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Heat kernel and Weyl anomaly of Schrödinger invariant theory

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We propose a method inspired by discrete light cone quantization to determine the heat kernel for a Schrödinger field theory (Galilean boost invariant with $z = 2$ anisotropic scaling symmetry) living in $d + 1$ dimensions, coupled to a curved Newton-Cartan background, starting from a heat kernel of a relativistic conformal field theory ($z = 1$) living in $d + 2$ dimensions. We use this method to show that the Schrödinger field theory of a complex scalar field cannot have any Weyl anomalies. To be precise, we show that the Weyl anomaly $\mathcal{A}^G_{d+1}$ for Schrödinger theory is related to the Weyl anomaly of a free relativistic scalar CFT $\mathcal{A}^R_{d+2}$ via $\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2}$, where m is the charge of the scalar field under particle number symmetry. We provide further evidence of the vanishing anomaly by evaluating Feynman diagrams in all orders of perturbation theory. We present an explicit calculation of the anomaly using a regulated Schrödinger operator, without using the null cone reduction technique. We generalize our method to show that a similar result holds for theories with a single time-derivative and with even $z > 2$.

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I. INTRODUCTION

The Weyl anomaly in relativistic conformal field theory (CFT) has a rich history [1–8]. In $1 + 1$ dimensions, irreversibility of RG flows has been established by Zamoldchikov [9] who showed monotonicity of a quantity C that equals the Weyl anomaly c at fixed points. Remarkably, the anomaly c equals the central charge of the CFT. In $3 + 1$ dimensions, there is a corresponding "a-theorem" [10–13] where a again appears in the Weyl anomaly, and there is strong evidence for a similar a-theorem in higher, even dimensions [14–17]. In contrast, much less is known in the case of nonrelativistic field theories admitting anisotropic scale invariance under the following transformation:

$$\boldsymbol{x} \to \lambda \boldsymbol{x}, \qquad t \to \lambda^z t. \tag{1}$$

Nonetheless, nonrelativistic conformal symmetry does emerge in various scenarios. For example, fermions at unitarity, in which the S-wave scattering length diverges, $|a| \to \infty$, exhibit nonrelativistic conformal symmetry. In ultracold atom gas experiments, the S-wave scattering length can be tuned freely along an RG flow and this has renewed interest in the study of the RG flow of such theories [18,19]. In fact, at $a^{-1} = -\infty$ the system behaves as a BCS superfluid while at $a^{-1} = \infty$ it becomes a BEC superfluid. The BCS-BEC crossover, at $a^{-1} = 0$, is precisely the unitarity limit, exhibiting nonrelativistic conformal symmetry [20,21]. In this regime, we expect universality, with features independent of any microscopic details of the atomic interactions. Other examples of nonrelativistic systems exhibiting scaling symmetry come with accidentally large scattering cross section. Examples include various atomic systems, like $^{85}\mathrm{R}b$ [22], $^{138}\mathrm{C}s$ [23], and a few nucleon systems like the deuteron [24,25].

Galilean CFT, which enjoys $z = 2$ scaling symmetry is special among nonrelativistic conformal field theories (NRCFTs). On group theoretic grounds, there is a special conformal generator for $z = 2$ that is not present for $z \neq 2$ theories [26,27]. The coupling of such theories to the Newton-Cartan (NC) structure is well understood [27–30]. The generic discussion of anomalies in such theories has been initiated by Jensen in [31]. Moreover, there have been recent works classifying and evaluating Weyl anomalies at fixed points [32–36] and even away from the fixed points; the latter have resulted in proposed C-theorem candidates [37,38].

It has been proposed in [31], using the fact that discrete light cone quantization (DLCQ) of a relativistic CFT living in $d + 2$ dimensions yields a nonrelativistic Galilean CFT in $d + 1$ dimensions with $z = 2$, that the Weyl anomaly of the relativistic CFT survives in the nonrelativistic theory. The conjecture states that the Weyl anomaly $\mathcal{A}^G$ for a Schrödinger field theory (Galilean boost invariant with $z = 2$ scale symmetry and special conformal symmetry) is given by

$$\mathcal{A}^G_{d+1} = aE_{d+2} + \sum_n c_n W_n, \tag{2}$$

where E_{d+2} is the $d + 2$-dimensional Euler density of the parent spacetime and W_n are Weyl covariant scalars with weight $(d + 2)$. The right-hand side is computed on a geometry given in terms of the $d + 2$-dimensional metric; this will be explained below; see Eq. (19). A specific example of particular interest is

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$$\mathcal{A}^G_{2+1} = aE_4 - cW^2, \tag{3}$$

where W^2 stands for the square of the Weyl tensor.

The purpose of this work is twofold. First, we show that these proposed relations must be corrected to include a factor of $\delta(m)$, when the Schrödinger invariant theory involves a single complex scalar field having charge m under the $U(1)$ symmetry. To be precise, we show that

$$\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2}, \tag{4}$$

where $\mathcal{A}^R_{d+2}$ is the Weyl anomaly of the corresponding relativistic CFT in $d+2$ dimensions. This is derived explicitly for the case of a bosonic (commuting) scalar field, but the derivation applies equally to the case of a fermionic (anticommuting) scalar field. The second purpose is to develop a framework inspired by DLCQ to evaluate the heat kernel of a theory with one time derivative kinetic term in a nontrivial curved background. This framework enables us to calculate not only the heat kernel but also the anomaly coefficients. In fact, using this method and its appropriately modified form enables us to generalize Eq. (4) to one time derivative theories with arbitrary even z, where the parent $d+2$ dimensional theory enjoys $SO(1,1)\times SO(d)$ symmetry with scaling symmetry acting as $t\to\lambda^{z/2}t$, $x^{d+2}\to\lambda^{z/2}x^{d+2}$, $x^i\to\lambda x^i$, $(i=1,\ldots,d+1)$.

The paper is organized as follows. We will briefly review coupling of a Schrödinger field theory to the Newton-Cartan structure in Sec. II. In Sec. III, we sketch how DLCQ can be used to obtain Schrödinger field theories following the procedure of [31] and propose its modified cousin, which we call lightcone reduction (LCR), to obtain a Schrödinger field theory. In Sec. IV, we determine the heat kernel for free Galilean CFT coupled to a flat NC structure in two different ways, on the one hand using LCR and on the other without the use of DLCQ, providing a check on our proposed method for determining the heat kernel for Galilean field theory coupled to a curved NC geometry. We then proceed to evaluate the heat kernel on curved spacetime according to the proposal and subsequently derive the Weyl anomaly for Schrödinger field theory of a single complex scalar. In Sec. V, we reconsider the computation using perturbation theory; we find that for a wide class of models on a curved background all vacuum diagrams vanish. In fact, we show that an anomaly is not induced in the more general case that $U(1)$ invariant dimensionless couplings are included, regardless of whether we are at a fixed point or away from it, in all orders of a perturbative expansion in the dimensionless coupling and metric. In Sec. VI, we give a formal proof of our prescription and generalize the framework to calculate the heat kernel and anomaly for theories with one time derivative and arbitrary even z. We conclude with a brief summary of the results obtained and discuss future directions of investigation. Technical aspects of defining heat kernel for one time derivative theory in flat spacetime are explored in Appendix A, and on a curved background in Appendix B. Finally, in Appendix C, we present an explicit calculation of the anomaly using a regulated Schrödinger operator, without using the null cone reduction technique.

II. NEWTON-CARTAN STRUCTURE & WEYL ANOMALY

The study of the Weyl anomaly necessitates coupling of nonrelativistic theory to a background geometry, which can potentially be curved. Generically, the prescription for coupling to a background can depend on the global symmetries of the theory on a flat background. Of interest to us are Galilean and Schrodinger field theories. The algebra of the Galilean generators is given by [26]

$$\begin{aligned}
&[M_{ij},N]=0, \qquad [M_{ij},P_k]=\imath(\delta_{ik}P_j-\delta_{jk}P_i),\\
&[M_{ij},K_k]=\imath(\delta_{ik}K_j-\delta_{jk}K_i),\\
&[M_{ij},M_{kl}]=\imath(\delta_{ik}M_{jl}-\delta_{jk}M_{il}+\delta_{il}M_{kj}-\delta_{jl}M_{ki}),\\
&[P_i,P_j]=[K_i,K_j]=0, \qquad [K_i,P_j]=\imath\delta_{ij}N,\\
&[H,N]=[H,P_i]=[H,M_{ij}]=0, \qquad [H,K_i]=-\imath P_i,
\end{aligned} \tag{5}$$

and the commutators of the dilatation generator aloong with the Galilean ones are given by

$$\begin{aligned}
&[D,P_i]=\imath P_i, \quad [D,K_i]=(1-z)\imath K_i,\\
&[D,H]=z\imath H, \quad [D,N]=\imath(2-z)N, \quad [M_{ij},D]=0
\end{aligned} \tag{6}$$

where $i,j=1,2,\ldots,d$ label the spatial dimensions, z is the anisotropic exponent, P_i, H, and M_{ij} are generators of spatial translations, time translation spatial rotations, respectively, K_i generates Galilean boosts along the x^i direction, N is the particle number (or rest mass) symmetry generator, and D is the generator of dilatations. The generators of Schrödinger invariance include, in addition, a generator of special conformal transformations, C. The Schrödinger algebra consists of the $z=2$ version of (5), (6) plus the commutators of C,

$$\begin{aligned}
&[M_{ij},C]=0, \qquad [K_i,C]=0,\\
&[D,C]=-2\imath C, \qquad [H,C]=-\imath D.
\end{aligned} \tag{7}$$

In what follows, by Schrödinger invariant theory, we mean a $z=2$ Galilean, conformally invariant theory. For $z\neq2$, we only discuss anisotropic scale invariant theories invariant under a group generated by P_i, M_{ij}, H, D, and N such that the kinetic term involves one time derivative only. The most natural way to couple Galilean (boost) invariant field theories to geometry is to use the Newton-Cartan (NC) structure [27–29]. In what follows, we briefly review NC geometry, following Ref. [31].

The NC structure defined on a $d+1$ dimensional manifold $\mathcal{M}_{d+1}$ consists of a one form n_μ, a symmetric positive semi-definite rank d tensor $h_{\mu\nu}$ and an $U(1)$ connection A_μ, such that the metric tensor

$$g_{\mu\nu} = n_\mu n_\nu + h_{\mu\nu} \tag{8}$$

is positive definite. The upper index data v^μ and $h^{\mu\nu}$ is defined by

$$\begin{aligned} v^\mu n_\mu &= 1, \qquad v^\nu h_{\mu\nu} = 0, \\ h^{\mu\nu} n_\nu &= 0, \qquad h^{\mu\rho} h_{\rho\nu} = \delta^\mu_\nu - v^\mu n_\nu \end{aligned} \tag{9}$$

Physically, v^μ defines a local time direction while $h_{\mu\nu}$ defines a metric on spatial slice of $\mathcal{M}_d$.

As prescribed in [27], while coupling a Galilean invariant field theory to a NC structure, we demand

(1) Symmetry under reparametrization of coordinates. Technically, this requirement boils down to writing the theory in a diffeomorphism invariant way.

(2) $U(1)$ gauge invariance. The fields belonging to some representation of Galilean algebra carry some charge under particle number symmetry, which is an $U(1)$ group. Promoting this to a local symmetry requires a gauge field A_μ that is sourced by the $U(1)$ current.

(3) Invariance under Milne boosts, under which $(n_\mu, h^{\mu\nu})$ remains invariant, while

$$\begin{aligned} v^\mu &\to v^\mu + \psi^\mu, \\ h_{\mu\nu} &\to h_{\mu\nu} - (n_\mu\psi_\nu + n_\nu\psi_\mu) + n_\mu n_\nu \psi^2, \\ A_\mu &\to A_\mu + \psi_\mu - \frac{1}{2} n_\mu \psi^2 \end{aligned} \tag{10}$$

where $\psi^2 = h^{\mu\nu}\psi_\mu\psi_\nu$ and $v^\nu\psi_\nu = 0$.

The action of a free Galilean scalar ϕ_m with charge m, coupled to this NC structure satisfying all the symmetry conditions listed above, is given by

$$\int d^{d+1}x\sqrt{g}[\imath m v^\mu(\phi_m^\dagger D_\mu\phi_m - \phi_m D_\mu\phi_m^\dagger) - h^{\mu\nu}D_\mu\phi_m^\dagger D_\nu\phi_m], \tag{11}$$

where $D_\mu = \partial_\mu - \imath m A_\mu$ is the appropriate gauge invariant derivative.

