dagger b_2. \tag{3.3}$$

Here, the creation and annihilation operators $b_i^\dagger$ and b_i also satisfy (3.2),

$$[b_i, b_j^\dagger] = \delta_{ij}, \qquad [b_i, b_j] = 0, \qquad i, j = 1, 2. \tag{3.4}$$

The diagonalization is achieved by the following Bogoliubov transformations:

$$\begin{aligned} b_1 &= \cosh\varphi a_1 + \sinh\varphi a_2^\dagger, \\ b_2 &= \sinh\varphi a_1^\dagger + \cosh\varphi a_2. \end{aligned} \tag{3.5}$$

After some trivial calculations one arrives at the following expressions for the new Hamiltonian coefficients:

$$E_1 = \frac{1}{2}\left(A - B + \sqrt{(A+B)^2 - 4C^2}\right), \tag{3.6}$$

$$E_2 = \frac{1}{2}\left(B - A + \sqrt{(A+B)^2 - 4C^2}\right), \tag{3.7}$$

$$E_0 = -(E_1 + E_2)\sinh^2\varphi, \tag{3.8}$$

where E_0 is the energy of the ground state. Now, let us focus on the calculation of the EREE for the system described by the Hamiltonian (3.3). First, we calculate the partition function

$$Z = \mathrm{Tr}_{1,2}(e^{-\beta H}) = \frac{e^{-\beta E_0}}{(1 - e^{-\beta E_1})(1 - e^{-\beta E_2})}. \tag{3.9}$$

The ordinary equilibrium density matrix is written by

$$\hat\rho_{eq} = \frac{e^{-\beta H}}{Z} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} e^{-\beta(E_1 n_1 + E_2 n_2 + E_0)} |n_1, n_2\rangle\langle n_1, n_2|, \tag{3.10}$$

where $n_i = b_i^\dagger b_i$, $i = 1, 2$, are the number operators of the Bogoliubov quasiparticles. The TFD statistical state, $|\Psi\rangle$, is defined as

$$\begin{aligned} |\Psi\rangle &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \sqrt{\hat\rho_{eq}} |n_1, n_2\rangle|\tilde n_1, \tilde n_2\rangle \\ &= \frac{1}{\sqrt{Z}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} e^{-\frac{\beta}{2}(E_1 n_1 + E_2 n_2 + E_0)} |n_1, n_2\rangle|\tilde n_1, \tilde n_2\rangle. \end{aligned} \tag{3.11}$$

Therefore, the extended density operator, $\hat\rho = |\Psi\rangle\langle\Psi|$, takes the form

$$\begin{aligned} \hat\rho = \frac{1}{Z}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} e^{-\frac{\beta}{2}(2E_0 + E_1(n_1+m_1) + E_2(n_2+m_2))} \\ \times |n_1, n_2\rangle\langle m_1, m_2||\tilde n_1, \tilde n_2\rangle\langle \tilde m_1, \tilde m_2|. \end{aligned} \tag{3.12}$$

Tracing out the states of the second system, one finds

$$\begin{aligned} \hat\rho_1 &= \mathrm{Tr}_2(\hat\rho) \\ &= \frac{1}{Z}\sum_{n_1=0}^{\infty}\sum_{m_1=0}^{\infty} \frac{\mathrm{e}^{-\beta(E_0 - E_2 + \frac{E_1(m_1+n_1)}{2})}}{\mathrm{e}^{\beta E_2} - 1} |n_1\rangle\langle m_1||\tilde n_1\rangle\langle \tilde m_1|. \end{aligned} \tag{3.13}$$

Finally, the renormalized extended entanglement entropy for the given bosonic system is written by

$$\begin{aligned} S_1(K_1) &= -\mathrm{Tr}_1(\hat\rho_1 \ln\hat\rho_1) \\ &= \frac{1}{2}\coth\frac{K_1}{4}\left\{\left(1 + \coth\frac{K_1}{4}\right)K_1 - 2\log(e^{K_1} - 1)\right\}. \end{aligned} \tag{3.14}$$

Here $K_1 = \beta E_1$ is the inverse scaled temperature. As expected the result agrees with Eq. (2.17) for $p = 1$. The dependence of the entropy on K_1 is illustrated in Fig. 1.

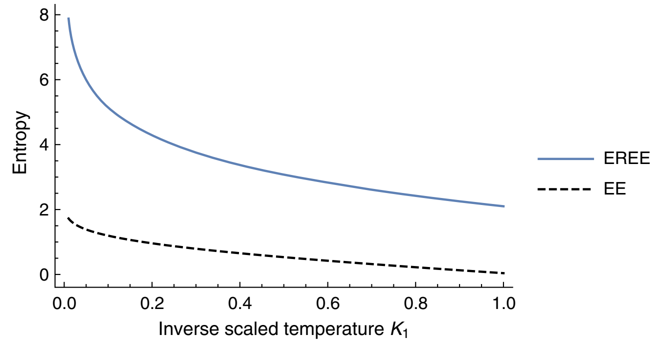

FIG. 1. The extended renormalized entanglement entropy S_1 (thick line) compared to the standard entanglement entropy Σ_1 (dashed line). As expected the EREE is bigger than the normal EE, but both diverge at the origin (at very high temperatures).

In this case the Fisher information (2.19) is a one parameter function given by

$$F(K_1) = \frac{K_1}{32}\mathrm{csch}^4\frac{K_1}{4}\left(4 + 2\cosh\frac{K_1}{2} + \sinh\frac{K_1}{2}\right) - \frac{1}{16}\mathrm{csch}^3\frac{K_1}{4}\left[3 + \log(e^{K_1} - 1) + \cosh\frac{K_1}{2}(2 + \log(e^{K_1} - 1))\right]\mathrm{sech}\frac{K_1}{4}. \quad (3.15)$$

It measures the amount of information that an observed random variable provides about an unknown parameter. It can be used in studying phase transitions, especially the second-order phase transitions, during which the Fisher information exhibits divergence. From Eq. (3.15) one notices that the Fisher information is singular at the origin, $K_1 = 0$. This suggest that at very high temperatures the Bogoliubov quasisystem undergoes a second-order phase transition, which is in agreement with the statement that the Fisher information is maximized at the phase transition points [25].

One can use the Fisher information (3.15) to define a distance between points on the statistical manifold, spanned by the inverse scaled temperatures K_μ, or in this case—only by $\theta = K_1$. The information-metric distance, or Fisher information distance [26] D_F between two distributions $f(\theta_1, x)$ and $f(\theta_2, x)$ in a single parameter family is defined by

$$D_F(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \sqrt{F(\theta)}d\theta, \quad (3.16)$$

where θ_1 and θ_2 are parameter values corresponding to the two probability distribution functions. In Fig. 2 are depicted several values of the Fisher distance D_F for several increasing positive values of the upper integral limit θ_2, while keeping the lower limit θ_1 fixed. This setup chooses different points on the statistical manifolds. One notices

