

**Energy in higher-dimensional spacetimes**Hamed Barzegar,<sup>\*</sup> Piotr T. Chruściel,<sup>†</sup> and Michael Hörzinger<sup>‡</sup>*Erwin Schrödinger Institute, Boltzmannngasse 9, A1090 Vienna, Austria and Faculty of Physics, University of Vienna, Boltzmannngasse 5, A1090 Vienna, Austria*

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We derive expressions for the total Hamiltonian energy of gravitating systems in higher-dimensional theories in terms of the Riemann tensor, allowing a cosmological constant  $\Lambda \in \mathbb{R}$ . Our analysis covers asymptotically anti-de Sitter spacetimes, asymptotically flat spacetimes, as well as Kaluza-Klein asymptotically flat spacetimes. We show that the Komar mass equals the Arnowitt-Deser-Misner (ADM) mass in stationary asymptotically flat spacetimes in all dimensions, generalizing the four-dimensional result of Beig, and that this is no longer true with Kaluza-Klein asymptotics. We show that the Hamiltonian mass does not necessarily coincide with the ADM mass in Kaluza-Klein asymptotically flat spacetimes, and that the Witten positivity argument provides a lower bound for the Hamiltonian mass—and not for the ADM mass—in terms of the electric charge. We illustrate our results on the five-dimensional Rasheed metrics, which we study in some detail, pointing out restrictions that arise from the requirement of regularity, which have gone seemingly unnoticed so far in the literature.

DOI: [10.1103/PhysRevD.96.124002](https://doi.org/10.1103/PhysRevD.96.124002)**I. INTRODUCTION**

A key notion in any physical theory is that of total energy, momentum, and similar global charges. The corresponding definitions, and their properties, depend very much upon the asymptotic conditions satisfied by the fields. There are various possibilities here, dictated by the physical problem at hand. For instance, the vanishing and the sign of the cosmological constant play a crucial role. Next, one may find it convenient to use direct coordinate methods [1–3] or conformal methods [4,5], or else [6], to define the asymptotic conditions and the objects at hand. Finally, one may want to use definitions arising from Hamiltonian techniques [7,8], or appeal to the Noether theorem [9], or use *ad hoc* conserved currents [10–14]. See also Ref. [15] for an excellent review of early work on the subject.

A natural class of asymptotic conditions arises when considering isolated systems in Kaluza-Klein-type theories; see Sec. II below. Much to our surprise, no systematic study of the notion of energy in this context appears to exist in the literature, and one of the aims of this work is to fill this gap. For this, we derive new expressions for the total Hamiltonian energy in higher dimensions in terms of the Riemann tensor, in asymptotically flat, asymptotically Kaluza-Klein (KK), or asymptotically anti-de Sitter (AdS) spacetimes. Our definitions arise from a Hamiltonian analysis of the fields and invoke direct coordinate- or tetrad-based asymptotic conditions. We relate these integrals to Komar-type integrals. We use Witten's argument to derive global inequalities between the Hamiltonian energy-momentum and the Kaluza-Klein

charges. We test our energy expressions on the Rasheed family of five-dimensional vacuum metrics, clarifying furthermore some aspects of the global structure of these solutions.

This paper is organized as follows. In Sec. II we make precise our notion of Kaluza-Klein asymptotic flatness. At the beginning of Sec. III we review the definition of energy within the Hamiltonian framework of Refs. [16,17]. In Sec. III A we apply the framework to spacetimes which are asymptotically flat in a Kaluza-Klein sense. In Sec. III B we derive general formulas which apply for a large class of asymptotic conditions. In Sec. IV we show how to rewrite the formulas derived so far in terms of the curvature tensor. This is done in Sec. IV A for KK-asymptotically flat solutions, and in Sec. IV B for general backgrounds. The formulas are then specialized in Sec. IV B 1 to asymptotically anti-de Sitter solutions, and in Sec. IV B 2 to a class of Kaluza-Klein solutions with vanishing cosmological constant which are not KK-asymptotically flat. In Sec. IV C we rewrite some of our Riemann-integral energy expressions in terms of a space-and-time decomposition of the metric. In Sec. V we show how to establish Komar-type expressions for energy in spacetimes with Killing vectors. In Sec. VI we show how a Witten-type positivity argument applies to obtaining global inequalities for KK-asymptotically flat metrics. Appendix A is devoted to a study of the geometry of Rasheed's Kaluza-Klein black holes, which provide a nontrivial family of examples for which our energy expressions can be explicitly calculated.

**II. KALUZA-KLEIN ASYMPTOTICS**

The starting point for our notion of Kaluza-Klein asymptotics is initial data surfaces in an  $(n + K + 1)$ -dimensional spacetime containing asymptotic ends of the form

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$$\mathcal{S}_{\text{ext}} := (\mathbb{R}^n \setminus B(0, R)) \times \underbrace{S^1 \times \cdots \times S^1}_{K \text{ factors}} := (\mathbb{R}^n \setminus B(0, R)) \times \mathbb{T}^K, \quad (2.1)$$

where  $S^1$  is the unit circle. We will say that the metric is KK-asymptotically flat if  $g$  has the following asymptotic form along  $\mathcal{S}_{\text{ext}} \equiv \{x^0 = 0\}$ :

$$g = \underbrace{\eta_{ab} dx^a dx^b + \delta_{AB} dx^A dx^B}_{=: \eta_{\mu\nu} dx^\mu dx^\nu} + o(r^{-\alpha}), \quad \partial_\mu g_{\nu\rho} = o(r^{-\alpha-1}), \quad (2.2)$$

where Greek indices run from 0 to  $n + K$ , uppercase Latin indices from the beginning of the alphabet run from  $n + 1$  to  $n + K$ , lowercase Latin indices from the beginning of alphabet run from 0 to  $n$ , and lowercase Latin indices from the middle of alphabet run from 1 to  $n$ . Finally, uppercase latin indices from the middle of the alphabet run from 1 to  $n + K$ . Summarizing:

$$(x^\mu) \equiv (x^0, x^i, x^A) \equiv (x^a, x^A) \equiv (x^0, x^I). \quad (2.3)$$

Last but not least,

$$r := \sqrt{(x^1)^2 + \cdots + (x^n)^2}.$$

The exponent  $\alpha$  will be chosen to be the optimal one for the purpose of a well-posed definition of the total energy, namely,

$$\alpha = \frac{n-2}{2}, \quad (2.4)$$

where, as in Eq. (2.1),  $n$  is the space dimension *without counting the Kaluza-Klein directions*.

In Kaluza-Klein theories it is often assumed that the vector fields  $\partial_A$  are Killing vectors, but we will not make this assumption unless explicitly indicated otherwise.

### III. HAMILTONIAN CHARGES

In this section we adapt the Hamiltonian analysis of Ref. [17] (based on Ref. [16], cf. Ref. [18]) to the asymptotically KK setting, which also provides convenient alternative expressions for the formulas for the Hamiltonians derived there. We use a background metric  $\bar{g}_{\mu\nu}$ , which is assumed to be asymptotically KK as defined in Sec. II, to determine the asymptotic conditions. The metric  $\bar{g}_{\mu\nu}$  should be thought of as being the metric  $\eta_{\mu\nu}$  of Sec. II at large distances, but it might be convenient in some situations to use coordinate systems where  $\bar{g}_{\mu\nu}$  does not take an explicitly flat form.

Every such metric  $\bar{g}_{\mu\nu}$  determines a family of metrics  $g_{\mu\nu}$  which asymptote to it in the sense of Eq. (2.2). We will

denote by  $\bar{\Gamma}^\alpha_{\beta\gamma}$  the Christoffel symbols of the Levi-Civita connection of  $\bar{g}_{\mu\nu}$ .

Given a vector field  $X$ , the calculations in Ref. [17] showed that the flow of  $X$  in the spacetime obtained by evolving the initial data on  $\mathcal{S}$  is Hamiltonian with respect to a suitable symplectic structure, with a Hamiltonian  $H(X, \mathcal{S})$  which, in vacuum, is given by the formula

$$H(X, \mathcal{S}) = \int_{\mathcal{S}} (p^\mu_{\alpha\beta} \mathcal{L}_X \mathfrak{g}^{\alpha\beta} - X^\mu L) d\Sigma_\mu, \quad (3.1)$$

where

$$L := \mathfrak{g}^{\mu\nu} \left[ (\Gamma^\alpha_{\sigma\mu} - \bar{\Gamma}^\alpha_{\sigma\mu})(\Gamma^\sigma_{\alpha\nu} - \bar{\Gamma}^\sigma_{\alpha\nu}) - (\Gamma^\alpha_{\mu\nu} - \bar{\Gamma}^\alpha_{\mu\nu})(\Gamma^\sigma_{\alpha\sigma} - \bar{\Gamma}^\sigma_{\alpha\sigma}) + \bar{\mathbf{R}}_{\mu\nu} - \frac{2}{d+K} \Lambda g_{\mu\nu} \right] - \frac{1}{16\pi} \sqrt{-\det \bar{g}} \bar{g}^{\mu\nu} \left( \bar{\mathbf{R}}_{\mu\nu} - \frac{2}{d+K} \Lambda \bar{g}_{\mu\nu} \right), \quad (3.2)$$

with  $\bar{\mathbf{R}}_{\mu\nu}$  being the Ricci tensor of the background metric  $\bar{g}_{\mu\nu}$ ,  $\Lambda$  the cosmological constant,  $d$  the dimension of the physical spacetime,  $K$  the number of Kaluza-Klein dimensions (possibly zero), and

$$\mathfrak{g}^{\mu\nu} \frac{1}{16\pi} \sqrt{-\det g} g^{\mu\nu}, \quad p^\lambda_{\mu\nu} := \frac{\partial L}{\partial \mathfrak{g}^{\mu\nu}_{,\lambda}} = (\bar{\Gamma}^\lambda_{\mu\nu} - \delta^\lambda_{(\mu} \bar{\Gamma}^\kappa_{\nu)\kappa}) - (\Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu} \Gamma^\kappa_{\nu)\kappa}). \quad (3.3)$$

Finally, the volume forms  $d\Sigma_\alpha$  and  $d\Sigma_{\alpha\beta}$  are defined as

$$d\Sigma_\alpha = \partial_\alpha \rfloor (dx^0 \wedge \cdots \wedge dx^{n+K}), \quad d\Sigma_{\alpha\beta} = \partial_\beta \rfloor d\Sigma_\alpha, \quad (3.4)$$

where  $\rfloor$  denotes the contraction: for any vector field  $X$  and skew-form  $\alpha$  we have  $X \rfloor \alpha(\cdot, \dots) := \alpha(X, \dots)$ .