From a group theory perspective, a Galilean group can be a subgroup of a larger group that includes dilatations. That is, besides the symmetries mentioned earlier, a Galilean invariant field theory coupled to the flat NC structure can also be scale invariant, *i.e.*, invariant under the following transformations

$$\boldsymbol{x} \to \lambda \boldsymbol{x}, \qquad t \to \lambda^z t, \tag{12}$$

where z is the dynamical critical exponent of the theory. As mentioned earlier, for $z=2$, the symmetry algebra may further be enlarged to contain a special conformal generator, resulting in the Schrödinger group. On coupling a Galilean CFT with arbitrary z to a nontrivial curved NC structure, the scale invariance can be thought of as invariance under following scaling of NC data (also known as anisotropic Weyl scaling; henceforth, we omit the word "*anisotropic*," and by Weyl transformation it should be understood that we mean the transformation with appropriate z):

$$n_\mu \to e^{z\sigma} n_\mu, \quad h_{\mu\nu} \to e^{2\sigma} h_{\mu\nu}, \quad A_\mu \to e^{(2-z)\sigma} A_\mu, \tag{13}$$

where σ is a function of space and time.

Even though classically a Galilean CFT may be scale invariant, it is not necessarily true that it remains invariant quantum mechanically. Renormalization may lead to anomalous breaking of scale symmetry much like in the Weyl anomaly in relativistic CFTs (where $z=1$). The anomaly $\mathcal{A}$ is defined from the infinitesimal Weyl variation (13) of the connected generating functional W:

$$\delta_\sigma W = \int d^{d+1}x\sqrt{g}\delta\sigma\mathcal{A},. \tag{14}$$

We mention in passing that away from the fixed point the coupling is scale dependent, that is, the running of the coupling under the RG must be accounted for, hence the variation δ_σ on the couplings needs to be incorporated. The generic scenario has been elucidated in Ref. [38].

In this work, we are interested in anomalies at a fixed point. Even in the absence of running of the coupling, the background metric can act as an external operator insertion on vacuum bubble diagrams leading to new UV divergences that are absent in flat spacetime. Removing these new divergences can potentially lead to anomalies. The anomalous ward identity for anisotropic Weyl transformation is given by [31]

$$z n_\mu \mathcal{E}^\mu - h^{\mu\nu} T_{\mu\nu} = \mathcal{A}, \tag{15}$$

where $n_\mu\mathcal{E}^\mu$ and $h^{\mu\nu}T_{\mu\nu}$ are respectively diffeomorphic invariant measure of energy density and trace of spatial stress-energy tensor.

In what follows, we will be interested in evaluating the quantity appearing on the right hand side of Eq. (15). A standard method is through the evaluation of the heat kernel in a curved background. Hence, our first task is to figure out a way to obtain the heat kernel for theories with kinetic term involving only one time derivative. In the next few sections we will introduce methods for computing heat kernels and arrive at the same result from different approaches.

III. DISCRETE LIGHT CONE QUANTIZATION (DLCQ) & ITS COUSIN LIGHTCONE REDUCTION (LCR)

One elegant way to obtain the heat kernel is to use discrete light cone quantization (DLCQ). This exploits the well-known fact that a $d+1$ Galilean invariant field theory can be constructed by starting from a relativistic theory in $d+2$ dimensional Minkowski space in light cone coordinates

$$ds^2 = 2dx^+dx^- + dx^i dx^i \tag{16}$$

where $i = 2, 3, \ldots, d+1$ and $x^{\pm} = \frac{x^1 \pm t}{\sqrt{2}}$ define light cone coordinates, followed by a compactification in the null coordinate x^- on a circle. From here on, by "*reduced*" theory we will mean the theory in $d+1$ dimensions while by "*parent*" theory we will mean the $d+2$ dimensional theory on which this DLCQ trick is applied. We first present a brief review of DLCQ.

The generators of $SO(d+1,1)$ which commute with P_-, the generator of translation in the x^- direction, generate the Galilean algebra. P_- is interpreted as the generator of particle number of the reduced theory. In light cone coordinates, the mass-shell condition for a massive particle becomes.[1]

$$p_+ = \frac{|\boldsymbol{p}|^2}{2(-p_-)} + \frac{M^2}{4(-p_-)} \tag{17}$$

Eq. (17) can be interpreted as the nonrelativistic energy of a particle, p_+, with mass $m = -p_-$ in a constant potential. The reduced mass-shell condition (17) is Galilean invariant, that is, invariant under boosts ($\boldsymbol{v}$) and rotations ($\mathbf{R}$):

$$\boldsymbol{p} \to \mathbf{R}\boldsymbol{p} - \boldsymbol{v}p_-, \qquad p_+ \to p_+ + \boldsymbol{v}\cdot(\mathbf{R}\boldsymbol{p}) - \frac{1}{2}|\boldsymbol{v}|^2 p_-$$

Setting $M = 0$, the dispersion relation is of the form

$$\omega = \frac{k^2}{2m} \tag{18}$$

and enjoys $z = 2$ scaling symmetry. To rephrase, setting $M = 0$ will allow one to append a dilatation generator, which acts as follows:

$$p_+ \to \lambda^2 p_+, \qquad p_- \to p_-, \qquad \boldsymbol{p} \to \lambda\boldsymbol{p}$$

Had we not compactified in the x^- direction, p_- would be a continuous variable. The parameter p_- can be changed using a boost in the $+-$ direction, but compactification in the x^- direction spoils relativistic boost symmetry and the eigenvalues of p_- become discretized, $p_- = \frac{n}{R}$, where R is the compactification radius. We note that Lorentz invariance is recovered in the $R \to \infty$ limit. For convenience, by appropriately rescaling the generators of spatial translations and of special conformal transformations, as well as P_-, we can set $R = 1$.

One can technically perform DLCQ even in a curved spacetime as long as the metric admits a null isometry. This guarantees that we can adopt a coordinate system with a null coordinate x^- such that all the metric components are independent of x^-. To be specific, we will consider the following metric:

$$ds^2 = G_{MN}dx^M dx^N, \qquad G_{\mu-} = n_\mu,$$
$$G_{\mu\nu} = h_{\mu\nu} + n_\mu A_\nu + n_\nu A_\mu, \qquad G_{--} = 0 \tag{19}$$

where $M, N = +, -, 1, 2, \ldots, d$ run over all the indices in $d+2$ dimensions, the index $\mu = +, 1, 2, \ldots, d$ runs over $d+1$ dimensions and $h_{\mu\nu}$ is a rank d tensor. Ultimately, $h_{\mu\nu}, n_\mu, A_\mu$ are to be identified with the NC structure, and just as above we can construct $h^{\mu\nu}$ and v^μ such that Eq. (9) holds. Moreover, these quantities transform under Milne boost symmetry as per Eq. (10). Hence, the boost invariant inverse metric is given by

$$G^{-\mu} = v^\mu - h^{\mu\nu}A_\nu, \qquad G^{\mu\nu} = h^{\mu\nu},$$
$$G^{--} = -2v^\mu A_\mu + h^{\mu\nu}A_\mu A_\nu. \tag{20}$$

Reduction on x^- yields a Galilean invariant theory coupled to an NC structure given by $(n_\mu, h^{\mu\nu}, A_\mu)$, with metric given by (8). Moreover, all the symmetry requirements listed above Eq. (10) are satisfied by construction.

This prescription allows us to construct Galilean QFT coupled to a nontrivial NC structure starting from a relativistic QFT placed in a curved background with one extra dimension. For example, we can consider DLCQ of a conformally coupled scalar field in $d+2$ dimensions,

$$S_R = \int d^{d+2}x\sqrt{-G}[-G^{MN}\partial_M\Phi^\dagger\partial_N\Phi - \xi\mathcal{R}\Phi^\dagger\Phi],$$
$$\xi = \frac{d}{4(d-1)} \tag{21}$$

where $\mathcal{R}$ stands for the Ricci scalar corresponding to the G_{MN} metric. We compactify x^- with periodicity 2π and expand Φ in Fourier modes as

$$\Phi = \frac{1}{\sqrt{2\pi}}\sum_m \phi_m(x^\mu)e^{\imath m x^-}, \qquad \phi_m = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} dx^- \Phi e^{-\imath m x^-}. \tag{22}$$

In terms of ϕ_m, we recast the action, Eq. (21), in the following form using Eq. (20),

[1]The unusual sign convention in our definition of x^- results in the peculiar sign in Eq. (17).

$$S_R = \sum_m \int d^{d+1}x\sqrt{g}[\imath m v^\mu(\phi_m^\dagger D_\mu\phi_m - \phi_m D_\mu\phi_m^\dagger) - h^{\mu\nu}D_\mu\phi_m^\dagger D_\nu\phi_m - \xi\mathcal{R}\phi_m^\dagger\phi_m], \tag{23}$$

where $D_\mu = \partial_\mu - \imath m A_\mu$ and where each of the ϕ_m carry charge m under the particle number symmetry and sit in distinct representations of the Schrödinger group. The theory described by Eq. (23) is not Lorentz invariant because we have a discrete sum over m, breaking the boost invariance along the null direction.

The point of DLCQ is to break Lorentz invariance to Galilean invariance. As explained above, one can work in the uncompactified limit, and still break the Lorentz invariance by dimensional reduction. In the uncompactified limit, the sum over eigenvalues of P_- becomes integration over the continuous variable p_-. Nonetheless, one can focus on any particular Fourier mode. Technically, we can implement this by performing a Fourier transformation with respect to x^- of quantities of interest. This procedure also yields a Galilean invariant field theory where the elementary field is the particular Fourier mode under consideration. Henceforth, we will refer to this modified version of DLCQ as lightcone reduction (LCR).

Taking a cue from the relation between the actions given by Eqs. (21) and (23), we propose the following prescription to extract the heat kernel in the reduced theory: The heat kernel operator K_G in $d+1$-dimensional Galilean theory is related to the heat kernel operator K_R of the parent $d+2$-dimensional relativistic theory via

$$\langle(\boldsymbol{x}_2, t_2)|K_G|(\boldsymbol{x}_1, t_1)\rangle = \int_{-\infty}^{\infty} dx^- \langle \boldsymbol{x}_2, x_2^-, x_2^+|K_R|\boldsymbol{x}_1, x_1^-, x_1^+\rangle e^{-\imath m x_{12}^-} \tag{24}$$

where $x_{12}^- = x_2^- - x_1^-$ and the time t in the reduced theory is to be equated with x^+ in the parent theory.

We will postpone the proof of our prescription to Sec. VI. In the next section, we will lend support to our prescription by verifying our claim using two different methods of calculating the heat kernel. We emphasize that the reduction prescription, described above, is applicable to the $z = 2$ case of Galilean and scale invariant theories. The generic reduction procedure for arbitrary z (though not Galilean boost invariant) is discussed later in sec. VI B.

IV. HEAT KERNEL FOR A GALILEAN CFT WITH $z = 2$

A. Preliminaries: Heat kernel, zeta regularization

We start by briefly reviewing the heat kernel and zeta function regularization method [11,16,39,40]. A pedagogical discussion can be found in [41,42]. Let us consider a theory with partition function $\mathcal{Z}$, formally given by

$$\mathcal{Z} = \int [\mathcal{D}\phi][\mathcal{D}\phi^\dagger] e^{-\int d^d x \phi^\dagger \mathcal{M}\phi}, \tag{25}$$

where the eigenvalues of the operator $\mathcal{M}$ have positive real part.[2] The path integral over the field variable ϕ suffers from ultraviolet (UV) divergences and requires proper regularization and renormalization to be rendered as a meaningful finite quantity. Similarly, the quantum effective action $W = -\ln\mathcal{Z}$ corresponding to this theory, given by a formal expression

$$W = \ln(\det(\mathcal{M}))$$

requires regularization and renormalization.[3]

The method of zeta-function regularization introduces several quantities; the heat kernel operator

$$\mathcal{G} = e^{-s\mathcal{M}}, \tag{26}$$

its trace K over the space L^2 of square integrable functions

$$K(s, f, \mathcal{M}) = \mathrm{Tr}_{L^2}(f\mathcal{G}) = \mathrm{Tr}_{L^2}(fe^{-s\mathcal{M}}), \tag{27}$$

where $f \in L^2$, and the zeta-function, defined as

$$\zeta(\epsilon, f, \mathcal{M}) = \mathrm{Tr}_{L^2}(f\mathcal{M}^{-\epsilon}). \tag{28}$$

K and ζ are related via Mellin transform,

$$K(s, f, \mathcal{M}) = \frac{1}{2\pi\imath}\int_{c-\imath\infty}^{c+\imath\infty} d\epsilon s^{-\epsilon}\Gamma(\epsilon)\zeta(\epsilon, f, \mathcal{M}) \quad \text{and}$$
$$\zeta(\epsilon, f, \mathcal{M}) = \frac{1}{\Gamma(\epsilon)}\int_0^\infty ds s^{\epsilon-1}K(s, f, \mathcal{M}). \tag{29}$$

As is customary, below we use $f = 1$. However this should be understood as taking the limit $f \to 1$ at the end of the computation to ensure all expressions in intermediate steps are well defined.

Formally W is given by the divergent expression

$$W = -\int_0^\infty ds \frac{1}{s} K(s, 1, \mathcal{M})$$

The regulated version, W_ϵ, is defined by shifting the power of s

$$W_\epsilon = -\tilde{\mu}^{2\epsilon}\int_0^\infty ds \frac{1}{s^{1-\epsilon}} K(s, 1, \mathcal{M}) = -\tilde{\mu}^{2\epsilon}\Gamma(\epsilon)\zeta(\epsilon, 1, \mathcal{M}), \tag{30}$$

where the parameter $\tilde{\mu}$ with length dimension -1 is introduced so that W_ϵ remains adimensional. In this context,

[2] Positivity is required for convergence of the Gaussian integral.

