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Relativistic effects due to gravimagnetic moment of a rotating body

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We compute the exact Hamiltonian (and corresponding Dirac brackets) for a spinning particle with gravimagnetic moment κ in an arbitrary gravitational background. The case $\kappa = 0$ corresponds to the Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations. $\kappa = 1$ leads to modified MPTD equations with improved behavior in the ultrarelativistic limit. So we study the modified equations in the leading post-Newtonian approximation. The rotating body with unit gravimagnetic moment has qualitatively different behavior as compared with the MPTD body: (A) If a number of gyroscopes with various rotation axes are freely traveling together, the angles between the axes change with time. (B) For specific binary systems, gravimagnetic moment gives a contribution to the frame-dragging effect with the magnitude that turns out to be comparable with that of Schiff frame dragging.

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I. INTRODUCTION

The rotating body in general relativity is usually described on the base of manifestly generally covariant Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations that prescribe the dynamics of both trajectory and spin of the body in an external gravitational field [1–6]. Starting from the pioneer works, these equations were considered as a Hamiltonian-type system. In the recent work [7], we realized this idea by constructing the minimal interaction with gravity in the vector model of the spinning particle, and showed that this indeed leads to MPTD equations in the Hamiltonian formalism (see also below). This allowed us to study the ultrarelativistic limit in exact equations for the trajectory of the MPTD particle in the laboratory time. Using the Landau-Lifshitz $(1+3)$ decomposition [8] we observed that, unlike a geodesic equation, the MPTD equations lead to the expression for three-acceleration, which contains divergent terms as $v \to c$ [9]. Fast test particles are now under intensive investigation [10–15], and represent an important tool in the study, for example, of near horizon geometry of black holes [16–24]. Readers may also consult (we are grateful to the reviewer for indicating these works) the very general treatment of these problems in [25]. So, it would be interesting to find a generalization of MPTD equations with improved behavior in the ultrarelativistic regime. This can be achieved if we add a nonminimal spin-gravity interaction through the gravimagnetic moment [26]. In the theory with unit gravimagnetic moment, both acceleration and spin torque have reasonable behavior in the ultrarelativistic limit. In the present work we study the modified equations in the regime of small velocities in the leading post-Newtonian approximation. In Schwarzschild and Kerr space-times, the modified equations imply a number of qualitatively new effects that could be used to test experimentally whether a rotating body in general relativity has null or unit gravimagnetic moment.

The work is organized as follows. In Sec. II we shortly describe Lagrangian and Hamiltonian formulations of the vector model of the spinning particle and compute Dirac brackets of the theory in an arbitrary gravitational background. In the formulation with use of Dirac brackets, the complete Hamiltonian acquires a simple and expected form, while an approximate $\frac{1}{c^2}$ Hamiltonian, further obtained in Sec. IV, strongly resembles that of the spinning particle in electromagnetic background. This is in correspondence with the known analogy between gravity and electromagnetism [27–30]. In Sec. III we introduce nonminimal spin-gravity interaction through the gravimagnetic moment and obtain the corresponding equations of motion. We show that constants of motion due to isometries of space-time for the MPTD and the modified equations are the same. In Sec. IV we compute the leading post-Newtonian corrections to the trajectory and spin of our particle with unit gravimagnetic moment, and present the corresponding effective Hamiltonian in $\frac{1}{c^2}$-approximation. The nonminimal interaction implies extra contributions into both trajectory and spin, as compared with MPTD equations in the same approximation. A number of effects due to nonminimal interaction are discussed in Sec. V.

Notation. Our variables are taken in arbitrary parametrization τ, then $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. The square brackets mean antisymmetrization, $\omega^{[\mu}\pi^{\nu]} = \omega^\mu\pi^\nu - \omega^\nu\pi^\mu$. For the four-dimensional quantities we suppress the contracted indices and use the notation $\dot{x}^\mu G_{\mu\nu}\dot{x}^\nu = \dot{x}G\dot{x}$, $N^\mu{}_\nu\dot{x}^\nu = (N\dot{x})^\mu$, $\omega^2 = g_{\mu\nu}\omega^\mu\omega^\nu$, μ, $\nu = 0$, 1, 2, 3. Notation for the scalar functions constructed from second-rank tensors is $\theta S = \theta^{\mu\nu}S_{\mu\nu}$, $S^2 = S^{\mu\nu}S_{\mu\nu}$. When we work in four-dimensional Minkowski space with coordinates $x^\mu = (x^0 = ct, x^i)$, we

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use the metric $\eta_{\mu\nu}=(-,+,+,+)$, then $\dot{x}\omega=\dot{x}^\mu\omega_\mu=-\dot{x}^0\omega^0+\dot{x}^i\omega^i$ and so on. Suppressing the indices of three-dimensional quantities, we use bold letters, $v^i\gamma_{ij}a^j=\mathbf{v}\gamma\mathbf{a}$, $v^iG_{i\mu}v^\mu=\mathbf{v}Gv$, i, $j=1$, 2, 3, and so on.

The covariant derivative is $\nabla\omega^\mu=\frac{d\omega^\mu}{d\tau}+\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\omega^\beta$. The tensor of Riemann curvature is $R^\sigma{}_{\lambda\mu\nu}=\partial_\mu\Gamma^\sigma{}_{\lambda\nu}-\partial_\nu\Gamma^\sigma{}_{\lambda\mu}+\Gamma^\sigma{}_{\beta\mu}\Gamma^\beta{}_{\lambda\nu}-\Gamma^\sigma{}_{\beta\nu}\Gamma^\beta{}_{\lambda\mu}$.

II. VECTOR MODEL OF SPIN AND MATHISSON-PAPAPETROU-TULCZYJEW-DIXON EQUATIONS

In the vector model of spin presented in [31], the configuration space consists of the position of the particle $x^\mu(\tau)$, and the vector $\omega^\mu(\tau)$ attached to the point $x^\mu(\tau)$. Minimal interaction with gravity is achieved by direct covariantization of the free action, initially formulated in Minkowski space. That is we replace $\eta_{\mu\nu}\to g_{\mu\nu}$ and usual derivatives of the vector ω^μ by the covariant derivative: $\dot{\omega}^\mu\to\nabla\omega^\mu$. The resulting Lagrangian action reads [7]

$$S=-\frac{1}{\sqrt{2}}\int d\tau\left[m^2c^2-\frac{\alpha}{\omega^2}\right]^{\frac{1}{2}}\times\sqrt{-\dot{x}N\dot{x}-\nabla\omega N\nabla\omega+T^{1/2}}. \quad (1)$$

We have denoted $T\equiv[\dot{x}N\dot{x}+\nabla\omega N\nabla\omega]^2-4(\dot{x}N\nabla\omega)^2$, and $N_{\mu\nu}\equiv g_{\mu\nu}-\frac{\omega_\mu\omega_\nu}{\omega^2}$. The matrix N is a projector on the plane orthogonal to ω: $N_{\mu\nu}\omega^\nu=0$. The parameter α determines the value of spin; in particular, $\alpha=\frac{3\hbar^2}{4}$ corresponds to the spin one-half particle. In the spinless limit, $\omega^\mu=0$ and $\alpha=0$, Eq. (1) reduces to the standard Lagrangian of a point particle, $-mc\sqrt{-\dot{x}^2}$.

The action is manifestly invariant under general-coordinate transformations as well as under reparametrizations of the evolution parameter τ. Besides, there is one more local symmetry, which acts in the spin sector and is called the spin-plane symmetry: the action remains invariant under rotations of the vectors ω^μ and $\pi_\mu=\frac{\partial L}{\partial\dot{\omega}^\mu}$ in their own plane [32]. Being affected by the local transformation, these vectors do not represent observable quantities. But their combination, $S^{\mu\nu}=2(\omega^\mu\pi^\nu-\omega^\nu\pi^\mu)$, is an invariant quantity, which represents the spin tensor of the particle. We decompose the spin tensor as follows:

$$S^{\mu\nu}=2(\omega^\mu\pi^\nu-\omega^\nu\pi^\mu)=(S^{i0}=D^i,S_{ij}=2\epsilon_{ijk}S_k), \quad (2)$$

where S_i is the three-dimensional spin vector, and D_i is the dipole electric moment [33].

Since we deal with a local-invariant theory and, furthermore, one of the basic observables is constructed from the phase-space variables, the Hamiltonian formalism is the most convenient for analyzing the dynamics of the theory. So, we first obtain the Hamiltonian equations of motion, and next, excluding momenta, we arrive at the Lagrangian equations for the physical-sector variables x and S.