We note that the last two,  $g$ -independent “renormalization” terms in Eq. (3.2) have been added for convergence of the integrals at hand.

We will write  $\det g \equiv \det(g_{\mu\nu})$  for the determinant of the full metric tensor, and explicitly write  $\det(g_{IJ})$  for the determinant of the metric  $g_{IJ} dx^I dx^J$  induced on the level sets of  $x^0$ , etc., when the need arises.

We emphasize that the formal considerations in Ref. [17] were quite general, and they apply regardless of the asymptotic conditions and of dimensions. However, the question of the convergence and well posedness of the resulting formulas appears to require a case-by-case analysis, once a set of asymptotic conditions has been imposed.

If  $X$  is a Killing vector field of  $\bar{g}_{\mu\nu}$  and if the Einstein equations with sources and with a cosmological constant  $\Lambda$  are satisfied,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (3.5)$$

the integrand (3.1) can be rewritten as the divergence of a ‘‘Freud-type superpotential,’’ up to source and renormalization terms:

$$\begin{aligned} H^\mu &\equiv p_{\alpha\beta}^\mu \mathcal{L}_X \mathfrak{g}^{\alpha\beta} - X^\mu L \\ &= \partial_\alpha \mathbb{U}^{\mu\alpha} - \sqrt{-\det g} T_{\alpha}^{\mu} X^\alpha + \frac{1}{16\pi} \sqrt{-\det \bar{g}} \bar{g}^{\alpha\beta} \\ &\quad \times \left( \bar{\mathbf{R}}_{\alpha\beta} - \frac{2}{d+K} \Lambda \bar{g}_{\alpha\beta} \right) X^\mu, \end{aligned} \quad (3.6)$$

with

$$\mathbb{U}^{\nu\lambda} = \mathbb{U}^{\nu\lambda}{}_{\beta} X^\beta - \frac{1}{8\pi} \sqrt{|\det g|} g^{\alpha[\nu} \delta_{\beta}^{\lambda]} \bar{\nabla}_\alpha X^\beta, \quad (3.7)$$

$$\mathbb{U}^{\nu\lambda}{}_{\beta} = \frac{2|\det \bar{g}|}{16\pi \sqrt{|\det g|}} g_{\beta\gamma} \bar{\nabla}_\kappa (e^2 g^{\gamma[\lambda} g^{\nu]\kappa}), \quad (3.8)$$

where  $\bar{\nabla}$  denotes the covariant derivative of the background metric  $\bar{g}_{\mu\nu}$  and

$$e^2 \equiv \frac{\det g}{\det \bar{g}}. \quad (3.9)$$

In vacuum this leads to the formula

$$H(X, S) = H_b(X, S) := \frac{1}{2} \int_{\partial S} (\mathbb{U}^{\nu\lambda} - \mathbb{U}^{\nu\lambda}|_{g=\bar{g}}) d\Sigma_{\nu\lambda}, \quad (3.10)$$

where the subscript ‘‘b’’ on  $H_b$  stands for ‘‘boundary.’’ For vector fields  $X$  which are not necessarily Killing vector fields of the background, the Hamiltonian might have some supplementary volume terms, cf. Refs. [18,19]. In non-vacuum Lagrangian diffeomorphism-invariant field theories, this formula for the total Hamiltonian of the coupled system of fields remains true after adding to  $H^\mu$  a contribution from the matter fields; cf., e.g., Refs. [16,19,20].

$$\begin{aligned} \mathbb{U}^{\nu\lambda} &= \mathbb{U}^{\nu\lambda}{}_{\beta} X^\beta \\ &= -\frac{1}{16\pi} (1 + o(r^{-\alpha})) (\eta_{\beta\gamma} + o(r^{-\alpha})) X^\beta [(\eta^{\nu\lambda} \eta^{\mu\kappa} \eta_{\rho\sigma} - \eta^{\nu\lambda} \eta^{\mu\kappa} \eta_{\rho\sigma}) g^{\rho\sigma}{}_{,\kappa} + g^{\nu\lambda}{}_{,\kappa} \eta^{\mu\kappa} - g^{\nu\lambda}{}_{,\kappa} \eta^{\mu\kappa} + \eta^{\nu\lambda} g^{\mu\kappa}{}_{,\kappa} - \eta^{\nu\lambda} g^{\mu\kappa}{}_{,\kappa} + o(r^{-2\alpha-1})] \\ &= -\frac{1}{16\pi} [(\eta^{\lambda\kappa} X^\nu - \eta^{\nu\kappa} X^\lambda) \eta_{\rho\sigma} g^{\rho\sigma}{}_{,\kappa} + \eta^{\nu\kappa} \eta_{\beta\gamma} g^{\gamma\lambda}{}_{,\kappa} X^\beta - \eta^{\lambda\kappa} \eta_{\beta\gamma} g^{\gamma\nu}{}_{,\kappa} X^\beta + g^{\nu\kappa}{}_{,\kappa} X^\lambda - g^{\lambda\kappa}{}_{,\kappa} X^\nu] + o(r^{-2\alpha-1}) \\ &= -\frac{1}{16\pi} \eta^{\delta\kappa} \eta_{\beta\gamma} g^{\gamma\tau}{}_{,\kappa} X^\xi (\delta_\xi^\lambda \delta_\delta^\nu \delta_\tau^\beta - \delta_\xi^\lambda \delta_\delta^\nu \delta_\tau^\beta + \delta_\tau^\lambda \delta_\delta^\nu \delta_\xi^\beta - \delta_\tau^\lambda \delta_\delta^\nu \delta_\xi^\beta + \delta_\xi^\lambda \delta_\tau^\nu \delta_\delta^\beta - \delta_\tau^\lambda \delta_\xi^\nu \delta_\delta^\beta) + o(r^{-2\alpha-1}) \\ &= \frac{3}{8\pi} \eta^{\delta\kappa} \eta_{\beta\gamma} g^{\gamma\tau}{}_{,\kappa} X^\xi \delta_{\tau\delta\xi}^{\nu\lambda\beta} + o(r^{-2\alpha-1}). \end{aligned} \quad (3.15)$$

## A. Kaluza-Klein asymptotics

For Kaluza-Klein asymptotically flat field configurations we have

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + o(r^{-\alpha}), & \partial_\sigma g_{\mu\nu} &= o(r^{-\alpha-1}), \\ \bar{g}_{\mu\nu} &= \eta_{\mu\nu} + o(r^{-\alpha}), & \partial_\sigma \bar{g}_{\mu\nu} &= o(r^{-\alpha-1}). \end{aligned} \quad (3.11)$$

In particular, this implies

$$\bar{\Gamma}^\alpha{}_{\beta\gamma} = o(r^{-\alpha-1}).$$

First, let us assume that  $X$  is  $\bar{g}$ -covariantly constant (and hence also a Killing vector of the background metric  $\bar{g}_{\mu\nu}$ ). One then checks that in the coordinates of Eq. (3.11) the vector field  $X$  has to be of the form

$$X^\mu = X_\infty^\mu + o(r^{-\alpha}), \quad \partial_\nu X_\infty^\mu = 0. \quad (3.12)$$

As  $\Lambda = 0$  in the current case, the convergence of the boundary integrals in vacuum will be guaranteed if one assumes, e.g.,

$$\sum_{\mu\alpha\beta} \int_{S \cap \{r \geq R\}} |\partial_\mu g_{\alpha\beta}|^2 d^{n+K}x < \infty. \quad (3.13)$$

This follows immediately from Stokes’ theorem together with Eqs. (3.1)–(3.3) and (3.6), keeping in mind that  $\Lambda = 0 = \bar{\mathbf{R}}_{\mu\nu}$  in the current context.

We note that Eq. (3.13) will hold if Eq. (2.4) is replaced by  $\alpha > (n-2)/2$ , which provides a sufficient but not a necessary condition.

While we are mostly interested in vacuum solutions, the analysis below applies to nonvacuum ones, provided that one also has

$$T_{\mu\nu} = o(r^{-n}) \quad \text{and} \quad \sum_{\alpha\beta} \int_{S \cap \{r \geq R\}} |T_{\alpha\beta}| d^{n+K}x < \infty. \quad (3.14)$$

Equations (3.13)–(3.14) will be assumed in the calculations that follow.