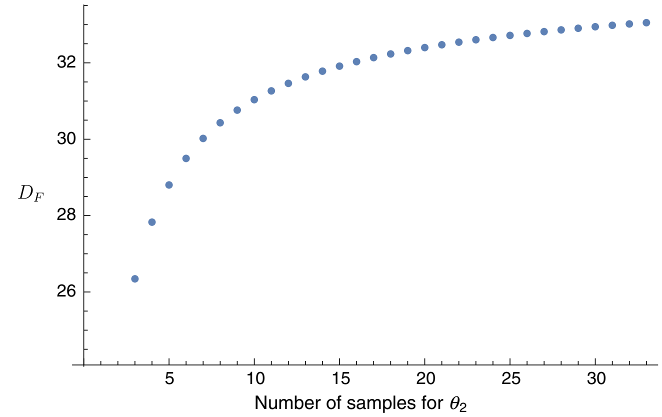

FIG. 2. Monotonously increasing Fisher information distance D_F between two distributions $f(\theta_1, x)$ and $f(\theta_2, x)$ in a single parameter family for $\theta_1 = 0.1$ and $\theta_2 \in [0.3, 10]$, with step size $\delta\theta_2 = 0.3$. One can interpret this within the framework of the fluctuation theory as follows: the less the probability of a fluctuation between two states, the further apart they are.

that D_F increases monotonously for increasing values of the upper limit θ_2. For nearby states, the square of the lengths of the geodesic paths gives the probability of a fluctuation between the states. In other words, the less the probability of a fluctuation between two states, the further apart they are [22].

B. Entanglement entropy for fermionic system

In this section we are going to consider a fermionic example, namely, an XY model in a magnetic field. It is a generalization of the Ising model in which an anisotropy is introduced with respect to the x and y directions by means of a real deformation parameter γ. In what follows we are going to shortly sketch the diagonalization of the Hamiltonian, which is given by [27]

$$H = -\sum_{\ell=-M}^{M}\left[\left(\frac{1+\gamma}{2}\right)\sigma_\ell^x\sigma_{\ell+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_\ell^y\sigma_{\ell+1}^y + h\sigma_\ell^z\right]. \quad (3.17)$$

Here $N = 2M + 1$ gives the total odd number of spins and h is the transverse magnetic field. In the $\gamma = 1$ case the system reduces to the one-dimensional Ising model with a transverse magnetic field. In order to diagonalize the Hamiltonian we begin by defining the following operators:

$$\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y), \qquad \sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y). \quad (3.18)$$

Next we perform the Jordan-Wigner transformation, which relates the spin operators σ_ℓ to a set of fermionic operators a_ℓ and $a_\ell^\dagger$ via

$$\sigma_\ell^+ = \left(\prod_{j=1}^{\ell-1}\sigma_j^z\right)a_\ell, \quad \sigma_\ell^- = \left(\prod_{j=1}^{\ell-1}\sigma_j^z\right)a_\ell^\dagger, \quad \sigma_\ell^z = 1-2a_\ell^\dagger a_\ell. \tag{3.19}$$

Here the operators a_ℓ and $a_\ell^\dagger$ satisfy the standard fermionic anticommutation relations:

$$\{a_\ell^\dagger, a_m\} = \delta_{\ell m}, \qquad \{a_\ell^\dagger, a_m^\dagger\} = \{a_\ell, a_m\} = 0. \tag{3.20}$$

The Hamiltonian, written in terms of these fermionic operators, assumes the form

$$H = -\sum_{\ell=-M}^{M}\left[\frac{1+\gamma}{2}(a_{\ell+1}a_\ell + a_{\ell+1}^\dagger a_\ell + a_\ell^\dagger a_{\ell+1} + a_\ell^\dagger a_{\ell+1}^\dagger) + \frac{\gamma-1}{2}(a_{\ell+1}a_\ell - a_\ell^\dagger a_{\ell+1} - a_\ell^\dagger a_{\ell+1} + a_\ell^\dagger a_{\ell+1}^\dagger) + h(1-2a_\ell^\dagger a_\ell)\right]. \tag{3.21}$$

Now we Fourier transform the creation and annihilation operators by

$$a_\ell = \frac{1}{\sqrt{N}}\sum_k e^{-ik\ell}d_k, \qquad a_\ell^\dagger = \frac{1}{\sqrt{N}}\sum_k e^{ik\ell}d_k^\dagger, \qquad \delta_{kk'} = \frac{1}{N}\sum_\ell e^{i\ell(k-k')}, \tag{3.22}$$

where $k = 2\pi/N, 4\pi/N\ldots, 2\pi$. The Hamiltonian is expressed as

$$H = -\sum_k [2(\cos k - h)d_k^\dagger d_k - i\gamma\sin k(d_k d_{-k} + d_k^\dagger d_{-k}^\dagger)] - hN. \tag{3.23}$$

After applying the following Bogoliubov transformations:

$$d_k = \cos\frac{\theta_k}{2}b_k + i\sin\frac{\theta_k}{2}b_{-k}^\dagger, \qquad d_k^\dagger = \cos\frac{\theta_k}{2}b_k^\dagger - i\sin\frac{\theta_k}{2}b_{-k}, \tag{3.24}$$

one finds

$$H = -\sum_k 2\gamma\sin k\sin\theta_k b_k^\dagger b_k - 2\cos\theta_k(\cos k - h)b_k^\dagger b_k - i\sum_k(\sin\theta_k(\cos k - h) - \gamma\sin k\sin\theta_k) \times (b_k b_{-k} + b_k^\dagger b_{-k}^\dagger) + \text{const}. \tag{3.25}$$

Imposing that the cross-terms be zero we arrive at the expressions relating θ_k with the original parameters (k, h, γ):

$$\cos\theta_k = \frac{\cos k - h}{\sqrt{(\cos k - h)^2 + \gamma^2\sin^2 k}}, \qquad \sin\theta_k = -\frac{\gamma\sin k}{\sqrt{(\cos k - h)^2 + \gamma^2\sin^2 k}}. \tag{3.26}$$

Finally, the Hamiltonian assumes the form of a quasifree fermionic system:

$$H = \sum_k \Lambda_k(n_k - 1), \tag{3.27}$$

where $n_k = b_k^\dagger b_k$ defines number operators for the quasiparticles and Λ_k sets the following dispersion relation:

$$\Lambda_k = \sqrt{(\cos k - h)^2 + \gamma^2\sin^2 k}. \tag{3.28}$$

One can repeat the TFD analysis from Sec. II to calculate the extended entanglement entropy, which, for arbitrary number of spins N, coincides with Eq. (2.17):

$$S_A(K_\mu) = \frac{1}{2}\left(\prod_{\mu=1}^{p}\coth\frac{K_\mu}{4}\right)\sum_{\mu=1}^{p}\left\{K_\mu\left(1+\coth\frac{K_\mu}{4}\right) - 2\ln(e^{K_\mu}-1)\right\}, \tag{3.29}$$

where the inverse scaled temperatures are given by $K_\mu = \beta\Lambda_\mu$. Here we used natural numbers to count the number of spins involved. In such notations one has to be careful with the expressions for Λ_μ and Λ_k, where $1 \le \mu \le p \le N-1, N > 1$, and the angle $k = 2\pi/p, 4\pi/p\ldots, 2\pi$. This implies the following relation $\mu \to k = 2\pi/\mu$, thus,

$$\Lambda_\mu \to \Lambda_{2\pi/\mu} = \sqrt{\left(\cos\left(\frac{2\pi}{\mu}\right) - h\right)^2 + \gamma^2\sin^2\left(\frac{2\pi}{\mu}\right)}. \tag{3.30}$$

The Fisher information metric in this case is the same as in Sec. II for particular values of k.