Conjugate momenta for x^μ and ω^μ are $p_\mu=\frac{\partial L}{\partial\dot{x}^\mu}$ and $\pi_\mu=\frac{\partial L}{\partial\dot{\omega}^\mu}$ respectively. Because of the presence of $\dot{x}^\mu$ in $\nabla\omega^\mu$, the conjugated momentum p_μ does not transform as a vector, so it is convenient to define the canonical momentum

$$P_\mu\equiv p_\mu-\Gamma^\beta_{\alpha\mu}\omega^\alpha\pi_\beta, \quad (3)$$

which transforms as a vector under general-coordinate transformations. The full set of phase-space coordinates consists of the pairs x^μ, p_μ, and ω^μ, π_μ. They fulfil the fundamental Poisson brackets $\{x^\mu,p_\nu\}=\delta^\mu_\nu$, $\{\omega^\mu,\pi_\nu\}=\delta^\mu_\nu$; then

$$\{P_\mu,\omega^\nu\}=\Gamma^\nu_{\mu\alpha}\omega^\alpha, \qquad \{P_\mu,\pi_\nu\}=-\Gamma^\alpha_{\mu\nu}\pi_\alpha,$$
$$\{P_\mu,\omega^2\}=\{P_\mu,\pi^2\}=\{P_\mu,\omega\pi\}=0. \quad (4)$$

For the quantities x^μ, P^μ, and $S^{\mu\nu}$, the basic Poisson brackets imply the typical relations used by people for spinning particles in Hamiltonian formalism,

$$\{x^\mu,P_\nu\}=\delta^\mu_\nu, \qquad \{P_\mu,P_\nu\}=-\frac{1}{4}R_{\mu\nu\alpha\beta}S^{\alpha\beta},$$
$$\{P_\mu,S^{\alpha\beta}\}=\Gamma^\alpha_{\mu\sigma}S^{\sigma\beta}-\Gamma^\beta_{\mu\sigma}S^{\sigma\alpha},$$
$$\{S^{\mu\nu},S^{\alpha\beta}\}=2(g^{\mu\alpha}S^{\nu\beta}-g^{\mu\beta}S^{\nu\alpha}-g^{\nu\alpha}S^{\mu\beta}+g^{\nu\beta}S^{\mu\alpha}). \quad (5)$$

Applying the Dirac-Bergman procedure for a singular system to the theory (1), we arrive at the Hamiltonian [9]

$$H=\frac{\lambda_1}{2}[T_1+4a(\pi\theta P)T_3-4a(\omega\theta P)T_4+T_5]+\lambda_2T_2, \quad (6)$$

composed of the constraints

$$T_1\equiv P^2+m^2c^2=0, \quad (7)$$

$$T_2\equiv\omega\pi=0, \qquad T_3\equiv P\omega=0,$$
$$T_4\equiv P\pi=0, \qquad T_5\equiv\pi^2-\frac{\alpha}{\omega^2}=0. \quad (8)$$

In the expression for H we have denoted

$$\theta_{\mu\nu}\equiv R_{\alpha\beta\mu\nu}S^{\alpha\beta}, \qquad a\equiv\frac{2}{16m^2c^2+(\theta S)}. \quad (9)$$

The antisymmetric tensor $\theta_{\mu\nu}$ turns out to be a gravitational analogy of the electromagnetic field strength $F_{\mu\nu}$; see below. $T_1,\ldots,T_4$ appear as the primary constraints in the course of the Dirac-Bergmann procedure, T_5 is the only secondary constraint of the theory, and λ_1, λ_2 are the Lagrangian multipliers associated to T_1 and T_2. Poisson brackets of the constraints are summarized in Table I. The table implies that T_3 and T_4 represent a pair of second-class

TABLE I. Poisson brackets of constraints.

	T_1	T_5	T_2	T_3	T_4
$T_1 = P^2 + m^2c^2$	0	0	0	$\frac{1}{2}(\omega\theta P)$	$\frac{1}{2}(\pi\theta P)$
$T_5 = \pi^2 - \frac{\alpha}{\omega^2}$	0	0	$-2T_5$	$-2T_4$	$-2\alpha T_3/(\omega^2)^2$
$T_2 = \omega\pi$	0	$2T_5$	0	$-T_3$	T_4
$T_3 = P\omega$	$-\frac{1}{2}(\omega\theta P)$	$2T_4$	T_3	0	$P^2 - \frac{(\theta S)}{16} \approx -\frac{1}{8a}$
$T_4 = P\pi$	$-\frac{1}{2}(\pi\theta P)$	$2\alpha T_3/(\omega^2)^2$	$-T_4$	$-P^2 + \frac{(\theta S)}{16} \approx \frac{1}{8a}$	0

constraints, while T_2, T_5 and the combination $T_1 + 4a(\pi\theta P)T_3 - 4a(\omega\theta P)T_4$ are the first-class constraints. So the Hamiltonian (6) consists of the first-class constraints.

Taking into account that each second-class constraint rules out one phase-space variable, whereas each first-class constraint rules out two variables, we have the right number of spin degrees of freedom, $8 - (2+4) = 2$. The meaning of the constraints becomes clear if we consider their effect over the spin tensor. The second-class constraints $T_3 = 0$ and $T_4 = 0$ imply the spin supplementary condition

$$S^{\mu\nu}P_\nu = 0, \tag{10}$$

while the first-class constraints T_2 and T_5 fix the value of the square of the spin tensor

$$S^{\mu\nu}S_{\mu\nu} = 8\alpha. \tag{11}$$

Equations (10) and (11) imply that only two components of spin tensor are independent, as it should be for an elementary spin one-half particle.

We could use Poisson brackets to obtain the Hamiltonian equations, $\dot{z} = \{z, H\}$, for the variables of physical sector $z = (x, P, S)$. But in this case we are forced to work with the rather inconvenient Hamiltonian (6). Instead, we construct the Dirac bracket associated with second-class constraints T_3 and T_4. It is convenient to denote $\{T_3, T_4\} = -\frac{1}{8\triangle}$, where $\triangle = \frac{-2}{16P^2-(\theta S)}$, then $\triangle \approx a$ on the surface of mass-shell constraint $T_1 = 0$. The Dirac bracket reads

$$\{A,B\}_D = \{A,B\} - 8\triangle[\{A,T_3\}\{T_4,B\} - \{A,T_4\}\{T_3,B\}]. \tag{12}$$

By construction, the Dirac bracket of any variable with the constraints vanishes, so T_3 and T_4 can be omitted from the Hamiltonian. The first-class constraints T_2 and T_5 can be omitted as well, since brackets of the variables x, P, and S with them vanish on the constraint surface. In the result we arrive at a simple Hamiltonian

$$H_0 = \frac{\lambda_1}{2}(P^2 + m^2c^2), \tag{13}$$

which looks like that of a free point particle. All the information on spin and interaction is encoded now in the Dirac bracket. In particular, equations of motion are obtained according to the rule $\dot{z} = \{z, H_0\}_D$.

Poisson brackets of our variables with T_3 and T_4 are

$$\begin{aligned}
&\{x^\mu, T_3\} = \omega^\mu, \qquad \{x^\mu, T_4\} = \pi^\mu, \\
&\{P_\alpha, T_3\} = -\frac{1}{4}\theta_{\alpha\beta}\omega^\beta + \Gamma^\lambda_{\alpha\beta}P_\lambda\omega^\beta, \\
&\{P_\alpha, T_4\} = -\frac{1}{4}\theta_{\alpha\beta}\pi^\beta + \Gamma^\lambda_{\alpha\beta}P_\lambda\pi^\beta, \\
&\{S^{\mu\nu}, T_3\} = 2P^{[\mu}\omega^{\nu]} + \Gamma^{[\mu}_{\alpha\beta}S^{\nu]\alpha}\omega^\beta, \\
&\{S^{\mu\nu}, T_4\} = 2P^{[\mu}\pi^{\nu]} + \Gamma^{[\mu}_{\alpha\beta}S^{\nu]\alpha}\pi^\beta.
\end{aligned} \tag{14}$$

Using these expressions in (12), we obtain the manifest form of the Dirac brackets

$$\begin{aligned}
\{x^\mu, x^\nu\}_D &= 4\triangle S^{\mu\nu}, \\
\{P_\mu, P_\nu\}_D &= -\frac{1}{4}\theta_{\mu\nu} + 4\triangle(\Gamma P)_{\mu\alpha}S^{\alpha\beta}(\Gamma P)_{\beta\nu} \\
&\quad - \frac{\triangle}{8}(\theta_{\mu\alpha}S^{\alpha\beta}[\theta_{\beta\nu} + 4(\Gamma P)_{\beta\nu}] - (\mu \leftrightarrow \nu)), \\
\{x^\mu, P_\alpha\}_D &= \delta^\mu_\alpha + \triangle S^{\mu\beta}[\theta_{\beta\alpha} + 4(\Gamma P)_{\beta\alpha}], \\
\{x^\mu, S^{\alpha\beta}\}_D &= -8\triangle\left[S^{\mu[\alpha}P^{\beta]} - \frac{1}{2}S^{\mu\sigma}\Gamma^{[\alpha}_{\sigma\lambda}S^{\beta]\lambda}\right], \\
\{P_\alpha, S^{\mu\nu}\}_D &= -\Gamma^{[\mu}_{\alpha\sigma}S^{\nu]\sigma} \\
&\quad + \triangle[\theta_{\alpha\beta} + 4(\Gamma P)_{\alpha\beta}](2S^{\beta[\mu}P^{\nu]} - S^{\beta\eta}\Gamma^{[\mu}_{\eta\lambda}S^{\nu]\lambda}), \\
\{S^{\mu\nu}, S^{\alpha\beta}\}_D &= \{S^{\mu\nu}, S^{\alpha\beta}\} - 8\triangle\Big[2(P^\mu P^\alpha S^{\beta\nu} - P^\mu P^\beta S^{\alpha\nu} \\
&\quad - P^\nu P^\alpha S^{\beta\mu} + P^\nu P^\beta S^{\alpha\mu}) - P^{[\mu}S^{\nu]\lambda}\Gamma^{[\alpha}_{\lambda\sigma}S^{\beta]\sigma} \\
&\quad + P^{[\alpha}S^{\beta]\lambda}\Gamma^{[\mu}_{\lambda\sigma}S^{\nu]\sigma} - \frac{1}{2}\Gamma^{[\mu}_{\sigma\lambda}S^{\nu]\sigma}S^{\lambda\rho}\Gamma^{[\alpha}_{\rho\epsilon}S^{\beta]\epsilon}\Big].
\end{aligned} \tag{15}$$

Their right-hand sides do not contain explicitly the variables ω and π, so the brackets form a closed algebra for the set (x, P, S).