Since the last term in Eq. (3.7) drops out when  $\bar{\nabla}_\beta X_\alpha = 0$ , we obtain

Plugging the result into Eq. (3.7) and renaming indices, in the limit  $r \rightarrow \infty$ , we obtain the following form of Eq. (3.10), which will be seen to be convenient in our further considerations:

$$H_b(X, \mathcal{S}) = \frac{3}{16\pi} \lim_{R \rightarrow \infty} \int_{S(R) \times \mathbb{T}^K} \delta_{\lambda\mu}^{\alpha\beta\gamma} X^\nu \eta^{\lambda\rho} \eta_{\gamma\sigma} \partial_\rho g^{\sigma\mu} dS_{\alpha\beta}, \quad (3.16)$$

where  $S(R)$  denotes a sphere of radius  $R$  in the  $\mathbb{R}^n$  factor of  $\mathcal{S}_{\text{ext}}$ , and

$$\delta_{\lambda\mu}^{\alpha\beta\gamma} := \delta_{[\lambda}^\alpha \delta_{\mu]}^\beta \delta_{\nu]}^\gamma. \quad (3.17)$$

We see from Eq. (3.12) that  $H_b(X, \mathcal{S})$  can be written as

$$H_b(X, \mathcal{S}) =: p_\mu X^\mu. \quad (3.18)$$

When  $K = 0$ , the coefficients  $p_\mu$  are called the Arnowitt-Deser-Misner (ADM) four-momentum of  $\mathcal{S}$  [1].

If  $X = \partial_0$ , we find a formula somewhat resembling the usual one:

$$\begin{aligned} p_0 &:= H_b(\partial_t, \mathcal{S}) \\ &= \frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} \sum_{I=1}^{n+K} (\partial_I g_{II} - \partial_I g_{II}) \frac{x^I}{R} d^{n+K-1} \mu \\ &= |\mathbb{T}^K| p_{0,\text{ADM}} \\ &\quad + \frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} \sum_{A=n+1}^{n+K} (\partial_A g_{iA} - \partial_i g_{AA}) \frac{x^i}{R} d^{n+K-1} \mu. \end{aligned} \quad (3.19)$$

Here  $d^{n+K-1} \mu$  is the measure induced on  $S(R) \times \mathbb{T}^K$  by the flat metric,  $|\mathbb{T}^K|$  denotes the volume of  $\mathbb{T}^K$ , and  $p_{0,\text{ADM}}$  is the usual (total) ADM energy of the physical-space metric  $g_{ij} dx^i dx^j$ . Perhaps not unexpectedly, the ADM energy  $p_{0,\text{ADM}}$  does *not* coincide with the Hamiltonian generating time translations in general.

Next, when  $X^0 = 0$ , after using Stokes' theorem in the integral

$$\int_{S(R) \times \mathbb{T}^K} \partial_J (g_{J0} \delta_I^L - g_{L0} \delta_I^J) \partial_L (dx^1 \wedge \cdots \wedge dx^{K+n}) = 0, \quad (3.20)$$

we obtain the formula

$$p_I := H_b(\partial_I, \mathcal{S}) = \frac{1}{8\pi R} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} P_{II} x^I d^{n+K-1} \mu. \quad (3.21)$$

Here,  $P_{IJ}$  is the usual canonical ADM momentum

$$\begin{aligned} P_{IJ} &:= g^{LM} k_{LM} g_{IJ} - k_{IJ}, \\ k_{IJ} &:= \frac{1}{2} \mathcal{L}_T g_{IJ} = \frac{1}{2} (\partial_0 g_{IJ} - \partial_I g_{0J} - \partial_J g_{0I}) + o(r^{-2\alpha-1}), \end{aligned} \quad (3.22)$$

while  $\mathcal{L}_T$  denotes the Lie derivative in the direction of the unit-timelike future-directed field  $T$  of normals to the level sets of  $x^0$ .

As an example, we compute the above integrals for the Rasheed metrics, described in Appendix A, with  $P = 0$ :

$$p_0 = 2\pi M, \quad p_i = 0, \quad p_4 = 2\pi Q. \quad (3.23)$$

Equation (3.23) includes a  $2\pi$  factor arising from a normalization in which the Kaluza-Klein coordinate  $x^4$  in the Rasheed solutions runs over a circle of length  $2\pi$ .

This should be compared with the ADM four-momentum  $p_{\mu,\text{ADM}}$  of the  $n$ -dimensional space metric  $g_{ij} dx^i dx^j$ , which reads

$$p_{0,\text{ADM}} = M - \frac{\Sigma}{\sqrt{3}}, \quad p_{i,\text{ADM}} = 0. \quad (3.24)$$

## B. General backgrounds

As discussed in detail in Appendix A 3, the Rasheed solutions with  $P \neq 0$  are not KK-asymptotically flat in the sense set forth above. To cover this case we need to generalize the calculations so far to the case where the background metric is not flat, with an asymptotic region  $\mathcal{S}_{\text{ext}} \subset \mathcal{S}$  diffeomorphic to

$$\mathcal{S}_{\text{ext}} \approx E(R_0), \quad \text{where } E(R) := (\mathbb{R}^n \setminus B(R)) \times {}^K \mathcal{N}, \quad (3.25)$$

with some  $K$ -dimensional compact manifold  ${}^K \mathcal{N}$ , for some  $R_0 \geq 0$ . We therefore have an associated global coordinate system  $x^i$  on  $\mathbb{R}^n \setminus B(R_0)$ , as well as the dilation vector field  $Z = x^i \partial_i \equiv r \partial_r$  which will play a key role in some calculations below.

Somewhat more generally, in order to be able to include general ‘‘Birmingham-Kottler-Schwarzschild anti-de Sitter’’ metrics, we will consider ends  $E(R)$  equipped with a radial function  $r$  so that

$$\mathcal{S}_{\text{ext}} \approx E(R_0), \quad \text{with } E(R) := \{r \geq R\} \equiv [R, \infty) \times \mathcal{K}, \quad (3.26)$$

where  $\mathcal{K}$  is a compact manifold. Here  $r$  is a coordinate running along the  $[R_0, \infty)$  factor of  $\mathcal{S}_{\text{ext}}$ , and the dilation vector  $Z$  is defined as  $Z := r \partial_r$ .

For the usual  $(n+1)$ -dimensional Schwarzschild-anti de Sitter metric the manifold  $\mathcal{K}$  will be an  $(n-1)$ -dimensional sphere, but it can be an arbitrary compact

manifold admitting Einstein metrics in the case of metrics (B1)–(B3) below.

Along  $\mathcal{S}_{\text{ext}}$  we are given two Lorentzian metrics  $g$  and  $\bar{g}$ , with  $g$  asymptotic to the background  $\bar{g}$  in a sense which we make precise now. Denoting by  $\bar{\nabla}$  the Levi-Civita connection associated with  $\bar{g}$ , we assume the existence of a  $\bar{g}$ -orthonormal frame  $\{\bar{e}_{\hat{\mu}}\}$  defined along  $\mathcal{S}_{\text{ext}}$  such that (decorating frame indices with hats)

$$\begin{aligned} g_{\hat{\mu}\hat{\nu}} &:= g(\bar{e}_{\hat{\mu}}, \bar{e}_{\hat{\nu}}) = \bar{g}_{\hat{\mu}\hat{\nu}} + o(r^{-\alpha}), \\ \bar{\nabla}_{\hat{\lambda}} g_{\hat{\mu}\hat{\nu}} &= o(r^{-\beta}). \end{aligned} \quad (3.27)$$

It seems that the specific values of  $\alpha$  and  $\beta$  as needed for our mass formulas can only be chosen after a case-by-case study of the background metric  $\bar{g}$ ; cf. Eqs. (3.31)–(3.32) below.

In what follows we will use the following convention: given two tensor fields  $u$  and  $v$ , we will write

$$u = v + o(r^{-\alpha}) \quad (3.28)$$

if the frame components of  $u - v$ , within the class of  $\bar{g}$ -ON frames chosen, decay as  $o(r^{-\alpha})$ . If  $\bar{e}_{\hat{0}}$  is orthogonal to  $\mathcal{S}_{\text{ext}}$  (which will often be assumed) then, if we denote by

$$\begin{aligned} \alpha \text{ and } \beta \text{ are such that the subleading terms } o(|X|r^{-\alpha-\beta}) \text{ in Eq. (III.29) give} \\ \text{a vanishing contribution to the boundary integrals after passing to the limit.} \end{aligned} \quad (3.31)$$

This will be the case, e.g., for all Rasheed metrics when  $\alpha = (n-2)/2$  as in Eq. (2.4),  $\beta = \alpha + 1$ , with  $X$  asymptotic to  $\partial_{\hat{\mu}}$  in coordinates as in Eq. (A33).

Similarly, Eq. (3.31) will be satisfied for asymptotically anti-de Sitter metrics with

$$\alpha = \beta = n/2, \quad (3.32)$$

where  $r$  is the area coordinate for the anti-de Sitter metric. Note that in this case we have  $|X| = O(r)$ .