IV. ENTANGLEMENT ENTROPY FOR CLOSED BOSONIC STRINGS IN HOMOGENEOUS PLANE-WAVE BACKGROUNDS

In this section we consider the closed bosonic string vibrating in regular homogeneous plane wave backgrounds. The given curved backgrounds have nonvanishing Neveu-Schwarz three-form field strength and a dilaton. We will closely follow [28], where the authors develop a general procedure for solving linear, but nondiagonal equations for the string coordinates, and determine the corresponding oscillator frequencies and the light-cone Hamiltonian. In this setup the Hamiltonian is automatically diagonalized and time independent. Therefore, finding the entanglement entropy in the framework of TFD naturally follows the steps shown in the previous sections. Bellow we will sketch the relevant result of [28].

A. String equations of motion and quantization

We begin by considering a closed relativistic string in nonsingular $2 + d$ dimensional homogeneous plane wave backgrounds with a metric of the following form:

$$ds^2 = 2dudv + k_{ij}x^i x^j du^2 + 2f_{ij}x^i dx^j du + dx^i dx^j. \tag{4.1}$$

Here k_{ij} and f_{ij} are constant, and the B field is given by $B_{iu} = -h_{ij}x^j$. Our aim is to solve the classical equations of motion for this string sigma model. We denote the string embedding coordinates as $X^M = (U, V, X^i)$. Choosing the orthogonal gauge for the world-sheet metric, the standard sigma model Lagrangian is written by

$$L = \frac{1}{2\pi}(G_{MN}(X) + B_{MN}(X))\partial_+ X^M \partial_- X^N. \tag{4.2}$$

The equations of motion for the bosonic field U are easily obtained:

$$\partial_+\partial_- U = 0. \tag{4.3}$$

Similarly, for the fields X^i, $i = 1, \ldots, d$, one finds

$$\begin{aligned} &-\partial_+\partial_- X_i + (f_{ij} + h_{ij})\partial_- U \partial_+ X^j + (f_{ij} - h_{ij})\partial_+ U \partial_- X^j \\ &\quad + k_{ij}X^j \partial_+ U \partial_- U = 0, \end{aligned} \tag{4.4}$$

where $\sigma^{\pm} = \tau \pm \sigma$ and $\partial_{\pm} = \partial_\tau \pm \partial_\sigma$. In the light-cone gauge U becomes

$$U = p_+\sigma^+ + p_-\sigma^- = \frac{p_v}{2}, \tag{4.5}$$

where the condition of periodicity of U in σ implies that $p_+ = p_- = p_v/2$. To solve Eq. (4.4) one makes the following mode expansion of the transverse coordinates:

$$X^i(\tau,\sigma) = \sum_{n=-\infty}^{\infty} X_n^i(\tau)e^{2in\sigma}, \qquad X_n^i = (X_{-n}^i)^*, \qquad 0 < \sigma \le \pi. \tag{4.6}$$

The substitution of the mode expansion in Eq. (4.4) leads to

$$-\ddot{X}_n^i + 2p_v f_{ij}\dot{X}_n^j + (p_v^2 k_{ij} - 4n^2\delta_{ij})X_n^j + 4inp_v h_{ij}X_n^j = 0. \tag{4.7}$$

For simplicity one can set $p_v = 1$ and assume that k_{ij} is diagonal, $k_{ij} = k_i\delta_{ij}$. As explained in [28], the general method to solve systems like (4.7) is to rewrite it as a set of $2d$ first-order equations and then use the appropriate methods available at hand. Fortunately, the authors noted that for generic values of the parameters in (4.7) one can use a much simpler procedure. Namely, to solve these equations, one makes the following ansatz:

$$X_n^i(\tau) = \sum_{J=1}^{2d} \zeta_J^{(n)} a_{iJ}^{(n)} e^{i\omega_J^{(n)}\tau}, \tag{4.8}$$

with the frequencies $\omega_J^{(n)}$ and their eigendirections $a_{iJ}^{(n)}$ to be determined. This frequency based ansatz for the modes leads to the matrix equation in the form

$$M_{ik}(\omega_J^{(n)}, n)a_{kJ}^{(n)} = 0, \tag{4.9}$$

where (for short $\omega = \omega_J^{(n)}$):

$$M_{ik} = (\omega^2 + k_i - 4n^2)\delta_{ik} + 2i\omega f_{ik} + 4inh_{ik}. \tag{4.10}$$

The matrix equation (4.9) is a homogeneous algebraic system. The necessary condition for finding a nontrivial solution is

$$\det M(\omega, n) = 0. \tag{4.11}$$

The later equation has $2d$ roots $\omega = \omega_J^{(n)}$, $J = 1, \ldots, 2d$, which are the frequencies from (4.8). The ansatz (4.8) is justified only if all the roots are distinct, or if equal roots are associated with linearly independent null eigenvectors, because it involves all the $2d$ linearly independent solutions of the equation (4.7). The degenerate case requires separate considerations. In what follows we will always assume distinct roots. From (4.10) one immediately notes that $M^T(\omega, n) = M(-\omega, -n)$. This property means that $M(\omega, n)$ and $M(-\omega, -n)$ have the same determinant and hence the same roots. This leads to the situation where for $n = 0$ the frequencies come in pairs, $\{\omega_J\} = \{\pm\omega_j, j = 1, \ldots, d\}$. It is then convenient to rewrite the expansion of the zero mode as

$$X_0^i(\tau) = \sum_{j=1}^{d} (\zeta_j^+ a_{ij}^+ e^{i\omega_j\tau} + \zeta_j^- a_{ij}^- e^{-i\omega_j\tau}). \tag{4.12}$$

For the higher modes ($n \ne 0$) the $\pm n$ modes are paired, $\omega_J^{(n)} = -\omega_j^{(n)}$, $J = 1, \ldots, 2d$. It is useful to chose the eigendirections $a_{iJ}^{(n)}$ in the following way:

$$a_{iJ}^{(n)} = (-1)^i m_{1i}(\omega_J^{(n)}), \tag{4.13}$$

where $m_{ij}(\omega_J^{(n)})$, $i,j=1,\ldots,d$, are the minors m_{ij} of the matrix $M(\omega,n)$, evaluated for $\omega=\omega_J^{(n)}$. Therefore one can rewrite the solution for the string modes explicitly as

$$X_0^i(\tau)=(-1)^i\sum_{j=1}^{d}(\zeta_j^+ m_{1i}(\omega_j)e^{i\omega_j\tau}+\zeta_j^- m_{i1}(\omega_j)e^{-i\omega_j\tau}),\quad n=0, \tag{4.14}$$

$$X_n^i(\tau)=(-1)^i\sum_{J=1}^{2d}\zeta_J^{(n)}m_{1i}(\omega_J^{(n)})e^{i\omega_J^{(n)}\tau},\quad n\neq 0. \tag{4.15}$$

In order to find the Hamiltonian we promote the ζ's to operators with commutation relations given by

$$C_j=[\zeta_j^-,\zeta_j^+],\qquad C_J^{(n)}=[\zeta_J^{(-n)},\zeta_J^n],\qquad C_J^{(-n)}=-C_J^{(n)}, \tag{4.16}$$

where one has

$$C_j=\frac{1}{2m_{11}(\omega_j)\omega_j\prod_{k\neq j}(\omega_j^2-\omega_k^2)},\qquad C_J^{(n)}=\frac{1}{m_{11}(\omega_J^{(n)})\prod_{K\neq J}(\omega_J^{(n)}-\omega_K^{(n)})}. \tag{4.17}$$