[3] For anticommuting fields $W = -\ln(\det(\mathcal{M}))$; the minus sign is the only difference between commuting and anticommuting cases, so that in what follows we restrict our attention to the case of commuting fields.

the parameter ϵ behaves like a regulator, the divergences reappearing as $\epsilon \to 0$. In this limit,

$$W_\epsilon = -\left(\frac{1}{\epsilon} - \gamma_E + \ln(\tilde{\mu}^2)\right)\zeta(0,1,\mathcal{M}) - \zeta'(0,1,\mathcal{M}) + O(\epsilon),$$

so that subtracting the $\frac{1}{\epsilon}$ term gives the renormalized effective action

$$W^{\rm ren} = -\zeta'(0,1,\mathcal{M}) - \ln(\mu^2)\zeta(0,1,\mathcal{M}). \quad (31)$$

where $\mu^2 = \tilde{\mu}^2 e^{-\gamma_E}$ and γ_E is the Euler constant. On a compact manifold $\zeta(\epsilon,1,\mathcal{M})$ is finite as $\epsilon \to 0$ and the renormalized effective action given by (31) is finite, as it should. For noncompact manifolds the standard procedure for computing a renormalized effective action is to subtract a reference action that does not modify the physics. One may, for example, define $W = \ln(\det(\mathcal{M})/\det(\mathcal{M}_0))$, where the operator $\mathcal{M}_0$ is defined on a trivial (flat) background. This amounts to replacing $K(s,1,\mathcal{M}) \to K(s,1,\mathcal{M}) - K(s,1,\mathcal{M}_0)$ in Eq. (30) and correspondingly $\zeta(\epsilon,1,\mathcal{M}) \to \zeta(\epsilon,1,\mathcal{M}) - \zeta(\epsilon,1,\mathcal{M}_0)$. The expression for $W^{\rm ren}$ in (31) remains valid if it is understood that this subtraction is made before the $\epsilon \to 0$ limit is taken.

Classical symmetry under Weyl variations (both in the relativistic case and the anisotropic one) guarantees $\mathcal{M}$ transforms homogeneously, *i.e.*, $\delta_\sigma\mathcal{M} = -\Delta\sigma\mathcal{M}$ under $\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}$ where Δ is the scaling dimension of $\mathcal{M}$. Hence, we have

$$\delta_\sigma\zeta(\epsilon,1,\mathcal{M}) = -\epsilon{\rm Tr}_{L^2}(\delta\mathcal{M}\mathcal{M}^{-\epsilon-1}) = \Delta\epsilon\zeta(\epsilon,\sigma,\mathcal{M}). \quad (32)$$

Consequently, the anomalous variation of W is given by

$$\delta_\sigma W^{\rm ren} = -\Delta\zeta(0,\sigma,\mathcal{M}). \quad (33)$$

In the relativistic case, using the fact that

$$\delta_\sigma W = \frac{1}{2}\int d^{d+1}x\sqrt{g}T_{\mu\nu}\delta g^{\mu\nu} = -\int d^{d+1}x\sqrt{g}T^\mu{}_\mu\delta\sigma, \quad (34)$$

one has the trace anomaly equation

$$\mathcal{A} = -T^\mu{}_\mu = -\frac{1}{\sqrt{g}}\Delta\left(\frac{\delta\zeta(0,\sigma,\mathcal{M})}{\delta\sigma}\right)_{\sigma=0}. \quad (35)$$

In the nonrelativistic case, the Weyl anisotropic scaling is given by $h_{\mu\nu} \to e^{2\sigma}h_{\mu\nu}$ and $n_\mu \to e^{z\sigma}n_\mu$. We have

$$\delta_\sigma W = \int d^{d+1}x\sqrt{g}\left(\frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} - \mathcal{E}_\mu\delta n^\mu\right)$$
$$= \int d^{d+1}x\sqrt{g}(h^{\mu\nu}T_{\mu\nu} - zn^\mu\mathcal{E}_\mu)\delta\sigma \quad (36)$$

leading to

$$\mathcal{A} = zn_\mu\mathcal{E}^\mu - h^{\mu\nu}T_{\mu\nu} = -\frac{1}{\sqrt{g}}\Delta\left(\frac{\delta\zeta(0,\sigma,\mathcal{M})}{\delta\sigma}\right)_{\sigma=0}. \quad (37)$$

One can evaluate $\delta\zeta(0,\sigma,\mathcal{M})/\delta\sigma|_{\sigma=0}$ using the asymptotic form ($s \to 0$) of the heat kernel, K. The asymptotic expansion depends on the operator $\mathcal{M}$ and its scaling dimension. Schematically, one has

$$K(s,1,\mathcal{M}) = \frac{1}{s^{d_\mathcal{M}}}\sum_{n=0}^\infty s^{\kappa(n)}\sqrt{g}a_n,$$

where $\kappa(n)$ is a linear function of n. The singular pre-factor, $\frac{1}{s^{d_\mathcal{M}}}$, is determined by the heat kernel in the background-free, flat spacetime limit while the expansion accounts for corrections from background fields or geometry. The asymptotic expansion is guaranteed to exist if the heat kernel is well behaved for $s > 0$ in the flat spacetime limit, that is, if $\sum_i e^{-s\lambda_i}$, with λ_i, the eigenvalues of the operator $\mathcal{M}$, is convergent. The convergence requires that λ_i have, at worst, a power law growth and positive real part [43].

We are interested in operators $\mathcal{M}$ of generic form

$$\mathcal{M} = 2\imath m\partial_{t'} - (-1)^{z/2}(\partial_i\partial_i)^{z/2},$$

for which the heat kernel has a small s expansion of the following form

$$K(s,1,\mathcal{M}) = \frac{1}{s^{1+d/z}}\sum_{n=0}^\infty s^{2n/z}\int d^{d+1}x\sqrt{g}a_n, \quad (38)$$

where d is number of spatial dimension and z is dynamical exponent.[4] Then the zeta function is given by

$$\zeta(0,f,\mathcal{M}) = \int d^{d+1}x\sqrt{g}fa_{(d+z)/2}, \quad (39)$$

so that we arrive at an expression for the Weyl anomaly

$$\mathcal{A} = -\Delta a_{(d+z)/2}. \quad (40)$$

Hence, in order to determine the Weyl anomaly, one has to calculate the coefficient $a_{(d+z)/2}$ of the heat kernel expansion (38).[5] In subsequent sections, we will find out a way to evaluate the heat kernel in flat spacetime and then in curved spacetime for a Schrödinger invariant field theory. We will be doing this first without using DLCQ/LCR, and then again with LCR (modified cousin of DLCQ) using the prescription introduced above.

[4]In next few sections, we explicitly find this asymptotic form for $z = 2$ while the arbitrary z case is handled separately in VI B.
[5]Incidentally, this shows that the anomaly is absent when $d + z$ is odd.

B. Heat kernel in flat spacetime

1. Direct calculation (without use of DLCQ)

The action for a free Galilean CFT on a flat spacetime (which is in fact invariant under the Schrödinger group) is given by

$$S = \int dt d^d x \phi^\dagger [2m\imath\partial_t + \nabla^2]\phi \tag{41}$$

In order to improve convergence of the functional integral defining the partition function we perform a continuation to imaginary time:

$$e^{\imath \int dt d^d x \phi^\dagger [2m\imath\partial_t + \nabla^2]\phi} \underset{t=-\imath\tau}{\mapsto} e^{-\int d\tau d^d x \phi^\dagger [2m\partial_\tau - \nabla^2]\phi} \tag{42}$$

Hence, the Euclidean version of $\mathcal{M} = 2m\imath\partial_t + \nabla^2$ is given by

$$\mathcal{M}_E = 2m\partial_\tau - \nabla^2, \tag{43}$$

and it is this operator for which we will compute the heat kernel. The prescription $t = -\imath\tau$ is equivalent to adding $+\imath\epsilon$ to the propagator in Minkowskian flat space. In fact, the same $+\imath\epsilon$ prescription is obtained by deriving the nonrelativistic propagator as the nonrelativistic limit of the relativistic propagator.

The heat kernel for $\mathcal{M}_E$ is a solution to the equation[6]

$$(\partial_s + \mathcal{M}_E)\mathcal{G} = 0, \tag{44}$$

that is

$$(\partial_s + 2m\partial_{\tau_2} - \nabla^2_{x_2})\mathcal{G}(s; (\boldsymbol{x}_2, \tau_2), (\boldsymbol{x}_1, \tau_1)) = 0, \tag{45}$$

with boundary condition $\mathcal{G}(0;(\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1)) = \delta(\tau_2-\tau_1)\delta^d(\boldsymbol{x}_2-\boldsymbol{x}_1)$. Equation (45) is solved by

$$\mathcal{G}(s; (\boldsymbol{x}_2, \tau_2), (\boldsymbol{x}_1, \tau_1)) = \delta(2ms - (\tau_2 - \tau_1)) \frac{e^{-\frac{|\boldsymbol{x}_2-\boldsymbol{x}_1|^2}{4s}}}{(4\pi s)^{\frac{d}{2}}} \tag{46}$$

Consequently, the Eulcidean two point correlator is given by

$$G((\boldsymbol{x}_2, \tau_2), (\boldsymbol{x}_1, \tau_1)) = \int_0^\infty ds \mathcal{G}(s) = \frac{\theta(\tau)}{2m} \frac{e^{-\frac{m|\boldsymbol{x}|^2}{2\tau}}}{(2\pi\frac{\tau}{m})^{\frac{d}{2}}} \tag{47}$$

where $\tau = \tau_2 - \tau_1$ and $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. The same two point correlator can be obtained by Fourier transform from the Minkowski momentum space propagator G_M, or its imaginary time version,

$$G_M(p, \omega) = \frac{\imath}{2m\omega - |\boldsymbol{p}|^2 + i0^+} \underset{\substack{t=-\imath\tau \\ \omega=\imath\omega_E}}{\mapsto} G = \frac{1}{2m\omega_E + \imath|\boldsymbol{p}|^2} \tag{48}$$

In the coincidence limit the heat kernel of (46) contains a Dirac-delta factor, $\delta(ms)$. Since this nonanalytic behavior is unfamiliar, it is useful to rederive this result by directly computing the trace K, Eq. (26). One can conveniently choose the test function $f = e^{-|\eta\omega|}$. Hence

$$\begin{aligned} K(s, f, \mathcal{M}_{E,g}) &= \mathrm{Tr}(f e^{-s\mathcal{M}_{E,g}}) \\ &= \int \left(\frac{d^d k}{(2\pi)^d} e^{-sk^2}\right)\left(\int \frac{d\omega}{2\pi} e^{-2m\imath s\omega - |\eta\omega|}\right) \end{aligned}$$

The integral over k gives the factor of $1/s^{d/2}$, while the integral over ω gives

$$\frac{1}{\pi}\frac{\eta}{4m^2 s^2 + \eta^2}$$

that tends to $\delta(2ms)$ as $\eta \to 0$. Before taking the limit, this factor gives a well-behaved function for which the Mellin transform that defines ζ, Eq. (29), is well defined for $d/2 < \mathrm{Re}(\epsilon) < d/2 + 2$ and can be analytically continued to $\epsilon = 0$.

One may be concerned that the derivation above is only formal as it does not involve an elliptic operator. This is easily remedied by considering the elliptic operator[7] $\mathcal{M}' = \imath\eta\sqrt{-\partial_t^2} + \imath(2m)\partial_t + \nabla^2$. Its spectrum, $(2m\omega - k^2 + \imath\eta|\omega|)$, tends to that of the Minkowskian Schrödinger operator $\mathcal{M}$ as $\eta \to 0$. Consequently, the spectrum for the Euclidean avatar[8] $(\mathcal{M}'_{E,g})$ of $\mathcal{M}'$ becomes $(k^2 + 2m\imath\omega + |\eta\omega|)$ and the heat kernel for that operator is given by

$$\begin{aligned} K(s, 1, \mathcal{M}'_{E,g}) &= \mathrm{Tr}(e^{-s\mathcal{M}'_{E,g}}) \\ &= \int \left(\frac{d^d k}{(2\pi)^d} e^{-sk^2}\right)\left(\int \frac{d\omega}{2\pi} e^{-2m\imath s\omega - s|\eta\omega|}\right) \end{aligned}$$

The integral over k gives the factor of $1/s^{d/2}$ as before, while the integral over ω gives

[6]Even though $\mathcal{M}_E$ is not a hermitian operator, the heat kernel is well defined for any operator as long as $\mathrm{Re}(\lambda_k) > 0$ where λ_k are its eigenvalues. We explore this technical aspect in Appendix A.

[7]The choice of regulator is suggested naturally, as it can ultimately be linked to the Minkowski form of the propagator $G = \frac{\imath}{2m\omega - k^2 + \imath|\eta\,\omega|} \to \frac{\imath}{2m\omega - k^2 + \imath 0^+}$.

