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# Analytic topologically nontrivial solutions of the (3+1)-dimensional $U(1)$ gauged Skyrme model and extended duality

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We construct the first analytic examples of topologically nontrivial solutions of the $(3+1)$-dimensional $U(1)$ gauged Skyrme model within a finite box in $(3+1)$-dimensional flat space-time. There are two types of gauged solitons. The first type corresponds to gauged Skyrmions living within a finite volume. The second corresponds to gauged time crystals (smooth solutions of the $U(1)$ gauged Skyrme model whose periodic time dependence is protected by a winding number). The notion of electromagnetic duality can be extended for these two types of configurations in the sense that the electric and one of the magnetic components can be interchanged. These analytic solutions show very explicitly the Callan-Witten mechanism (according to which magnetic monopoles may "swallow" part of the topological charge of the Skyrmion) since the electromagnetic field contributes directly to the conserved topological charge of the gauged Skyrmions. As it happens in superconductors, the magnetic field is suppressed in the core of the gauged Skyrmions. On the other hand, the electric field is strongly suppresed in the core of gauged time crystals.

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## I. INTRODUCTION

It is impossible to underestimate the relevance of the Skyrme theory [1] in high energy physics (for instance see [2–7]). Not only does such a model correspond to the low energy QCD [8], but it also discloses beautifully the very important role of topology in theoretical physics. In particular, the solitons (*Skyrmions*) of this theory (made of bosonic degrees of freedom) describe baryons, with the baryon charge being expressed as a topological invariant (see [8–15] and references therein).

Such tools are, nowadays, very important also in many other areas of physics such as semiconductors (see [16] and references therein), Bose-Einstein condensates (see [17] and references therein), magnetic materials (see [18] and references therein), gravitational physics (see [19–26] and references therein) and so on.

It is possible to consider the original Skyrme model as a prototype of nonintegrable systems. Until very recently, basically no analytic solution with nontrivial topological properties was known. In particular, the lack of explicit solutions with topological charge on flat space-times made the analysis of the corresponding phase diagram very difficult. Early important results (based on the original spherical Skyrmion[1]) analyzing finite density effects as well as the role of the isospin chemical potential can be found in [27–31].

Due to the importance of the Skyrme model as a low energy limit of QCD, it is a mandatory task to analyze the effects of the coupling of a $U(1)$ gauge field with the Skyrme theory. The so-called gauged Skyrme model is able to describe the decay of nuclei due to the coupling with weak interactions. Such a model also describes electric and magnetic properties of baryons as well as allowing us to study the decay of nuclei in the vicinity of a monopole (classic references are [8,32–36]). The gauged Skyrme model is expected to have very interesting applications in nuclear and particle physics, as well as in astrophysics, when the coupling of baryons with strong electromagnetic fields cannot be neglected.

On the other hand, the field equations of the $U(1)$ gauged Skyrme model are even more complicated than the field equations of the original Skyrme model. There are no analytic topologically nontrivial solutions. At a first glance, one could guess that the task to construct analytic and topologically nontrivial configurations of the gauged Skyrme model is hopeless. This is quite unfortunate as, until now, it has been impossible to construct analytic solutions of the gauged Skyrme model disclosing explicitly the Callan-Witten mechanism [32] (according to which the baryon charge of the gauge

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[1]It is worth emphasizing that both finite volume effects and isospin chemical potential are expected to break spherical symmetry.

Skyrme model can be "swallowed" by the magnetic field). Another interesting issue is whether or not Skyrmions tend to suppress the magnetic field within their cores. More generally, very little is known about the effects of the interactions between $U(1)$ gauge fields and Skyrmions. Nice numerical studies of topological configurations in the gauged Skyrme model can be found in [37,38] and references therein.

In fact, recently, in ([39–46] and references therein) an approach has been introduced in order to build a more general hedgehog ansatz allowing a departure from spherical symmetry both in Skyrme and Yang-Mills theories (see [47–49] and references therein). Such an approach gave rise to the first $(3+1)$-dimensional analytic and topologically nontrivial solutions of the Skyrme-Einstein system in [44] as well as of the Skyrme model without spherical symmetry living within a finite box in flat space-times in [50]. In the last reference, it has been possible to derive also the critical isospin chemical potential beyond which the Skyrmion living in the box ceases to exist. Due to the similarity of the minimal $U(1)$ gauge coupling with the introduction of the isospin chemical potential, it is natural to wonder whether the results of [40,44,50] can be extended to the $U(1)$ gauged Skyrme model.

Remarkably, using the above approach, it is possible to construct the first analytic and topologically nontrivial solutions of the $U(1)$ gauged Skyrme model. There are two types of gauged solitons. Firstly, gauged Skyrmions living within a finite volume appear as the natural generalization of the usual Skyrmions living within a finite volume. Secondly, there are smooth solutions of the $U(1)$ gauged Skyrme model whose periodic time dependence is protected by a topological conservation law. These solitons manifest very interesting similarities with superconductors as well as with dual superconductors.

This paper is organized as follows: In Sec. II, the model and notations are introduced. In Sec. III, a short review of the properties of the $(3+1)$-dimensional Skyrme model at finite volume both without and with isospin chemical potential is presented (such a review is very useful to understand the novel results in the following sections). In Sec. IV, the gauged solitons are constructed and their main physical properties are discussed. In Sec. V, it is discussed how electromagnetic duality can be extended to include these gauged solitons. In Sec. VI, a physically interesting approximation is discussed in which the Skyrme field is considered as fixed and the electromagnetic field is slowly turned on. In Sec. VII, we draw some concluding ideas.

## II. THE $U(1)$ GAUGED SKYRME MODEL

We consider the $U(1)$ gauged Skyrme model in four dimensions with global $SU(2)$ isospin internal symmetry. The action of the system is

$$S = \int d^4x\sqrt{-g}\left[\frac{K}{2}\left(\frac{1}{2}\mathrm{Tr}(R^\mu R_\mu) + \frac{\lambda}{16}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu})\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{1}$$

$$R_\mu = U^{-1}D_\mu U, \quad G_{\mu\nu} = [R_\mu, R_\nu], \quad D_\mu = \nabla_\mu + A_\mu[t_3, .], \tag{2}$$

$$U \in SU(2), \qquad R_\mu = R_\mu^j t_j, \qquad t_j = i\sigma_j, \tag{3}$$

where $\sqrt{-g}$ is the (square root of minus) the determinant of the metric, $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the electromagnetic field strength, $\nabla_\mu$ is the partial derivative, the positive parameters $K$ and $\lambda$ are fixed experimentally and $\sigma_j$ are the Pauli matrices. In our conventions $c = \hbar = \mu_0 = 1$, the space-time signature is $(-,+,+,+)$ and Greek indices run over space-time. The stress-energy tensor is

$$T_{\mu\nu} = -\frac{K}{2}\mathrm{Tr}\left[R_\mu R_\nu - \frac{1}{2}g_{\mu\nu}R^\alpha R_\alpha + \frac{\lambda}{4}\left(g^{\alpha\beta}G_{\mu\alpha}G_{\nu\beta} - \frac{g_{\mu\nu}}{4}G_{\sigma\rho}G^{\sigma\rho}\right)\right] + \bar{T}_{\mu\nu},$$

with

$$\bar{T}_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu} \tag{4}$$

being the part emanating from the electromagnetic action. The matter field equations are

$$D^\mu\left(R_\mu + \frac{\lambda}{4}[R^\nu, G_{\mu\nu}]\right) = 0, \tag{5}$$

$$\nabla_\mu F^{\mu\nu} = J^\nu, \tag{6}$$

where $J^\nu$ is the variation of the Skyrme action [the first two terms in Eq. (1)] with respect to $A_\nu$. A direct computation shows that

$$J^\mu = \frac{K}{2}\mathrm{Tr}\left[\hat{O}R^\mu + \frac{\lambda}{4}\hat{O}[R_\nu, G^{\mu\nu}]\right], \tag{7}$$

where

$$\hat{O} = U^{-1}t_3U - t_3.$$

It is worth to note that when the gauge potential reduces to a constant along the time-like direction, the field equations (5) describe the Skyrme model at a finite isospin chemical potential.

Hence, the term *gauged Skyrmions* [or, more generically, *gauged topological configurations* of the $U(1)$ gauged

Skyrme model in $(3+1)$ dimensions] refers to smooth regular solutions of the coupled system in Eqs. (5) and (6) possessing a nonvanishing winding number [defined here below in Eq. (12)]. The aim of the present work is to construct the first (to the best of the authors' knowledge) analytic configurations of this type and to disclose their intriguing physical properties.