The expressions for these coefficients follow from the canonical equal-time commutation relations between the string modes X. The relations between the ζ's and the canonically normalized operators $a_j^{\pm}$, $[a_j^-,a_k^+]=\delta_{jk}$, are given by

$$a_j^{\pm\sigma}=\frac{\zeta_j^{\pm}}{\sqrt{|C_j|}}, \tag{4.18}$$

where $\sigma=\mathrm{sign}(C_j)$. With this choice for the a's the bilinear combination $\zeta_j^+\zeta_j^-+\zeta_j^-\zeta_j^+$ is related to the number operator, $\mathcal{N}_j=a_j^+a_j^-$, by

$$\frac{1}{2}(\zeta_j^+\zeta_j^-+\zeta_j^-\zeta_j^+)=|C_j|\left(\mathcal{N}_j+\frac{1}{2}\right). \tag{4.19}$$

Similar relation holds for the higher modes,

$$\frac{1}{2}(\zeta_J^{(n)}\zeta_J^{(-n)}+\zeta_J^{(-n)}\zeta_J^{(n)})=|C_J^{(n)}|\left(\mathcal{N}_J^{(n)}+\frac{1}{2}\right), \tag{4.20}$$

which connects the number operator $\mathcal{N}_J^{(n)}=a_J^{(n)}a_J^{(-n)}$ with the ζ operators. Hence, one has an explicit string mode expansion. Thus, the string Hamiltonian,

$$H=\frac{1}{2\pi}\int_0^{\pi}d\sigma[\delta_{ij}(\dot{X}^i\dot{X}^j+X^{i\prime}X^{j\prime}-k_iX^iX^j)-2h_ijX^iX^{j\prime}], \tag{4.21}$$

can be written as a sum of n-level harmonic oscillator Hamiltonians

$$H=\sum_{n=0}^{\infty}H^{(n)}. \tag{4.22}$$

Here the zero-mode part Hamiltonian assumes the form

$$H^0=\sum_{j=1}^{d}\mathrm{sign}(C_j)\Omega_j\left(\mathcal{N}_j+\frac{1}{2}\right), \tag{4.23}$$

with frequencies

$$\Omega_j=\frac{\sum_{i=1}^{d}(\omega_j^2-k_i)m_{ii}(\omega_j)}{2\omega_j\prod_{k\neq j}(\omega_j^2-\omega_k^2)}. \tag{4.24}$$

Likewise, the Hamiltonians for higher modes of the string are given by

$$H^{(n)}=\sum_{J=1}^{2d}\mathrm{sign}(C_J^{(n)})\Omega_J^{(n)}\left(\mathcal{N}_J^{(n)}+\frac{1}{2}\right),\quad n>0, \tag{4.25}$$

where the frequency $\Omega_J^{(n)}$ is a sum of two terms—one coming from the plane wave metric, and the other coming from the Kalb-Ramond B field:

$$\Omega_J^{(n)}=2\omega_J^{(n)}C_J^{(n)}m_{11}(\omega_J^{(n)})\times\sum_{i,j}(\omega_J^{(n)}\delta_{ij}+i(-1)^{i+j}f_{ij})m_{ij}(\omega_J^{(n)}). \tag{4.26}$$

B. Extended entanglement entropy in the ground state of the bosonic string

We are now ready to apply the TFD technique for the entanglement entropy on every energy level of the string spectrum. Here, for convenience, we consider only the $n=0$ Hamiltonian of the string from Eq. (4.23). Assume the following two subsystems:

$$\{\mathcal{N}_j\}_{j=1}^d=\{\mathcal{N}_\mu\}_{\mu=1}^p\bigcup\{\mathcal{N}_k\}_{k=p+1}^d,\quad p\leq d-1,\qquad 2\leq d\leq 9, \tag{4.27}$$

the resulting entanglement entropy agrees with Eq. (2.17):

$$S_A(\tilde{K}_\mu)=\frac{1}{2}\left(\prod_{\mu=1}^{p}\coth\frac{\tilde{K}_\mu}{4}\right)\sum_{\mu=1}^{p}\left\{K_\mu\left(1+\coth\frac{\tilde{K}_\mu}{4}\right)-2\ln(e^{\tilde{K}_\mu}-1)\right\}. \tag{4.28}$$

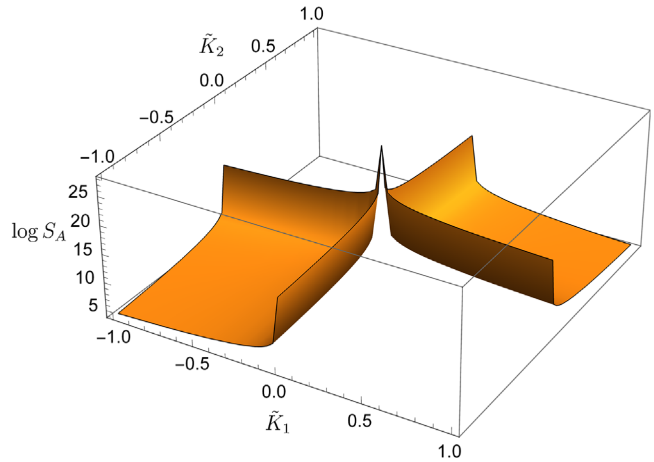

FIG. 3. The renormalized entanglement entropy for closed string in five-dimensional regular plane wave background, $k_B = 1$. At the origin (at very high temperatures) EREE diverges.

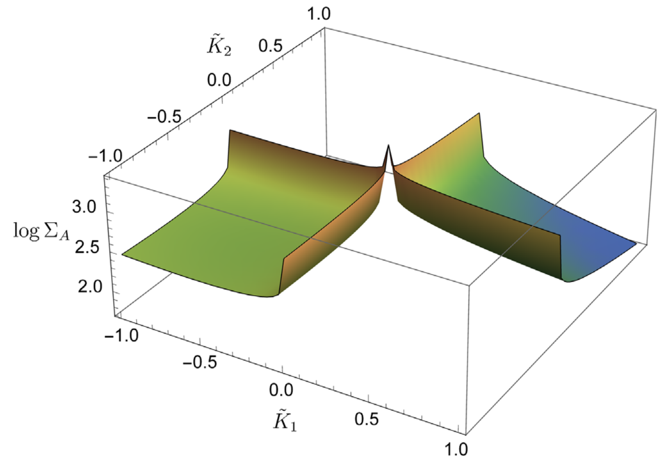

FIG. 4. The standard entanglement entropy for closed bosonic string in the five-dimensional regular plane wave background.