The Dirac brackets remain different from the Poisson brackets even in the limit of a free theory, $g_{\mu\nu} \to \eta_{\mu\nu}$. In particular, in the sector of canonical variables x and p we have

$$\{x^\mu, x^\nu\}_D = -\frac{S^{\mu\nu}}{2p^2}, \qquad \{x^\mu, p^\nu\}_D = \eta^{\mu\nu}, \qquad \{p^\mu, p^\nu\}_D = 0. \tag{16}$$

Hence, the account of spin leads to deformation of the phase-space symplectic structure: the position variables of the relativistic spinning particle obey the noncommutative bracket, with the noncommutativity parameter being proportional to the spin tensor. This must be taken into account in the construction of quantum mechanics of a spinning particle [34,35]. In particular, for an electron in electromagnetic field, the spin-induced noncommutativity explains the famous one-half factor in the Pauli equation without appeal to the Thomas precession, Dirac equation, or Foldy-Wouthuysen transformation; see [36]. Besides, for a spinning body in gravitational field, the spin-induced noncommutativity clarifies the discrepancy in expressions for three-acceleration obtained by different methods; see [37].

Using the Dirac brackets together with the Hamiltonian (13), we obtain equations of motion

$$\begin{aligned}
\dot{x}^\mu &= \{x^\mu, H_0\}_D = \lambda_1[P^\mu + aS^{\mu\beta}\theta_{\beta\alpha}P^\alpha], \\
\dot{P}_\mu &= \{P_\mu, H_0\}_D = \left(-\frac{1}{4}\theta_{\mu\nu} + (\Gamma P)_{\mu\nu}\right)\lambda_1[P^\nu + aS^{\nu\beta}\theta_{\beta\alpha}P^\alpha] \\
&= -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu + \Gamma^\alpha_{\mu\nu}P_\alpha\dot{x}^\nu, \\
\dot{S}^{\mu\nu} &= \{S^{\mu\nu}, H_0\}_D \\
&= (2P^\mu\delta^\nu{}_\alpha - \Gamma^\mu_{\alpha\sigma}S^{\sigma\nu})\lambda_1[P^\alpha + aS^{\alpha\beta}\theta_{\beta\gamma}P^\gamma] - (\mu \leftrightarrow \nu) \\
&= 2P^{[\mu}\dot{x}^{\nu]} - \Gamma^\mu_{\alpha\sigma}S^{\sigma\nu}\dot{x}^\alpha - \Gamma^\nu_{\alpha\sigma}S^{\mu\sigma}\dot{x}^\alpha.
\end{aligned} \tag{17}$$

They can be rewritten in a manifestly general-covariant form as follows:

$$\dot{x}^\mu = \lambda_1(\delta^\mu{}_\nu + aS^{\mu\beta}\theta_{\beta\nu})P^\nu, \tag{18}$$

$$\nabla P_\mu = -\frac{1}{4}R_{\mu\nu\alpha\beta}S^{\alpha\beta}\dot{x}^\nu \equiv -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu, \tag{19}$$

$$\nabla S^{\mu\nu} = 2(P^\mu\dot{x}^\nu - P^\nu\dot{x}^\mu). \tag{20}$$

Some relevant comments are in order.

(1) *Comparison with MPTD equations.* Despite the fact that the vector model has been initially constructed as a theory of an elementary particle of spin one-half, it turns out to be suitable to describe a rotating body in general relativity in the pole-dipole approximation [5,38]. Indeed, Eqs. (19) and (20) coincide with Dixon equations of the body (our spin is twice that of Dixon), while our constraint (10) is just the Tulczyjew spin supplementary condition.[1] Besides, the Hamiltonian equation (18) can be identified with the velocity-momentum relation, implied by MPTD equations; see [26] for a detailed comparison. The only difference is that values of momentum and spin are conserved quantities of MPTD equations, while in the vector model they are fixed by constraints. In summary [26], to study the class of trajectories of a body with $\sqrt{-P^2} = k$ and $S^2 = \beta$, we can use our spinning particle with $m = \frac{k}{c}$ and $\alpha = \frac{\beta}{8}$.

(2) *Ultrarelativistic limit.* Using the Landau-Lifshitz $1+3$-decomposition [8], we showed in [26] that MPTD equations yield a paradoxical behavior in the ultrarelativistic limit: three-dimensional acceleration of the particle grows with its speed, and diverges as $|\mathbf{v}| \to c$. In the next section, we improve this by adding a nonminimal spin-gravity interaction through the gravimagnetic moment.

(3) *Analogy between gravitation and electromagnetism.* Many people mentioned remarkable analogies between gravitation and electromagnetism in various circumstances [17,27–30]. Here we observe an analogy, comparing (18)–(20) with equations of motion of the spinning particle (with null gyromagnetic ratio) [31] in electromagnetic field with the strength $F_{\mu\nu}$,

$$\dot{x}^\mu = \lambda_1(\delta^\mu{}_\nu + aS^{\mu\beta}F_{\beta\nu})P^\nu, \qquad \text{where } a = \frac{-2e}{4m^2c^3 - e(SF)}, \tag{21}$$

$$\dot{P}_\mu = \frac{e}{c}F_{\mu\nu}\dot{x}^\nu, \tag{22}$$

$$\dot{S}^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]}. \tag{23}$$

One system just turns into another if we identify $\theta_{\mu\nu} \equiv R_{\mu\nu\alpha\beta}S^{\alpha\beta} \sim F_{\mu\nu}$, and set $e = -\frac{c}{4}$. That is a curvature influences the trajectory of a spinning particle in the same way as an electromagnetic field with the strength $\theta_{\mu\nu}$. We now use this analogy to construct a nonminimal spin-gravity interaction.

[1] While the variational problem dictates [39] Eq. (10), in the multipole approach there is a freedom in the choice of a spin supplementary condition, related with the freedom in the choice of a representative point x^μ describing position of the body [3,4,6]. Different conditions lead to the same results in $\frac{1}{c^2}$-approximation; see [5,40,41].

III. ROTATING BODY WITH GRAVIMAGNETIC MOMENT

The Hamiltonian (6) is a combination of constraints, so the Hamiltonian formulation of our model is completely determined by the set of constraints (7) and (8), and by the expression (3) for canonical momentum P^μ through the conjugated momentum p^μ. We observe that algebraic properties of the constraints do not change, if we replace the mass-shell constraint $T_1 = P^2 + m^2c^2$ by $\tilde{T}_1 = P^2 + f(x,P,S) + m^2c^2$, where $f(x^\mu, P^\nu, S^{\mu\nu})$ is an arbitrary scalar function. Indeed, in the modified theory T_3 and T_4 remain the second-class constraints, while T_2, T_5, and the combination $\tilde{T}_1 - \{T_3,T_4\}^{-1}\{\tilde{T}_1,T_4\}T_3 + \{T_3,T_4\}^{-1}\{\tilde{T}_1,T_3\}T_4$ form a set of first-class constraints. If we confine ourselves to the linear in curvature and quadratic in spin approximation, the only scalar function f, which can be constructed from the quantities at our disposal, is $\frac{\kappa}{16}R_{\mu\nu\alpha\beta}S^{\mu\nu}S^{\alpha\beta} \equiv \kappa R_{\mu\nu\alpha\beta}\omega^\mu\pi^\nu\omega^\alpha\pi^\beta$, where κ is a dimensionless parameter. The resulting constraint,

$$\tilde{T}_1 = P^2 + \frac{\kappa}{16}(\theta S) + m^2c^2 = 0, \tag{24}$$

is similar to the Hamiltonian $\frac{\lambda_1}{2}(P^2 - \frac{eg}{c}(FS) + m^2c^2)$ of a spinning particle interacting with electromagnetic field through the gyromagnetic ratio g; see [31]. In view of this similarity, the interaction constant κ is called the gravimagnetic moment [16,17], and we expect that nonminimally interacting theory with the Hamiltonian (24) could be a consistent generalization of MPTD equations. The consistency has been confirmed in [26], where we presented the Lagrangian action of a spinning particle that implies the constraints (24) and (8) in Hamiltonian formalism.