Instead of Eq. (3.16) we now obtain

$$H_b(X, \mathcal{S}) = \frac{3}{16\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} \delta_{\lambda\mu\nu}^{\alpha\beta\gamma} X^\nu \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_{\rho} g^{\sigma\mu} dS_{\alpha\beta}, \quad (3.33)$$

where the two-forms  $dS_{\alpha\beta}$  in  $d+K \equiv n+1+K$  space-time dimensions take the form

$$\begin{aligned} dS_{\alpha\beta} &= \frac{1}{(n+K-1)!} \epsilon_{\alpha\beta\xi_1 \dots \xi_{n+K-1}} dx^{\xi_1} \wedge \dots \wedge dx^{\xi_{n+K-1}} \\ &\equiv \partial_{\beta} \rfloor \partial_{\alpha} \rfloor \underbrace{\sqrt{|\det g|} dx^0 \wedge \dots \wedge dx^{n+K}}_{=: d\mu_g}. \end{aligned} \quad (3.34)$$

$\bar{g}_{\mathcal{S}} := \bar{g}_{IJ} dx^I dx^J$  the Riemannian metric induced by  $\bar{g}$  on  $\mathcal{S}_{\text{ext}}$ , and by  $|\cdot|_{\bar{g}_{\mathcal{S}}}$  the associated norm, we have, e.g.,

$$u_{\mu\nu} = o(r^{-\alpha}) \Leftrightarrow |u_{\hat{0}\hat{0}}| + |u_{\hat{0}I} dx^I|_{\bar{g}_{\mathcal{S}}} + |u_{IJ} dx^I dx^J|_{\bar{g}_{\mathcal{S}}} = o(r^{-\alpha}).$$

Assuming again that  $X$  is  $\bar{g}$ -covariantly constant, the second term of Eq. (3.7) vanishes and for the first term we have the same expression as in the KK-asymptotically flat case, with the difference that instead of  $\eta_{\mu\nu}$  we have  $\bar{g}_{\mu\nu}$  and instead of partial derivatives we have covariant derivatives of the background metric, i.e.,

$$\begin{aligned} \mathbb{U}^{\nu\lambda} &= \mathbb{U}^{\nu\lambda}{}_{\xi} X^{\xi} \\ &= \left( \frac{3}{8\pi} \delta_{\tau\delta\xi}^{\nu\lambda\sigma} \bar{g}^{\delta\kappa} \bar{g}_{\sigma\gamma} X^{\xi} \bar{\nabla}_{\kappa} g^{\gamma\tau} + o(|X|r^{-\alpha-\beta}) \right) \sqrt{|\det g|}, \end{aligned} \quad (3.29)$$

where

$$|X|^2 := \sum_{\mu} (X^{\hat{\mu}})^2. \quad (3.30)$$

In order to control the error terms appearing in Eq. (3.29) we will assume that

We can now compute the Hamiltonian charges for this general case. We have

$$\begin{aligned} &\frac{3}{16\pi} \delta_{\lambda\mu\nu}^{\alpha\beta\gamma} X^\nu \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_{\rho} g^{\sigma\mu} dS_{\alpha\beta} \\ &= \frac{1}{16\pi} (\delta_{\lambda\mu}^{\alpha\beta} \delta_{\nu}^{\gamma} + \delta_{\mu\nu}^{\alpha\beta} \delta_{\lambda}^{\gamma} + \delta_{\nu\lambda}^{\alpha\beta} \delta_{\mu}^{\gamma}) X^\nu \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_{\rho} g^{\sigma\mu} dS_{\alpha\beta} \\ &= \frac{1}{16\pi} (X^\gamma \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_{\rho} g^{\sigma\mu} dS_{\lambda\mu} + X^\nu \bar{\nabla}_{\sigma} g^{\sigma\mu} dS_{\mu\nu} \\ &\quad + X^\nu \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_{\rho} g^{\sigma\gamma} dS_{\nu\lambda}). \end{aligned} \quad (3.35)$$

To continue, it is best to use a  $\bar{g}$ -orthonormal frame  $\bar{e}_{\hat{i}}$  with  $\bar{e}_{\hat{0}}$  orthogonal to  $\mathcal{S}$  and  $\bar{e}_{\hat{a}}$  tangent to  $\partial E(R)$ . Then only the forms  $dS_{\hat{0}\hat{i}}$  give a nonvanishing contribution to the boundary integral. In the calculations that follow we will write “n.c.” for the sum of those terms which do not contribute to the integral either because of the integration domain, or by Stokes’ theorem, or by passage to the limit.

If  $X = \partial_0$ , and assuming that

$$\partial_0 = X^{\hat{0}} \bar{e}_{\hat{0}}, \quad (3.36)$$

one finds, using frame indices throughout the calculation,

$$\begin{aligned}
 & \frac{3}{16\pi} \delta_{\lambda\mu}^{\alpha\beta\gamma} X^\nu \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_\rho g^{\sigma\mu} dS_{\alpha\beta} \\
 & \equiv \frac{3}{8\pi} X^{\hat{0}} \delta_{\lambda\hat{\mu}\hat{0}}^{\hat{0}\hat{k}\hat{j}} \bar{g}^{\lambda\hat{\rho}} \bar{g}_{\hat{j}\hat{\nu}} \bar{\nabla}_{\hat{\rho}} g^{\hat{\nu}\hat{\mu}} dS_{\hat{0}\hat{k}} \\
 & \equiv \frac{1}{16\pi} X^{\hat{0}} [\bar{\nabla}^{\hat{k}} (\bar{g}_{\hat{j}\hat{L}} g^{\hat{j}\hat{L}}) - \bar{\nabla}_{\hat{j}} g^{\hat{j}\hat{k}}] d\sigma_{\hat{k}} + \text{n.c.}, \quad (3.37)
 \end{aligned}$$

where

$$\begin{aligned}
 d\sigma_{\hat{i}} & := dS_{\hat{0}\hat{i}} \\
 & \equiv \bar{e}_{\hat{i}} \lrcorner \bar{e}_{\hat{0}} \lrcorner (\sqrt{|\det g|} dx^0 \wedge \cdots \wedge dx^{n+K}) \\
 & \equiv \bar{e}_{\hat{i}} \lrcorner (\sqrt{|\det g_{IJ}|} dx^1 \wedge \cdots \wedge dx^{n+K}) + \text{n.c.} \quad (3.38)
 \end{aligned}$$

Hence, we obtain the following generalization of the ADM energy:

$$\begin{aligned}
 p_0 & = H_b(\partial_t, \mathcal{S}) \\
 & = \frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} X^{\hat{0}} [\bar{\nabla}^i (\bar{g}_{JK} g^{JK}) - \bar{\nabla}_J g^{Ji}] d\sigma_i. \quad (3.39)
 \end{aligned}$$

The existence of the limit in Eq. (3.39) will be guaranteed if, instead of Eqs. (3.13)–(3.14), one now assumes, e.g.,

$$\begin{aligned}
 & \int_{\mathcal{S} \cap \{r \geq R\}} |X| \left( \sum_{\hat{\mu}\hat{\alpha}\hat{\beta}} |\bar{\nabla}_{\hat{\mu}} g_{\hat{\alpha}\hat{\beta}}|^2 + \sum_{\hat{\alpha}\hat{\beta}} |T_{\hat{\alpha}\hat{\beta}}| + |\Lambda| |g^{\mu\nu} (g_{\mu\nu} - \bar{g}_{\mu\nu})| \right) \\
 & \times d\mu_{\bar{g}_S} < \infty, \quad (3.40)
 \end{aligned}$$

where  $d\mu_{\bar{g}_S}$  is the  $(n+K)$ -dimensional Riemannian measure induced on  $\mathcal{S}$  by  $\bar{g}$ . A condition on the metric and the energy-momentum tensor of matter fields naturally associated with Eq. (3.40) is

$$\begin{aligned}
 & \frac{1}{2} (U^{\alpha\beta} - U^{\alpha\beta}|_{g=\bar{g}}) d\Sigma_{\alpha\beta} = \frac{1}{2} \left( \mathbb{U}^{\alpha\beta}{}_{\lambda} X^\lambda - \frac{1}{8\pi} \sqrt{|\det g|} g^{\mu[\alpha} \delta_{\nu}^{\beta]} \bar{\nabla}_\mu X^\nu - U^{\alpha\beta}|_{g=\bar{g}} \right) d\Sigma_{\alpha\beta} \\
 & = \frac{3}{16\pi} \delta_{\lambda\mu\nu}^{\alpha\beta\gamma} X^\nu \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_\rho g^{\sigma\mu} dS_{\alpha\beta} - \frac{1}{16\pi} \left( \sqrt{|\det g|} g^{\mu[\alpha} - \sqrt{|\det \bar{g}|} \bar{g}^{\mu[\alpha} \right) \bar{\nabla}_\mu X^{\beta]} d\Sigma_{\alpha\beta} + \text{n.c.} \\
 & = \frac{1}{8\pi} (3\delta_{\lambda\mu\nu}^{0i\gamma} X^\nu \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_\rho g^{\sigma\mu} - (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]}) N d\sigma_i + \text{n.c.}
 \end{aligned}$$

Here we have used  $dS_{0i} = N d\sigma_i$ , where  $N$  is the lapse function of the foliation by the level sets of  $t$ , defined by writing the metric as  $g = -N^2 dt^2 + g_{IJ} (dx^I + N^I dt)(dx^J + N^J dt)$ . We conclude that

$$\begin{aligned}
 H_b(X, \mathcal{S}) & = \frac{1}{8\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} (3\delta_{\lambda\mu\nu}^{0i\gamma} X^\nu \bar{g}^{\lambda\rho} \bar{g}_{\gamma\sigma} \bar{\nabla}_\rho g^{\sigma\mu} \\
 & \quad - (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]}) N d\sigma_i. \quad (3.43)
 \end{aligned}$$

We can apply the last formula to the background Killing vectors  $\partial_i$  and  $\partial_4$  for Rasheed metrics with  $P \neq 0$ . A calculation gives

$$\begin{aligned}
 & \lim_{R \rightarrow \infty} |X| |\partial E(R)| T_{\hat{\mu}\hat{\nu}} = 0, \\
 & \lim_{R \rightarrow \infty} |X| |\partial E(R)| |\Lambda| |g^{\mu\nu} (g_{\mu\nu} - \bar{g}_{\mu\nu})| = 0, \quad (3.41)
 \end{aligned}$$

where  $|\partial E(R)|$  denotes the area of  $\partial E(R)$ ; cf. Eq. (3.14). This will be assumed whenever relevant.