[8]Alternatively, one can think of introducing the regulator, only after going over to the Euclidean version. The unregulated Euclidean operator, $\mathcal{M}_{E,g} = 2m\partial_\tau - \nabla^2$ is regulated to $\mathcal{M}'_{E,g} = 2m\partial_\tau - \nabla^2 + \eta\sqrt{-\partial_\tau^2}$.

$$\frac{1}{\pi s}\left(\frac{\eta}{4m^2+\eta^2}\right)$$

that tends to $\frac{1}{s}\delta(2m)$ as $\eta \to 0$. As we will see later, the light cone reduction technique indeed reproduces this factor of $\delta(2m)$.

2. Derivation using LCR

In Euclidean, flat $d+2$-dimensional spacetime, the heat kernel $\mathcal{G}_{R,E}$ of a relativistic scalar field at free fixed point is given by [44]

$$\mathcal{G}_{R,E}(s;x_2^M,x_1^M) = \frac{1}{(4\pi s)^{d/2+1}} e^{-\frac{(x_1-x_2)^2}{4s}} \tag{49}$$

where the superscript reminds us that this is the relativistic case and $(x_1-x_2)^2 = (x_1^M - x_2^M)(x_1^N - x_2^N)\delta_{MN}$.

In preparation for using LCR, we rewrite the expression (49) by first reverting to Minkowski space, $t = -\imath x^0$, and then switching to light-cone coordinates.[9] Using $x^\pm = x_2^\pm - x_1^\pm$ we have:

$$\begin{aligned}&\mathcal{G}_{R,M}(s;(x_2^+,x_2^-,\boldsymbol{x}_2),(x_1^+,x_1^-,\boldsymbol{x}_1))\\&\quad = \frac{1}{(4\pi s)^{d/2+1}} e^{-\frac{x^+x^-}{2s}-\frac{|\boldsymbol{x}|^2}{4s}}\end{aligned} \tag{50}$$

where $\mathcal{G}_{R,M}$ is the heat kernel in Minkowski space. Now, in the reduced theory, the co-ordinate x^+ becomes the time coordinate t. Going to imaginary time, $t \to \tau = \imath t$, and Fourier transforming we obtain the heat kernel $\mathcal{G}_{g,E}$ for the Galilean invariant theory in Euclidean space:

$$\begin{aligned}&\mathcal{G}_{g,E}(s;(\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1)))\\&\quad = \int_{-\infty}^{\infty} \frac{1}{(4\pi s)^{d/2+1}} e^{\frac{\imath\tau x^-}{2s}-\frac{|\boldsymbol{x}|^2}{4s}} e^{-\imath m x^-} dx^-\\&\quad = 2\pi\delta\left(\frac{\tau}{2s}-m\right)\frac{1}{(4\pi s)^{d/2+1}} e^{-\frac{|\boldsymbol{x}|^2}{4s}}\end{aligned} \tag{51}$$

where $\tau = \tau_2 - \tau_1$, in detailed agreement with Eq. (46). For later use we note that in the coincidence limit we have

$$\mathcal{G}_{g,E}((\boldsymbol{x},\tau),(\boldsymbol{x},\tau))) = \frac{2\pi\delta(m)}{(4\pi s)^{d/2+1}}. \tag{52}$$

It is interesting to note that LCR directly gives $\sim\delta(m)/s^{d/2+1}$ while the direct computations gives $\sim\delta(ms)/s^{d/2}$. Our main result, below, follows from the coincidence limit of the heat kernel expansion in Eq. (57), which is useful only for $s \neq 0$, since it is used to extract the coefficients of powers of s in the expansion. The limiting behavior as $s \to 0$ of the function $\mathcal{G}_{g,E}$ is a delta function enforcing coincidence of the points, by construction (and this is why $a_0 = 1$ at coincidence), and therefore the behavior as $s \to 0$ is correct but of no significance.

The spectral dimension of the operator $\mathcal{M}_E$ is given by

$$d_{\mathcal{M}} = -\frac{d\ln(K)}{d\ln(s)} = \frac{d}{2}+1 \tag{53}$$

which explains why there can not be any trace anomaly when the spatial dimension d is odd. This has to be contrasted with the relativistic case where the spectral dimension of the laplacian operator is given by $\frac{d+1}{2}$, so that in the relativistic case the anomaly is only present when the spatial dimension d is odd.

C. Heat kernel in curved spacetime

Now that we know that LCR works in flat spacetime, we can go ahead and implement it in curved spacetime exploiting the known fact that for relativistic field theories coupled to a curved geometry, the heat kernel can be obtained as an asymptotic series. The method is explained in, *e.g.*, Refs. [16,39,44].

The method, first worked out by DeWitt [45], starts with an ansatz for the form of the heat kernel taking a cue from the form of the solution in flat spacetime for the heat equation. For small enough s, the ansatz for the heat kernel, corresponding to a relativistic theory in $d+2$ dimensions, reads

$$\begin{aligned}&\mathcal{G}_{R,E}(x_2,x_1;s) = \frac{\Delta_{\mathrm{VM}}^{1/2}(x_2,x_1)}{(4\pi s)^{d/2+1}} e^{-\sigma(x_2,x_1)/2s}\sum_{n=0}^{\infty} a_n(x_2,x_1)s^n,\\&a_0(x_1,x_2) = 1\end{aligned} \tag{54}$$

with $a_n(x_2,x_1)$ the so-called Seeley-DeWitt coefficients and where $\sigma(x_2,x_1)$ is the biscalar distance-squared measure (also known as the geodetic interval, as named by DeWitt), defined by

$$\begin{aligned}&\sigma(x_2,x_1) = \frac{1}{2}\left(\int_0^1 d\lambda\sqrt{G_{MN}\frac{dy^M}{d\lambda}\frac{dy^N}{d\lambda}}\right)^2,\\&y(0) = x_1, \qquad y(1) = x_2,\end{aligned} \tag{55}$$

with $y(\lambda)$ a geodesic. The bifunction $\Delta_{\mathrm{VM}}(x_2,x_1)$ is called the van Vleck-Morette determinant; this biscalar describes the spreading of geodesics from a point and is defined by

$$\begin{aligned}&\Delta_{\mathrm{VM}}(x_2,x_1)\\&\quad = G(x_2)^{-1/2}G(x_1)^{-1/2}\det\left(-\frac{\partial^2}{\partial x_2^M \partial x_1^{N'}}\sigma(x_2,x_1)\right),\end{aligned} \tag{56}$$

where G is the negative of determinant of metric G_{MN}.

[9] Recall, in the parent theory $x^\pm = \frac{1}{\sqrt{2}}(x^1 \pm t)$. Note that we are using a nonstandard sign convention in the definition of x^-.

Now, to implement LCR, recall that a Schrödinger invariant theory coupled to a generic curved NC structure is obtained by reducing from the $d+2$-dimensional metric G_{MN} in Eq. (19). In taking the coincident limit we must keep x_1^- and x_2^- arbitrary in order to Fourier transform with respect to x^- per the prescription (24). Therefore, we work in the coincident limit where $x_1^\mu = x_2^\mu$, with $\mu = +, 1, 2, \ldots, d$. Now, since x^- is a null direction, in this limit we have $\sigma((x_1^-, x^\mu), (x_2^-, x^\mu)) = 0$ or $[\sigma] = 0$ for brevity. Furthermore, null isometry guarantees that metric components are independent of x^- and so are $[a_n]$ and $[\Delta_{VM}]$. Thus the coincident limit is equivalent to the coincident limit of the parent theory, hence $[\Delta_{VM}] = 1$. We refer to Appendix B for details.

Thus, in the coincidence limit, we have the following expression for the heat kernel corresponding to the reduced theory:

$$\mathcal{G}_{g,E}(s; (\tau, \boldsymbol{x}), (\tau, \boldsymbol{x})) = \frac{2\pi\delta(m)}{(4\pi s)^{d/2+1}} \sum_{n=0}^{\infty} a_n((\tau, \boldsymbol{x}), (\tau, \boldsymbol{x}))s^n,$$
$$a_0((\tau_1, \boldsymbol{x}_1), (\tau_2, \boldsymbol{x}_2)) = 1, \tag{57}$$

where to define τ, we have proceeded just as in flat space: first revert to a Minkowski metric, then switch to light cone coordinates, and finally go over to imaginary x^+ time, τ. Subsequently, using Eq. (40) the anomaly is given by

$$\mathcal{A}^G_{d+1} = -4\pi\delta(m)\frac{a_{d/2+1}}{(4\pi)^{d/2+1}}. \tag{58}$$

From Eq. (57) it is clear that only the zero mode of P_- can contribute to the anomaly; the anomaly vanishes for fields with nonzero $U(1)$ charge. We already know that the anomaly for the relativistic complex scalar case is given by

$$\mathcal{A}^R_{d+2} = -\frac{2a_{d/2+1}}{(4\pi)^{d/2+1}}. \tag{59}$$

Thereby we establish the result advertised in the introduction, giving the Weyl anomaly of a $d+1$-dimensional Schrödinger invariant field theory of a single complex scalar field carrying charge m under $U(1)$ symmetry), $\mathcal{A}^G_{d+1}$, in terms of the anomaly in the relativistic theory in $d+2$ dimensions, $\mathcal{A}^R_{d+2}$:

$$\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2}, \tag{60}$$

computed on the class of metrics given in Eq. (19).

At this point, we pause to remark on the interpretation of the $\delta(m)$ factor. While it trivially shows that the anomaly is absent for $m \neq 0$, the interpretation becomes subtle when $m = 0$. The apparent divergence in the anomaly is just an artifact of the usual zero mode problem associated with null reduction. A similar issue has been pointed out in [27] in reference to [46,47]. The reduced theory in the $m \to 0$ limit becomes infrared divergent; the fields become nondynamical in that limit. The infrared divergence is also evident from Eq. (24). One may further understand the presence of $\delta(m)$ by letting m be a continuous parameter and considering a continuous set of fields ϕ_m, of charge m. The anomaly arising from the continuous set of fields is given by summing over their contributions:

$$\frac{1}{2\pi}\int dm \mathcal{A}^G_{d+1} = \mathcal{A}^R_{d+2}\int dm\delta(m) = \mathcal{A}^R_{d+2}.$$

The right hand side is exactly what we expect since allowing the parameter m to continuously vary restores the Lorentz invariance: consulting Eq. (23), we see that this continuous sum corresponds to restoring the relativistic theory of Eq. (21).

That the constant of proportionality relating $\mathcal{A}^R_{d+2}$ to $\mathcal{A}^G_{d+1}$ vanishes for $m \neq 0$ can be verified by an all-orders computation of $\mathcal{A}^G_{d+1}$, to which we now turn our attention.

V. PERTURBATIVE PROOF OF VANISHING ANOMALY

The fact that the anomaly vanishes for nonvanishing m can be shown perturbatively taking the background to be slightly curved. In flat spacetime, wave-function renormalization and coupling constant renormalization are sufficient to render a quantum field theory finite. Defining composite operators requires further renormalization. Therefore, when the model is placed on a curved background additional short distance divergences appear since the background metric can act as a source of operator insertions. To cure these divergences, new counter-terms are required that may break scaling symmetry even at a fixed point of the renormalization group flow. In this section, we will treat the background metric as a small perturbation of a flat metric so that we compute in a field theory in flat spacetime with the effect of curvature appearing as operator insertions of the perturbation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. To be specific, we will look at the vacuum bubble diagrams with external metric insertions. It turns out that all of these Feynman diagrams vanish at all orders of perturbation theory, leading to a vanishing anomaly. In fact, we will show that these anomalies vanish even away from the fixed point as long as the theory satisfies some nice properties.

Suppose we have a rotationally invariant field theory such that

(1) The theory includes only rotationally invariant ("scalar") fields.
(2) At free fixed point, the theory admits an $U(1)$ symmetry under which the scalar fields are charged.
(3) The free propagator is of the form $\frac{\imath}{2m\omega - f(|\boldsymbol{k}|) + \imath\epsilon}$, where, generically, $f(|\boldsymbol{k}|) = |\boldsymbol{k}|^z$.
(4) The interactions are perturbations about the free fixed point by operators of the form $g(\phi, \phi^*)|\phi|^2$, where g is a polynomial of the scalar field ϕ.

An elementary argument presented below shows that, under these conditions, all the vacuum bubble diagrams vanish to all orders in perturbation theory.

Before showing this, a few comments are in order. First, the argument is valid in any number of spatial dimensions. Second, assumption 4 precludes terms like $\phi^4 + (\phi^*)^4$ or $K\phi^2$ in the Lagrangian. To be precise, $F(\phi) + \mathrm{H.c.}$ can evade this theorem for any holomorphic function F of ϕ. This is because assumption 4 implies that each vertex of the Feynman diagrams of the theory has at least one incoming scalar field into it and one outgoing scalar field line from it; having both incoming and outgoing lines at each vertex is at the heart of this result. Thirdly, it should be understood that all interactions that can be generated via renormalization, that is, not symmetry protected, are to be included. For example, were we to consider a single scalar field with only the interaction $\phi^3\phi^* + \mathrm{H.c.}$, the interactions $\phi^4 + (\phi^*)^4$ and $(\phi\phi^*)^2$ will be generated along the RG flow. Nonetheless, $U(1)$ symmetry will always prohibit a holomorphic interaction $F(\phi) + \mathrm{H.c.}$ Lastly, assumption 3 can be relaxed to include a large class of functions $f(|\boldsymbol{k}|^2)$; this means one can recast this result in terms of perturbation theory along the RG-flow rather than about fixed points.