Poisson brackets of the constraints $\tilde{T}_1$, T_3, and T_4 read

$$\{\bar{T}_1, T_3\} = \frac{1}{2}(1-\kappa)(\omega\theta P) + \kappa\omega^\sigma(\nabla_\sigma R_{\mu\nu\alpha\beta})\omega^\mu\pi^\nu\omega^\alpha\pi^\beta, \tag{25}$$

$$\{\bar{T}_1, T_4\} = \frac{1}{2}(1-\kappa)(\pi\theta P) + \kappa\pi^\sigma(\nabla_\sigma R_{\mu\nu\alpha\beta})\omega^\mu\pi^\nu\omega^\alpha\pi^\beta, \tag{26}$$

$$\{T_3, T_4\} = P^2 - \frac{1}{16}(\theta S) \approx -8\bar{a}, \quad \text{where } \bar{a} = \frac{2}{16m^2c^2 + (\kappa+1)(\theta S)}. \tag{27}$$

These expressions must be substituted in place of terms $\frac{1}{2}(\omega\theta P)$, $\frac{1}{2}(\omega\theta P)$, and a in Table I. The Dirac brackets (15), being constructed with the help of T_3 and T_4, remain valid in the modified theory. Our new Hamiltonian is $H = \frac{\lambda}{2}H_0 + \frac{\lambda}{2}H_\kappa$, with H_0 from (13) and $H_\kappa = \frac{\kappa}{16}(\theta S)$. Hence, to obtain the manifest form of equations of motion $\dot{z} = \{z,H_0\}_D + \{z,H_\kappa\}_D$, we only need to compute the brackets $\{z,H_\kappa\}_D$. They are

$$\{x^\mu, H_\kappa\}_D = -\lambda_1\kappa\bar{a}\left[S^{\mu\alpha}\theta_{\alpha\beta}P^\beta - \frac{1}{8}S^{\mu\nu}(\nabla_\nu R_{\alpha\beta\sigma\lambda})S^{\alpha\beta}S^{\sigma\lambda}\right], \tag{28}$$

$$\{P_\mu, H_\kappa\}_D = -\frac{1}{4}\theta_{\mu\alpha}\{x^\alpha, H_\kappa\}_D + \Gamma^\beta_{\mu\alpha}P_\beta\{x^\alpha, H_\kappa\}_D - \frac{\lambda_1\kappa}{32}(\nabla_\mu R_{\alpha\beta\sigma\lambda})S^{\alpha\beta}S^{\sigma\lambda}, \tag{29}$$

$$\{S^{\mu\nu}, H_\kappa\}_D = \frac{\kappa\lambda_1}{4}\theta^{[\mu}_\alpha S^{\nu]\alpha} + 2P^{[\mu}\{x^{\nu]}, H_\kappa\}_D - (\Gamma^\mu_{\alpha\beta}S^{\alpha\nu} + \Gamma^\nu_{\alpha\beta}S^{\mu\alpha})\{x^\beta, H_\kappa\}_D. \tag{30}$$

Adding them to the equations $\dot{z} = \{z,H_0\}_D$ given in (18)–(20), we arrive at the dynamical equations

$$\dot{x}^\mu = \lambda_1[\delta^\mu{}_\nu - \bar{a}(\kappa-1)S^{\mu\alpha}\theta_{\alpha\nu}]P^\nu + \frac{\lambda_1\kappa\bar{a}}{8}S^{\mu\nu}(\nabla_\nu R_{\alpha\beta\sigma\lambda})S^{\alpha\beta}S^{\sigma\lambda}, \tag{31}$$

$$\nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \frac{\lambda_1\kappa}{32}(\nabla_\mu R_{\alpha\beta\sigma\lambda})S^{\alpha\beta}S^{\sigma\lambda}, \tag{32}$$

$$\nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]} + \frac{\lambda_1\kappa}{4}\theta^{[\mu}{}_\alpha S^{\nu]\alpha}. \tag{33}$$

Together with the constraints (10), (11), and (24), they give a complete system of Hamiltonian equations of spinning particle with gravimagnetic moment κ. As it should be, our equations reduce to MPTD Eqs. (18)–(20) when $\kappa = 0$. Comparing the two systems, we see that the nonminimal interaction yields quadratic and cubic in spin corrections to MPTD equations.

The Eqs. (31)–(33) are greatly simplified for a particle with unit gravimagnetic moment, $\kappa = 1$ (gravimagnetic particle). It has a qualitatively different behavior as compared with the MPTD particle. First, the gravimagnetic particle has an expected behavior in the ultrarelativistic limit [9,26]: three-dimensional acceleration of the particle and angular velocity of precession remain finite as $|\mathbf{v}| \to c$, while the longitudinal acceleration vanishes in the limit. Second, at low velocities, taking $\kappa = 1$ and keeping only the terms which may give a contribution in the leading post-Newton approximation, $\sim\frac{1}{c^2}$, we obtain from (31)–(33) the approximate equations

$$\dot{x}^\mu = \lambda_1 P^\mu, \qquad \nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \frac{\lambda_1}{32}(\nabla_\mu R_{\alpha\beta\sigma\lambda})S^{\alpha\beta}S^{\sigma\lambda},$$
$$\nabla S^{\mu\nu} = \frac{\lambda_1}{4}\theta^{[\mu}{}_\alpha S^{\nu]\alpha}, \tag{34}$$

while MPTD equations ($\kappa = 0$) in the same approximation read

$$\dot{x}^\mu = \lambda_1 P^\mu, \qquad \nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu, \qquad \nabla S^{\mu\nu} = 0. \tag{35}$$

In Sec. IV, we compute $\frac{1}{c^2}$ corrections due to the extra terms appearing in (34).

A. Conserved charges

In curved space that possesses some isometry, MPTD equations admit a constant of motion (see, for example, [7,25])

$$J^{(\xi)} = P^\mu \xi_\mu - \frac{1}{4}S^{\mu\nu}\nabla_\nu \xi_\mu, \tag{36}$$

where ξ_μ is the Killing vector that generates the isometry, i.e., $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$. Let us show that $J^{(\xi)}$ remains a constant of motion when the gravimagnetic interaction is included. Using (32) and (33), we obtain by direct calculation

$$\dot{J}^{(\xi)} = \frac{\kappa\lambda_1}{8}\left[S^{\alpha\beta}R^\mu{}_{\sigma\alpha\beta}S^{\sigma\nu}\nabla_\nu\xi_\mu - \frac{1}{4}S^{\alpha\beta}S^{\sigma\lambda}\xi^\mu\nabla_\mu R_{\alpha\beta\sigma\lambda}\right]. \tag{37}$$

Using the Bianchi identities we find the relation

$$S^{\alpha\beta}S^{\sigma\lambda}\xi^\mu\nabla_\mu R_{\alpha\beta\sigma\lambda} = 2S^{\alpha\beta}S^{\sigma\nu}\xi^\mu\nabla_\sigma R_{\mu\nu\alpha\beta}. \tag{38}$$

The derivative of a curvature tensor is related with the derivative of a Killing vector by the formula $\xi^\mu\nabla_\sigma R_{\alpha\beta\nu\mu} - \xi^\mu\nabla_\nu R_{\alpha\beta\sigma\mu} = R_{\alpha\beta\sigma}{}^\mu\nabla_\nu\xi_\mu - R_{\alpha\beta\nu}{}^\mu\nabla_\sigma\xi_\mu + R_{\sigma\nu\alpha}{}^\mu\nabla_\beta\xi_\mu - R_{\sigma\nu\beta}{}^\mu\nabla_\alpha\xi_\mu$. Contracting twice with the spin tensor we obtain

$$S^{\alpha\beta}S^{\sigma\nu}\xi^\mu\nabla_\sigma R_{\mu\nu\alpha\beta} = 2S^{\alpha\beta}R^\mu{}_{\sigma\alpha\beta}S^{\sigma\nu}\nabla_\nu\xi_\mu. \tag{39}$$

Using this expression in (38), we obtain $S^{\alpha\beta}S^{\sigma\lambda}\xi^\mu\nabla_\mu R_{\alpha\beta\sigma\lambda} = 4S^{\alpha\beta}R^\mu{}_{\sigma\alpha\beta}S^{\sigma\nu}\nabla_\nu\xi_\mu$. This implies that the right-hand side of (37) vanishes, so $\dot{J}^{(\xi)} = 0$. Thus, the quantity (36) represents a constant of motion of a spinning particle with gravimagnetic moment.

B. Lagrangian system of equations of motion

Since we are interested in the influence of nonminimal spin-gravity interaction on the trajectory and spin of the particle, we eliminate the momenta P^μ and the auxiliary variable λ_1 from Eqs. (31)–(33), obtaining their Lagrangian form. In Eq. (31), which relates velocity and momentum, there appears the matrix

$$T^\alpha{}_\nu \equiv \delta^\alpha{}_\nu - (\kappa - 1)\bar{a}S^{\alpha\sigma}\theta_{\sigma\nu}. \tag{40}$$

Using the identity $(S\theta S)^{\mu\nu} = -\frac{1}{2}(S^{\alpha\beta}\theta_{\alpha\beta})S^{\mu\nu}$, we find the inverse[2] of the matrix T,

$$\tilde{T}^\alpha{}_\nu \equiv \delta^\alpha{}_\nu + (\kappa - 1)bS^{\alpha\sigma}\theta_{\sigma\nu}, \quad b = \frac{1}{8m^2c^2 + \kappa(S\theta)}. \tag{41}$$

Using (41), we solve (31) with respect to P^μ. Using the resulting expression in the constraint (24), we obtain $\lambda_1 = \frac{\sqrt{-\dot{x}G\dot{x}}}{m_r c}$, where $m_r^2 \equiv m^2 + \frac{\kappa}{16c^2}(S\theta) - \kappa^2 Z^2$ is the radiation mass in gravitational field. By Z^μ we have denoted the vector, which vanishes in spaces with covariantly constant curvature, $Z^\mu = \frac{b}{8c}S^{\mu\sigma}(\nabla_\sigma R_{\alpha\beta\rho\delta})S^{\alpha\beta}S^{\rho\delta}$. Besides, in the expression for λ_1 appeared a kind of effective metric G induced by spin-gravity interaction along the worldline, $G_{\mu\nu} = \tilde{T}^\alpha{}_\mu g_{\alpha\beta}\tilde{T}^\beta{}_\nu$. Only for the gravimagnetic particle ($\kappa = 1$), the effective metric reduces to the original one. Using (31) and (41), we obtain expression for momentum in terms of velocity

$$P^\mu = \frac{m_r c}{\sqrt{-\dot{x}G\dot{x}}}\tilde{T}^\mu{}_\nu\dot{x}^\nu - \kappa c Z^\mu. \tag{42}$$