As an example, we consider the Rasheed metrics of Appendix A with  $P \neq 0$ , which are vacuum. The  $\bar{g}$ -Killing vector  $X = \partial_t$  is  $\bar{g}$ -covariantly constant so that Eq. (3.39) applies. The asymptotic behavior of the metric coefficients in the frame (A39) coincides with the asymptotic behavior of the metric coefficients in manifestly asymptotically Minkowskian coordinates we have seen in the case  $P = 0$ , and is given by Eq. (A33). One obtains

$$p_0 = 4\pi P M, \quad (3.42)$$

where the extra factor  $4P$ , as compared to Eq. (3.23), is due to the  $8P\pi$  periodicity of the coordinate  $x^4$  [cf. Eq. (A37)], as enforced by the requirement of the smoothness of the metric. Note that the formulas (3.24) for the ADM four-momentum remain unchanged.

We emphasize that the calculations above are done at fixed  $P$ , since every  $P$  defines its own class of asymptotic backgrounds. As a result, the phase space of all configurations considered above splits into sectors parametrized by  $P$ . It would be interesting to investigate the question of the existence of a Hamiltonian in a phase space where  $P$  is allowed to vary. We leave this question to future work.

If  $X$  is not  $\bar{g}$ -covariantly constant, the second term of Eq. (3.7) does not vanish. Thus, disregarding those terms which do *not* involve the forms  $d\Sigma_{0i}$ , we obtain (keeping in mind that  $X$  is a Killing vector field of  $\bar{g}$ )

$$p_i = 0, \quad p_4 = 4\pi P Q. \quad (3.44)$$

Here one can note that  $\partial_z$  is  $\bar{g}$ -covariantly constant so that the last term in Eq. (3.43) certainly does not contribute, while  $p_x = p_y = 0$  follows from the axisymmetry of the Rasheed metrics. [In fact,  $\bar{\nabla} X = O(r^{-2})$  or better for these Killing vectors so that the last term never contributes in the current case.]

Equation (3.43) applies for completely general background metrics  $\bar{g}$ , assuming that Eqs. (3.40) and (3.31) hold, for a large class of field equations. In particular, it applies to asymptotically Kottler (“anti-de Sitter”) metrics, cf. Refs. [19,21–23].

#### IV. ENERGY-MOMENTUM AND THE CURVATURE TENSOR

For our further purposes it is convenient to rewrite Eq. (3.10) in terms of the Christoffel symbols. As a first step towards this we note the following consequence of Eq. (3.34):

$$\delta_{\lambda\mu\nu}^{\alpha\beta\gamma} dS_{\alpha\beta} = \frac{1}{3} \cdot \frac{1}{(n+K-2)!} \epsilon_{\lambda\mu\nu\xi_1 \dots \xi_{n+K-2}} dx^\gamma \wedge dx^{\xi_1} \wedge \dots \wedge dx^{\xi_{n+K-2}}. \quad (4.1)$$

##### A. KK-asymptotic flatness

We assume again that  $X$  is  $\bar{g}$ -covariantly constant; of course, it would suffice to assume that  $\bar{\nabla}X$  falls off fast enough to provide a vanishing contribution to the integral defining the Hamiltonian in the limit.

In the KK-asymptotically flat case, Eq. (3.16) can be rewritten as

$$p_\mu X_\infty^\mu = \frac{(-1)^{n+K-1}}{16\pi(n+K-2)!} \lim_{R \rightarrow \infty} \int_{S(R) \times \mathbb{T}^n} \epsilon_{\lambda\mu\nu\xi_1 \dots \xi_{n+K-2}} X^\nu g^{\lambda\rho} \Gamma_{\gamma\rho}^\mu dx^{\xi_1} \wedge \dots \wedge dx^{\xi_{n+K-2}} \wedge dx^\gamma. \quad (4.2)$$

In the standard asymptotically flat case, without Kaluza-Klein directions, Eq. (4.2) can be used to obtain an expression for the ADM energy-momentum in terms of the Riemann tensor, generalizing a similar formula derived by Ashtekar and Hansen in spacetime dimension four [5] (cf. Refs. [4,24]), as follows. We can write

$$\begin{aligned} & \epsilon_{\lambda\mu\nu\xi_1 \dots \xi_{n-2}} X^\nu g^{\lambda\rho} \Gamma_{\gamma\rho}^\mu dx^{\xi_1} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\ &= d(\epsilon_{\lambda\mu\nu\xi_1 \dots \xi_{n-2}} X^\nu x^{\xi_1} g^{\lambda\rho} \Gamma_{\gamma\rho}^\mu dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma) \\ & \quad - (-1)^{n-3} \epsilon_{\lambda\mu\nu\xi_1 \dots \xi_{n-2}} X^\nu g^{\lambda\rho} dx^{\xi_1} dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \\ & \quad \wedge \underbrace{(\partial_\sigma \Gamma_{\gamma\mu\rho} dx^\sigma \wedge dx^\gamma)}_{= \frac{1}{2} R^\mu{}_{\rho\sigma\gamma} dx^\sigma \wedge dx^\gamma} + \text{n.c.} \end{aligned} \quad (4.3)$$

Inserting this into Eq. (4.2) and applying Stokes' theorem, one obtains

$$\begin{aligned} p_\mu X_\infty^\mu &= \frac{1}{32\pi(n-2)!} \lim_{R \rightarrow \infty} \int_{S(R)} \epsilon_{\nu\xi_1 \dots \xi_{n-2}\lambda\mu} X^\nu x^{\xi_1} R^{\lambda\mu}{}_{\delta\gamma} \\ & \quad \times \underbrace{dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\delta \wedge dx^\gamma}_{= \frac{1}{2} \epsilon^{\xi_2 \dots \xi_{n-2} \delta\gamma\mu\nu} dS_{\mu\nu}} \\ &= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \int_{S(R)} X^\mu x^\nu R_{\mu\nu\rho\sigma} dS^{\rho\sigma}, \end{aligned} \quad (4.4)$$

which is the desired new formula.

Let us now pass to a derivation of a version of Eq. (4.4) relevant for Kaluza-Klein asymptotically flat spacetimes. In this case we will be integrating the integrand of Eq. (4.2) over

$$S^{n-1} \times \mathbb{T}^K = S^{d-2} \times \mathbb{T}^K.$$

So only those forms in the sum which contain a  $dx^d \wedge \dots \wedge dx^{d+K-1}$  factor will survive integration. We will use the following symbols:

- (1)  $\mathbf{R}^\alpha{}_{\beta\gamma\delta}$  denotes the Riemann tensor of the  $(d+K)$ -dimensional metric  $g_{\mu\nu} dx^\mu dx^\nu$ .
- (2)  $R^a{}_{bcd}$  denotes the Riemann tensor of the  $d$ -dimensional metric  $g_{ab} dx^a dx^b$ .
- (3)  $\mathbf{R}^I{}_{JKL}$  denotes the Riemann tensor of the  $(n+K)$ -dimensional metric  $g_{IJ} dx^I dx^J$ .
- (4)  $\mathcal{R}^I{}_{jkl}$  denotes the Riemann tensor of the  $n$ -dimensional metric  $g_{ij} dx^i dx^j$ .