To prove this claim, notice first that a vacuum diagram is a connected graph without external legs (hanging edges). Moreover, since we are considering a complex scalar field, the vertices are connected by directed line segments. These directed segments form directed closed paths. To see this, recall that by assumption each vertex has at least one ingoing and one outgoing path. Starting from any vertex, we have at least one outgoing path. Any one of these paths must have a second vertex at its opposite end, since by assumption there are not hanging edges. Take any one outgoing path and follow it to the next vertex. Now, at this second vertex repeat this argument: follow the outward path to a third vertex. And so on. Since a finite graph has a finite number of vertices, at some point in the process we have to come back to a vertex we have already visited. For example, assume that we first revisit the ith vertex. This means that starting from the vertex i we have a directed path which loops back to the ith vertex itself. The simplest example is that of a path starting and ending on the first vertex, corresponding to a self-contraction of the elementary field in the operator insertion.

Let us call this directed loop Γ. We use the freedom in the choice of loop energy and momentum in the evaluation of the Feynman diagram to assign a loop energy ω in a way such that ω loops around Γ. In performing the integral over ω it suffices to consider the Γ subdiagram only. The resulting integration is of the form:

$$\int d\omega P(\omega,\boldsymbol{k},\{\omega_n,\boldsymbol{k}_n\})\prod_{n\in\Gamma}\frac{1}{(\omega+\omega_n-f(|\boldsymbol{k}+\boldsymbol{k}_n|)/2m+i\epsilon)} \tag{61}$$

where the product is over all vertices in Γ and correspondingly over all line segments in Γ out of these vertices. Energy ω_n and momentum $\boldsymbol{k}_n$ enter Γ at the vertex n. The factor $P(\omega,\boldsymbol{k},\{\omega_n,\boldsymbol{k}_n\})$ is polynomial in momentum and energy and may arise if there are derivative interactions. Note that every propagator factor has the same sign $i\epsilon$ prescription, that is, all poles in complex-ω lie in the lower half plane (have negative imaginary part). The integral over the real ω axis can be turned into an integral over a closed contour in the complex plane, by closing the contour on an infinite radius semicircle on the upper half plane, using the fact that for two or more propagators the integral over the semicircle at infinity vanishes. Then Cauchy's theorem gives that the integral over the closed contour vanishes as there are no poles inside the contour.

This proves the claim, except for the singular case of a self-contraction, that is, a propagator from one vertex to itself. Self-contractions can be removed by normal ordering, again giving a vanishing result. For an alternative way of seeing this note that this integral is independent of external momentum and energy, and is formally divergent in the ultraviolet (as $|\omega| \to \infty$). The integral results in a constant (independent of external momentum and energy) that must be subtracted to render it finite, and can be chosen to be subtracted completely, to give a vanishing result.

The computation in the case of anticommuting fields differs only in that a factor of -1 is introduced for each closed fermionic loop. Hence the claim applies equally to the case of anticommuting scalar fields.

We now return to the derivation of our main result, Eq. (4). The conditions above are satisfied for the theories considered in Sec. IV C, namely, free theories of complex scalars, with the free propagator given by $\frac{\imath}{2m\omega-|\boldsymbol{k}|^2+\imath 0^+}$. Recall that we are to put the theory on a curved background which is assumed to be a small perturbation from flat background. The perturbations act as insertions on vacuum bubble diagrams, but since they preserve the $U(1)$ symmetry the model still satisfies the assumptions above. Hence all the bubble diagrams vanish, and we conclude there are no divergences coming from metric insertion on bubble diagrams. Consequently, there is no scale anomaly. We emphasize that the absence of the Weyl anomaly is valid in all orders of perturbation in both the coupling and the metric. The result holds true even if we make the couplings to be spacetime dependent so that every coupling insertion injects additional momentum and energy to the bubble diagram. Physically, the anomaly vanishes because the absence of antiparticles in nonrelativistic field theories and the conservation of $U(1)$ charge forbid pair creation, necessary for vacuum fluctuations that may give rise to the anomaly.

This perturbative proof holds for theories which need not be Galilean invariant, and the question arises as to whether one may use LCR to make statements about anomalies for theories with kinetic term involving one time derivative and

$z \neq 2$. We will take up this task in following section, starting by giving the promised proof of our prescription in Eq. (24).

We remark that perturbative proof works for $m \neq 0$. For $m = 0$, the integrand becomes independent of ω, and one can not perform the contour integral to argue the diagrams vanish. In fact, the integral over ω is divergent, as expected from our earlier expectation that at $m = 0$ one encounters IR divergences. One way to see the presence of $\delta(m)$, as explained earlier, is to take a continuous set of fields ϕ_m, labelled by continuous parameter m. If we exchange the sum over (1-loop) bubble diagrams and the integral over m, then each of the propagator can be thought of as a relativistic propagator with m, playing the role of p_-. Thus the whole calculation formally becomes that of the relativistic anomaly.

One can verify our results by explicit calculation in specific cases. In a slightly curved spacetime, one can treat the deviation from flatness as background field sources. This also serves the purpose of checking that the η-regularization is appropriate, obtaining the anomaly as a function of η. Since, as $\eta \to 0$, for $m \neq 0$, the flat space heat kernel vanishes, one expects the anomaly to be vanishing. In fact, one can check that a $\delta(m)$ is recovered as $\eta \to 0$. We refer to the App. C for an explicit calculation; it verifies our results in detail, and shows the vanishing anomaly regardless of the order of limits $\eta \to 0$ and $m \to 0$.

VI. MODIFIED LCR AND GENERALIZATION

A. Proving the heat kernel prescription

In this subsection we will explain why our proposed method to determine the heat kernel for Schrödinger field theory ($z = 2$) worked in a perfect manner, as evidenced by the agreement between Eqs. (46) and (51). We will see that one can use LCR to relate the heat kernel of a theory living in $d+1$ dimensions with that of a parent theory living in $d+2$ dimensions, as long as the parent theory has $SO(1,1)$ invariance.[10] Furthermore, if the parent theory has a dynamical scaling exponent given by z, then the theory living in $d+1$ dimension has $2z$ as its dynamical exponent. We will make these statements precise in what follows.

Suppose the operator D defined in $d+2$-dimensional spacetime is diagonal in the eigenbasis of P_-, the conjugate momenta to x^-:

$$\langle x_2^+, x_2^i, m_2 | D | x_1^+, x_1^i, m_1 \rangle = \langle x_2^+, x_2^i | D_{m_2} | x_1^+, x_1^i \rangle \delta(m_2 - m_1), \tag{62}$$

where $m_{1,2}$ label the eigenvalues of P_-. The example worked out in Sec. IV B had $D = \mathcal{M}$, and it does satisfy this requirement. It follows that

$$\begin{aligned}
&\langle x_2^+, x_2^i, x_2^- | e^{-sD} | x_1^+, x_1^i, x_1^- \rangle \\
&= \frac{1}{2\pi} \int dm_1 dm_2 e^{-\imath m_1 x_1^- + \imath m_2 x_2^-} \langle x_2^+, x_2^i, m_2 | e^{-sD} | x_1^+, x_1^i, m_1 \rangle \\
&= \frac{1}{2\pi} \int dm_1 e^{\imath m_1 x_{\bar{1}2}^-} \langle x_2^+, x_2^i | e^{-sD_{m_1}} | x_1^+, x_1^i \rangle,
\end{aligned} \tag{63}$$

from which we obtain

$$\begin{aligned}
&\langle x_2^+, x_2^i | e^{-sD_m} | x_1^+, x_1^i \rangle \\
&\quad = \int dx^- e^{-\imath m x_{\bar{1}2}^-} \langle x_2^+, x_2^i, x_2^- | e^{-sD} | x_1^+, x_1^i, x_1^- \rangle.
\end{aligned} \tag{64}$$

This is precisely the prescription we gave in Eq. (24).

B. Generalization

Since the LCR (or DLCQ) trick requires null cone reduction, it may seem necessary that the parent theory have $SO(d+1,1)$ symmetry, and that this will result necessarily in a Galilean invariant reduced theory, that is, with $z = 2$. This is not quite right: one may relax the condition of $SO(d+1,1)$ symmetry and obtain reduced theories with $z \neq 2$. The key observation is that for null cone reduction only two null coordinates are needed, with the rest of the coordinates playing no role. Hence, we consider null cone reduction of a $d+2$-dimensional theory which enjoys $SO(1,1) \times SO(d)$ symmetry. The reduced theory will be a $d+1$-dimensional theory with $SO(d)$ rotational symmetry and a residual $U(1)$ symmetry that arises from the null reduction. The point is that the theory can enjoy anisotropic scaling symmetry. Consider, for example, the following class of operators,

$$\mathcal{M}_{rc;d+2} = (-\partial_t^2 + \partial_x^2) - (-1)^{z/2} (\partial_i \partial^i)^{z/2}, \tag{65}$$

where $t = x^0$ and $x = x^{d+1}$ and for the reminder of this section there is an implicit sum over repeated latin indices, over the range $i = 1, \ldots, d$. These operators transform homogeneously under

$$x^i \to \lambda x^i, \qquad t \to \lambda^{z/2} t \quad \text{and} \quad x \to \lambda^{z/2} x. \tag{66}$$

Introducing null coordinates as before, $x^\pm = \frac{1}{\sqrt{2}}(x \pm t)$, null reduction of this operator yields

$$\mathcal{M}_{gc;d+1} = 2\imath m \partial_{t'} - (-1)^{z/2} (\partial_i \partial_i)^{z/2}, \tag{67}$$

where $t' = x^+$ is the time coordinate of the reduced theory. From the dispersion relation of the reduced theory, $2m\omega = |\boldsymbol{k}|^z$, we read off that the dynamical exponent is z. Here we are interested in even z to insure that the operator $\mathcal{M}_{gc;d+1}$ is local. For $z = 2$, we recover the case discussed in earlier sections with the parent theory being Lorentz

[10] One may as well assume that both parent and reduced theories have, in addition, $SO(d)$ rotational symmetry.

invariant and the reduced theory being Schrödinger invariant.

Following the prescription (64), we can relate the matrix element of the heat kernel operator for $\mathcal{M}_{r;d+2}$ to that of $\mathcal{M}_{g;d+1}$, via[11]

$$\mathcal{G}_{\mathcal{M}_{gc;d+1}} = \int_{-\infty}^{\infty} dx^- e^{-\imath m x^-} \langle x_0^- + x^- | \mathcal{G}_{\mathcal{M}_{rc;d+2}} | x_0^- \rangle. \quad (68)$$

This should be viewed as an operator relation: thinking of the basis on which the operator $\mathcal{G}_{\mathcal{M}_{r;d+2}}$ acts as given by the tensor product of $|x^+\rangle$, $|x^-\rangle$ and $|x^i\rangle$ for $i = 1, 2, \ldots, d$, then $\langle x_0^- + x^- | \mathcal{G}_{\mathcal{M}_{r;d+2}} | x_0^- \rangle$ is an operator acting on the complement of the space spanned by $|x^-\rangle$. Taking the trace on both sides of Eq. (68), we obtain the heat kernel of the reduced theory:

$$K_{\mathcal{M}_{gc;d+1}} = \int_{-\infty}^{\infty} dx^- e^{-\imath m x^-} \mathrm{Tr}_{x^+, x^i} \langle x_0^- + x^- | \mathcal{G}_{\mathcal{M}_{rc;d+2}} | x_0^- \rangle \quad (69)$$

Equations (68) or (69) are useful in practice only when we know either left or right hand sides by some other means. Hence, the next meaningful question to be asked is whether we can calculate $\mathcal{G}_{\mathcal{M}_r}$ explicitly for a curved spacetime for any z. The case for $z = 2$, that in which the parent theory is relativistic and the reduced theory is Schrödinger invariant, is well known and was presented in Sec. IV B. For generic z, the answer is yes to some extent. We will find a closed form expression when the slice of constant (t, x) in spacetime is described by a metric that does not depend on t or x:

$$ds^2 = -dt^2 + (dx)^2 + h_{ij}(x^i) dx^i dx^j \quad (70)$$

With this choice, the heat kernel equation for the curved background version of the operator $\mathcal{M}_{rc;d+2}$ of Eq. (65) admits a solution by separation of variables, into the product of the relativistic heat kernel in $1+1$ dimensions and the heat kernel for an operator acting only on the d-dimensional slice [32]. Specifically, we consider operators

$$\mathcal{M}_{rc;d+2} = \nabla_{t,x}^2 - D^{z/2} \quad (71)$$

where $\nabla_{t,x}^2 = (-\partial_t^2 + \partial_x^2)$ and D is a second order scalar differential operator on the slice of constant (t, x), *e.g.*, $D = -\nabla^2 = -1/\sqrt{h}\partial_i \sqrt{h} h^{ij} \partial_j$. With these choices,

$$\mathcal{G}_{\mathcal{M}_{rc;d+2}} = \mathcal{G}_{\nabla_{t,x}^2} \mathcal{G}_{D^{z/2}}. \quad (72)$$