### A. Topological charge

If we adopt the standard parametrization of the $SU(2)$-valued scalar $U(x^\mu)$

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu)\mathbf{I} \pm Y^i(x^\mu)t_i, \quad (Y^0)^2 + Y^iY_i = 1, \tag{8}$$

where $\mathbf{I}$ is the $2\times 2$ identity and

$$Y^0 = \cos C, \qquad Y^i = n^i \cdot \sin C, \tag{9}$$

$$n^1 = \sin F \cos G, \quad n^2 = \sin F \sin G, \quad n^3 = \cos F. \tag{10}$$

In the original Skyrme model, without coupling with the $U(1)$ gauge field, the Skyrme field possesses a nontrivial conserved topological charge. Such a charge can be expressed as an integral over a suitable three-dimensional hypersurface $\Sigma$

$$W = \frac{1}{24\pi^2}\int_\Sigma \epsilon^{ijk}\mathrm{Tr}(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U) = \frac{1}{24\pi^2}\int_\Sigma \rho_B. \tag{11}$$

A direct computation shows that the charge density is $\rho_B = 12\sin^2 C \sin F dC \wedge dF \wedge dG$. Obviously, in order for the topological charge density to be nonvanishing, one has to require $dC \wedge dF \wedge dG \neq 0$.

The usual situation considered in the literature corresponds to a spacelike $\Sigma$, in which case $W$ is the baryon charge. On the other hand, recently [50] it has been proposed to also consider cases in which $\Sigma$ is timelike or lightlike. If $W \neq 0$ (whether $\Sigma$ is spacelike, timelike or lightlike), then one cannot deform continuously the corresponding ansatz into the trivial vacuum $U = \mathbf{I}$. Consequently [50], when $\Sigma$ is timelike and $W \neq 0$, one gets a Skyrmionic configuration whose time dependence is topologically protected as it cannot decay in static solutions. This kind of solitons has been named topologically protected time crystals in [50].

#### *1. Gauged topological charge*

Obviously, when the coupling to a $U(1)$ gauge field is considered, the expression in Eq. (11) cannot be correct since it is not gauge invariant. The simplest solution to replace in Eq. (11) all the derivatives with covariant derivatives is wrong as well (since it leads to a gauge invariant expression which is, however, not conserved). The correct solution has been constructed in [32] (see also the pedagogical analysis in [37]): the expression for the gauge invariant and conserved topological charge reads

$$W = \frac{1}{24\pi^2}\int_\Sigma \epsilon^{ijk}\mathrm{Tr}\{(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U) - \partial_i[3A_j t_3(U^{-1}\partial_k U + \partial_k U U^{-1})]\}. \tag{12}$$

Hence, the topological charge gets one extra contribution which, at the end, is responsible for the Callan-Witten effect [32]. The computations below show that such an effect (according to which, roughly speaking, a magnetic monopole may "swallow" part of the topological charge) is more general and, in principle, strong magnetic fields may be able to support it even without magnetic monopoles.

## III. REVIEW OF THE SKYRMIONS AT FINITE VOLUME

In [50] an extension of the method introduced in [44] which also works in situations in which the Skyrme model is analyzed within a finite volume in a flat metric has been constructed. It is based on the following ansatz:

$$G = \frac{\gamma+\phi}{2}, \quad \tan F = \frac{\tan H}{\cos A}, \quad \tan C = \frac{\sqrt{1+\tan^2 F}}{\tan A}, \tag{13}$$

where

$$A = \frac{\gamma - \phi}{2}, \qquad H = H(r,z). \tag{14}$$

It can be verified directly that the topological density $\rho_B$ is nonvanishing. From the standard parametrization of $SU(2)$ [51] it follows that

$$0 \leq \gamma \leq 4\pi, \qquad 0 \leq \phi \leq 2\pi, \tag{15}$$

while the boundary condition for $H$ will be discussed below.

### A. Sine-Gordon and Skyrmions

A quite efficient way to put the Skyrme model within a flat region of finite volume is to introduce the following metric:

$$ds^2 = -dz^2 + l^2(dr^2 + d\gamma^2 + d\phi^2) \tag{16}$$

(here $z$ is the time variable). The size of the volume of this region is of order $l^3$ (the parameter $l$ has dimension of length). On the other hand, $r$, $\gamma$ and $\phi$ are angular coordinates (so that they are adimensional); the domain of $\gamma$ and $\phi$ is given by (15), while for $r$ we choose the finite interval $0 \leq r \leq 2\pi$.

In the case in which $A_\mu = 0$, the gauged Skyrme model reduces to the original Skyrme configuration and the corresponding field equations (5) can be simplified (without loosing the topological charge) using the ansatz in Eqs. (9), (10), (13) and (14) as has been shown in [50]. Indeed, one gets *just one scalar differential equation for the profile H*

$$\Box H - \frac{\lambda}{8l^2(\lambda + 2l^2)} \sin(4H) = 0, \tag{17}$$

where $\Box$ is the two-dimensional d'Alambert operator.

When $A_\mu = 0$, the topological baryon charge $B$ and charge density $\rho_B$ become, respectively,

$$B = \frac{1}{24\pi^2} \int_{t=\mathrm{const}} \rho_B, \qquad \rho_B = 3\sin(2H) dH d\gamma d\phi. \tag{18}$$

If we replace the topologically nontrivial ansatz in Eqs. (9), (10), (13) and (14) in the original action (1), we obtain an effective action given by

$$\mathcal{L}(H) = 16l^2(\lambda + 2l^2)\nabla_\mu H \nabla^\mu H - \lambda\cos(4H), \tag{19}$$

which reproduces equation of motion (17). The boundary conditions for the function $H$ are

$$H(0) = 0, \qquad H(2\pi) = \pm\frac{\pi}{2}, \tag{20}$$

which corresponds to $B = \pm 1$ and

$$H(0) - H(2\pi) = 0, \tag{21}$$

which leads to $B = 0$. The sector $B = 0$ is relevant in the construction of Skyrmion–anti-Skyrmion bound states.

Hence, the original $(3+1)$-dimensional Skyrme field equations, energy density and effective action in a topologically nontrivial sector (as $\rho_B \neq 0$) can be reduced to the corresponding quantities of the $(1+1)$-dimensional sine-Gordon model (a well-known example of integrable models; see [52]). Following [50], this allows us to construct Skyrmions as well as Skyrmion–anti-Skyrmion bound states.[2] The effective coupling sine-Gordon action and coupling constants (following [53]) read

$$\mathcal{L}(\Phi) = -\frac{1}{2}\nabla^\mu\Phi\nabla_\mu\Phi + \frac{\alpha}{\beta^2}(\cos(\beta\Phi) - 1), \tag{22}$$

$$\alpha = \frac{\lambda}{2l^2(\lambda + 2l^2)}, \qquad \beta = \frac{4l}{\sqrt{\lambda + 2l^2}}. \tag{23}$$

Therefore, the Skyrme model within the finite volume defined above always satisfies the Coleman bound $\beta^2 < 8\pi$.

It is worth emphasizing that Skyrme and Perring [54] used sine-Gordon in $(1+1)$-dimensions as a "toy model" for the $(3+1)$-dimensional Skyrme model. The analogies between (a simplified version of) the Skyrme model and the sine-Gordon model have also been emphasized in [55] and references therein. The very surprising feature of the results in [50] is that there is a nontrivial topological sector of the full $(3+1)$-dimensional Skyrme model in which it is *exactly equivalent to the sine-Gordon model* in $(1+1)$-dimensions.[3]

### 1. An interesting function

As is well known, in the usual case the energy of the spherical Skyrmion (found numerically by Skyrme himself) exceeds the bound in terms of the baryon charge by 23%. One can ask a similar question of the Skyrmions living at finite volume constructed in [50]. Only in this subsection, we will adopt the convention that $K = 2$ and $\lambda = 1$ (see page 25 of [11]), according to which lengths are measured in fm and energy in GeV. Let us consider the function

$$\Delta = E - 12\pi^2|B| = E - 12\pi^2, \tag{24}$$

where $E$ is the energy of the (anti-)Skyrmion constructed in [50] and $B$ is its baryon charge. The nice pedagogical review [57] clarifies that Eq. (24) corresponds to the right hand side of the Skyrme Bogomol'nyi-Prasad-Sommerfield (BPS) bound in terms of the baryon charge. The function $\Delta$ (once $K$ and $\lambda$ have been fixed) is a function of the size of the box $l$ in which these configurations live. We have to note that in the relevant equation in [50] there appears a $\sqrt{2}$ multiplying the term that is substracted from the energy, which is a typo. One can see that for high densities (when $l$ is around 10 fm or less) the Skyrmions constructed exceed the topological bound by $\sim$53% and when the size is large, their energy increases very rapidly.

### 2. Inclusion of chemical potential and infinite volume limit

As is well known, the presence of the isospin chemical potential is encoded in the following covariant derivative (see [28–31]):

$$D_\mu = \nabla_\mu + \bar{\mu}[t_3, ]\delta_{\mu 0}. \tag{25}$$

[2] Indeed, a quite remarkable prediction of the Skyrme model at finite volume (as discussed in [50]) is that the model possesses around $8\pi/\beta^2 - 1$ Skyrmion–anti-Skyrmion bound states where $\beta$ is in Eq. (23). When the size of the box is large compared with $fm$, one gets that the number of these bound states is between 5 and 6 (in good agreement with the number of baryon antibaryon resonances appearing in particle physics).