We substitute this P^μ into (32) and (33), arriving at the Lagrangian equations of our spinning particle with gravimagnetic moment κ,

$$\nabla\left[\frac{m_r}{\sqrt{-\dot{x}G\dot{x}}}\tilde{T}^\mu{}_\nu\dot{x}^\nu\right] = -\frac{1}{4c}\theta^\mu{}_\nu\dot{x}^\nu - \kappa\frac{\sqrt{-\dot{x}G\dot{x}}}{32m_r c^2}\nabla^\mu(S\theta) + \kappa\nabla Z^\mu, \tag{43}$$

$$\nabla S^{\mu\nu} = -\frac{\kappa\sqrt{-\dot{x}G\dot{x}}}{4m_r c}(\theta S)^{[\mu\nu]} - \frac{2m_r c(\kappa - 1)b}{\sqrt{-\dot{x}G\dot{x}}}\dot{x}^{[\mu}(S\theta\dot{x})^{\nu]} + 2\kappa c\dot{x}^{[\mu}Z^{\nu]}. \tag{44}$$

IV. LEADING POST-NEWTONIAN CORRECTIONS DUE TO UNIT GRAVIMAGNETIC MOMENT

Taking $\kappa = 1$ in (43) and (44), we obtain equations of our gravimagnetic body

$$\nabla\left[\frac{m_r\dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}}\right] = -\frac{1}{4c}\theta^\mu{}_\nu\dot{x}^\nu - \frac{\sqrt{-\dot{x}g\dot{x}}}{32m_r c^2}\nabla^\mu(S\theta) + \nabla Z^\mu, \tag{45}$$

$$\nabla S^{\mu\nu} = -\frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c}(\theta S)^{[\mu\nu]} + 2c\dot{x}^{[\mu}Z^{\nu]}. \tag{46}$$

To test these equations, we compute the leading relativistic corrections due to unit gravimagnetic moment to the trajectory and precession of a gyroscope, orbiting around

[2]We point out that the analogous matrix, present in MPTD equations, cannot be explicitly inverted in the multipole approach.

a rotating spherical body of mass M and angular momentum $\mathbf{J}$. To this aim, we write equations of motion implied by (45) and (46) for the three-dimensional position $x^i(t)$ and for the spin vector

$$\mathbf{S} = \frac{1}{2}(S^{23}, S^{31}, S^{12}), \quad \text{or} \quad S_i(t) = \frac{1}{4}\epsilon_{ijk}S^{jk}(t),$$
$$S^{ij} = 2\epsilon^{ijk}S_k, \tag{47}$$

as functions of the coordinate time $t = \frac{x^0}{c}$. Because of the reparametrization invariance, the desired equations are obtained by setting $\tau = t$ in (45) and (46). We consider separately the trajectory and the spin.

A. Trajectory

We denote $v^\mu \equiv \frac{dx^\mu}{dt} = (c, \mathbf{v})$, so $\sqrt{-\dot{x}g\dot{x}} = \sqrt{-vgv} = \sqrt{-c^2 g_{00} - 2cg_{0i}v^i - g_{ij}v^i v^j}$. The temporal and spatial parts of Eq. (45) read

$$\frac{d}{dt}\left[\frac{m_r}{\sqrt{-vgv}}\right] + \frac{m_r}{c\sqrt{-vgv}}\Gamma^0{}_{\mu\nu}v^\mu v^\nu$$
$$= -\frac{1}{4c^2}\theta^0{}_\nu v^\nu - \frac{\sqrt{-vgv}}{32m_r c^3}\nabla^0(S\theta) + \frac{1}{c}\nabla_t Z^0,$$
$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\mu\nu}v^\mu v^\nu + \frac{v^i\sqrt{-vgv}}{m_r}\frac{d}{dt}\left[\frac{m_r}{\sqrt{-vgv}}\right]$$
$$= -\frac{\sqrt{-vgv}}{4m_r c}\theta^i{}_\nu v^\nu + \frac{vgv}{32m_r^2c^2}\nabla^i(S\theta) + \frac{\sqrt{-vgv}}{m_r}\nabla_t Z^i.$$

Using the first equation in the second one, we avoid the necessity of computing the time derivative in the second term, and obtain

$$\frac{d^2x^i}{dt^2} = -\Gamma^i{}_{\mu\nu}v^\mu v^\nu + \frac{v^i}{c}\Gamma^0{}_{\mu\nu}v^\mu v^\nu - \frac{\sqrt{-vgv}}{4m_r c}\left[\theta^i{}_\nu v^\nu - \frac{v^i}{c}\theta^0{}_\nu v^\nu\right]$$
$$+ \frac{vgv}{32m_r^2c^2}\left[\nabla^i(S\theta) - \frac{v^i}{c}\nabla^0(S\theta)\right]$$
$$+ \frac{\sqrt{-vgv}}{m_r}\left[\nabla_t Z^i - \frac{v^i}{c}\nabla_t Z^0\right]. \tag{48}$$

Now we assume a nonrelativistic motion, $\frac{v}{c} \ll 1$, and expand all quantities in (48) in series with respect to $\frac{1}{c}$. The typical metric of stationary spaces has the series of the form [42]

$$g_{00} = -1 + {}^2g_{00} + {}^4g_{00} + \cdots,$$
$$g_{ij} = \delta_{ij} + {}^2g_{ij} + {}^4g_{ij} + \cdots,$$
$$g_{i0} = {}^3g_{i0} + {}^5g_{i0} + \cdots, \tag{49}$$

where ${}^n g_{\mu\nu}$ denotes the term in $g_{\mu\nu}$ of order $1/c^n$. As a consequence, the series of connection, curvature, and its covariant derivative starts from $\frac{1}{c^2}$ or from higher order. In some details, we have

$$\Gamma^\mu_{\nu\alpha} = {}^2\Gamma^\mu_{\nu\alpha} + {}^4\Gamma^\mu_{\nu\alpha} + \cdots \quad \text{for } \Gamma^i_{00}, \Gamma^i_{mn}, \Gamma^0_{0m}, \tag{50}$$

$$\Gamma^\mu_{\nu\alpha} = {}^3\Gamma^\mu_{\nu\alpha} + {}^5\Gamma^\mu_{\nu\alpha} + \cdots \quad \text{for } \Gamma^i_{0m}, \Gamma^0_{00}, \Gamma^0_{mn}, \tag{51}$$

$$R^\mu{}_{\nu\alpha\beta} = {}^2R^\mu{}_{\nu\alpha\beta} + {}^4R^\mu{}_{\nu\alpha\beta} + \cdots$$
$$\text{for } R^0{}_{mn0}, R^0{}_{0mn}, R^i{}_{0m0}, R^i{}_{jmn}, \tag{52}$$

$$R^\mu{}_{\nu\alpha\beta} = {}^3R^\mu{}_{\nu\alpha\beta} + {}^5R^\mu{}_{\nu\alpha\beta} + \cdots$$
$$\text{for } R^0{}_{imn}, R^0{}_{0m0}, R^i{}_{0mn}, R^i{}_{jm0}. \tag{53}$$

Besides, for various quantities that appear in Eqs. (45) and (46), we have the estimations

$$\sqrt{-vgv} \sim c + \frac{1}{c} + \cdots, \qquad -vgv \sim c^2 + 1 + \frac{1}{c^2} + \cdots,$$
$$m_r^2 \sim m^2 + \frac{1}{c^4} + \cdots, \qquad \theta_{\mu\nu} \sim \frac{1}{c^2} + \cdots,$$
$$b \sim \frac{1}{c^2} + \cdots, \qquad Z^\mu \sim \frac{1}{c^5} + \cdots. \tag{54}$$

At last, the spin supplementary condition implies

$$S^{i0} = \frac{1}{c}S^{ij}v^j + \cdots. \tag{55}$$

Keeping only the terms that may contribute up to order $\frac{1}{c^2}$ in Eq. (48), we obtain

$$\frac{d^2x^i}{dt^2} = -\Gamma^i{}_{\mu\nu}v^\mu v^\nu + \frac{v^i}{c}\Gamma^0{}_{\mu\nu}v^\mu v^\nu$$
$$+ \frac{1}{4m}[v^i\theta^0{}_0 - c\theta^i{}_0 - \theta^i{}_j v^j] - \frac{1}{32m^2}\nabla^i(S\theta). \tag{56}$$

The terms on the right-hand side of this equation are conveniently grouped according to their origin

$$\frac{d^2\mathbf{x}}{dt^2} = \mathbf{a}_\Gamma + \mathbf{a}_R + \mathbf{a}_{\nabla R}. \tag{57}$$