The thermal parameters, $\tilde{K}_\mu = \beta \mathrm{sign}(C_\mu)\Omega_\mu$, depend on the frequencies of the classical string modes ω_μ, $\mu = 1, \ldots, p$. In four-dimensional spacetime $d = 2$, thus $p = 1$, the space of parameters is one-dimensional spanned by the values of $\tilde{K}_1$. The entanglement entropy behaves as shown in Fig. 1. In five dimensions ($d = 3$, $p = 2$), the space of parameters is a two-dimensional Riemannian manifold spanned by $(\tilde{K}_1, \tilde{K}_2)$. The entanglement entropy is given by

$$S_A(\tilde{K}_1, \tilde{K}_2) = \frac{1}{2}\coth\frac{\tilde{K}_1}{4}\coth\frac{\tilde{K}_2}{4} \times \left[\tilde{K}_1\left(1 + \coth\frac{\tilde{K}_1}{4}\right) + \tilde{K}_2\left(1 + \coth\frac{\tilde{K}_2}{4}\right) - 2\log[(e^{\tilde{K}_1} - 1)(e^{\tilde{K}_2} - 1)]\coth\frac{\tilde{K}_1}{4}\right]. \tag{4.29}$$

The inverse scaled temperatures, $\tilde{K}_{1,2}$, depend on the sign of the coefficients $C_{1,2}$, which leads to two regions on the plot (Fig. 3)—one for positive values of the K's, and one for negative ones.

As expected for very high temperatures ($\tilde{K}_i \to 0$) the entropy diverges. One finds similar situation for the standard entanglement entropy,

$$\Sigma_A(\tilde{K}_1, \tilde{K}_2) = \frac{4\tilde{K}_1}{e^{\tilde{K}_1} - 1} + \frac{4\tilde{K}_2}{e^{\tilde{K}_2} - 1} - \log[(1 - e^{-\tilde{K}_1})(1 - e^{-\tilde{K}_2})], \tag{4.30}$$

shown in Fig. 4 below,

The Fisher metric, $g_{\mu\nu} = \partial_\mu \partial_\nu S$, at the point ($\tilde{K}_1 = 0$, $\tilde{K}_2 = 0$), is also singular, as can be seen from the following expressions for the metric coefficients:

$$g_{11} = \frac{1}{64}\coth\frac{\tilde{K}_2}{4}\mathrm{csch}^2\frac{\tilde{K}_1}{4}\left[\tilde{K}_1\left(3 + 5\coth^2\frac{\tilde{K}_1}{4} + 7\mathrm{csch}^2\frac{\tilde{K}_1}{4}\right) + 4\tanh\frac{\tilde{K}_1}{4} + 4\coth\frac{\tilde{K}_1}{4}\left(\tilde{K}_1 + \tilde{K}_2 - 5 + \tilde{K}_2\coth\frac{\tilde{K}_2}{4} - 2\log[(e^{\tilde{K}_1} - 1)(e^{\tilde{K}_2} - 1)]\right)\right], \tag{4.31}$$

$$g_{12} = g_{21} = \frac{1}{32}\mathrm{csch}^2\frac{\tilde{K}_1}{4}\mathrm{csch}^2\frac{\tilde{K}_2}{4}\left[\tilde{K}_1\left(1 + 2\coth\frac{\tilde{K}_1}{4}\right) + \tilde{K}_2\left(1 + 2\coth\frac{\tilde{K}_2}{4}\right) - 4 - 2\log\left[(e^{\tilde{K}_1} - 1)(e^{\tilde{K}_2} - 1)\right]\right], \tag{4.32}$$

$$g_{22} = \frac{1}{64}\coth\frac{\tilde{K}_1}{4}\mathrm{csch}^2\frac{\tilde{K}_2}{4}\left[\tilde{K}_2\left(3 + 5\coth^2\frac{\tilde{K}_2}{4} + 7\mathrm{csch}^2\frac{\tilde{K}_2}{4}\right) + 4\tanh\frac{\tilde{K}_2}{4} + 4\coth\frac{\tilde{K}_2}{4}\left(\tilde{K}_1 + \tilde{K}_2 - 5 + \tilde{K}_1\coth\frac{\tilde{K}_1}{4} - 2\log\left[(e^{\tilde{K}_1} - 1)(e^{\tilde{K}_2} - 1)\right]\right)\right]. \tag{4.33}$$

This result is already familiar [19]. The singular point at the origin is a signal of a phase transition. As shown from the geometric analysis of the Fisher metric in Sec. IV C its Ricci scalar is regular at the origin, which suggests that the point $(K_1 = 0,\ K_2 = 0)$ is not a second-order phase transition. Furthermore, the scalar curvature at that point is zero, corresponding to a free quasisystem at very high temperatures.

C. Geometric analysis of the Fisher metric and phase transitions

A well-known fact is that the Fisher information metric defines a Riemannian metric on the space of parameters [29–31] for variety of statistical systems. Such geometrization is often useful in the analysis of the phase structure for a given statistical model [22,32]. Here the scalar curvature, R, plays a central role, e.g., a noninteracting model shows a flat geometry ($R = 0$), while R diverges at the critical points of an interacting one, thus effectively preventing geodesics from crossing into the nonphysical area of phase space [33–36]. The specific critical points, where the phase transition occurs, lie on the spinodal curve. An advantage of the probabilistic description of the system's phase structure is that one does not require the definition of order parameters. This is useful for systems where an order parameter is difficult to identify, or does not exist.

In what follows we analyze the scalar curvature R of the Fisher metric from (4.31), (4.32), and (4.33). The scalar curvature is independent of the chosen coordinates, so, for convenience, we perform a change of variables from K_1 and K_2 to $t_1 = e^{K_1}$ and $t_2 = e^{K_2}$. The Fisher metric in the new coordinates is given by

$$g_{11} = \frac{1}{4t_1^{3/2}(T_1^-)^4 T_1^+ (T_2^-)^2}\{T_1^+(1+8\sqrt{t_1}+3t_1)(t_2-1)\log t_1 + 2T_1^-(2(1+3\sqrt{t_1}+t_1)(1-t_2) + (T_1^+)^2((1-t_2)\log[(t_1-1)(t_2-1)] + (\sqrt{t_2}+t_2)\log t_2))\}, \tag{4.34}$$

$$g_{12} = g_{21} = \frac{1}{2t_2\sqrt{t_1}(T_1^-)^3(T_2^-)^3}\{(t_2-\sqrt{t_2})\times((1+3\sqrt{t_1})\log t_1 - 2T_1^-(2+\log[(t_1-1)(t_2-1)])) + T_1^-(\sqrt{t_2}+3t_2)\log t_2\}, \tag{4.35}$$

$$g_{22} = \frac{1}{4t_2^{3/2}T_2^+(T_1^-)^2(T_2^-)^4}\{2T_1^+T_2^-\sqrt{t_1}(T_2^+)^2\log t_1 + (t_1-1)(-2T_2^-(2(1+3\sqrt{t_2}+t_2) + (T_2^+)^2\log[(t_1-1)(t_2-1)]) + T_2^+(1+8\sqrt{t_2}+3t_2)\log t_2)\}, \tag{4.36}$$

where $1 \le t_1, t_2 \le \infty$ and

$$T_1^- = \sqrt{t_1} - 1, \qquad T_1^+ = 1 + \sqrt{t_1},$$
$$T_2^- = \sqrt{t_2} - 1, \qquad T_2^+ = 1 + \sqrt{t_2}. \tag{4.37}$$