[3] The semiclassical quantization in the present sector of the Skyrme model can be analyzed following [13,14,56] since *principle of symmetric criticality* applies (see [50]).

This has a correspondence to a special case of a coupling with an electromagnetic field with a potential of the form $A_\mu = \bar{\mu}\delta_{\mu 0}$. As has been shown in [50], for static configurations $H(r)$ *the full Skyrme field equations with isospin chemical potential reduce to the following ordinary differential equation (ODE) for*

$$(\lambda + 2l^2 - 8\lambda l^2 \bar{\mu}^2 \sin^2(H(r)))H''(r) - 4\lambda l^2 \bar{\mu}^2 \sin(2H(r))H'^2 + \lambda\left(\bar{\mu}^2 l^2 - \frac{1}{8}\right)\sin(4H(r)) + 4\bar{\mu}^2 l^4 \sin(2H(r)) = 0, \quad (26)$$

which can be further reduced to

$$Y(H)\frac{(H')^2}{2} + V(H) = E_0, \quad (27)$$

where

$$Y(H) = \lambda + 2l^2 - 8\lambda l^2 \bar{\mu}^2 \sin^2(H),$$
$$V(H) = -\frac{\lambda}{4}\left(\bar{\mu}^2 l^2 - \frac{1}{8}\right)\cos(4H) - 2\bar{\mu}^2 l^4 \cos(2H).$$

In order to determine the integration constant $E_0$, one has to require the relation here below,

$$\int_0^{\pi/2} \frac{[Y(H)]^{1/2}}{[E_0 - V(H)]^{1/2}} dH = \sqrt{2}2\pi. \quad (28)$$

Consequently, the critical isospin chemical potential $\bar{\mu}_c$ can be defined as the value of $\mu$ beyond which Eq. (28) cannot be satisfied anymore,

$$(\bar{\mu}_c)^2 = \frac{\lambda + 2l^2}{8\lambda l^2}. \quad (29)$$

It is also easy to see that, before reaching the critical value defined above, the presence of the chemical potential suppresses the energy peak of the Skyrmion, making it flatter. This comment will be useful to provide us with a simple interpretation of the physical effects of the $U(1)$ gauge coupling.

It is worth emphasizing that if one considers the infinite volume limit of the above expression for the critical isospin chemical potential, one gets a value which is consistent with the value computed in the literature (see Refs. [30,31]) in the standard infinite volume case. We hope to come back to the relations between our finite-volume results and the infinite volume limit in a future publication.

### B. Time crystals

In [58–60], Wilczek and Shapere made the following deep observation: One can construct simple models in which it is *possible to break spontaneously time translation symmetry.*

Although it is well known that no-go theorems [61,62] ruled out the original proposals, new research fields started trying to realize in a concrete system the ideas presented in [58–60] (a nice review is [63]). Many examples have been found since then in condensed matter physics [64–69]. The first example in nuclear and particles physics has been found in [50] in the Skyrme model at finite volume.

Namely, the $(3+1)$-dimensional Skyrme model supports exact time-periodic configurations which cannot be deformed continuously to the trivial vacuum as they possess a nontrivial winding number. Consequently, these time crystals are only allowed to decay into other time-periodic configurations, hence the name *topologically protected time crystals*.

Needless to say, there are many time-periodic solutions of the Skyrme model which cannot be considered time crystals.[4]

Following [50], a good choice to describe the finite box is the line element

$$ds^2 = -d\gamma^2 + l^2(dz^2 + dr^2 + d\phi^2), \quad (30)$$

where $\gamma$ plays the role of time. We have to make the following modification to ansatz (13), (14):

$$A = \frac{\omega\gamma - \phi}{2}, \qquad G = \frac{\omega\gamma + \phi}{2}, \quad (31)$$

where $0 \leq \omega\gamma \leq 4\pi$ and the frequency $\omega$ is necessary to keep $A$ and $G$ dimensionless. Note that, with the above ansatz, the Skyrme configuration $U$ is periodic in time.

The profile $H$ depends on two spacelike coordinates. In the case in which the coupling with the $U(1)$ gauge field is neglected, the Skyrme configurations $U$ defined in Eqs. (9), (10), (13), (14) and (31) are necessarily time-periodic. The full Skyrme field equations (5) reduce in this case to

$$\triangle H - \frac{\lambda\omega^2}{4(l^2(\lambda\omega^2 - 4) - \lambda)}\sin(4H) = 0, \quad (32)$$

$$\omega^2 \neq \omega_c^2 = \frac{\lambda + 4l^2}{l^2\lambda}, \quad (33)$$

where $\triangle$ is the two-dimensional Laplacian in $z$ and $r$. Equation (32) is the Euclidean sine-Gordon equation. Exact solutions of Eq. (32)[5] can easily be constructed taking, for instance, $H = H(r)$.

Time crystal configuration can be constructed explicitly considering $H = H(r)$. These configurations have a nontrivial winding number. The topological density is given by

[4]As an example, consider the Skyrmion–anti-Skyrmion bound state at finite volume which corresponds to breather (so that they are time-periodic). In fact, they are not topologically protected since, if one "pays" the corresponding binding energies, they decay into the trivial vacuum.

[5]Previous literature on the analogies between sine-Gordon and Skyrme models can be found in [70–72] and references therein. As has been emphasized previously, sine-Gordon theory was believed to be just a "toy model" for the $(3+1)$-dimensional Skyrme model. In fact, we proved that in a nontrivial topological sector they exactly coincide.

$\rho_B = 3\sin(2H)dH \wedge d(\omega\gamma) \wedge d\phi$, and thus the winding number can be evaluated to

$$W = \frac{\omega}{8\pi^2}\int_{z=\text{const}} \sin(2H)dHd\gamma d\phi = \pm 1. \qquad (34)$$

Hence, there are smooth time-periodic regular configurations of the Skyrme model living at finite volume possessing a nontrivial winding number along a three-dimensional time-like surface. Consequently, these configurations can only decay into other configurations which are also time-periodic (as for static configurations the above winding number vanishes). Thus, the time periodicity of these configurations is topologically protected by their winding number.

It is worth stressing the following fact: In the well-known case of non-Abelian gauge theories admitting BPS monopoles, the ground state in the sector with unit non-Abelian magnetic charge is the BPS monopole. Such a configuration cannot be deformed continuously to the trivial vacuum: in particular, it is not invariant under spatial rotations (unless they are compensated by internal rotations). The famous "spin from isospin effect" disclosed in the 1970s is a consequence of this lack of invariance. In the topologically protected time crystals constructed in [50], the ground state is time-periodic, and consequently the theorems in [61,62] (which assume that the ground state is static) do not apply, in complete analogy with what happens for BPS monopoles.

#### *1. The chemical potential*

One can introduce the isospin chemical potential also for these time crystals. The analysis in [50] shows that critical chemical potential $\bar{\mu}^*$ can be determined by requiring

$$\lambda + l^2[4 - \lambda\omega^2 + 8\lambda\bar{\mu}^*(\omega - 2\bar{\mu}^*)] \leq 0. \qquad (35)$$

The latter condition implies

$$\bar{\mu}^* \leq \frac{\omega}{4} - \frac{\sqrt{\frac{4l^2}{\lambda}+1}}{4l} \quad \text{or} \quad \bar{\mu}^* \geq \frac{\omega}{4} + \frac{\sqrt{\frac{4l^2}{\lambda}+1}}{4l}, \qquad (36)$$

which leads us to consider as critical values

$$\bar{\mu}^*_{cr} = \frac{\omega}{4} \pm \frac{\sqrt{\frac{4l^2}{\lambda}+1}}{4l}. \qquad (37)$$

## IV. GAUGED SKYRMIONS AND TIME CRYSTALS

In this section, we extend the Skyrmions and the time crystals constructed above to the cases in which the minimal coupling with the $U(1)$ gauge field cannot be neglected: namely, we will construct analytic examples of gauged Skyrmions as well as gauged time crystals. Then, in the following sections we analyze the most interesting physical properties of these gauged configurations.

For what follows we start by considering the following parametrization of the $SU(2)$-valued scalar $U$:

$$U = e^{t_3\alpha}e^{t_2\beta}e^{t_3\rho}, \qquad (38)$$

where $\alpha$, $\beta$ and $\rho$ are the Euler angles which in a single covering of space take the values $\alpha \in [0, 2\pi]$, $\beta \in [0, \frac{\pi}{2}]$ and $\rho \in [0, \pi]$.