Gilkey has shown that the heat kernel expansion for D^k can be computed from that for D [43] for $k > 0$. The argument is based on the observation that the ζ-functions for the two operators are related:

$$\zeta(\epsilon, f, D^k) = \mathrm{Tr}_{L^2}(f(D^k)^{-\epsilon}) = \mathrm{Tr}_{L^2}(f D^{-k\epsilon}) = \zeta(k\epsilon, f, D).$$

Gilkey's result is as follows: If D has heat kernel expansion

$$K_D = \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{n\geq 0} s^{n-\frac{d}{2}} a_n^{(d)} \quad (73)$$

then the heat kernel expansion of D^k is

$$\begin{aligned} K_{D^k} &= \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{n\geq 0} s^{\frac{2n-d}{2k}} \frac{\Gamma(\frac{d-2n}{2k})}{k\Gamma(\frac{d}{2}-n)} a_n^{(d)} \\ &= \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{\substack{n\geq 0 \\ 2n\neq d (\mathrm{mod}\, 2k)}} s^{\frac{2n-d}{2k}} \frac{\Gamma(\frac{d-2n}{2k})}{k\Gamma(\frac{d}{2}-n)} a_n^{(d)} \\ &\quad + \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{\substack{n\geq 0 \\ 2n=d (\mathrm{mod}\, 2k)}} s^{\frac{2n-d}{2k}} (-1)^{\frac{(2n-d)(1-k)}{2k}} a_n^{(d)} \end{aligned} \quad (74)$$

Hence, $\mathcal{M}_{rc;d+2} = (-\partial_t^2 + \partial_x^2) - (-\nabla^2)^{z/2}$ has heat kernel expansion

$$\begin{aligned} &\langle x_2^+, x_2^-, x^i | \mathcal{G}_{\mathcal{M}_{rc;d+2}} | x_1^+, x_1^-, x^i \rangle \\ &= \frac{e^{\frac{-x_{12}^+ x_{12}^-}{2s}}}{4\pi s} \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{n\geq 0} s^{\frac{2n-d}{z}} \frac{\Gamma(\frac{d-2n}{z})}{\frac{z}{2}\Gamma(\frac{d}{2}-n)} a_n^{(d)} \end{aligned} \quad (75)$$

where $x_{12}^{\pm} = x_2^{\pm} - x_1^{\pm}$ and $a_n^{(d)}$ are the well-known coefficients of the heat kernel expansion of $-\nabla^2$.

Now, the reduced theory lives on $d+1$-dimensional spacetime with curved spatial slice; *i.e.*, the background metric is given by

$$ds^2 = -dt^2 + h_{ij} dx^i dx^j, \quad (76)$$

where i runs from 1 to d. In order to extract the heat kernel of $\mathcal{M}_{gc;d+1} = 2\imath m \partial_t + (-\nabla^2)^{z/2}$, we need partial tracing of heat kernel of $\mathcal{M}_{rc;d+2}$,

$$\begin{aligned} &\langle x_0^- + x^- | \mathrm{Tr}_{x^+, x^i} \mathcal{G}_{\mathcal{M}_{rc;d+2}} | x_0^- \rangle \\ &= \left(\frac{1}{\sqrt{4\pi}}\right)^d \frac{1}{4\pi s} \sum_{n\geq 0} s^{\frac{2n-d}{z}} \frac{\Gamma(\frac{d-2n}{z})}{\frac{z}{2}\Gamma(\frac{d}{2}-n)} a_n^{(d)}, \end{aligned} \quad (77)$$

leading to

$$K_{\mathcal{M}_{gc;d+1}} = 2\pi\delta(m) \frac{1}{4\pi s} \left(\frac{1}{\sqrt{4\pi}}\right)^d \sum_{n\geq 0} s^{\frac{2n-d}{z}} \frac{\Gamma(\frac{d-2n}{z})}{\frac{z}{2}\Gamma(\frac{d}{2}-n)} a_n^{(d)}. \quad (78)$$

Adding conformal coupling modifies $a_n^{(d)}$ but the prefactor stays $2\pi\delta(m)\frac{1}{4\pi s}(\frac{1}{\sqrt{4\pi}})^d$. Hence, we have the generalized result,

$$\mathcal{A}_{d+1}^g = 2\pi\delta(m)\mathcal{A}_{d+2}^r, \quad (79)$$

[11]Provided these heat kernels are well defined. We save this technical aspect for Appendix A.

where $\mathcal{A}^g_{d+1}$ is the Weyl anomaly of a theory of a single complex scalar field of charge m under a $U(1)$ symmetry living in $d+1$ dimensions with dynamical exponent z and $\mathcal{A}^r_{d+2}$ is the Weyl anomaly of a field theory living in $d+2$ dimension such that it admits a symmetry under $t\to\lambda^{z/2}t$, $x^{d+2}\to\lambda^{z/2}x^{d+2}$ and $x^i\to\lambda x^i$ for $i=1,\ldots,d+1$. Thus we have shown that theories with one time derivative on a time independent curved background do not have any Weyl anomalies. This is consistent with the perturbative result obtained previously.

It deserves mention that the operator $\mathcal{M}_{rc;d+2}$ of Eq. (71) does not transform homogeneously under Weyl transformations. In order to construct a Weyl covariant operator consider generalizing the metric (70) to the following form

$$ds^2 = N dx^+ dx^- + h_{ij}dx^i dx^j. \tag{80}$$

If N is independent of x^- the metric for the reduced theory will include a general lapse function N. Then we replace $(\nabla^2)^{\frac{z}{2}}$ by $\mathcal{O}^{(d+2z-4)}\mathcal{O}^{(d+2z-8)}\ldots\mathcal{O}^{(d+4)}\mathcal{O}^{(d)}$ with $\mathcal{O}^{(p)}$ defined as

$$\mathcal{O}^{(p)} \equiv \nabla^2 - \frac{p}{4(d-1)}R + \frac{2+p-d}{z}\frac{\partial_i N}{N}h^{ij}\partial_j + \frac{d}{4z^2}(2+p-d)\frac{\partial_i N}{N}h^{ij}\frac{\partial_j N}{N} \tag{81}$$

Under $h_{ij}\to e^{2\sigma}h_{ij}$, $N\to e^{z\sigma}N$ and $\psi\to e^{-\frac{p}{2}\sigma}\psi$, this operator transforms covariantly, in the sense that

$$\mathcal{O}^{(p)}\psi \to e^{-(\frac{p}{2}+2)\sigma}\mathcal{O}^{(p)}\psi. \tag{82}$$

Therefore, under the Weyl rescaling $h_{ij}\to e^{2\sigma}h_{ij}$, $N\to e^{z\sigma}N$ and $\phi\to e^{-\frac{d}{2}\sigma}\phi$ we have that

$$N\sqrt{h}\phi^*\mathcal{O}^{(d+2z-4)}\mathcal{O}^{(d+2z-8)}\ldots\mathcal{O}^{(d+4)}\mathcal{O}^{(d)}\phi \tag{83}$$

is invariant under Weyl transformations.

Adding the conformal coupling will modify the expressions for $a_n^{(d)}$, but scaling with respect to s will remain unmodified. Hence we can enquire about existence or absence of potential Weyl anomalies. To have a nonvanishing Weyl anomaly, we need to have an s independent term in the heat kernel expansion. This is possible only when $\frac{2n-d}{z}=1$, *i.e.*, when $d+z$ is even; see Eqs. (75) and (78). Since for a local Lagrangian z must be even, this condition corresponds to even d[12] This is expected because of the following reason: the scalars we can construct out of geometrical data (that can potentially appear as a trace anomaly) have even dimensions and the volume element scales like λ^{d+z}, so that in order to form a scale invariant quantity $d+z$ has to be even. Now when d is even, we have s independence for $n=(d+z)/2$ and the coefficient of s^0 is given by $(\frac{1}{\sqrt{4\pi}})^d(-1)^{1-\frac{z}{2}}a^d_{\frac{d+z}{2}}$. Hence, the result relating anomalies in the parent and reduced theory, Eq. (79), still holds.

[12]Giving up on the requirement of locality allows z to be any positive real number. In this case, the anomaly is expected to be present whenever $d+z$ is even. It might be of potential interest to look at these cases carefully and make sure that nonlocality does not provide any obstruction in the anomaly calculation and that the renormalization process can be done in a consistent manner.

VII. SUMMARY, DISCUSSION AND FUTURE DIRECTIONS

We have shown that for a $d+1$-dimensional Schrödinger invariant field theory of a single complex scalar field carrying charge m under $U(1)$ symmetry, the Weyl anomaly, $\mathcal{A}^G_{d+1}$, is given in terms of that of a relativistic free scalar field living in $d+2$ dimensions, $\mathcal{A}^R_{d+2}$, via

$$\mathcal{A}^G_{d+1} = 2\pi\delta(m)\mathcal{A}^R_{d+2}. \tag{84}$$

Here the parent $d+2$ theory lives in a spacetime with null isometry generated by the Killing vector ∂_- so that the metric can be given in terms of a $d+1$-dimensional Newton-Cartan structure. The result is shown to be generalized to

$$\mathcal{A}^g_{d+1} = 2\pi\delta(m)\mathcal{A}^r_{d+2}, \tag{85}$$

where $\mathcal{A}^g_{d+1}$ is the Weyl anomaly of a theory of a single complex scalar field of charge m under an $U(1)$ symmetry living in $d+1$ dimensions with dynamical exponent z, while $\mathcal{A}^r_{d+2}$ is the Weyl anomaly of an $SO(1,1)\times SO(d)$ invariant theory living in $d+2$ dimensions such that it admits symmetry under $t\to\lambda^{z/2}t$, $x^{d+2}\to\lambda^{z/2}x^{d+2}$ and $x^i\to\lambda x^i$ for $i=1,\ldots,d+1$.

To obtain information regarding the anomaly, we introduced a method to systematically handle the heat kernel for a theory with kinetic term involving one time derivative only. We provided crosschecks and consistency checks on our heat kernel prescription. One may worry that to properly define a heat kernel the square of the derivative operator must be considered. This would also be the case for, say, the Dirac operator. In fact, one can properly define it this way; see, for example, Ref. [48].

The result obtained regarding the anomaly of Schrödinger field theory is consistent with the one by Jensen [27]. Auzzi *et al*, [49] have studied the anomaly for a Euclidean operator given by

$$\mathcal{M}'_{E,g} = 2m\sqrt{-\partial_t^2} - \nabla^2, \tag{86}$$

with eigenspectra given by $|\boldsymbol{k}|^2 + 2m|\omega| \geq 0$. One can define the heat kernel for this operator as well, but the eigenspectra of this operator is not analytically related to that of $\mathcal{M}_{M,g} = 2\imath m\partial_t + \nabla^2$, which is $-k^2 + 2m\omega$. As a result the propagator in ω-$\boldsymbol{k}$ space has a cut on the complex ω plane with branch point at the origin, making the analytic continuation to Minkowski space problematic. It is known that the two point correlator of Schrödinger field theory is constrained and has a particular form as elucidated in Ref. [20,50]. While our prescription and the resulting Euclidean correlator conforms to that form, it is not clear how the Euclidean Schrödinger operator defined in Ref. [49] does, if at all. Finally, we note that the operator $\sqrt{-\partial_t^2}$ is nonlocal (in the sense that the kernel, defined by $\sqrt{-\partial_t^2}f(t) = \int dt' K(t-t')f(t')$, has nonlocal support, $K(t) = 2\partial_t \mathrm{P}\frac{1}{t}$).

There are several avenues of investigation suggested by this work:

(1) What happens in the case of several scalar fields with different charge interacting with each other while preserving Schrödinger invariance in flat spacetime? How is the pre-factor $\delta(m)$ modified?

(2) It is not obvious how null reduction of a theory of a Dirac spinor in $d+2$ dimensions can result in a Lagrangian in $d+1$ dimensions of the form $\mathcal{L} = 2\imath m\psi^\dagger\partial_t\psi + \psi^\dagger\nabla^2\psi$, let alone one with $\mathcal{L} = 2\imath m\psi^\dagger\partial_t\psi - \psi^\dagger(-\nabla^2)^{z/2}\psi$ for $z \neq 2$. On the other hand, as we have seen, the functional integral over nonrelativistic anticommuting fields yields the same determinant as that of commuting fields (only a positive power). Hence, the anomaly of the anticommuting field is the negative of that of the commuting field.

(3) Calculations using the same Euclidean operator as in Ref. [49] give a nonvanishing entanglement entropy in the ground state [51]. By contrast, for the operator $\mathcal{M}_{M,g} = 2\imath m\partial_t + \nabla^2$, the entanglement entropy in the ground state vanishes, since for this local nonrelativistic field theory $\phi(x)|0\rangle = 0$ and hence the ground state is a product state. It would be of interest to verify this result by direct computation using a method based on our prescription.