The explicit expression for R is too lengthy to be presented here. However, its functional dependence on $(t_1,\ t_2)$ near the origin (1,1) is shown on Fig. 5. One notes that the Ricci scalar is positive defined and shows local maximum near the point $(t_1 = 1.3,\ t_2 = 1.3)$. The positive values of the scalar curvature suggest elliptic geometry in the thermodynamic parameter space, while the local maximum corresponds to the strongest interaction between the constituents of the quasisystem. There is a level curve $k(t_1, t_2)$ for which $R = 0$, corresponding to free noninteracting system (Fig. 6). At the origin the scalar curvature is also regular and tends to zero, which implies that the singular point $(K_1 = 0, K_2 = 0)$ in the Fisher metric is not a second-order phase transition and also shows that at very high temperatures the system is free. One notes that the values of R do not deviate much from zero, which makes the entire quasisystem almost noninteracting. This kind of behavior is expected due to the properties of the Bogoliubov transformation, which smoothens out the

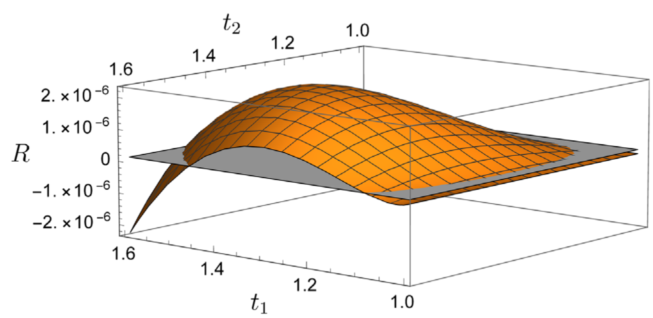

FIG. 5. Visualization of the scalar curvature R in terms of t_1, t_2, near the origin ($t_1 = 1$, $t_2 = 1$). The curvature is positive defined implying elliptic geometry on the statistical manifold of thermodynamic parameters. The local maximum corresponds to the strongest interaction in the quasisystem.

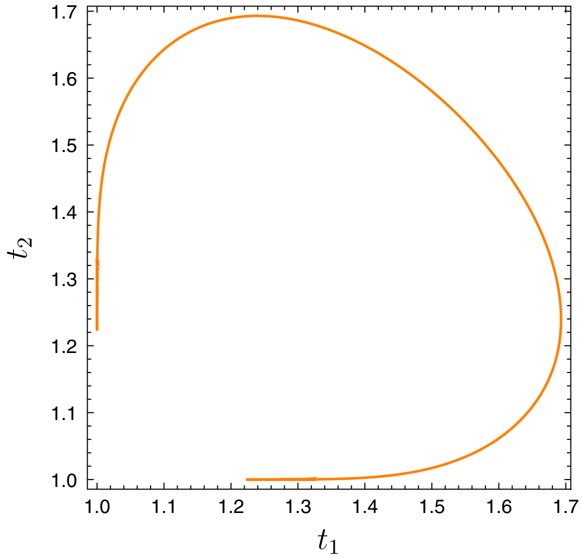

FIG. 6. The contour plot of the level curve $R = 0$, corresponding to the values of the parameters for which the quasisystem is effectively free. In this picture one should also include the origin (high temperatures), $t_1 = t_2 = 1$, and infinity (low temperatures), t_1, $t_2 \to \infty$.

strength of the interactions in the original quantum system and effectively produces a noninteracting quasisystem.

For larger values of the parameters, the Ricci scalar is not positive defined as shown in Fig. 7. In this case the geometry in the space of parameters is hyperbolic. The nonzero values of the scalar curvature suggest also an interacting system. There is a local minimum, corresponding to the highest strength of the interactions in the hyperbolic case. For low temperatures ($K_{1,2} \to \infty$) the scalar curvature tends to zero once again corresponding to a free noninteracting quasisystem. The TFD scalar curvature is regular for all points from the two-dimensional space of parameters. Therefore, one concludes that the closed bosonic string system in a five-dimensional

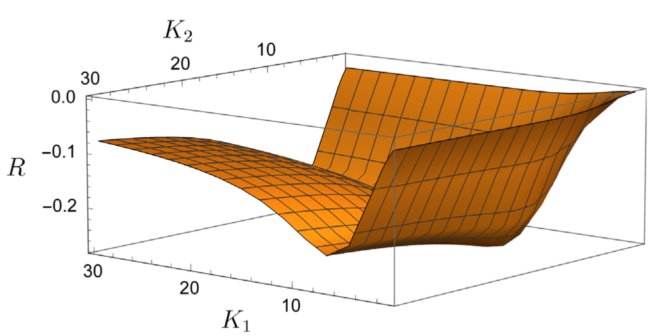

FIG. 7. The behavior of the scalar curvature R in terms of K_1, K_2 for large values of the inverse scale temperatures. There is an obvious local minimum, corresponding to the strongest interaction in the hyperbolic case.

homogeneous plane wave background does not show any second-order phase transitions.

V. NONEQUILIBRIUM ENTANGLEMENT ENTROPY FOR DISSIPATIVE SYSTEMS

A. Extended entanglement entropy for a dissipative system

Following [14], one can consider the Hamiltonian from Eq. (2.7) as a nonequilibrium system with dissipations. In this case the time-dependent density operator $\hat{\rho}_{\rm neq}(t)$ satisfies the dissipative von Neumann equation:

$$i\partial_t\hat{\rho}_{\rm neq}(t) = [\hat{H}, \hat{\rho}_{\rm neq}(t)] - \varepsilon(\hat{\rho}_{\rm neq}(t) - \hat{\rho}_{\rm eq}), \quad (5.1)$$

where ε is a dissipation parameter and $\hat{\rho}_{\rm eq}$ is defined in Eq. (2.9). The solution to this equation is formally given by

$$\hat{\rho}_{\rm neq}(t) = e^{-\varepsilon t}\hat{U}^{\dagger}(t)\hat{\rho}^{(0)}\hat{U}(t) + (1 - e^{-\varepsilon t})\hat{\rho}_{\rm eq}, \quad (5.2)$$

where $\hat{\rho}^{(0)}$ is an arbitrary initial density matrix and $\hat{U}(t) \coloneqq e^{it\hat{H}}$. The diagonal form of the Hamiltonian allows one to write

$$\hat{U}(t) = \sum_{\{n_i\}=0}^{\infty} e^{it(\sum_{i=1}^{N} E_i n_i + E_0)} |\{n_i\}\rangle\langle\{n_i\}|. \quad (5.3)$$

The case of arbitrary initial conditions significantly complicates the calculations. Therefore one can consider only initial conditions in the ground state as suggested in Ref. [14]:

$$\hat{\rho}^{(0)} = \frac{e^{-K_0}}{Z} |\{0\}\rangle\langle\{0\}|. \quad (5.4)$$

Here $K_0 = \beta E_0$, E_0 is the energy in the ground state, and $Z(K_i)$ is given by Eq. (2.8). Once again we use the simple bipartition of the bulk system, namely,

$$\{n_i\}_{i=1}^{N} = \{n_A\}_{A=1}^{p} \bigcup \{n_B\}_{B=p+1}^{N}, \quad p \leq N-1, \quad N \geq 2. \quad (5.5)$$