### A. Gauged Skyrmions

Like in the case without an electromagnetic field, we start by considering metric (16), where the ordering of the coordinates that we assume is

$$x^\mu = (z, r, \gamma, \phi),$$

and where again we fix the dimension of the spatial box by requiring

$$0 \leq r \leq 2\pi, \qquad 0 \leq \gamma \leq 4\pi, \qquad 0 \leq \phi \leq 2\pi. \qquad (39)$$

As before, $l$ represents the size of the box while $r$, $\gamma$ and $\phi$ are adimensional angular coordinates and $z$ represents the time coordinate. It is possible to choose the ansatz for the Skyrme configuration in the following manner:

$$\alpha = p\frac{\gamma}{2}, \quad \beta = H(r), \quad \rho = q\frac{\phi}{2}, \quad p, \quad q \in \mathbb{N}, \qquad (40)$$

where $p$ and $q$ must be integer in order to cover $SU(2)$ an integer number of times. We restrict ourselves to the study of a static profile $H = H(r)$. In this context we assume an electromagnetic potential of the form

$$A_\mu = (b_1(r), 0, b_2(r), b_3(r)). \qquad (41)$$

Under the previous setting, the ensuing Maxwell equations (6) become

$$b_I''(r) = \frac{K}{2}(M_{IJ}b_J(r) + N_I), \qquad I, J = 1, 2, 3 \qquad (42)$$

where

$$M_{11} = 4\sin^2(H)\left(2\lambda H'^2 + \frac{\lambda(p^2+q^2)}{2}\cos^2(H) + 2l^2\right), \qquad (43a)$$

$$M_{23} = M_{32} = -\frac{pq}{2}\lambda\sin^2(2H), \qquad (43b)$$

$$M_{22} = M_{11} + \frac{p}{q}M_{23}, \qquad (43c)$$

$$M_{33} = M_{11} + \frac{q}{p}M_{23}, \qquad (43d)$$

the rest of $M_{IJ}$'s zero and

$$N = \left(0, \frac{p}{4}M_{11} - \frac{q^2 - p^2}{4q}M_{23}, -\frac{q}{4}M_{11} - \frac{q^2 - p^2}{4p}M_{23}\right). \tag{44}$$

Quite remarkably, the hedgehog property is not destroyed by the coupling to the above $U(1)$ gauge field since *the Skyrme equations lead to a single equation for the profile* $H(r)$ (see the appendix for more details)

$$\begin{aligned} &4\left(X_1\sin^2(H) + \frac{\lambda(p^2+q^2)}{2} + 2l^2\right)H'' + 2X_1\sin(2H)H'^2 \\ &\quad + 4\sin^2(H)X_1'H' + \left(2\lambda(pb_2+qb_3)\left(pb_2+qb_3 \right.\right.\\ &\quad \left.\left. + \frac{p^2-q^2}{2}\right) - \frac{\lambda p^2q^2}{2} - \frac{p^2+q^2}{2}X_1\right) \\ &\quad \times\sin(4H) - \frac{2l^2X_1}{\lambda}\sin(2H) = 0, \end{aligned} \tag{45}$$

where

$$X_1(r) = 4\lambda(-2l^2b_1^2 + b_2(2b_2+p) + b_3(2b_3-q)). \tag{46}$$

Still, at a first glance, it is a quite hopeless task to find analytic solutions to the coupled system corresponding to Eqs. (42) and (45).

In fact, a closer look at the simpler situation (described in the previous section) in which one wants to describe the effects of the isospin chemical potential offers a surprising solution.

*Firstly*, one has to observe that the Skyrme field equations (26) and (27) are integrable (as they are reduced to quadratures).

*Secondly*, one can ask the following question: under which circumstances does Eq. (45) for the Skyrme profile interacting with the $U(1)$ gauge field become as similar as possible to (the much easier) Eq. (26)?

The answer is that this happens in the special cases where

$$X_1 = -\frac{\lambda(p^2+q^2)}{2} = \text{const}, \tag{47}$$

and

$$b_2(r) = -\frac{q}{p}b_3(r) - \frac{p^2-q^2}{4p}. \tag{48}$$

Thus, when Eqs. (47) and (48) are satisfied, Eq. (45) for the Skyrme profile interacting with the $U(1)$ gauge field becomes integrable [as it can be reduced to a quadrature using the same step to go from Eq. (26) to Eq. (27)].

However, we are not done yet since Eqs. (47) and (48) could be incompatible with Maxwell equations Eq. (42). In other words, it could happen that there is no solution of Eq. (42) in which Eqs. (47) and (48) are satisfied.

In fact, a direct computation shows that if one places Eqs. (47) and (48) into Maxwell equations Eq. (42), then the full system reduces to the following single scalar equation for $b_3(r)$:

$$\begin{aligned} &b_3'' + \frac{K}{4}(q-4b_3)\sin^2(H)(4l^2 + 4\lambda H'^2 + \lambda(p^2+q^2) \\ &\quad \times\cos^2(H)) = 0 \end{aligned} \tag{49}$$

and the corresponding equation for the profile $H$ reads

$$\left(\frac{8l^2}{p^2+q^2} + 2\lambda\cos^2(H)\right)H'' + \sin(2H)(l^2 - \lambda H'^2) = 0. \tag{50}$$

Interestingly enough, the above equation for the profile interacting with a $U(1)$ gauge field is equivalent to the Skyrme field equation with a chemical potential possessing a value $\bar{\mu}_0^2 = \frac{p^2q^2}{4l^2(p^2+q^2)}$.

This is a quite remarkable result *since the full coupled Skyrme Maxwell system* made by Eqs. (5) and (6) in a topologically nontrivial sector (as it will be shown below) in the finite box defined in Eq. (16) *can be reduced consistently to a solvable system of two ODEs* [namely, Eqs. (49) and (50)]. Hence, gauged Skyrmions can be constructed explicitly. In the appendix there can be found the details of the derivation of this result.

The recipe is to use the static ansatz in Eq. (40) for the Skyrme configuration and the ansatz in Eqs. (41), (47) and (48) for the $U(1)$ gauge field. Thus, one can determine the Skyrme profile $H(r)$ from Eq. (50) and then Eq. (49) for the gauge potential $b_3(r)$ becomes a simple linear nonhomogeneous equation in which there is an effective potential which depends on $H(r)$. The other components of the gauge potential are determined by solving the simple algebraic conditions (47) and (48). The above system allows us to clearly disclose many features of the $U(1)$ gauged Skyrme model which are close to superconductivity.

In comparison with the chemical potential (26), where $p = q = 1$, the corresponding value $\bar{\mu}_0^2 = \frac{1}{8l^2}$ is lower than the upper critical bound set as $\bar{\mu}_c^2 = \frac{1}{8l^2} + \frac{1}{4\lambda}$ by (29). It can be shown that the same thing happens with the introduction of $p$ and $q$ since now—by following the same procedure—the critical value ends up being $\bar{\mu}_c^2 = \frac{1}{16l^2}(p^2+q^2) + \frac{1}{4\lambda}$ and again $\bar{\mu}_0 \leq \bar{\mu}_c$ for $\lambda > 0$ and any value of $p, q$. Thus, the first physical conclusion can be drawn. Since the equation for the Skyrme profile coupled with the $U(1)$ gauge field looks like the Skyrme field equations with isospin chemical potential and we know that a nonvanishing isospin chemical potential suppresses the Skyrmion (until it reaches the critical value when the Skyrmion completely disappears), we can conclude that the coupling with the Maxwell field suppresses (but without destroying) the Skyrmion.

As a consistency check, if one considers $b_i \to 0 \Rightarrow X_1 \to 0$, then (45) reduces to

$$H''(r) - \frac{\lambda p^2 q^2 \sin(4H(r))}{4(4l^2 + \lambda(p^2 + q^2))} = 0, \tag{51}$$

with a conserved quantity of

$$H'^2 + \frac{\lambda p^2 q^2 \cos(4H(r))}{8(4l^2 + \lambda(p^2 + q^2))} = I_0 = \text{const}. \tag{52}$$

The general solution of (51) is given in terms of the Jacobi amplitude function.

#### 1. Topological charge

Here we calculate the temporal component of the baryon density as it is modified by the Maxwell field following the steps of [32]. Due to $A_\mu$ and the introduction of a covariant derivative, there exists an additional term in the form of a total divergence and the full density reads

$$\begin{aligned} B_0 = &\frac{\epsilon_{0ijk}}{24\pi^2}[\mathrm{Tr}(R_i R_j R_k) \\ &-3\partial_i(A_j \mathrm{Tr}(\tau_3(U^{-1}\partial_k U + \partial_k U U^{-1})))] \\ = &-\frac{pq}{8\pi^2}\sin(2H)H' + \frac{pq}{4\pi^2}\partial_r(\cos^2(H)(b_2 - b_3)). \end{aligned} \tag{53}$$

The baryon number that we get with the help of $B_0$ is

$$\begin{aligned} B &= \int B_0 dr d\gamma d\phi \\ &= -pq \int \sin(2H) dH \\ &\quad + 2[\cos^2(H(r))(q b_2(r) - p b_3(r))]_0^{2\pi}, \end{aligned} \tag{54}$$

and it leads to

$$B = -pq - 2(q b_2(0) - p b_3(0)), \tag{55a}$$

$$B = pq + 2(q b_2(2\pi) - p b_3(2\pi)), \tag{55b}$$

depending on the boundary values that we assume: $H(2\pi) = \pi/2$, $H(0) = 0$ or $H(2\pi) = 0$, $H(0) = \pi/2$, respectively. Clearly, $B$ depends now on the size of the system through $p$ and $q$, as well as on the boundary values set for $b_2$ and $b_3$ which are related to the magnetic components of $F_{\mu\nu}$. The solutions we have found with $p = q = 1$ for $b_3$ (and the corresponding values of $b_2$) of Eq. (49) have $b_2(0) = b_3(0)$ so that the topological charge reduces to the usual integer value. However, it is clear that there are many more general possibilities, and one could try to construct configurations in which the topological charge is "shared" by the Skyrmion and the electromagnetic field. We hope to come back to this interesting issue in a future publication.