(4) The method described in Sec. VI B to compute Weyl anomalies in theories with $z \neq 2$ is not sufficiently general in that, by assuming the metric is time independent and has constant lapse, it neglects anomalies involving extrinsic curvature or gradients of the lapse function. A future challenge is to develop a more general computational method.

We hope to come back to these questions in the future.

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APPENDIX A: TECHNICAL ASPECTS OF HEAT KERNEL FOR ONE TIME DERIVATIVE THEORY

Here is one more perspective on why $\delta(m)$ appears in the heat kernel for one-time derivative theory using the eigenspectra of the operator $\mathcal{M}_g$ with one time derivative. The Minkowski $\mathcal{M}_{M,g}$ operator is given by

$$\mathcal{M}_{M,g} = 2\imath m\partial_t - (-\nabla^2)^{z/2} \tag{A1}$$

and the eigenspectra is given by $2m\omega - k^z$. Now, we can not directly define the heat kernel since the eigenvalues range from $-\infty$ to ∞, and therefore it blows up. A similar situation also arises in relativistic theory where the eigenspectra is given by $-\omega^2 + k^2$. There we define the heat kernel by Euclideanizing the time coordinate so that the eigenvalues become $\omega^2 + k^2 \geq 0$ and this positive definiteness allows for convergence. Technically, we can always define heat kernel for an operator M as long as the eigenvalues of M have positive real part. Building up on our experience to deal with the relativistic case, we use analytic continuation here as well. We define the Euclidean operator as

$$\mathcal{M}_{E,g} = 2m\partial_\tau + (-\nabla^2)^{z/2} \tag{A2}$$

with eigenspectra given by $\lambda_{k,\omega} = -2\imath m\omega + k^z$. Evidently, $\mathrm{Re}(\lambda_{k,\omega}) \geq 0$; hence, we have a well-defined heat kernel, given by

$$\begin{aligned} K_{\mathcal{M}_{E,g}} &= \mathrm{Tr}e^{-s\mathcal{M}_{E,g}} = \int \frac{d^dk}{(2\pi)^d} e^{-sk^z} \int \frac{d\omega}{2\pi} e^{-2m\imath s\omega} \\ &= \frac{\delta(m)}{2s} \frac{2}{\Gamma(\frac{d}{2})} \frac{\Gamma(\frac{d}{z}+1)}{d(\sqrt{4\pi s^{\frac{2}{z}}})^d} \end{aligned} \tag{A3}$$

Similarly, the Euclidean heat kernel is well defined for the operator $\mathcal{M}_{rc;d+2} = \nabla^2_{t,x} - (-\nabla^2_{x^i})^{z/2}$, where $i = 1, 2, \ldots d$ and $x \equiv x^{d+2}$. If we Wick rotate to Euclidean time τ, the eigenvalues of the operator $\mathcal{M}_{rc;d+2}$ are given by $\omega^2 + (k^{d+2})^2 + (|\boldsymbol{k}|^2)^{z/2} \geq 0$. The presence of $\delta(m)$ can more formally be treated with an extra regulizer η, as discussed in the last few paragraphs of IV B 1 for $z = 2$; a similar argument, using the regulator η, applies to any z.

APPENDIX B: RIEMANN NORMAL COORDINATE AND COINCIDENT LIMIT

In this appendix, we show x^- independence of quantities relevant to the computation of the coincidence limit of the heat kernel when the light cone reduction technique is used. We assume that the daughter theory is coupled to a Newton Cartan structure, satisfying the Frobenius condition; *i.e.*, $\boldsymbol{n} \wedge d\boldsymbol{n} = 0$ is satisfied. This condition allows a foliation of the manifold globally. Thus, without loss of generality, the metric is given by

$$g_{\mu\nu} = n_\mu n_\nu + h_{\mu\nu}$$
$$n_\mu = (n, 0, 0, \ldots, 0), \qquad h_{\tau\nu} = 0. \tag{B1}$$

Using (9) and the fact h_{ij} is a positive definite matrix, we thus have

$$h^{\tau\nu} = 0, \qquad v^\mu = \left(\frac{1}{n}, 0, 0, \ldots, 0\right). \tag{B2}$$

The form of the metric, to which the reduced theory is coupled, corresponds to a parent spacetime metric G_{MN}, with nonvanishing components given by

$$G_{-+} = n, \qquad G_{ij} = h_{ij}. \tag{B3}$$

In addition, we assume that the parent spacetime admits a null isometry so that h_{ij} and n are independent of x^-.

In what follows, we will work with this particular choice of metric G_{MN} (B3). Without loss of generality, we choose $x_1 = (0, 0, \ldots, 0)$ (we call it point P) and construct the Riemann normal coordinate with the origin as the base point. The Riemann normal coordinate y^M, is given in terms of the original co-ordinate x^M as follows [52]:

$$y^M = x^M + f^M_{AB} x^A x^B + f^M_{ABC} x^A x^B x^C + \cdots, \tag{B4}$$

where the index M runs over $+, -, 1, 2, 3, \ldots, d$. In the coincident limit of the reduced theory, *i.e.*, $x_2^\mu \to 0$, for $\mu = +, 1, 2, \ldots, d$ (with x_2^- possibly different from 0), we claim that

$$[y_2^\mu] = 0, \qquad [y_2^-] = x_2^-, \tag{B5}$$

where, henceforth, the square bracket is used to denote the coincident limit in the reduced theory.

We note that $[f^M_{ABC\ldots} x^A x^B x^C \cdots] = 0$ whenever any of the indices is not $-$. Recall that $f^M_{ABC\cdots}$ are constructed out of derivatives acting on metric. Thus, $f^M_{\underbrace{- - \cdots -}_{N\,\text{indices}}}$ can be nonzero only if it contains N factors of the metric tensor G_{-K_i}, where K_i is a running index with $i = 1, 2, \ldots, N$. This is because $G_{--} = 0$ and derivatives can not carry the "$-$" index as the metric components are x^--independent. Moreover, by dimensional analysis $f^M_{\underbrace{- - \cdots -}_{N}}$ has $N-1$ derivatives $f^M_{\underbrace{- - \cdots -}_{N}}$. Schematically, this assumes one of the following forms

$$\partial_{A_1} \cdots \partial_{A_{N-1}} G_{-K_1} \cdots G_{-K_N} G^{MA_i} G^{A_{i_1} K_{j_1}} G^{A_{i_2} A_{j_2}} \cdots G^{K_{i_3} K_{j_3}} \cdots, \tag{B6}$$

$$\partial_{A_1} \cdots \partial_{A_{N-1}} G_{-K_1} \cdots G_{-K_N} G^{MK_i} G^{A_{i_1} K_{j_1}} G^{A_{i_2} A_{j_2}} \cdots G^{K_{i_3} K_{j_3}} \cdots. \tag{B7}$$

Here the derivatives are assumed to act on all possible combinations, resulting in different possible terms. For example, for $N = 2$, one can have the following terms:

$$G^{MA_1} G^{K_1 K_2} G_{-K_2} \partial_{A_1} G_{-K_1},$$
$$G^{MK_2} G^{A_1 K_1} G_{-K_1} \partial_{A_1} G_{-K_2},$$
$$G^{MK_2} G^{A_1 K_1} G_{-K_2} \partial_{A_1} G_{-K_1}. \tag{B8}$$

There can not be any x^- derivative for a term to be nonvanishing. This implies the indices A_i are contracted among themselves, except possibly for one contracted with G^{MA_i}, and the indices K_i are contracted among themselves. But since $G_{-K} = 0$ except for G_{-+}, and $G^{++} = 0$, any term for which two factors of the metric tensor, $G_{-K_{i_1}}$ and $G_{-K_{i_2}}$, are contracted via $G^{K_{i_1} K_{i_2}}$ vanish.

Next, we show that $[\Delta_{VM}] = 1$. The expression for Δ_{VM}, Eq, (56), involves bi-derivatives of the geodetic interval, Eq. (55), and the determinant of the metric. To begin with, we turn our attention to the determinant of the metric and note that

$$[G'(y_2)] = J^2(0, x_2^-, 0, \ldots, 0) G(0, x_2^-, 0, \ldots, 0), \tag{B9}$$

where a prime indicates quantities in Riemann normal coordinate and J is the Jacobian associated with the coordinate transformation (B4). The x^- independence in the original coordinate guarantees that $G(0, x_2^-, 0, \ldots, 0) = G(0, 0, 0, \ldots, 0)$, hence we have

$$[G'(y_2)] = \left(\frac{J(0, x_2^-, 0, \ldots, 0)}{J(0, 0, 0, \ldots, 0)}\right)^2 G'(0). \tag{B10}$$

Next consider the geodetic interval from point P to point Q. In Riemann normal coordinates [52]

$$y_2^M = y^M(Q) = y_1^M + s_Q \left.\frac{dx^M}{ds}\right|_{s=0}, \tag{B11}$$

where s_Q is the value of the affine parameter at Q and $s = 0$ at P, with $y_1^M = y^M(P)$. Using Eq. (55), hence we have

$$\begin{aligned} 2\sigma(y_2, y_1) &= G_{MN}(0)(y_2^M - y_1^M)(y_2^N - y_1^N) \\ &= G'_{MN}(0)(y_2^M - y_1^M)(y_2^N - y_1^N) \end{aligned} \tag{B12}$$

where we have used $G'_{MN}(0) = G_{MN}(0)$. It follows that

$$\Delta_{VM} = \left(\frac{G'(y_2)}{G'(0)}\right)^{-1/2}. \tag{B13}$$

We have continued back to Minkowskian signature (the definition in Eq. (56) is for metric with Euclidean signature). Since Δ_{VM} is a biscalar, use of Eqs. (B10) and (B13) and of $J(0, 0, 0, \ldots, 0) = 1$ gives

$$[\Delta_{VM}] = \left(\frac{J(0,x_2^-,0,\ldots,0)}{J(0,0,0,\ldots,0)}\right)^{-1} = J^{-1}(0,x_2^-,0,\ldots,0) \tag{B14}$$

in the original coordinate system, x^M. Equation (B14) is consistent with the result that $\Delta_{VM} = 1$ when all the coordinates, including x^-, coincide, *i.e.*, when $x_2^- = 0$.

We aim to show that

$$\left[\det\left(\frac{\partial y^M}{\partial x^N}\right)\right] = \det\left(\left[\frac{\partial y^M}{\partial x^N}\right]\right) = 1 \tag{B15}$$

From Eq. (B4) we have

$$\left[\frac{\partial y^M}{\partial x^N}\right] = \delta^M_N + (f^M_{N-} + f^M_{-N})x^- + (f^M_{N--} + f^M_{-N-} + f^M_{--N})x^-x^- + \cdots \tag{B16}$$

Consider first the lowest two terms in the expansion. Explicitly, we have [52]

$$2f^M_{N-} = 2f^M_{-N} = \Gamma^M_{N-} = -\frac{1}{2}G^{Mi}\partial_i G_{N-} - \frac{1}{2}G^{M+}\partial_+ G_{N-} + \frac{1}{2}G^{M+}\partial_N G_{+-}. \tag{B17}$$

It follows that $f^M_{N-} \neq 0$ only for $M = -$ or $N = +$. Similarly, $f^M_{(N--)} \neq 0$ provided $M = -$ or $N = +$, since [52]

$$6f^M_{NIJ} = \Gamma^M_{NE}\Gamma^E_{IJ} + \partial_N \Gamma^M_{IJ} \tag{B18}$$

By an argument analogous to that below Eqs. (B8) one can show that $[f^M_{N---\ldots}] = 0$ (at least three $-$ subscripts). Schematically

$$\left[\left(\frac{\partial y^M}{\partial x^N}\right)\right] = \begin{pmatrix} 1 & * & * & \ldots & \ldots & * \\ 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & * & 0 & 0 & 1 & 0 \end{pmatrix}$$

where a "$*$" means a nonzero entry. Thus, the matrix has unit determinant and we have, using Eq. (B14),

$$[\Delta_{VM}] = 1. \tag{B19}$$

Lastly, we turn to the heat kernel expansion coefficients, a_n. They are determined by the recursive relation [44],

$$na_n + \partial_M\sigma\partial^M a_n = -\Delta_{VM}^{-1/2}\mathcal{M}(\Delta_{VM}^{1/2}a_{n-1}), \tag{B20}$$

and $a_0 = 1$, where $\mathcal{M}$ is the relativistic operator in the parent theory. The condition of x^- independence of $[a_n]$, $[\partial_i a_n]$ and $[\partial_i\partial_j a_n]$ can be imposed on the recursion self-consistently. To show this one uses x^- independence of $[\Delta_{VM}]$, $[\partial_i\Delta_{VM}]$ and $[\partial_i\partial_j\Delta_{VM}]$, which follows from an argument similar to the one used to establish Eq. (B19).