Following the steps shown in [17], one finds the renormalized extended entanglement entropy in the form ($k_B = 1$):

$$S_A(K_i, \varepsilon; t) = -e^{-\varepsilon t} a(t) \log a(t) + (1 - e^{-\varepsilon t}) \times \sum_{\mu=1}^{p} \left(S_\mu(t) \tanh\frac{K_\mu}{4} \right) \prod_{\alpha=1}^{p} \coth\frac{K_\alpha}{4}, \quad (5.6)$$

where $i = 1, \ldots, p, p+1, \ldots, N$, and

$$S_\mu(t) = -a_\mu(t)\log a_\mu(t) - \frac{1}{2}e^{-K_\mu}\coth\frac{K_\mu}{4}\left\{b(t)(e^{K_\mu/2}-1)\left[4\log b(t) + 4\log(e^{K_\mu}-1) - K_\mu\left(5+\coth\frac{K_\mu}{4}\right)\right]\right.$$
$$\left. + (1-e^{-\varepsilon t})\left[2\log(1-e^{-\varepsilon t}) + 2\log(e^{K_\mu}-1) - K_\mu\left(3+\coth\frac{K_\mu}{4}\right)\right]\right\}, \tag{5.7}$$

$$a(t) = \left((1-e^{-\varepsilon})\prod_{r=p+1}^{N}(1-e^{-K_r})^{-1} + e^{-\varepsilon t}\right)\prod_{\mu=1}^{p}(1-e^{-K_\mu})\prod_{r=p+1}^{N}(1-e^{-K_r}), \tag{5.8}$$

$$b(t) = \left(\sqrt{1-e^{-\varepsilon t}} - (1-e^{-\varepsilon t})\left(\prod_{r=p+1}^{N}(1-e^{-K_r})^{-1} - 1\right)\right)\prod_{r=p+1}^{N}(1-e^{-K_r}), \tag{5.9}$$

$$a_\mu(t) = (1-e^{-K_\mu})\left((1-e^{-\varepsilon t})\prod_{r=p+1}^{N}(1-e^{-K_r})^{-1} + e^{-\varepsilon t}\right)\prod_{r=p+1}^{N}(1-e^{-K_r}). \tag{5.10}$$

The limit, $t\to\infty$, coincides with the equilibrium case (2.16):

$$\lim_{t\to\infty} S_A(K_i,\varepsilon;t) = \frac{1}{2}\left(\prod_{\mu=1}^{p}\coth\frac{K_\mu}{4}\right)\sum_{\mu=1}^{p}\left\{K_\mu\left(1+\coth\frac{K_\mu}{4}\right) - 2\ln(e^{K_\mu}-1)\right\}. \tag{5.11}$$

The limit at $t\to 0$ reduces to the ground state:

$$\lim_{t\to 0} S_A(K_\mu,\varepsilon;t) = -\frac{e^{-K_0}}{Z}\log\frac{e^{-K_0}}{Z}. \tag{5.12}$$

In Fig. 8 is shown the time dependence of the entropy for several values of the dissipation parameter ε. Clearly, given enough time, the entropy reaches the equilibrium case from (2.17) (depicted as a black dashed line). This is consistent with the second law of thermodynamics.

One also depicts a generic behavior of the entropy, which has been observed for various systems prepared in a state with initially low entanglement entropy, $S_A(t_0)\leq S_{\mathrm{eq}}$: after some short transient period of time, which depends on the initial state of the system, the entanglement entropy goes through a phase of linear or almost linear growth, $S_A(t)\sim\Lambda_A t$, until it settles at the saturation phase. This kind of behavior is observed in the time evolution of various quantum systems that bear the signatures of quantum chaos [37–40], or in the study of thermalization in some holographic systems [41–44]. In particular, the rate of growth Λ_A of the entanglement entropy in the phase of linear growth is connected to the scrambling time in chaotic quantum systems, or the Kolmogorov-Sinai entropy rate [45], in cases where the quantum system has a classical counterpart.

Finally, in Fig. 9 we show the logarithm of the entanglement entropy as a function of the scaled inverse temperature parameter K_1 at a fixed finite moment of time t. As expected the entropy decreases with increasing K_1.

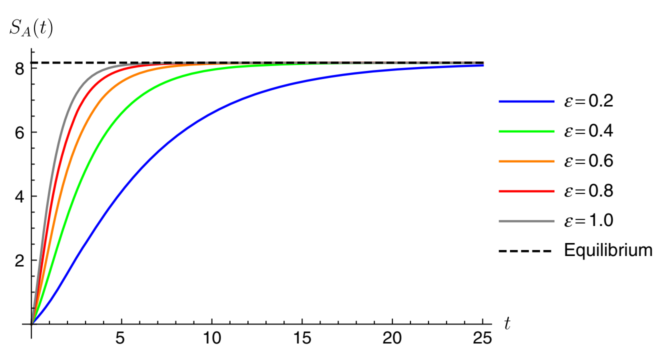

FIG. 8. Nonequilibrium EREE as a function of time in the $p=1$ case (also $k_B=1$ and $K_1=1$). The dashed line is the equilibrium value of the entropy saturated at $t\to\infty$. Different curves correspond to different values of ε in the interval [0,1].

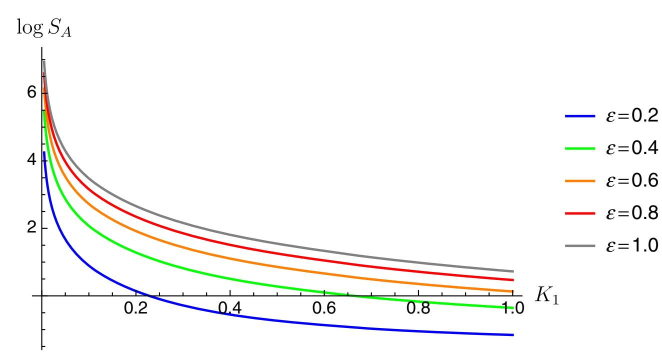

FIG. 9. Nonequilibrium EREE ($p=1$) as a function of the scaled inverse temperature parameter K_1 at fixed moment $t=0.5$, for different values of the dissipation parameter $\varepsilon\in[0,1]$.

B. Entanglement entropy production rate

We calculate and analyze the entanglement production during the evolution of a quantum mechanical dissipative system. Although entanglement entropy plays an essential role in the thermalization of isolated quantum systems [46–48], the central quantity in nonequilibrium systems is not the entropy, but the rate of change of the entropy, which is known as entropy production [49]. It serves as a measure of the irreversibility of a physical process.

One can obtain the TFD entanglement entropy production rate (EEPR) by taking the time derivative of the extended nonequilibrium EE (5.6):

$$\dot{S}_A(t) = -e^{-\varepsilon t}\left[\dot{a}(t)(\log a(t) - 1) - \varepsilon(a(t)\log a(t) + \sum_{\mu=1}^{p}\left(S_\mu(t)\tanh\frac{K_\mu}{4}\right)\prod_{\alpha=1}^{p}\coth\frac{K_\alpha}{4}\right)\right] + (1 - e^{-\varepsilon t})\sum_{\mu=1}^{p}\left(\dot{S}_\mu(t)\tanh\frac{K_\mu}{4}\right)\prod_{\alpha=1}^{p}\coth\frac{K_\alpha}{4}, \tag{5.13}$$

where the dot denotes derivative with respect to time t. The entanglement entropy production rate $\dot{S}_A(t)$ is shown in Fig. 10 for several values of the dissipation parameter ε. One notices that for a short time, the EEPR reaches peak value, after which it monotonously decreases with time. The local maximum of $\dot{S}_A(t)$ agrees with the Zeigler's principle of maximum entropy production [50–52]. Furthermore the EEPR is a positive quantity which is expected and confirms the statement of the second law of thermodynamics for nonequilibrium systems. One also notes that the peak entropy production is bigger for strongly dissipative systems (large values of ε).