It is worth noting that by combining tools developed in the present paper with the techniques introduced in [45] one can construct multilayered configurations of the gauged Skyrme model such that each layer corresponds to the present gauged-Skyrmion configuration with baryon charge the product $pq$, while the number of layers is related to the number of peaks of the (energy-density associated with the) profile $H(r)$. This observation suggests that the present formalism could be used to describe analytically the regular patterns which are known to appear in the Skyrme model when fine-density effects are taken into account. We hope to come back to this very interesting issue in a future publication.

### B. Gauged time crystal

As in [50], a very efficient choice to describe the finite box is the line element in Eq. (30) where $\gamma$ plays the role of time. The Skyrme configuration reads

$$\alpha = \frac{\phi}{2}, \qquad \beta = H(r), \qquad \rho = \frac{\omega\gamma}{2}, \tag{56}$$

where $\omega$ again is a frequency so that $\rho$ is dimensionless (as it should be). Once more we assume an electromagnetic potential of the form (41), but now we have to consider that the coordinate ordering is

$$x^\mu = (\gamma, r, z, \phi). \tag{57}$$

The Maxwell equations have the same form as (42). The entries of matrix $M$ read

$$M_{11} = 2\sin^2(H(r))(4\lambda H'^2 + \lambda\cos^2(H) + 4l^2), \tag{58a}$$

$$M_{13} = -\frac{\lambda\omega}{2}\sin^2(2H), \tag{58b}$$

$$M_{22} = M_{11} + \frac{l^2\omega^2 + 1}{\omega} M_{13}, \tag{58c}$$

$$M_{33} = M_{11} + l^2\omega M_{13}, \tag{58d}$$

$$M_{31} = -l^2 M_{13}, \tag{58e}$$

while

$$N = \left(\frac{1}{4}(M_{13} - \omega M_{11}), 0, \frac{1}{4}\left(\frac{(2l^2\omega^2 + 1)}{\omega} M_{13} + M_{11}\right)\right). \tag{59}$$

Interestingly enough, also in this case the hedgehog property is not lost. Namely, the full Skyrme field equations for the time crystal ansatz defined above coupled to the $U(1)$ gauge field in Eq. (41) [taking into account that the coordinates are as in Eq. (57)] reduce to a single ODE for the profile $H(r)$ (for more details see the appendix),

$$\begin{aligned}&4(l^2(4-\lambda\omega^2)+X_2\sin^2(H)+\lambda)H''+2X_2\sin(2H)H'^2\\&\quad+4\sin^2(H)X_2'H'+\bigg[\frac{1}{4}(l^2\omega^2-1)X_2+\lambda(2l^2\omega b_1\\&\quad-2b_3-1)(2l^2\omega b_1-2b_3-l^2\omega^2)\bigg]\sin(4H)\\&\quad-\frac{2l^2X_2}{\lambda}\sin(2H)=0,\end{aligned} \tag{60}$$

where

$$X_2(r)=8\lambda(l^2b_1(\omega-2b_1)+2b_2^2+b_3(1+2b_3)). \tag{61}$$

The closeness with the situation in which one has (instead of the dynamical Maxwell field) a nonvanishing chemical potential is useful in this case as well. Indeed, by requiring

$$X_2=\lambda(l^2\omega^2-1)=\text{const.} \tag{62}$$

and

$$b_3(r)=l^2\omega b_1(r)-\frac{l^2\omega^2}{4}-\frac{1}{4}, \tag{63}$$

not only does the equation for the time crystal profile become solvable (as it is reduced to a quadrature) but also the full Maxwell equations reduce consistently to a scalar ODE for $b_1(r)$.

All in all, using the ansatz in Eqs. (41), (31), (62) and (63) [in the line element in Eq. (30) with coordinates (57)] the *full coupled Skyrme Maxwell system* made by Eqs. (5) and (6) in a topologically nontrivial sector (as will be shown below) *can be reduced consistently to the following solvable system of two coupled ODEs for* $H(r)$ and $b_1(r)$ (a detailed derivation of this result can be encountered in the appendix),

$$\begin{aligned}&b_1''-\frac{K}{8}(\omega-4b_1)\sin^2(H)(l^2(\lambda\omega^2-8)-\lambda\\&\qquad+\lambda(l^2\omega^2-1)\cos(2H)-8\lambda H'^2)=0,\end{aligned} \tag{64}$$

$$\begin{aligned}&2(\lambda(l^2\omega^2-1)\cos^2(H)-4l^2)H''+(l^2\omega^2-1)\\&\quad\times\sin(2H)(l^2-\lambda H'^2)=0.\end{aligned} \tag{65}$$

Hence, also in this case the recipe is to determine the profile $H(r)$ (as the corresponding equation is solvable) and then to replace the result into the equation for $b_1(r)$ which becomes a simple linear nonhomogeneous equation in which there is an effective potential which depends on $H(r)$. The other components of the gauge potential are determined solving the simple algebraic conditions in Eqs. (62) and (63). The above system allows to clearly disclose many features of the gauged time crystals (and, more in general, of the $U(1)$ gauged Skyrme model) which are close to a "dual superconductivity."

The first integral of (65) which allows us to reduce it to quadratures is

$$\begin{aligned}&(4l^2+\lambda(1-l^2\omega^2)\cos^2(H))H'^2\\&\quad-\frac{1}{2}l^2(1-l^2\omega^2)\cos(2H)=I_0,\end{aligned} \tag{66}$$

where $I_0$ is determined by the boundary conditions. Once $H(r)$ is known, Eq. (64) can be analyzed with the standard tools of the theory of linear ordinary differential equations.

As we did in the previous section for the Skyrmion, we also calculate here for the time crystal, the nonvanishing winding number that is produced by

$$\begin{aligned}B_2&=\frac{\epsilon_{2ijk}}{24\pi^2}[\mathrm{Tr}(R_iR_jR_k)\\&\quad-3\partial_i(A_j\mathrm{Tr}(\tau_3(U^{-1}\partial_kU+\partial_kUU^{-1})))]\\&=\frac{\omega}{8\pi^2}\sin(2H)H'+\frac{1}{4\pi^2}\partial_r(\cos^2(H)(b_1-\omega b_3)),\end{aligned} \tag{67}$$

where the latin indices of the previous relation assume the values 0,1,3 and the resulting integral is

$$\begin{aligned}W&=\int B_2drd(\omega\gamma)d\phi\\&=1+2\left[\cos^2(H(r))\left(\frac{b_1(r)}{\omega}-b_3(r)\right)\right]_0^{2\pi}\\&=1-2\left(\frac{b_1(0)}{\omega}-b_3(0)\right),\end{aligned} \tag{68}$$

if we consider $r\in[0,2\pi]$, $\omega\gamma\in[0,4\pi]$, $\phi\in[0,2\pi]$ and $H(2\pi)=\pi/2$, $H(0)=0$.