Here $\mathbf{a}_\Gamma$ is the contribution due to connection, $\mathbf{a}_R$ comes from interaction of spin with space-time curvature, and $\mathbf{a}_{\nabla R}$ is the contribution that involves derivatives of the Riemann tensor. Using (50)–(53) we obtain

$$a^i_\Gamma \equiv -\Gamma^i_{\alpha\beta}v^\alpha v^\beta + \frac{v^i}{c}\Gamma^0_{\alpha\beta}v^\alpha v^\beta$$
$$= -c^{2\,2}\Gamma^i_{00} - {}^2\Gamma^i_{mn}v^n v^m + 2v^{i\,2}\Gamma^0_{m0}v^m$$
$$- c^{2\,4}\Gamma^i_{00} + cv^{i\,3}\Gamma^0_{00} - 2c^3\Gamma^i_{m0}v^m, \tag{58}$$

$$a^i_R \equiv \frac{1}{4m}[v^i\theta^0{}_0 - c\theta^i{}_0 - \theta^i{}_j v^j]$$
$$= -\frac{1}{4m}[2{}^2R^i{}_{0m0}S^{mn}v^n + {}^2R^i{}_{kmn}S^{mn}v^k - {}^2R^0{}_{0mn}S^{mn}v^i]$$
$$-\frac{c}{4m}{}^3R^i{}_{0mn}S^{mn}, \tag{59}$$

$$a^i_{\nabla R} \equiv -\frac{1}{32m^2}g^{i\sigma}\nabla_\sigma R_{\alpha\beta\mu\nu}S^{\alpha\beta}S^{\mu\nu}$$
$$= -\frac{1}{32m^2}\partial_i{}^2R_{jklm}S^{jk}S^{lm}. \tag{60}$$

As a concrete example of an external gravitational field, we take a stationary, asymptotically flat metric in the post-Newtonian approximation up to order $\frac{1}{c^4}$ [42],

$$ds^2 = \left(-1 + \frac{2GM}{c^2r} - \frac{2G^2M^2}{c^4r^2}\right)(dx^0)^2 - 4G\frac{\epsilon_{ijk}J^jx^k}{c^3r^3}dx^0dx^i$$
$$+\left(1 + \frac{2GM}{c^2r} + \frac{3G^2M^2}{2c^4r^2}\right)dx^idx^i. \tag{61}$$

It can be obtained taking the asymptotic form of the Kerr metric for a large radial coordinate [43]. With this metric, Eqs. (58)–(60) are[3]

$$\mathbf{a}_\Gamma = -\frac{MG}{r^2}\hat{\mathbf{r}} + \frac{4GM}{c^2r^2}(\hat{\mathbf{r}}\cdot\mathbf{v})\mathbf{v} - \frac{GM}{c^2r^2}v^2\hat{\mathbf{r}} + \frac{4G^2M^2}{c^2r^3}\hat{\mathbf{r}}$$
$$+ 2\frac{G}{c^2}\left[\frac{3(\mathbf{J}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{J}}{r^3}\right]\times\mathbf{v}, \tag{62}$$

$$\mathbf{a}_R = 3\frac{GM}{mc^2r^3}[(\hat{\mathbf{r}}\times\mathbf{v})(\hat{\mathbf{r}}\cdot\mathbf{S}) + \hat{\mathbf{r}}(\mathbf{S}\cdot(\hat{\mathbf{r}}\times\mathbf{v}))]$$
$$-\frac{1}{m}\boldsymbol{\nabla}\left[\frac{G}{c^2}\left(\frac{3(\mathbf{J}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{J}}{r^3}\right)\cdot\mathbf{S}\right], \tag{63}$$

$$\mathbf{a}_{\nabla R} = -\frac{1}{2m}\boldsymbol{\nabla}\left[\frac{G}{c^2}\left(\frac{M}{m}\right)\left(\frac{3(\mathbf{S}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{S}}{r^3}\right)\cdot\mathbf{S}\right]. \tag{64}$$

We denoted by $\hat{\mathbf{r}}$ the unit vector in the direction of $\mathbf{r}$.

B. Spin torque

Setting $\tau = t \equiv \frac{x^0}{c}$ in the spatial part of Eq. (46), this reads

$$\frac{dS^{ij}}{dt} = -\Gamma^i_{\alpha\beta}v^\alpha S^{\beta j} - \Gamma^j_{\alpha\beta}v^\alpha S^{i\beta} + \frac{\sqrt{-vgv}}{4m_rc}\theta^{[i}_\alpha S^{j]\alpha} + 2cv^{[i}Z^{j]}. \tag{65}$$

For the spin vector (47), this equation implies

$$\frac{dS^i}{dt} = -\frac{1}{2}\epsilon^{ijk}\Gamma^j{}_{\mu\nu}v^\mu S^{\nu k} - \frac{\sqrt{-vgv}}{8m_rc}\epsilon^{ijk}\theta^j{}_\nu S^{\nu k} + c\epsilon^{ijk}v^jZ^k. \tag{66}$$

Taking into account Eqs. (50)–(54), we keep only the terms that may contribute up to order $\frac{1}{c^2}$,

$$\frac{dS^i}{dt} = -\frac{1}{2}\epsilon^{ijk}[c\Gamma^j{}_{00}S^{0k} + v^n\Gamma^j{}_{nm}S^{mk} + c\Gamma^j{}_{0n}S^{nk}]$$
$$-\frac{1}{8m}\epsilon^{ijk}\theta^j{}_nS^{nk}$$
$$= S^n({}^2\Gamma^n_{00}v^i + {}^2\Gamma^n_{ik}v^k) - S^i({}^2\Gamma^k_{00} + {}^2\Gamma^l_{kl})v^k + c^3\Gamma^k_{0i}S^k$$
$$+\frac{1}{2m}\epsilon_{mnl}{}^2R^k{}_{imn}S^kS^l. \tag{67}$$

The total torque on the right-hand side of this equation can be conveniently grouped as follows:

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\tau}_v + \boldsymbol{\tau}_J + \boldsymbol{\tau}_R, \tag{68}$$

where $\boldsymbol{\tau}_v$ contains the velocity-dependent terms, $\boldsymbol{\tau}_J$ depends on inner angular momentum of the central body, and $\boldsymbol{\tau}_R$ is due to spin-curvature interaction. Computing these terms for the metric (61), we obtained

$$\boldsymbol{\tau}_v = \frac{GM}{c^2r^2}[2(\mathbf{S}\cdot\hat{\mathbf{r}})\mathbf{v} + (\hat{\mathbf{r}}\cdot\mathbf{v})\mathbf{S} - (\mathbf{S}\cdot\mathbf{v})\hat{\mathbf{r}}], \tag{69}$$

$$\boldsymbol{\tau}_J = \frac{G}{c^2}\left[\frac{3(\mathbf{J}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{J}}{r^3}\right]\times\mathbf{S}, \tag{70}$$

$$\boldsymbol{\tau}_R = \frac{G}{c^2}\left(\frac{M}{m}\right)\left[\frac{3(\mathbf{S}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}}{r^3}\right]\times\mathbf{S}. \tag{71}$$

The torque (68) does not represent a directly measurable quantity. Indeed, evolution of the gyroscope axis is observed in the frame comoving with the gyroscope, so the measurable quantity is $\frac{d\mathbf{S}'}{ds}$, where S'_i are components of spin vector in the rest frame of the gyroscope, and s is its proper time. Magnitudes of the two torques do not coincide, since $\mathbf{S}$ is not a covariant object. According to the classical work of Schiff [40], we can present $\mathbf{S}'$ through $\mathbf{S}$, and then use the resulting relation to compute $\frac{d\mathbf{S}'}{ds}$ through $\frac{d\mathbf{S}}{dt}$ given in (68). The procedure is as follows. First, we use the tetrad formalism, presenting an original metric along an infinitesimal arc of the gyroscope trajectory as $g_{\mu\nu} = \tilde{e}^A_\mu\tilde{e}^B_\nu\eta_{AB}$. Let e^μ_A be the inverse matrix of $\tilde{e}^A_\mu$. Applying a general-coordinate transformation $x^\mu \to x^A$ with the transition functions $\frac{\partial x^\mu}{\partial x^A} = e^\mu_A$, the metric acquires the Lorentz form, $\eta_{AB} = e^\mu_Ae^\nu_Bg_{\mu\nu}$. So the transformed spin tensor, $S^{CD} = \tilde{e}^C_\mu\tilde{e}^D_\nu S^{\mu\nu}$, represents the spin of the gyroscope in a free-falling frame. Second, we apply the Lorentz

[3]The first two terms in $\mathbf{a}_R$ can be written also as follows: $-3\frac{GM}{mc^2r^3}[(\mathbf{v}\times\mathbf{S}) - 2\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot(\mathbf{v}\times\mathbf{S})) - (\hat{\mathbf{r}}\cdot\mathbf{v})(\hat{\mathbf{r}}\times\mathbf{S})]$.

boost ${\Lambda^C}_A(\mathbf{v})$, where $\mathbf{v}$ is velocity of the gyroscope, to make the frame comoving with the gyroscope. This gives the spin tensor $S'^{CD} = {\Lambda^C}_A {\Lambda^D}_B \tilde{e}^A_\mu \tilde{e}^B_\nu S^{\mu\nu}$. Then three-dimensional spin (47) in the comoving frame can be presented through the quantities given in original coordinates as follows:

$$S'_i = \frac{1}{4}\epsilon_{ijk} {\Lambda^j}_A {\Lambda^k}_B \tilde{e}^A_\mu \tilde{e}^B_\nu S^{\mu\nu}. \tag{72}$$