No distinction between  $g_{ab} dx^a dx^b$  and  $g_{\mu\nu} dx^\mu dx^\nu$  will be made when  $K=0$ . Keeping in mind that n.c. denotes the sum of those terms which do not contribute to the integral either because of the integration domain, or by Stokes' theorem, or by passage to the limit, we find

$$\begin{aligned} & \epsilon_{\lambda\mu\nu\xi_1 \dots \xi_{d+K-3}} X^\nu g^{\lambda\rho} \Gamma_{\gamma\rho}^\mu dx^{\xi_1} \wedge \dots \wedge dx^{\xi_{d+K-3}} \wedge dx^\gamma \\ &= \frac{(d+K-3)!}{(d-3)!N!} \epsilon_{bcfa_1 \dots a_{d-3} A_1 \dots A_K} X^f g^{be} \Gamma_{ae}^c dx^{a_1} \wedge \dots \wedge dx^{A_K} \wedge dx^a \\ & \quad + \frac{(d+K-3)!}{(d-2)!(N-1)!} \epsilon_{\lambda\mu\nu a_1 \dots a_{d-2} A_1 \dots A_{K-1}} X^\nu g^{\lambda\rho} \Gamma_{A\rho}^\mu dx^{a_1} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A \\ &= \frac{(d+K-3)!}{(d-3)!N!} d(\epsilon_{bcfa_1 \dots a_{d-3} A_1 \dots A_K} X^f \eta^{be} \Gamma_{ae}^c x^{a_1} dx^{a_2} \wedge \dots \wedge dx^{A_K} \wedge dx^a) \\ & \quad - \frac{(d+K-3)!}{(d-3)!N!} \epsilon_{bcfa_1 \dots a_{d-3} A_1 \dots A_K} X^f \eta^{be} x^{a_1} d\Gamma_{ae}^c \wedge \dots \wedge dx^{A_K} \wedge dx^a \\ & \quad + \frac{(d+K-3)!}{(d-2)!(N-1)!} d(\epsilon_{\lambda\mu\nu a_1 \dots a_{d-2} A_1 \dots A_{K-1}} X^\nu \eta^{\lambda\rho} \Gamma_{A\rho}^\mu x^{a_1} dx^{a_2} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A) \\ & \quad - \frac{(d+K-3)!}{(d-2)!(N-1)!} \epsilon_{\lambda\mu\nu a_1 \dots a_{d-2} A_1 \dots A_{K-1}} X^\nu \eta^{\lambda\rho} x^{a_1} d\Gamma_{A\rho}^\mu \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A + \text{n.c.} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(d+K-3)!}{2(d-3)!N!} \epsilon_{bcfa_1\dots a_{d-3}A_1\dots A_K} X^f x^{a_1} \mathbf{R}^{bc}{}_{a_{d-1}a_{d-2}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \dots \wedge dx^{A_K} \wedge dx^{a_{d-2}} \\
 &+ \frac{(d+K-3)!}{(d-2)!(N-1)!} \epsilon_{\lambda\mu\nu a_1\dots a_{d-2}A_1\dots A_{K-1}} X^\nu x^{a_1} \mathbf{R}^{\lambda\mu}{}_{a_{d-1}A} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A + \text{n.c.} \quad (4.5)
 \end{aligned}$$

Using

$$\begin{aligned}
 \epsilon_{\lambda\mu\nu a_1\dots a_{d-2}A_1\dots A_{K-1}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A &= 3(N-1)!(-1)^{d+K-1} \delta_{\lambda\mu\nu}^{Aab} \epsilon_{aba_1\dots a_{d-2}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \dots \\
 &\wedge dx^{a_{d-2}} \wedge dx^{d+1} \wedge \dots \wedge dx^{d+K},
 \end{aligned}$$

after some reordering of indices one obtains

$$\begin{aligned}
 p_\mu X^\mu_\infty &= \frac{(-1)^n}{32\pi(n-1)!} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} x^{a_1} [(n-1) \epsilon_{a_1 a_2 \dots a_{n-2} abc} X^a \mathbf{R}^{bc}{}_{a_{n-1} a_n} \\
 &- \underbrace{\epsilon_{a_1 a_2 \dots a_{n-1} ab} (4X^a \mathbf{R}^{bA}{}_{a_n A} + 2X^A \mathbf{R}^{ab}{}_{a_n A})}_{(*)} dx^{a_2} \wedge \dots \wedge dx^{a_n} \wedge dx^{d+1} \wedge \dots \wedge dx^{d+K}. \quad (4.6)
 \end{aligned}$$

Using

$$dx^{a_2} \wedge \dots \wedge dx^{a_n} \wedge dx^{d+1} \wedge \dots \wedge dx^{d+K} = -\frac{1}{2} \epsilon^{a_2 \dots a_n ef} dS_{ef} \quad (4.7)$$

and

$$\mathbf{R}^{ad}{}_{bd} = \mathbf{R}^a{}_b - \mathbf{R}^{aA}{}_{bA},$$

one obtains for the first term of the Hamiltonian integral [where in the fourth line below we use Eq. (C3)]

$$\begin{aligned}
 &\epsilon_{a_1 a_2 \dots a_{n-2} abc} x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1} a_n} dx^{a_2} \wedge \dots \wedge dx^{a_n} \wedge dx^{d+1} \wedge \dots \wedge dx^{d+K} \\
 &= -\frac{1}{2} \epsilon_{a_1 a_2 \dots a_{n-2} abc} \epsilon^{a_2 \dots a_n ef} x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1} a_n} dS_{ef} \\
 &= \frac{1}{2} (-1)^{n-1} (n-3)! 4! \delta_{a_1 a b c}^{a_{n-1} a_n ef} x^{a_1} X^a \mathbf{R}^{bc}{}_{a_{n-1} a_n} dS_{ef} \\
 &= 2(-1)^{n-1} (n-3)! x^{a_1} X^a (\mathbf{R}^{ef}{}_{a_1 a} + \delta_{a_1 a}^{ef} \mathbf{R}^{bc}{}_{bc} - 4\delta_{[a_1}^{[e} \mathbf{R}^{f]c}{}_{a]c}) dS_{ef} \\
 &= 2(-1)^{n-1} (n-3)! x^{a_1} X^a [\mathbf{R}^{ef}{}_{a_1 a} + \delta_{a_1 a}^{ef} (\mathbf{R}^c{}_c - \mathbf{R}^{cA}{}_{cA}) - 4(\delta_{[a_1}^{[e} \mathbf{R}^{f]a}{}_{a]} - \delta_{[a_1}^{[e} \mathbf{R}^{f]A}{}_{a]A})] dS_{ef}. \quad (4.8)
 \end{aligned}$$

Now, recall that finiteness of the total energy of matter fields together with the dominant energy condition requires, essentially, that

$$T_{\mu\nu} = o(r^{-n}); \quad (4.9)$$

cf. Eq. (3.14). This, together with the Einstein equations, implies that the Ricci-tensor contribution to the integrals

will vanish in the limit  $R \rightarrow \infty$ . Nevertheless, we will keep the Ricci tensor terms for future reference.

Using

$$-\frac{1}{2} \epsilon_{a_1 a_2 \dots a_{n-1} ab} \epsilon^{a_2 \dots a_n ef} dS_{ef} = 3(-1)^n (n-2)! \delta_{a_1 ab}^{a_n ef} dS_{ef},$$

the terms involving (\*) in Eq. (4.6) can be manipulated as

$$\begin{aligned}
 &6(-1)^n (n-2)! \delta_{a_1 ab}^{a_n ef} x^{a_1} (2X^a \mathbf{R}^{bA}{}_{a_n A} + X^A \mathbf{R}^{ab}{}_{a_n A}) dS_{ef} \\
 &= 2(-1)^n (n-2)! x^{a_1} (\delta_{a_1 a}^{a_n e} \delta_b^f + \delta_{a_1 a}^{ef} \delta_b^{a_n} + \delta_{a_1 a}^{f a_n} \delta_b^e) (2X^a \mathbf{R}^{bA}{}_{a_n A} + X^A \mathbf{R}^{ab}{}_{a_n A}) dS_{ef} \\
 &= 2(-1)^n (n-2)! [2(x^{[a_n} X^e] \mathbf{R}^{fA}{}_{a_n A} + x^{[e} X^f] \mathbf{R}^{a_n A}{}_{a_n A} + x^{[f} X^{a_n]} \mathbf{R}^{eA}{}_{a_n A}) \\
 &+ X^A (x^{[a_n} \mathbf{R}^{e]f}{}_{a_n A} + x^{[e} \mathbf{R}^{f]a_n}{}_{a_n A} + x^{[f} \mathbf{R}^{a_n]e}{}_{a_n A})] dS_{ef}.
 \end{aligned}$$

Renaming the indices, rearranging terms, and plugging the results into the integral, one obtains our final expression

$$\begin{aligned}
p_\mu X^\mu &= \frac{1}{32\pi(n-2)} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} \left\{ -2x^b X^a [\mathbf{R}^{ef}{}_{ba} + \delta_{ba}^{ef} (\mathbf{R}^c{}_c - \mathbf{R}^{cA}{}_{cA}) - 4(\delta_{[b}^{[e} \mathbf{R}^{f]a]} - \delta_{[b}^{[e} \mathbf{R}^{f]A}{}_{a]A})] \right. \\
&\quad \left. - 2 \frac{n-2}{n-1} [2(2x^{[b} X^e] \mathbf{R}^{fA}{}_{ba} + x^e X^f \mathbf{R}^{bA}{}_{ba}) + 3X^A x^e \mathbf{R}^{fb}{}_{ba}] \right\} dS_{ef} \\
&= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} \left( X^a x^b \mathbf{R}_{ab}{}^{ef} + 4x^{[e} X^a] \mathbf{R}^f{}_a - x^e X^f \mathbf{R}^c{}_c \right. \\
&\quad \left. - \frac{1}{n-1} [(n-3)x^e X^f \mathbf{R}^{bA}{}_{ba} + 4x^{[e} X^a] \mathbf{R}^{fA}{}_{aA} + 3(n-2)X^A x^e \mathbf{R}^{fb}{}_{ba}] \right) dS_{ef}. \tag{4.10}
\end{aligned}$$

Some special cases are of interest:

- (1) Suppose that  $X^\mu = \delta_0^\mu$ ; thus,  $X$  has only a time component. At  $x^0 = 0$  we have

$$\begin{aligned}
p_0 &= \frac{1}{8(n-2)\pi} \\
&\times \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} \left[ x^j \mathbf{R}_{0j}{}^{0i} + \frac{1}{2} x^i (\mathbf{R}^j{}_j - \mathbf{R}^0{}_0) \right. \\
&\quad \left. - x^j \mathbf{R}^i{}_j - \frac{1}{n-1} \left( \frac{1}{2} x^i \mathbf{R}^{0A}{}_{0A} - x^j \mathbf{R}^{iA}{}_{jA} \right) \right] dS_{0i}, \tag{4.11}
\end{aligned}$$

where the terms involving the Ricci tensor give a vanishing contribution in view of Eq. (4.9) [and similarly for Eq. (4.12) below].