However, a "normal" topological charge is also present here due to the correction from the electromagnetic potential. By taking $B_0$ as defined in (53) as an integral over spatial slices, we obtain

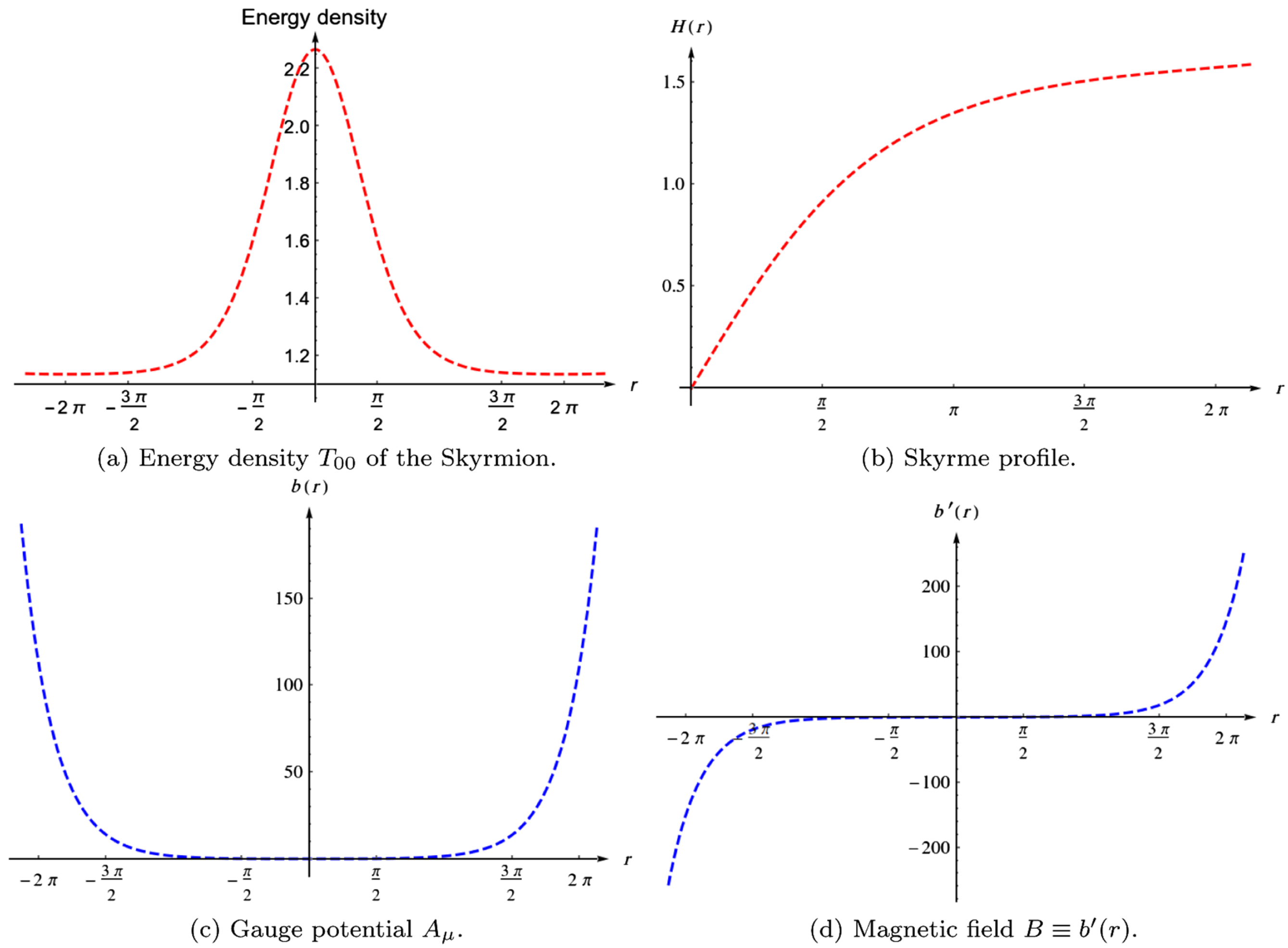
$$B=\int B_0drdzd\phi=-2[\cos^2(H(r))b_2(r)]_0^{2\pi}=2b_2(0),$$

with the same boundary values used as in (68) with the difference now that we have $z$ in place of $\gamma$, for which we consider $z\in[0,2\pi]$. The charge $B$ is nonzero as long as $b_2(0)\neq0$.

## V. EXTENDED DUALITY

In this section we show that an extended electromagnetic duality exists between the gauged Skyrmion and the gauged time crystal constructed above. This means that, in order to disclose such duality, one not only needs to interchange electric and magnetic components in a suitable way, but also to transform certain parameters of the gauged solitons.

In other words, the question we want to answer in this section is: how do the usual duality transformations of the




(a) Energy density $T_{00}$ of the Skyrmion.

(b) Skyrme profile.

(c) Gauge potential $A_\mu$.

(d) Magnetic field $B \equiv b'(r)$.

FIG. 1. The solutions for Eqs. (49) and (50) correspond to the values: $\lambda = 0.04$, $l = 0.47$, $K = 1.0$, $p = 1.0$, and $q = 1.0$. Solving for $b \equiv -b_2 = b_3$. The above plots clearly show the suppression of the magnetic field (which is nonvanishing only in the $\gamma$ and $\phi$ directions) in the core of the Skyrmion.

electromagnetic field have to be generalized so as to act on the Skyrmions and time crystals considered here in such a way that the field equations Eqs. (49) and (50) (corresponding to the gauged Skyrmion) are mapped into the field equations Eqs. (64) and (65) (corresponding to the gauged time crystal)?

Let us take[6] the simplest nontrivial cases of gauged configurations we examined above. For the Skyrmion we have the profile equation (50), which for $p = q = 1$ reduces to

$$2(2l^2 + \lambda\cos^2(H))H'' + \sin(2H)(l^2 - \lambda H'^2) = 0, \qquad (69)$$

while the for the electromagnetic potential components we get—from relations (47) and (48)—

$$b_1 = \pm\frac{1 - 4b_3}{2\sqrt{2}l}, \qquad b_2 = -b_3, \qquad (70)$$

with $b_3$ being given by the differential equation

$$b_3'' + \frac{K}{2}(1 - 4b_3)\sin^2(H)(2l^2 + 2\lambda H'^2 + \lambda\cos^2(H)) = 0. \qquad (71)$$

Let us now consider the corresponding time crystal equations, where—in order to avoid confusion—we denote the potential as $A_\mu = (a_1(r), 0, a_2(r), a_3(r))$ (namely, we label differently the components). The profile equation is, of course, given by (65) with the potential components related as

$$a_2 = \pm\frac{1}{4}l\sqrt{1 - l^2\omega^2}(\omega - 4a_1),$$
$$a_3 = l^2\omega a_1 - \frac{l^2\omega^2}{4} - \frac{1}{4}, \qquad (72)$$

[6] We will analyze here only how to extend electromagnetic duality in the integrable sectors considered above. However, we hope to come back to the intriguing issue of how to extend duality to more general configurations of the $U(1)$ gauged Skyrme model in a future publication.

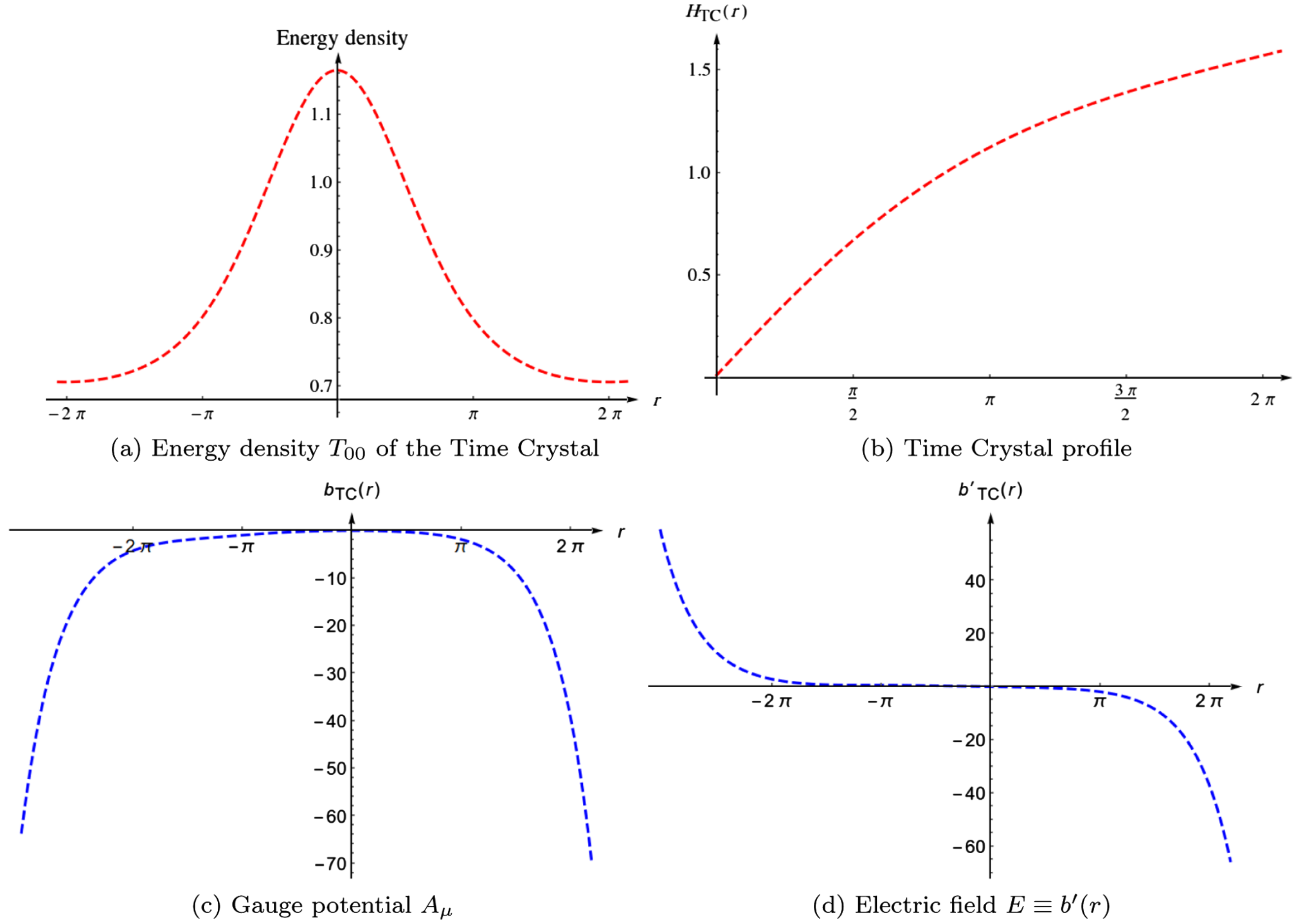



FIG. 2. The solutions for the Eqs. (64) and (65) correspond to the values: $\lambda = 0.04$, $l = 0.47$, $\omega = 0.95$, $K = 1.00$, $p = 1.00$ and $q = 1.00$. Solving for $b_{TC} \equiv b_1$. The above plots show clearly the suppression of the electric field in the core of the time crystal.