Since our metric is diagonal in $\frac{1}{c^2}$-approximation, the tetrad field is diagonal as well, and reads $\tilde{e}^0_0 = 1 - \frac{GM}{c^2 r}$, $\tilde{e}^i_i = 1 + \frac{GM}{c^2 r}$, $i = 1, 2, 3$, again to $\frac{1}{c^2}$ order. The Lorentz boost is given by the matrix with components ${\Lambda^0}_0 = \gamma$, ${\Lambda^i}_0 = {\Lambda^0}_i = -\gamma\frac{v^i}{c}$, ${\Lambda^i}_j = {\delta^i}_j + \frac{\gamma - 1}{\mathbf{v}^2} v^i v_j$, where $\gamma = (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}$. Using these expressions in Eq. (72), we write it in $\frac{1}{c^2}$-approximation,

$$\mathbf{S}' = \mathbf{S} + \frac{2GM}{c^2 r}\mathbf{S} - \frac{1}{2c^2}[\mathbf{v}^2\mathbf{S} - (\mathbf{v}\cdot\mathbf{S})\mathbf{v}]. \tag{73}$$

To compute derivative $\frac{d}{ds}$ of this expression, we note that the difference between ds and dt can be neglected, being of order $\frac{1}{c^2}$, so we can replace $\frac{d}{ds}$ on $\frac{d}{dt}$ on the right-hand side of (73). For $\frac{d\mathbf{v}}{dt}$ we use its expression (57) in the leading approximation, $\frac{d\mathbf{v}}{dt} = -\frac{MG}{r^2}\hat{\mathbf{r}}$. The result is

$$\frac{d\mathbf{S}'}{ds} = \frac{d\mathbf{S}}{dt} - \frac{GM}{c^2 r^2}\left[(\hat{\mathbf{r}}\cdot\mathbf{v})\mathbf{S} + \frac{1}{2}(\hat{\mathbf{r}}\cdot\mathbf{S})\mathbf{v} + \frac{1}{2}(\mathbf{v}\cdot\mathbf{S})\hat{\mathbf{r}}\right]. \tag{74}$$

We substitute (68) into (74), and then replace $\mathbf{S}$ on $\mathbf{S}'$ in the resulting expression, since according to (73), $\mathbf{S}$ differs from $\mathbf{S}'$ only by terms of order $\frac{1}{c^2}$. The final result for total torque in the rest frame of the gyroscope is

$$\frac{d\mathbf{S}'}{ds} = \boldsymbol{\tau}'_v + \boldsymbol{\tau}'_J + \boldsymbol{\tau}'_R, \tag{75}$$

where

$$\boldsymbol{\tau}'_v = \frac{3GM}{2c^2 r^2}[\hat{\mathbf{r}}\times\mathbf{v}]\times\mathbf{S}', \tag{76}$$

while $\boldsymbol{\tau}'_J$ and $\boldsymbol{\tau}'_R$ are given by (70) and (71), where $\mathbf{S}$ must be replaced on $\mathbf{S}'$.

Comment 1. Curiously enough, spin torque in original coordinates, being averaged over a revolution along an almost closed orbit, almost coincides with instantaneous torque in the comoving frame. This has been observed by direct computation of the mean value of $\frac{d\mathbf{S}}{ds}$; see [44,45]. The same result is implied by Eq. (74), $\langle\frac{d\mathbf{S}'}{ds}\rangle - \langle\frac{d\mathbf{S}}{dt}\rangle \sim \frac{1}{c^2}\langle\frac{d\mathbf{S}}{dt}\rangle \sim \frac{1}{c^4}$, and since $\langle\frac{d\mathbf{S}'}{ds}\rangle \approx \frac{d\mathbf{S}'}{ds}$, we have $\langle\frac{d\mathbf{S}}{dt}\rangle \approx \frac{d\mathbf{S}'}{ds}$.

2. The spin tensor subject to the condition $S^{\mu\nu}P_\nu = 0$ can be used to construct the four-dimensional Pauli-Lubanski vector

$$s_\mu = \frac{\sqrt{-\det g_{\mu\nu}}}{4\sqrt{-P^2}}\epsilon_{\mu\alpha\beta\gamma}P^\alpha S^{\beta\gamma}, \quad \text{where } \epsilon_{0123} = -1. \tag{77}$$

In a free theory, where P^α does not depend on $S^{\beta\gamma}$, this equation can be inverted, so $S^{\beta\gamma}$ and s_μ are mathematically equivalent. Hence spatial components $\mathbf{s}$ could be equally used to describe the spin of a gyroscope [42]. In $\frac{1}{c^2}$-approximation we have $P^\alpha = m\dot{x}^\alpha$, and (77) implies $\mathbf{s} = \mathbf{S}$ in the rest frame of the gyroscope. Under general-coordinate transformations, $\mathbf{S}$ transforms as the spatial part of a tensor, while $\mathbf{s}$ transforms as a part of the four-vector. So the two spins differ in all frames except the rest frame. Let us find the relation between them in $\frac{1}{c^2}$-approximation. Using the approximate equalities $(-vgv)^{-\frac{1}{2}} = \frac{1}{c}(1 + \frac{v^2}{2c^2} + \frac{GM}{c^2 r})$ and $\sqrt{-\det g_{\mu\nu}} = 1 + \frac{2GM}{c^2 r}$ together with Eqs. (42), (54), and (55), we obtain for the spatial part of (77)

$$\mathbf{s} = \frac{1}{\gamma}\mathbf{S} + \frac{1}{c^2}(\mathbf{v}\cdot\mathbf{S})\mathbf{v} + \frac{3GM}{c^2 r}\mathbf{S}. \tag{78}$$

Computing the derivative of this equality and using (68)–(71), we arrive at the following expression for variation rate of $\mathbf{s}$:

$$\frac{d\mathbf{s}}{dt} = \frac{GM}{c^2 r^2}[(\mathbf{s}\cdot\hat{\mathbf{r}})\mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{r}})\mathbf{s} - 2(\mathbf{s}\cdot\mathbf{v})\hat{\mathbf{r}}] + \boldsymbol{\tau}_J + \boldsymbol{\tau}_R. \tag{79}$$

The first term coincides with that of Weinberg [42].

C. Post-Newtonian Hamiltonian

Let us obtain an effective Hamiltonian, which yields Eqs. (57) and (68) in $\frac{1}{c^2}$-approximation. According to the procedure described in [36], the complete Hamiltonian for dynamical variables as functions of the coordinate time t is $H = -cp_0$, where p_0 is a solution to the mass-shell constraint (24) with P_μ given in (3). Solving the constraint, we obtain

$$H = \frac{c}{\sqrt{-g^{00}}}\sqrt{(mc)^2 + \gamma^{ij}P_iP_j + \frac{1}{16}(\theta S)} - c\pi_\mu {\Gamma^\mu}_{0\nu}\omega^\nu + \frac{cg^{0i}}{g^{00}}P_i, \tag{80}$$

where $\gamma^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}$. After tedious computations, this gives the following expression up to $\frac{1}{c^2}$-order:

$$H = mc^2 + \frac{1}{2m}\left[\mathbf{p} + \frac{m}{c}\left(\frac{2G}{c}\left[\mathbf{J}\times\frac{\mathbf{r}}{r^3}\right] + 2\frac{M}{m}\frac{G}{c}\left[\mathbf{S}\times\frac{\mathbf{r}}{r^3}\right]\right)\right]^2 - \frac{(\mathbf{p}^2)^2}{8m^3c^2} - \frac{3GM}{2mc^2r}\mathbf{p}^2 - m\frac{GM}{r} + m\frac{(MG)^2}{2c^2r^2} + \frac{1}{2c}\left(\frac{2G}{c}\left[\nabla\times\left[\mathbf{J}\times\frac{\mathbf{r}}{r^3}\right]\right] + \frac{M}{m}\frac{G}{c}\left[\nabla\times\left[\mathbf{S}\times\frac{\mathbf{r}}{r^3}\right]\right]\right)\cdot\mathbf{S}. \tag{81}$$

Together with the Dirac brackets (15), also taken in $\frac{1}{c^2}$-approximation, this gives Hamiltonian equations of motion. Excluding from them the momentum $\mathbf{p}$, we arrive at the Lagrangian Eqs. (57) and (68).

To write the Hamiltonian in a more convenient form, we introduce[4] vector potential $A_{Ji} = -c^2 g_{0i}$ for the gravitomagnetic field $\mathbf{B}_J$, produced by rotation of the central body (we use the conventional factor $\frac{2G}{c}$, different from that of Wald [28]. In the result, our $\mathbf{B}_J = 4\mathbf{B}_{\mathrm{Wald}}$)

$$\mathbf{A}_J = \frac{2G}{c}\left[\mathbf{J}\times\frac{\mathbf{r}}{r^3}\right],$$

$$\text{then}\ \ \mathbf{B}_J = [\nabla\times\mathbf{A}_J] = \frac{2G}{c}\frac{3(\mathbf{J}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{J}}{r^3}. \tag{82}$$

Then Eq. (81) prompts us to introduce also the vector potential $\mathbf{A}_S$ of fictitious gravitomagnetic field $\mathbf{B}_S$ due to rotation of a gyroscope

$$\mathbf{A}_S = \frac{M}{m}\frac{G}{c}\left[\mathbf{S}\times\frac{\mathbf{r}}{r^3}\right],$$

$$\text{then}\ \mathbf{B}_S = [\nabla\times\mathbf{A}_S] = \frac{M}{m}\frac{G}{c}\frac{3(\mathbf{S}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{S}}{r^3}, \tag{83}$$

as well as the extended momentum

$$\mathbf{\Pi} \equiv \mathbf{p} + \frac{m}{c}(\mathbf{A}_J + 2\mathbf{A}_S). \tag{84}$$

With these notations, the Hamiltonian (81) becomes similar to that of the spinning particle in a magnetic field,

$$H = mc^2 + \frac{1}{2m}\mathbf{\Pi}^2 - \frac{(\mathbf{\Pi}^2)^2}{8m^3c^2} - \frac{3GM}{2mc^2r}\mathbf{\Pi}^2 \tag{85}$$

$$-\frac{mGM}{r} + \frac{m(MG)^2}{2c^2r^2} + \frac{1}{2c}(\mathbf{B}_J + \mathbf{B}_S)\cdot\mathbf{S} = \frac{c}{\sqrt{-g^{00}}}\sqrt{(mc)^2 + g^{ij}\Pi_i\Pi_j} + \frac{1}{2c}(\mathbf{B}_J + \mathbf{B}_S)\cdot\mathbf{S}. \tag{86}$$

[4]We recall [46] that vector potential, produced by a localized current distribution $\mathbf{J}(\mathbf{x}')$ in electrodynamics, is determined, in the leading order, by the vector of magnetic moment $\boldsymbol{\mu} = \frac{1}{2c}\int[\mathbf{x}'\times\boldsymbol{J}(\mathbf{x}')]d^3x$ as follows: $\mathbf{A} = [\boldsymbol{\mu}\times\frac{\mathbf{r}}{r^3}]$, and the corresponding magnetic field is $\mathbf{B} = [\nabla\times\mathbf{A}] = \frac{3(\boldsymbol{\mu}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \boldsymbol{\mu}}{r^3}$.