- (2) Suppose that  $X^A = 0$ ; thus,  $X$  has only spacetime components. Then

$$\begin{aligned}
p_a X^a &= \frac{1}{16(n-2)\pi} \\
&\times \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} \left( X^a x^b \mathbf{R}_{ab}{}^{ef} + 4x^{[e} X^a] \mathbf{R}^f{}_a \right. \\
&\quad \left. - x^e X^f \mathbf{R}^c{}_c - \frac{1}{n-1} [(n-3)x^e X^f \mathbf{R}^{bA}{}_{ba} \right. \\
&\quad \left. + 4x^{[e} X^a] \mathbf{R}^{fA}{}_{aA}] \right) dS_{ef}. \tag{4.12}
\end{aligned}$$

We will see below that the first term on the right-hand side is related to the Komar integral. It is not clear whether or not the remaining terms vanish in general. However, when  $X^0 = 0$ , at  $t = 0$  the third term in the integrand gives a vanishing contribution, so that the generators of space translations read

$$\begin{aligned}
p_i X^i &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} \left[ X^i x^k \mathbf{R}_{ik}{}^{0j} \right. \\
&\quad \left. + 2x^{[i} X^j] \left( \frac{2}{n-1} \mathbf{R}^{0A}{}_{iA} + \mathbf{R}^0{}_i \right) \right] dS_{0j}. \tag{4.13}
\end{aligned}$$

We also note that when  $K = 1$  the contribution of the fourth term in the integrand in Eq. (4.12) always vanishes because then, denoting by  $x^4$  the Kaluza-Klein coordinate,

$$\mathbf{R}^{bA}{}_{bA} = \mathbf{R}^{b4}{}_{b4} = \mathbf{R}^{\mu 4}{}_{\mu 4} = \mathbf{R}^4{}_4 = o(r^{-n}),$$

which gives a zero contribution in the limit.

- (iii) Suppose instead that  $X^a = 0$ ; thus,  $X$  has only components tangential to the Kaluza-Klein fibers. Then, again at  $x^0 = 0$ ,

$$\begin{aligned}
p_A X^A &= \frac{3}{16(n-1)\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} X^A x^e \mathbf{R}_{Ab}{}^{fb} dS_{ef} \\
&= \frac{3}{16(n-1)\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^K} X^A x^j \mathbf{R}_{AB}{}^{0B} dS_{0i}, \tag{4.14}
\end{aligned}$$

where the decay  $o(r^{-n})$  of the Ricci tensor of the  $(n+K+1)$ -dimensional metric has been used.

## B. General case

For general background metrics, still assuming a covariantly-constant  $\bar{g}$ -Killing vector, we start by rewriting Eq. (3.33) as

$$\begin{aligned}
H_b(X, \mathcal{S}) &= \frac{(-1)^{n+K-1}}{16\pi(n+K-2)!} \lim_{R \rightarrow \infty} \int_{\partial E(R)} \epsilon_{\lambda\mu\nu\xi_1 \dots \xi_{n+K-2}} \\
&\quad \times X^\nu g^{\lambda\rho} \delta \Gamma^\mu{}_{\nu\rho} dx^{\xi_1} \wedge \dots \wedge dx^{\xi_{n+K-2}} \wedge dx^\nu, \tag{4.15}
\end{aligned}$$

where

$$\Gamma \delta^\alpha{}_{\beta\gamma} := \Gamma^\alpha{}_{\beta\gamma} - \bar{\Gamma}^\alpha{}_{\beta\gamma} = o(r^{-\beta}), \tag{4.16}$$

with the last equality following from Eq. (3.27).

In order to obtain a version of Eq. (4.3) suitable to the current setting, we will assume that there exists a vector field  $Z$  with  $Z^A = 0$  and a real number  $\gamma > 0$  such that

$$\bar{\nabla}_a Z^b = \delta_a^b + O(r^{-\gamma}) \pmod{(\delta_\mu^r, \delta_\mu^t)}. \quad (4.17)$$

Here we write “ $\pmod{(\delta_\mu^r, \delta_\mu^t)}$ ” for a tensor which has the form  $\delta_\mu^{\overset{\circ}{\alpha}} + \delta_\mu^{\overset{\circ}{\beta}}$  for some tensor fields  $\overset{\circ}{\alpha}$  and  $\overset{\circ}{\beta}$ . That is to say, if  $X$  is a vector field tangent to the submanifolds of constant  $t$ ,  $r$ , and if “ $u_{\mu\dots} = 0 \pmod{(\delta_\mu^r, \delta_\mu^t)}$ ,” then  $X^\mu u_{\mu\dots} = 0$ .

We show in Appendix B that the vector field defined in appropriate coordinates as

$$Z = r\partial_r \quad (4.18)$$

satisfies Eq. (4.17) for a) asymptotically anti-de Sitter metrics and b) general Rasheed metrics, in both cases without the error term  $O(r^{-\gamma})$ ; equivalently, the exponent  $\gamma$  can be taken as large as desired. We have introduced the  $O(r^{-\gamma})$  term for possible future generalizations.

We further assume that

$$\bar{\nabla}_\mu X^\nu = O(|X|r^{-\beta}) \pmod{(\delta_\mu^r, \delta_\mu^t)}, \quad (4.19)$$

which will certainly be the case if  $X$  is  $\bar{g}$ -covariantly constant. Last but not least, we replace Eq. (3.31) by the requirement that

$$\begin{aligned} & \text{terms } o(|X|r^{-\alpha-\beta}), o(|Z||X|r^{-2\beta}), \text{ and } o(|X|r^{-\beta-\gamma}) \text{ give a vanishing contribution} \\ & \text{to boundary integrals at fixed } r \text{ and } t, \text{ after passing to the limit } r \rightarrow \infty. \end{aligned} \quad (4.20)$$

Now, the identity that we are about to derive will be integrated on submanifolds of fixed  $r$  and  $t$ , so that any forms containing a factor  $dr$  or  $dt$  will give zero integral. Assuming that there are no Kaluza-Klein directions ( $K = 0$ ), we find

$$\begin{aligned} & d(\epsilon_{\lambda\mu\nu\xi_1\dots\xi_{n-2}} X^\nu Z^{\xi_1} g^{\lambda\rho} \delta\Gamma^\mu_{\gamma\rho} dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma) \\ &= \bar{\nabla}_\sigma (\sqrt{|\det g|} \epsilon_{\lambda\mu\nu\xi_1\dots\xi_{n-2}} X^\nu Z^{\xi_1} g^{\lambda\rho} \delta\Gamma^\mu_{\gamma\rho}) dx^\sigma \wedge dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\ &= Z^{\xi_1} \epsilon_{\lambda\mu\nu\xi_1\dots\xi_{n-2}} g^{\lambda\rho} \delta\Gamma^\mu_{\gamma\rho} \underbrace{\bar{\nabla}_\sigma X^\nu dx^\sigma}_{\text{n.c.}} \wedge dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\ &\quad + \epsilon_{\lambda\mu\nu\xi_1\dots\xi_{n-2}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu_{\gamma\rho} \underbrace{\bar{\nabla}_\sigma Z^{\xi_1} dx^\sigma}_{dx^{\xi_1} + \text{n.c.}} \wedge dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\ &\quad + (-1)^{n-3} \epsilon_{\lambda\mu\nu\xi_1\dots\xi_{n-2}} X^\nu g^{\lambda\rho} Z^{\xi_1} dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge \underbrace{(\bar{\nabla}_\sigma \delta\Gamma^\mu_\gamma dx^\sigma \wedge dx^\gamma)}_{= (\frac{1}{2} \delta R^\mu_{\rho\sigma\gamma} + o(r^{-2\beta})) dx^\sigma \wedge dx^\gamma} + \text{n.c.} \\ &= \epsilon_{\lambda\mu\nu\xi_1\dots\xi_{n-2}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu_{\gamma\rho} dx^{\xi_1} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\gamma \\ &\quad + (-1)^{n-3} \frac{1}{2} \epsilon_{\lambda\mu\nu\xi_1\dots\xi_{n-2}} X^\nu g^{\lambda\rho} \delta R^\mu_{\rho\sigma\gamma} Z^{\xi_1} dx^{\xi_2} \wedge \dots \wedge dx^{\xi_{n-2}} \wedge dx^\sigma \wedge dx^\gamma + \text{n.c.} \end{aligned} \quad (4.21)$$