and with $a_1$ determined by the following equation:

$$a_1'' - \frac{K}{8}(\omega - 4a_1)\sin^2(H)(l^2(\lambda\omega^2 - 8) - \lambda + \lambda(l^2\omega^2 - 1)\cos(2H) - 8\lambda H'^2) = 0. \quad (73)$$

These are just equations (62), (63) and (64) with the new labeling of the components.

An immediate observation is that profile equations (65) and (69) become identical if we set $\omega = -\frac{i}{l}$. Then, it is an easy task to see that (73) and (72) are mapped to (71) and (70) under the linear transformation

$$a_1 = \frac{i}{l}b_2, \qquad a_2 = ilb_1, \qquad a_3 = -b_3. \quad (74)$$

The appearance of the imaginary units is not alarming, since one also needs a suitable imaginary scaling in the relevant coordinates to map one space-time metric to the other. Notice that the imaginary part of the transformation involves only the $y$ and $z$ components of $A_\mu$. Hence, the end result after performing such a transformation is a real electromagnetic tensor of the Skyrmion case.

We have to note, however, that transformation (74) is not unique. There are other linear transformations that map the two sets of equations to each other by mixing the electric and magnetic components. However, (74) belongs to a smaller class of transformations that associates the electric component of the time crystal potential $a_1$ with the purely magnetic components of the Skyrmion, namely $b_2$ and $b_3$. In particular, this property is respected by any linear transformation of the form $a_i = L_{ij}b_j$ as long as the following set conditions hold:

$$L_{13} = L_{12} - \frac{i}{l}, \quad L_{21} = il, \quad L_{23} = L_{22},$$
$$L_{11} = L_{31} = 0, \quad L_{33} = L_{32} - 1.$$

Of course, the free parameters appearing in the above transformation must be chosen each time in such a way so that the end result is strictly real. In Table I, we can see how the gauged Skyrmion and the time crystal (T.C.) components of the electromagnetic field are mapped into each other as well as $H$, $A$ and $G$ [in relations (13) and (14)] that are involved in the generalized hedgehog ansatz. Thus, we can see that the two configurations correspond to an interchange between the electric and one of the magnetic components that looks like a duality relation as seen in the

TABLE I. In this table we demonstrate how the components of the electromagnetic fields are interchanged under the mapping (74), while the profile and Euler angles remain the same.

| Gauged Skyrmion and gauged time crystal correspondence | |
|---|---|
| $E_1$ | $-B_3$ |
| $B_2$ | $-B_2$ |
| $B_3$ | $E_1$ |
| $(H, A, G)$ | $(H, A, G)$ |

plane formed by the $x^1$ and $x^3$ components. One can call this transformation an extended or generalized duality.

It is a very surprising result that a sort of a duality symmetry exists, which maps the gauged Skyrmion into the gauged time crystal. Thus, if such extended duality transformations discussed here would have been known in advance, one could have found that time crystals exist just by applying such transformations to the gauged Skyrmion. Moreover, the plots in Fig. 1 and Fig. 2 clearly show that as the magnetic field is suppressed in the gauged Skyrmion core, the electric field is suppressed in the gauged time crystal core. Thus, as gauged Skyrmions have some features in common with superconductors, gauged time crystals have some features in common with dual superconductors. We hope to come back to this very interesting issue and to its possible relevance in Yang-Mills theories in a future publication.

## VI. EXTERNAL PERIODIC FIELDS

In this section, we will discuss an approximation which can be of practical importance in many applications from nuclear physics to astrophysics.

We have been able to construct analytically two different types of topologically nontrivial configurations of the full $(3+1)$-dimensional $U(1)$ gauged Skyrme model (which are dual to each other in the electromagnetic sense). Thus, it is natural to ask why we should analyze approximated solutions when we have the exact ones.

The obvious reason is that, in this way, we will be able to discuss electromagnetic fields more general than the ones leading to the exact solutions discussed in the previous sections. In particular, it is interesting to discuss the physical effects of time-periodic electromagnetic fields (which do not belong to the class leading to the above exact solutions).

Here we will consider the case in which the Skyrme configuration is fixed and not affected by the electromagnetic field (as in [50]) which is slowly turned on to get a tiny time-periodic electromagnetic field in these Skyrme background solutions. In this case, the Skyrme background plays the role of an effective medium for the Maxwell equations. Very interesting is the situation in which the background is a time crystal, as the reaction of the time-dependent Maxwell perturbation to the presence of the time crystal critically depends on the ratio between the frequency of the perturbation and the frequency of the time crystal.

### A. Tiny time-periodic fields in Skyrme background

Let us consider the approximate situation where we introduce a small enough electromagnetic field so as not to consider its effect on the profile equations. Additionally, we demand that it be periodic in time in one of its components,

$$A_\mu = (b_1(r), 0, b_2(r)\cos(\Omega\gamma), b_3(r)). \tag{75}$$

The charge is conserved $\partial_\mu J^\mu = 0$, while the Maxwell equations constitute a compatible system of differential equations. The one that corresponds to $b_2$ is

$$\frac{b_2''}{b_2} = \frac{K}{2}[\lambda(8H'^2 + 2(1 - l^2\omega^2)\cos^2(H)) + 8l^2] \times \sin^2(H) - l^2\Omega^2, \tag{76}$$

and by considering the approximation $b_2 \ll 1$ we are led to the single profile equation

$$H'' = \frac{l^2\lambda\omega^2\sin(4H)}{4(l^2(\lambda\omega^2 - 4) - \lambda)}, \tag{77}$$

with the corresponding constant of motion

$$\frac{l^2\lambda\omega^2\cos(4H)}{16(l^2(\lambda\omega^2 - 4) - \lambda)} + \frac{1}{2}H'^2 = I_0. \tag{78}$$

With the help of the latter, we can express (76) as

$$\frac{b_2''}{b_2} = \frac{K}{2}(x^2 - 1)\left(\frac{2l^4(\lambda\omega^2 - 4)(\lambda x^2\omega^2 - 4) + l^2\lambda(\lambda(8x^4 - 10x^2 + 1)\omega^2 + 8)}{l^2(\lambda\omega^2 - 4) - \lambda} - 2\lambda(8I_0 + x^2)\right) - l^2\Omega^2, \tag{79}$$

where $x = \cos(H)$. From the form of (79) we can deduce that the nature of the solution strongly depends on the sign of the right-hand side. If the sign is negative, one expects a periodic type of behavior. On the other hand, if it is positive, we rather expect an exponential kind of behavior.

Clearly, the appearance of these two possibilities has to do with the value of the frequency $\Omega$ of the field and its relation to the rest of the parameters of the model.

In general, one can consider the function

$$f(x) = (x^2 - 1)\left(\frac{2l^4(\lambda\omega^2 - 4)(\lambda x^2\omega^2 - 4) + l^2\lambda(\lambda(8x^4 - 10x^2 + 1)\omega^2 + 8)}{l^2(\lambda\omega^2 - 4) - \lambda} - 2\lambda(8I_0 + x^2)\right), \tag{80}$$

which at most possesses five extrema. The value $x = 0$ is always a global extremum; for the rest of the values of $x$ in $[-1, 1]$ one may have from none up to four extrema depending on the parameters $\lambda$, $l$, $\omega$ and $I_0$.

For example, when $l = \omega = \lambda = 1$, $I_0 = -1/2$, one gets five extrema in the region $x \in [-1, 1]$, of which $x = 0$ is a global maximum; on the other hand, when $l = \omega = \lambda = 1$, $I_0 = -1$, one gets only one extremum, $x = 0$, which now is a minimum.