VI. CONCLUSION

In this paper we study the extended entanglement entropy of a nontrivial quantized system, namely, the closed bosonic string in a homogeneous plane wave geometry.

The EREE dependence on the inverse scaled temperatures in the ground state of the string has been explicitly shown analytically and graphically in four and five dimensions. The parameter space in these cases is one- and two-dimensional manifold correspondingly. The result shows that EREE increases with increasing temperature, while at very high temperatures (zero inverse scaled temperatures) the entropy diverges. This is also true for the components of the Fisher metric, which is an indication of an instability of the system or a phase transition. To address this problem, we have analyzed analytically the singularities in the scalar curvature of the metric. We have shown that the Ricci scalar is regular everywhere, including at very high temperatures. This suggests that the considered bosonic string system does not possess any critical points, representing a second-order phase transition, although a first-order phase transition is not excluded at high temperature.

The graphical and analytical study of the scalar curvature showed that there are three geometrically different regions, corresponding to different types of interactions. Near the origin (Fig. 5) the scalar curvature is nonzero and positive, thus defining elliptical geometry on the statistical manifold of thermodynamic parameters. In this case the absolute value of the local maximum of the curvature corresponds to the strongest interaction in the quasisystem. For lower temperatures (Fig. 7) the curvature becomes negative, thus defining a hyperbolic geometry. Here, the strongest interaction is given by the absolute value of the local minimum of the curvature. The value of the scalar curvature on the level curve (Fig. 6), which separates the different geometric regions, is zero, thus corresponding to a free, noninteracting theory. The limit value of the Ricci scalar at the origin (high temperatures) and at infinity (low temperatures) is also zero, which again leads to a free theory in these cases.

In the one-parameter case the Fisher distance has been derived, which is a measure of dissimilarity between two probability distribution functions. We have shown graphically that in this case the Fisher distance increases monotonously for increasing values of the upper limit of the defining integral. This is interpreted by the theory of fluctuations due to Ruppeiner [22] in the following manner:

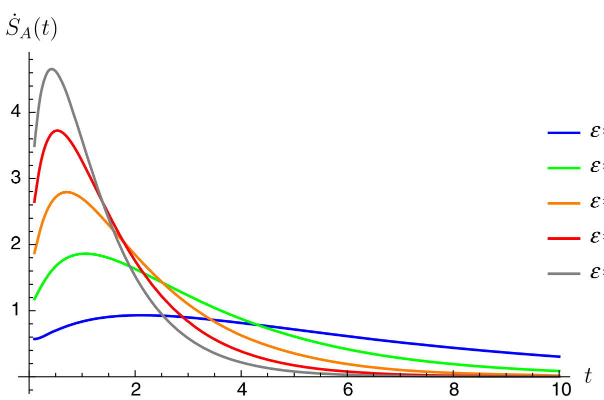

FIG. 10. The entropy increase rate $\dot{S}_A(t)$ for several values of the dissipation parameter ε. One notes that for a short time the entropy production rate reaches peak value, after which it monotonously decreases with time. Also the peak entropy production is bigger for strongly dissipative systems (large values of ε).

the probability of a fluctuation between two equilibrium string states decreases with the increasing of the distance between them.

We have also shown that the general expressions for the entanglement entropy (2.18) and the Fisher information matrix (2.19) work also for other quantum models. Some particular examples were considered, namely, BCS-type bosonic systems and the XY model in a magnetic field, which is a generalization of the Ising model.

We also manage to derive explicit expressions for the entanglement entropy in the nonequilibrium case for a system with dissipation. We showed that the time-dependent entropy increases with time, until it settles at equilibrium (thermalization of the system). The equilibrium value of the entanglement entropy coincides with the thermal equilibrium entanglement entropy derived for a time-independent system. This is also in agreement with the second law of thermodynamics.

We also depicted one generic behavior of the entanglement entropy, namely, after a transient time which depends on the details of the initial state of the system, the entanglement entropy goes through a phase of linear or almost linear growth, $S_A(t) \sim \Lambda_A t$, until it settles at the saturation phase. It is also interesting to point out that this kind of behavior of the entanglement entropy is observed in the time evolution of various quantum systems that signify quantum chaos [37–40], or in the study of thermalization in some holographic systems [41–44].

The entropy increase rate $\dot{S}_A(t)$ has also been studied for several values of the dissipation parameter ε. We have shown that shortly after the initial moment the entanglement entropy production rate reaches peak value, it monotonously decreases with time. The local maximum of $\dot{S}_A(t)$ confirms Zeigler's principle of maximum entropy production [50–52]. Furthermore, the EEPR is a positive quantity which is just the statement of the second law of thermodynamics.

Another interesting question is how to reconstruct a parametric family of probability distributions corresponding to the given Fisher metric and to define under what conditions such reconstruction is possible. The Fisher information metric can be straightforwardly calculated once a probability distribution has been chosen. A set of distributions $f(\vec{x},\vec{\theta})$, parametrized by $\vec{\theta}$, forms a statistical manifold. The Riemannian metric on this manifold is the Fisher information metric defined by the following Lebesgue integral:

$$g_{\mu\nu}(\vec{\theta}) = \int_X \mathcal{D}f(\vec{x},\vec{\theta}) \frac{\partial \ln f(\vec{x},\vec{\theta})}{\partial\theta^\mu} \frac{\partial \ln f(\vec{x},\vec{\theta})}{\partial\theta^\nu}. \qquad (6.1)$$

Here $\vec{x} \in X$ is a point from the sample space X. It can be proved that the only Riemannian metric is the Fisher metric for which the geometry is invariant under coordinate transformations of $\vec{\theta}$ and also under one-to-one transformations of the random variable $\vec{x}$ [30,31]. The Fisher metric is also a solution to the Einstein field equations, which can be useful in finding the corresponding family of probability distributions $f(\vec{x},\vec{\theta})$. Unfortunately the equations are highly nonlinear and cumbersome. The defining integral (6.1) only imposes nontrivial constraints on the probability distribution.

Although this survey is instigated more or less by the fact that superstring theory on pp-wave backgrounds is exactly solvable, the TFD framework is powerful enough to treat more complicated supergravity background solutions. Extending the scope of the present research, the next natural step is to initiate a more thorough investigation of the geometric, thermodynamic, and information-theoretic aspects of some certain holographic models. Such models, for instance, are the $\mathcal{N}=1$ and $\mathcal{N}=2^*$ Pilch-Warner solutions [53–55], Lunin-Maldacena background [56], some recent non-Abelian T-dual solutions [57–62], and their Penrose-Güven limit [63,64] or pp-wave limit [62,65,66], the latter being easier to study with the techniques used in this paper. Such investigations are expected to shed light on the interplay between spacetime, global, and local properties in holography.

ACKNOWLEDGMENTS

The authors would like to thank D. Marvakov for fruitful discussions. This work was partially supported by the Bulgarian NSF Grant No. T02/6 and Sofia University Research Fund Grants No. 80-10-116, No. 80-10-118, and No. 80-10-148.

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