APPENDIX C: EXPLICIT PERTURBATIVE CALCULATION OF THE η-REGULATED HEAT KERNEL

In this appendix, we give an explicit perturbative computation that shows the vanishing of the anomaly for a class of curved backgrounds. This serves to verify the general arguments presented in the body of the manuscript in a specific, simple example, and allows us to study explicitly the η regulated heat kernel asking in particular whether the $\eta \to 0$ limit is a well defined limit as $m \neq 0$. To be specific, we compute the heat kernel on a curved background, characterized by

$$n_\mu = \left(\frac{1}{1-n(x)},0,0\right), \qquad v^\mu = (1-n(x),0,0) \tag{C1}$$

$$h_{ij} = \delta_{ij}, \qquad \sqrt{g} = \sqrt{\det(n_\nu n_\nu + h_\mu h_\nu)} = \frac{1}{1-n(x)}. \tag{C2}$$

where $n(x)$ is a function of space only and $h_{i0} = 0$. The special choice is inspired by [49] and additionally serves the purpose of affording a direct comparison with that work. We will perform a perturbative calculation as an expansion in $n(x)$. We will specialize to a $2+1$-dimensional Schrödinger field theory coupled to this background. The action is given by

$$S = \int dt d^2x N\left(2m\phi^\dagger\imath\frac{1}{N}\partial_t\phi - h^{ij}\partial_i\phi^\dagger\partial_j\phi - \xi R\phi^\dagger\phi\right), \tag{C3}$$

where $N(x) = \frac{1}{1-n(x)}$ and R is the Ricci scalar of the $3+1$-dimensional geometry, on which the parent theory lives.

As we will see, the result of this calculation is that the Weyl anomaly, corresponding to the theory described by Eq. (C3) is given by

$$\mathcal{A}_G = 2\pi\delta(m)(-aE_4 + cW^2 + bR^2 + dD_M D^M R) \tag{C4}$$

where the coefficients a, b, c, d are given by

$$a = \frac{1}{8\pi^2}\frac{1}{360}, \qquad b = \frac{1}{8\pi^2}\frac{1}{2}\left(\xi - \frac{1}{6}\right)^2,$$
$$c = \frac{1}{8\pi^2}\frac{1}{120}, \qquad d = \frac{1}{8\pi^2}\left(\frac{1-5\xi}{30}\right). \tag{C5}$$

These are exactly the same as in the expression for the Weyl Anomaly of a relativistic complex scalar field theory[13] living in one higher dimension [2–8]:

$$\mathcal{A}_R = (-aE_4 + cW^2 + bR^2 + dD_M D^M R). \tag{C6}$$

To arrive at this result, we proceed by considering the heat kernel of the following Euclidean operator, corresponding to the action in Eq. (C3), namely

$$\mathcal{M}_{E,c} = 2m\frac{1}{N}\partial_\tau - \mathcal{D}^2 + \xi R, \tag{C7}$$

where we have

$$\mathcal{D}^2 = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}h^{ij}\partial_j) = \partial^2 + (1+n)(\partial_i n)\partial_i, \tag{C8}$$

$$R = -2\partial^2 n - 2n\partial^2 n - \frac{7}{2}\partial_i n\partial_i n + \cdots, \tag{C9}$$

$$-g^{1/4}\mathcal{D}^2(g^{-1/4}\delta(x)) = -\partial^2\delta(x) + \delta(x)\left(\frac{1}{2}\partial^2 n + \frac{1}{2}n\partial^2 n + \frac{3}{4}\partial_i n\partial_i n\right). \tag{C10}$$

The Euclidean operator can be expressed as the one in flat spacetime, perturbed by the background field $n(x)$:

$$\langle \boldsymbol{x},\tau|\mathcal{M}_{E,c}|\boldsymbol{x}',\tau'\rangle = \langle \boldsymbol{x},\tau|\mathcal{M}_{E,f}|\boldsymbol{x}',\tau'\rangle + mP_1(x)\partial_\tau\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau') + P_2(x)\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau'), \tag{C11}$$

where the subscript c and f denote the curved and flat spacetime, respectively, while E denote the Euclidean nature of the operator. Here we have introduced

$$P_1(x) = 2n(x),$$
$$P_2(x) = \left(\frac{1}{2}\partial^2 n + \frac{1}{2}n\partial^2 n + \frac{3}{4}\partial_i n\partial_i n\right) - \xi\left(2\partial^2 n + 2n\partial^2 n + \frac{7}{2}\partial_i n\partial_i n\right). \tag{C12}$$

The heat kernel can be obtained as a perturbative expansion of the background fields as follows:

$$K(s) = \exp[-s(\mathcal{M}_{E,f} + P)] = \sum_{N=0}^{\infty}(-1)^N K_N(s). \tag{C13}$$

The $K_N(s)$ is defined as follows:

$$K_N(s) = \int_0^s ds_N \int_0^{s_N} ds_{N-1}\cdots \times \int_0^{s_2} ds_1 G(s-s_N)PG(s_N - s_{N-1})P\cdots \times G(s_2 - s_1)PG(s_1). \tag{C14}$$

where $G(s) = e^{-s\mathcal{M}_{E,f}}$ and P is the perturbation (C11), explicitly given by

$$\langle \boldsymbol{x},\tau|P|\boldsymbol{x}',\tau'\rangle = mP_1(x)\partial_\tau\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau') + P_2(x)\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\tau-\tau'). \tag{C15}$$

One can now complete the calculation by using the matrix element of $G(s)$ as given by

$$\mathcal{G}_{g,E}(s;(\boldsymbol{x}_2,\tau_2),(\boldsymbol{x}_1,\tau_1)) \equiv \langle \boldsymbol{x}_2,\tau_2|G(s)|\boldsymbol{x}_1,\tau_1\rangle = \frac{1}{\pi}\left(\frac{1}{4\pi s}\right)^{d/2}\left[\frac{s\eta}{(2ms - \tau_2 + \tau_1)^2 + s^2\eta^2}\right]e^{-\frac{(x_2-x_1)^2}{4s}}, \tag{C16}$$

which corresponds to the heat kernel expression for the η-regulated Euclidean operator: $\mathcal{M}'_{E,g} = 2m\partial_\tau - \nabla^2 + \eta\sqrt{-\partial_\tau^2}$, as discussed in the last few paragraphs of IV B 1.[14] This reproduces Eq. (46) as $\eta \to 0$.

The evaluation of Eq. (C14) follows the procedure sketched out in the appendix of [49]. We separate the contributions from P_1 and P_2 to K_1 as follows:

$$K_{1P_1}(s) = \left(\frac{\frac{\eta}{2}}{m^2 + \frac{\eta^2}{4}}\right)\left(\frac{-1}{4m^2+\eta^2}\right)\frac{8m^2}{(4\pi s)^2} \times \left(P_1 + \frac{s}{6}\partial^2 P_1 + \frac{s^2}{60}\partial^2\partial^2 P_1 + \cdots\right), \tag{C17}$$

$$K_{1P_2}(s) = \left(\frac{\frac{\eta}{2}}{m^2 + \frac{\eta^2}{4}}\right)\frac{2}{(4\pi s)^2}\left(sP_2 + \frac{s^2}{6}\partial^2 P_2 + \cdots\right), \tag{C18}$$

[13]The Weyl anomaly of a complex scalar field is twice of that of a real scalar field.

[14]In curved spacetime, $\mathcal{M}'_{E,g}$ includes a perturbation $n(x)\eta\sqrt{-\partial_\tau^2}$, that, however, does not contribute to the anomaly in the $\eta \to 0$ limit. This term's contribution to K_1 is proportional to $\frac{\eta(\eta^2-4m^2)}{(\eta^2+4m^2)^2}$ that vanishes as $\eta \to 0$, without giving a $\delta(m)$ (or any derivative of $\delta(m)$). This term's contributions to K_2 also vanish as $\eta \to 0$. We omit these terms for simplicity for rest of the appendix.

and for K_2, which gets contributions quadratic in P_1 and P_2, as follows:

$$K_{2P_1P_1}(s) = \frac{(24m^2 - 2\eta^2)}{(\eta^2+4m^2)^2}\left(\frac{2m^2\eta}{4m^2+\eta^2}\right)\frac{1}{(4\pi s)^2}\left(P_1^2 + \frac{s}{3}P_1\partial^2 P_1 + \frac{s}{6}\partial_i P_1\partial_i P_1 + \frac{s^2}{180}(6P_1\partial^2\partial^2 P_1 + 5\partial^2 P_1\partial^2 P_1 + 12\partial_i\partial^2 P_1\partial_i P_1 + 4(\partial_i\partial_j P_1)(\partial_i\partial_j P_1))\right) \tag{C19}$$

$$K_{2P_1P_2}(s) = \left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right)\left(\frac{-1}{4m^2+\eta^2}\right)\frac{8m^2}{(4\pi s)^2}\left(\frac{s}{2}P_1P_2 + \frac{s^2}{12}(P_2\partial^2 P_1 + P_1\partial^2 P_2 + \partial_i P_1\partial_i P_2) + \cdots\right) \tag{C20}$$

$$K_{2P_2P_1}(s) = K_{2P_1P_2}(s) \tag{C21}$$

$$K_{2P_2P_2}(s) = \left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right)\frac{2}{(4\pi s)^2}\left(\frac{s^2}{2}P_2^2 + \cdots\right) \tag{C22}$$

The anomaly is determined by the s-independent terms in K_N. In $\eta \to 0$ limit, factors of $\delta(m)$ arise, after use of the following easily verifiable limits

$$\lim_{\eta\to 0}\left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right)\left(\frac{8m^2}{4m^2+\eta^2}\right) = \pi\delta(m),$$
$$\lim_{\eta\to 0}\left(\frac{\frac{\eta}{2}}{m^2+\frac{\eta^2}{4}}\right) = \pi\delta(m),$$
$$\lim_{\eta\to 0}\frac{24m^2-2\eta^2}{(\eta^2+4m^2)^2}\left(\frac{2\eta m^2}{m^2+\frac{\eta^2}{4}}\right) = 2\pi\delta(m).$$

In $\eta \to 0$ limit, the s independent terms are given by

$$K_{1P_1}(s) \ni \frac{\delta(m)}{16\pi}\left[-\frac{1}{30}\partial^2\partial^2 n\right],$$
$$K_{1P_2}(s) \ni \frac{\delta(m)}{16\pi}\left[\frac{1}{3}\partial^2 P_2\right]$$
$$= \frac{\delta(m)}{16\pi}\left[\frac{1}{3}\left(\left(\frac{1}{2}-2\xi\right)\partial^2\partial^2 n + \left(\frac{1}{2}-2\xi\right)\partial^2 n\partial^2 n + \left(\frac{1}{2}-2\xi\right)n\partial^2\partial^2 n + \left(\frac{5}{2}-11\xi\right)\partial_i n\partial_i\partial^2 n + \left(\frac{3}{2}-7\xi\right)(\partial_i\partial_j n)(\partial_i\partial_j n)\right)\right],$$
$$K_{2P_1P_1} \ni \frac{\delta(m)}{16\pi}\left[\frac{1}{90}(6n\partial^2\partial^2 n + 5\partial^2 n\partial^2 n + 12\partial_i\partial^2 n\partial_i n + 4(\partial_i\partial_j n)(\partial_i\partial_j n))\right],$$
$$K_{2P_1P_2} + K_{2P_1P_2} \ni \frac{\delta(m)}{16\pi}\left[\frac{-1}{3}(P_2\partial^2 n + n\partial^2 P_2 + \partial_i n\partial_i P_2)\right] = \frac{\delta(m)}{16\pi}\left[\frac{-1}{3}\left(\frac{1}{2}-2\xi\right)(\partial^2 n\partial^2 n + n\partial^2\partial^2 n + \partial_i n\partial_i\partial^2 n)\right],$$
$$K_{2P_2P_2} \ni \frac{\delta(m)}{16\pi}[P_2^2 + \cdots] = \frac{\delta(m)}{16\pi}\left[\left(\frac{1}{2}-2\xi\right)^2\partial^2 n\partial^2 n + \cdots\right].$$

Using

$$R = -2\partial^2 n - 2n\partial^2 n - \frac{7}{2}\partial_i n\partial_i n + \cdots, \tag{C23}$$

$$R^2 = 4(\partial^2 n)^2 + \cdots, \qquad W^2 = \frac{1}{3}(\partial^2 n)^2 + \cdots, \tag{C24}$$

$$E_4 = 2(\partial^2 n)^2 - 2(\partial_i\partial_j n)(\partial_i\partial_j n) + \cdots, \tag{C25}$$

$$D_M D^M R = -2\partial^4 n - 2(\partial^2 n)^2 - 2n\partial^4 n - 13(\partial_j n)(\partial_j\partial^2 n) - 7(\partial_i\partial_j n)(\partial_i\partial_j n) + \cdots. \tag{C26}$$

one verifies the anomaly expression in Eqs. (C4) and (C5). Since our calculation only fixes the value of $12b + c$, in order to break the degeneracy we use the fact that for $\xi = \frac{1}{6}$ the Wess-Zumino consistency condition precludes an R^2 anomaly [49] and assume c is ξ-independent.

We emphasize that the calculation carried out here does not rely on any null cone reduction technique, hence, this lends further credence to the LCR prescription, which has correctly produced the $\delta(m)$ factor, as elucidated before.

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