Note that the Hamiltonian $\frac{c}{\sqrt{-g^{00}}}\sqrt{(mc)^2 + g^{ij}p_ip_j}$ corresponds to the usual Lagrangian $L = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ describing a particle propagating in the Schwarzschild metric $g_{\mu\nu}$. So, the approximate Hamiltonian (86) can be thought of as describing a gyroscope orbiting in the field of Schwarzschild space-time and interacting with the gravitomagnetic field.

The effective Hamiltonian for MPTD equations turns out to be less symmetric: it is obtained from (86) excluding the term $\frac{1}{2c}(\mathbf{B}_S\cdot\mathbf{S})$, while keeping the potential $\mathbf{A}_S$ in (84). Hence the only effect of nonminimal interaction is the deformation of gravitomagnetic field of central body according to the rule

$$\mathbf{B}_J \to \mathbf{B}_J + \mathbf{B}_S. \tag{87}$$

V. CONCLUSION

Starting from a variational problem, we have studied the relativistic spinning particle with nonminimal spin-gravity interaction through the gravimagnetic moment κ. Hamiltonian equations for an arbitrary κ are presented in (31)–(33). When $\kappa = 0$, our variational problem yields MPTD Eqs. (19) and (20), accompanied by the momentum-velocity relation (18) and by the expected constraints (7), (10), and (11). When $\kappa \neq 0$, the MPTD equations are modified by extra terms; see Eqs. (31)–(33) above. The case $\kappa = 1$ is of special interest [compare (34) with MPTD Eqs. (35)], since only for unit gravimagnetic moment, both acceleration and spin torque have reasonable behavior in the ultrarelativistic limit [9].

We have computed, in the coordinate-time parametrization $t = \frac{x^0}{c}$, the acceleration (88)–(90) and the spin torque (69)–(71) of our gravimagnetic particle in the field of a rotating central body (61) in the leading post-Newtonian approximation. We also obtained the approximate Hamiltonian (86), which implies these expressions in the Hamiltonian formulation with use of Dirac brackets. As it should be expected, the expressions (62) and (63) and (69) and (70) coincide with those known from analysis of MPTD equations [41,44,45,47–54]. The new terms due to the nonminimal interaction are (64) and (71). While they are presented in the theory with any $\kappa \neq 0$, we continue with the case $\kappa = 1$, in which the model is consistent in the ultrarelativistic regime.

Using the notation (82) and (83), the total acceleration of the spinning particle in $\frac{1}{c^2}$-approximation reads

$$\mathbf{a} = -\frac{MG}{r^2}\hat{\mathbf{r}} + \frac{4GM}{c^2r^2}(\hat{\mathbf{r}}\cdot\mathbf{v})\mathbf{v} - \frac{GM}{c^2r^2}v^2\hat{\mathbf{r}} + \frac{4G^2M^2}{c^2r^3}\hat{\mathbf{r}} \tag{88}$$

$$+\frac{1}{c}(\mathbf{B}_J + \mathbf{B}_S)\times\mathbf{v} + \frac{GM}{mc^2r^3}[\mathbf{S}\times\mathbf{v} + 3(\mathbf{S}\cdot(\hat{\mathbf{r}}\times\mathbf{v}))\hat{\mathbf{r}}] \tag{89}$$

$$-\frac{1}{2mc}\nabla([\mathbf{B}_J + \mathbf{B}_S]\cdot\mathbf{S}). \tag{90}$$

The first term in (88) represents the standard limit of Newtonian gravity and implies an elliptical orbit. The next three terms represent an acceleration in the orbital plane and are responsible for the precession of perihelia [42,47,48]. The term $\frac{1}{c}\mathbf{B}_J \times \mathbf{v}$ represents the acceleration due to Lense-Thirring rotation of the central body, while the remaining terms in (89) and (90) describe the influence of the gyroscopes spin on its trajectory. The first term in (89) has been computed by Lense and Thirring [50–52]; the remaining terms in (89) have been discussed in [18,28,37]. The gravitational dipole-dipole force $\frac{1}{2mc}\boldsymbol{\nabla}(\mathbf{B}_J \cdot \mathbf{S})$ has been computed by Wald [28]. The new contribution due to nonminimal interaction, $\frac{1}{2mc}\boldsymbol{\nabla}(\mathbf{B}_S \cdot \mathbf{S})$, is similar to the Wald term. The acceleration (89) comes from the second term of effective Hamiltonian (85), while (90) comes from the last term.

The geodetic precession (69) comes from the second term of effective Hamiltonian (85), while the frame-dragging precession (70) is produced by the term $\frac{1}{2c}(\mathbf{B}_J \cdot \mathbf{S})$. So they are the same for both the gravimagnetic and MPTD particle. They have been first computed by Schiff [40], and measured during the Stanford Gravity Probe B experiment [55,56]. The term (71) is due to nonminimal interaction, and appears only for the gravimagnetic particle.

Comparing the expressions (70) and (71), we conclude that precession of spin $\mathbf{S}$ due to nonminimal interaction is equivalent to that caused by rotation of the central body with the momentum $\mathbf{J} = \frac{M}{m}\mathbf{S}$.

The effective Hamiltonian for the case of nonrotating central body (Schwarzschild metric) is obtained from (86) by setting $\mathbf{A}_J = \mathbf{B}_J = 0$. We conclude that, due to the term $\frac{1}{2c}\mathbf{B}_S \cdot \mathbf{S}$, the spin of the gravimagnetic particle will experience the frame-dragging effect (71) even in the field of a nonrotating central body.

In a comoving frame, the gravimagnetic particle experiences the precession $\frac{d\mathbf{S}}{dt} = [\boldsymbol{\Omega} \times \mathbf{S}]$ with angular velocity

$$\boldsymbol{\Omega} = \frac{3GM}{2c^2r^2}[\hat{\mathbf{r}} \times \mathbf{v}] + \frac{1}{2c}\mathbf{B}_J + \frac{1}{c}\mathbf{B}_S, \tag{91}$$

which depends on gyroscopes spin $\mathbf{S}$. Hence, two gyroscopes with different magnitudes and directions of spin precess around different rotation axes. Then the angle between their own rotation axes changes with time in Schwarzschild or Kerr space-time. Since the variation of the angle can be measured with high precision, this effect could be used to find out whether a rotating body has unit or null gravimagnetic moment.

To estimate the relative magnitude of spin torques due to $\mathbf{B}_J$ and $\mathbf{B}_S$, we represent them in terms of angular velocities. Assuming that both bodies are spinning spheres of uniform density, we write $\mathbf{J} = I_1\boldsymbol{\omega}_1$ and $\mathbf{S} = I_2\boldsymbol{\omega}_2$, where $\boldsymbol{\omega}_i$ is angular velocity and $I_i = (2/5)m_i r_i^2$ is moment of inertia. Then the last two terms in (91) read

$$\boldsymbol{\Omega}_{fd} = \frac{2Gm_1r_1^2}{5c^2r^3}[3([\boldsymbol{\omega}_1 + \rho^2\boldsymbol{\omega}_2]\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - (\boldsymbol{\omega}_1 + \rho^2\boldsymbol{\omega}_2)], \tag{92}$$

where $\rho \equiv (r_2/r_1)$. Note that $\boldsymbol{\Omega}_{fd}$ does not depend on mass of the test particle. The ratio $\rho^2 \equiv (r_2/r_1)^2$ is extremely small for the case of the Gravity Probe B experiment, so the MPTD and gravimagnetic bodies are indistinguishable in this experiment. For a system like Sun Mercury $\rho^2 \sim 10^{-5}$. For a system like Sun Jupiter $\rho^2 \sim 10^{-2}$. The new effect could be relevant to the analysis of binary pulsars with massive companions, where the geodetic spin precession has been observed [57–59]. Besides, the two torques could have comparable magnitudes in a binary system with stars of the same size (so $\rho = 1$), but one of them much heavier than the other (neutron star or white dwarf). Then our approximation of a central field is reasonable and, according to Eq. (92), the frame-dragging effect due to the gravimagnetic moment becomes comparable with the Schiff frame-dragging effect.

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