In any case, it is possible to arrange the external field frequency $\Omega$ so that the right-hand side of (79) has a clearly positive or negative sign. The critical value for this is $\Omega_{cr} = \frac{K}{2l^2} f(0)$, where

$$f(0) = \frac{\lambda(4l^2 + \lambda)}{l^2(4 - \lambda\omega^2) + \lambda} + 8l^2 + \lambda(16I_0 - 1). \quad (81)$$

If $f(0)$ is a maximum, we need to have $\Omega > \Omega_{cr}$ in order to obtain a periodic type of behavior. Alternatively if $f(0)$ is a minimum, the condition $\Omega < \Omega_{cr}$ leads to an exponential type behavior.

This simple analysis shows that the reaction of a time-periodic Maxwell perturbation to the presence of a time crystal strongly depends on the relations between the frequency of the Maxwell perturbation and the parameters characterizing the time crystal.

## VII. CONCLUSIONS AND PERSPECTIVES

Using the generalized hedgehog approach, we have constructed the first analytic and topologically nontrivial solutions of the $U(1)$ gauged Skyrme model in $(3+1)$-dimensional flat space-times at finite volume. There are two types of gauged solitons. Firstly, gauged Skyrmions living within a finite volume appear as the natural generalization of the usual Skyrmions living within a finite volume. The second type of gauged solitons corresponds to gauged time crystals. These are smooth solutions of the $U(1)$ gauged Skyrme model whose periodic time dependence is protected by a topological conservation law. Interestingly enough, electromagnetic duality can be extended to include these two types of solitons. Gauged Skyrmions manifest very interesting similarities with superconductors, while gauged time crystals do so with dual superconductors.

Due to the relations of the Skyrme model with the low energy limit of QCD, the present results can be useful in many situations in which the backreaction of baryons on a Maxwell field (and vice versa) cannot be neglected (this is especially true in plasma physics and astrophysics).

It is a very interesting issue (which we hope to come back to in a future publication) to understand the relevance of the present results in Yang-Mills theory. From the technical point of view, the tools which allowed the construction of the present gauged configurations have been extended to the Yang-Mills case as well (see [47–49] and references therein). Thus, it is natural to wonder whether time crystals can be defined in the Yang-Mills case as well. The present analysis suggests that this construction could shed some light on the dual superconductor picture.

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## APPENDIX: DERIVATION OF THE REDUCED SYSTEM OF EQUATIONS (42), (45) AND (60)

In this section we demonstrate how equations (42), (45) and (60) are obtained from the general field equations (5) and (6) thanks to the generalized hedgehog ansatz [40–43], which remarkably enough still holds when the Skyrme field is coupled to Maxwell theory.

It can be easily seen that, under the choice (40) for the Euler angles and (41) for the electromagnetic potential, the three components of $R_\mu = R^i_\mu t_i$, $i = 1, 2, 3$ in Eq. (3) read [the order of the space-time coordinates in the gauged Skyrmion case is $x^\mu = (z, r, \gamma, \phi)$]

$$R^1_\mu = \left(b_1 \cos(q\phi)\sin(2H), -\sin(q\phi)H', \left(\frac{p}{2} + b_2\right)\cos(q\phi)\sin(2H), b_3\cos(q\phi)\sin(2H)\right), \quad (A1a)$$

$$R^2_\mu = \left(b_1 \sin(q\phi)\sin(2H), \cos(q\phi)H', \left(\frac{p}{2} + b_2\right)\sin(q\phi)\sin(2H), b_3\sin(q\phi)\sin(2H)\right), \quad (A1b)$$

$$R^3_\mu = \left(-2b_1\sin^2(H), 0, \frac{p}{2}\cos(2H) - 2b_2\sin^2(H), \frac{q}{2} - 2b_3\sin^2(H)\right). \quad (A1c)$$

With the help of the latter, the electromagnetic current vector can be computed through (7) to be

$$J^\mu = \frac{K}{2l^2}(M_{1I}b_I + N_1, 0, M_{2I}b_J + N_2, M_{3I}b_J + N_3), \qquad I = 1, 2, 3 \quad (A2)$$

with the expressions $M_{IJ}$ and $N_J$ being given by (43) and (44), respectively. It can be easily verified that $\nabla_\mu J^\mu = 0$ holds as an identity for the previous expression.

By using (A1), in the three gauged Skyrme equations

$$D^\mu\left(R^i_\mu t_i + \frac{\lambda}{4}[R^\nu, G_{\mu\nu}]^i t_i\right) =: E^i t_i = 0, \tag{A3}$$

the latter become

$$E^1 = -\frac{\sin(q\phi)}{16l^4}\mathcal{A}(r) = 0, \tag{A4a}$$

$$E^2 = \frac{\cos(q\phi)}{16l^4}\mathcal{A}(r) = 0, \tag{A4b}$$

$$E^3 \equiv 0, \tag{A4c}$$

where

$$\begin{aligned}\mathcal{A}(r) = {}& 4(8\lambda \sin^2(H)(-2l^2b_1^2 + b_2(2b_2+p) - b_3(q-2b_3)) \\ & + 4l^2 + \lambda(p^2+q^2))H'' - 16\lambda\sin(2H)(2l^2b_1^2 \\ & - b_2(2b_2+p) + b_3(q-2b_3))(H')^2 \\ & - 32\lambda\sin^2(H)(4l^2b_1b_1' - (4b_2+p)b_2' \\ & + (q-4b_3)b_3')H' + \lambda(4l^2b_1^2(p^2+q^2) \\ & - (2qb_2 + p(q-2b_3))^2)\sin(4H) \\ & + 16l^2(2l^2b_1^2 - b_2(2b_2+p) + b_3(q-2b_3))\sin(2H).\end{aligned} \tag{A5}$$

We can see that the $t_3$ component becomes identically zero, while the other two are proportional after the substitution of all the involved quantities. The remaining $\phi$ variable is decoupled from $r$ and the system is reduced to the single equation, $\mathcal{A} = 0$, for $H(r)$, which we have expressed in a more compact form in (45). At the same time, the current $J^\mu$, as given by (A2), is only $r$-dependent and leads to the Maxwell set of equations (42).

The exact same thing can be repeated for the profile equation of the gauged time crystal (60). This time we have to consider (56) for the Euler angles, with the help of which the three $R_\mu$ components are written as [remember that now $x^\mu = (\gamma, r, z, \phi)$]

$$\begin{aligned}R^1_\mu = {}& \Big(b_1\cos(\omega\gamma)\sin(2H), -\sin(\omega\gamma)H', \\ & b_2\cos(\omega\gamma)\sin(2H), \left(\frac{1}{2}+b_3\right)\cos(\omega\gamma)\sin(2H)\Big),\end{aligned} \tag{A6a}$$

$$\begin{aligned}R^2_\mu = {}& \Big(b_1\sin(\omega\gamma)\sin(2H), \cos(\omega\gamma)H', \\ & b_2\sin(\omega\gamma)\sin(2H), \left(\frac{1}{2}+b_3\right)\sin(\omega\gamma)\sin(2H)\Big),\end{aligned} \tag{A6b}$$

$$\begin{aligned}R^3_\mu = {}& \Big(\frac{\omega}{2} - 2b_1\sin^2(H), 0, -2b_2\sin^2(H), \\ & \frac{1}{2}\cos(2H) - 2b_3\sin^2(H)\Big).\end{aligned} \tag{A6c}$$

In the same manner the variables are decoupled in the three profile equations (A3) and the system once more reduces to the single equation. The $t_3$ component is identically zero, while the other two are proportional to each other leading to a single equation for $H(r)$, which is given by (60). In particular, we obtain

$$E^1 = -\frac{\sin(\omega\gamma)}{16l^4}\mathcal{B}(r) = 0, \tag{A7a}$$

$$E^2 = \frac{\cos(\omega\gamma)}{16l^4}\mathcal{B}(r) = 0, \tag{A7b}$$

$$E^3 \equiv 0, \tag{A7c}$$

with

$$\begin{aligned}\mathcal{B} = {}& 4(8\lambda\sin^2(H)(l^2b_1(\omega-2b_1) + 2b_2^2 + b_3(1+2b_3)) \\ & + l^2(4-\lambda\omega^2) + \lambda)H'' + 16\lambda\sin(2H)(l^2b_1(\omega-2b_1) \\ & + 2b_2^2 + b_3(1+2b_3))(H')^2 + 32\lambda\sin^2(H)(l^2(\omega-4b_1)b_1' \\ & + 4b_2b_2' + (4b_3+1)b_3')H' + \lambda(l^2\omega^2 + 4l^2b_1(b_1-\omega) \\ & + 4b_2^2(l^2\omega^2-1) + 4l^2\omega b_3(-2b_1+\omega b_3+\omega))\sin(4H) \\ & - 16l^2(l^2b_1(\omega-2b_1) + 2b_2^2 + 2b_3^2 + b_3)\sin(2H).\end{aligned} \tag{A8}$$

It is easy to verify that $\mathcal{B} = 0$ is equivalent to (60). Of course, the same considerations are also true for the Maxwell equations and relation (A2) still holds for the current, where now the $M_{IJ}$ and $N_I$ are given by expressions (58) and (59), respectively.

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