This identity replaces Eq. (4.3) in the current setting. One can now repeat the remaining calculations of Sec. IV A by replacing every occurrence of the Christoffel symbols by the difference of those of  $g$  and  $\bar{g}$ , every occurrence of the Riemann tensor by the difference of the Riemann tensors of  $g$  and  $\bar{g}$ , and every occurrence of an undifferentiated  $x^\alpha$  by  $Z^\alpha$ . Some care must be taken when generalizing Eq. (4.10) when passing from the background Riemann tensor to the background Ricci tensor, because in Eq. (C1) all indices are lowered and raised with  $g$ . Thus, Eq. (C1) is now replaced by

$$\begin{aligned} & 3! \delta_{\lambda\mu\nu\xi}^{\sigma\gamma\alpha\beta} (\mathbf{R}^\mu_{\rho\sigma\gamma} - \bar{\mathbf{R}}^\mu_{\rho\sigma\gamma}) g^{\lambda\rho} \\ &= (\mathbf{R}^{\alpha\beta}_{\xi\nu} - \bar{\mathbf{R}}^{\alpha\beta}_{\rho\xi\nu} g^{\beta\rho}) + (\mathbf{R} - \bar{\mathbf{R}}_{\rho\lambda} g^{\rho\lambda}) \delta_{\xi\nu}^{\alpha\beta} \\ &\quad - 4\delta_{[\xi}^{\alpha} \mathbf{R}^{\beta]}_{\nu]} + 2\delta_{[\xi}^{\alpha} g^{\beta]\rho} \bar{\mathbf{R}}_{\nu]\rho} - 2\bar{\mathbf{R}}^{\alpha}{}_{\rho\lambda[\xi} \delta_{\nu]}^{\beta]} g^{\rho\lambda}. \end{aligned} \quad (4.22)$$

The simplest situation is obtained when  $K = 0$  so that  $K\mathcal{N}$  is reduced to a point, and Eq. (3.43) becomes

$$\begin{aligned} H_b(X, \mathcal{S}) &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \{ X^\nu Z^\xi (R^{0i}_{\nu\xi} - \bar{R}^{[0}_{\rho\nu\xi} g^{i]\rho}) \\ &\quad + X^{[0} Z^i] (R - \bar{R}_{\rho\lambda} g^{\rho\lambda}) + 2X^\nu Z^{[0} R^i]_{\nu} \\ &\quad - 2Z^\nu X^{[0} R^i]_{\nu} + (Z^\nu X^{[0} g^{i]\rho} - X^\nu Z^{[0} g^{i]\rho}) \bar{R}_{\nu\rho} \\ &\quad - X^{[\nu} Z^i] \bar{R}^0_{\rho\lambda\nu} g^{\rho\lambda} + X^{[\nu} Z^0] \bar{R}^i_{\rho\lambda\nu} g^{\rho\lambda} \\ &\quad - (n-2)(g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^i] \} Nd\sigma_i. \end{aligned} \quad (4.23)$$

### I. $\Lambda \neq 0$

We wish to analyze Eq. (4.23) for metrics  $g$  which asymptote a maximally symmetric background  $\bar{g}$  with

$\Lambda \neq 0$ . This case requires separate attention as then the background curvature tensor does not approach zero as we recede to infinity. We note that the calculations in this section are formally correct independently of the sign of  $\Lambda$ ,

but to the best of our knowledge they are only relevant in the case  $\Lambda < 0$ .

It is useful to decompose the Riemann tensor into its irreducible components,

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= W_{\alpha\beta\gamma\delta} + \frac{1}{d-2}(R_{\alpha\gamma}g_{\beta\delta} - R_{\alpha\delta}g_{\beta\gamma} + R_{\beta\delta}g_{\alpha\gamma} - R_{\beta\gamma}g_{\alpha\delta}) - \frac{R}{(d-1)(d-2)}(g_{\beta\delta}g_{\alpha\gamma} - g_{\beta\gamma}g_{\alpha\delta}) \\ &= W_{\alpha\beta\gamma\delta} + \frac{1}{d-2}(P_{\alpha\gamma}g_{\beta\delta} - P_{\alpha\delta}g_{\beta\gamma} + P_{\beta\delta}g_{\alpha\gamma} - P_{\beta\gamma}g_{\alpha\delta}) + \frac{R}{d(d-1)}(g_{\beta\delta}g_{\alpha\gamma} - g_{\beta\gamma}g_{\alpha\delta}), \end{aligned}$$

where  $W_{\alpha\beta\gamma\delta}$  is the Weyl tensor and  $P_{\alpha\beta}$  is the trace-free part of the Ricci tensor,

$$P_{\mu\nu} = R_{\mu\nu} - \frac{R}{d}g_{\mu\nu}.$$

This leads to the following rewriting of Eq. (4.23):

$$\begin{aligned} H_b(X, \mathcal{S}) &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} \left\{ X^\nu Z^\xi \left( W^{0i}{}_{\nu\xi} - \bar{W}^{[0}{}_{\rho\nu\xi} g^{i]\rho} + \frac{2R}{n(n+1)} \delta_{[\nu}^{[0} \delta_{\xi]}^{i]} - \frac{2\bar{R}}{n(n+1)} \delta_{[\nu}^{[0} \bar{g}_{\xi]\rho} g^{i]\rho} \right) \right. \\ &\quad + X^{[0} Z^i] (R - \bar{R}_{\rho\lambda} g^{\rho\lambda}) + 2X^\nu Z^{[0} R^i]{}_{\nu} - 2Z^\nu X^{[0} R^i]{}_{\nu} + (Z^\nu X^{[0} g^{i]\rho} - X^\nu Z^{[0} g^{i]\rho}) \bar{R}_{\nu\rho} \\ &\quad \left. - \frac{2\bar{R}}{n(n+1)} (X^{[\nu} Z^i] \delta_{[\lambda}^0 \bar{g}_{\nu]\rho} - X^{[\nu} Z^0] \delta_{[\lambda}^i \bar{g}_{\nu]\rho}) g^{\rho\lambda} - (n-2)(g^{\mu 0} - e^{-1} \bar{g}^{\mu 0}) \bar{\nabla}_\mu X^i \right\} N d\sigma_i. \end{aligned} \quad (4.24)$$

Assuming that the background Weyl tensor falls off sufficiently fast so that it does not contribute to the integrals (e.g., vanishes, when the background is a space-form such as the anti-de Sitter metric), that both the energy-momentum tensor of matter and  $e-1$  decay fast enough [cf. Eq. (3.41)], and setting

$$\Delta^{\mu\nu} := g^{\mu\nu} - \bar{g}^{\mu\nu},$$

we obtain

$$\begin{aligned} H_b(X, \mathcal{S}) &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} \left\{ X^\nu Z^\xi \left( W^{0i}{}_{\nu\xi} - \frac{2\bar{R}}{n(n+1)} \delta_{[\nu}^{[0} \bar{g}_{\xi]\rho} \Delta^{i]\rho} \right) - X^{[0} Z^i] \bar{R}_{\rho\lambda} \Delta^{\rho\lambda} + (Z^\nu X^{[0} \Delta^{i]\rho} - X^\nu Z^{[0} \Delta^{i]\rho}) \bar{R}_{\nu\rho} \right. \\ &\quad \left. - \frac{2\bar{R}}{n(n+1)} (X^{[\nu} Z^i] \delta_{[\lambda}^0 \bar{g}_{\nu]\rho} - X^{[\nu} Z^0] \delta_{[\lambda}^i \bar{g}_{\nu]\rho}) \Delta^{\rho\lambda} - (n-2) \Delta^{\mu 0} \bar{\nabla}_\mu X^i \right\} N d\sigma_i, \end{aligned} \quad (4.25)$$

where we have also used the hypothesis (3.31) that terms such as  $|X||Z|\Delta^{\mu\nu}\Delta_{\rho\sigma}$  and  $|X||Z|g_{\mu\nu}\Delta^{\mu\nu}$  fall off fast enough so that they give no contribution to the integral in the limit. With some further work, one gets

$$H_b(X, \mathcal{S}) = \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} \left\{ X^\nu Z^\xi W^{0i}{}_{\nu\xi} + (n-2) \Delta^{\mu 0} \left[ \frac{\bar{R}}{n(n+1)} (X_\mu Z^i - Z_\mu X^i) - \bar{\nabla}_\mu X^i \right] \right\} N d\sigma_i. \quad (4.26)$$

To continue, we assume the Birmingham-Kottler form (B1)–(B3) of the background metric  $\bar{g}$ . If  $X$  is the  $\bar{g}$ -Killing vector field  $\partial_t$  then, writing momentarily  $X_\nu$  for  $\bar{g}_{\nu\mu}X^\mu$ ,

$$\begin{aligned} \bar{\nabla}_\sigma X_\nu dx^\sigma \otimes dx^\nu &= \bar{\nabla}_{[\sigma} X_{\nu]} dx^\sigma \otimes dx^\nu = \partial_{[\sigma} X_{\nu]} dx^\sigma \otimes dx^\nu \\ &= \partial_{[\sigma} \bar{g}_{\nu]0} dx^\sigma \otimes dx^\nu = \frac{1}{2} \partial_r \bar{g}_{00} dx^r \wedge dx^0 \\ &= \frac{1}{2} \partial_r \bar{g}_{00} \bar{\Theta}^1 \wedge \bar{\Theta}^0. \end{aligned}$$

Using this, one checks that all terms linear in  $\Delta$  in Eq. (4.26) cancel out, leading to the elegant formulas

$$\begin{aligned} H_b(X, \mathcal{S}) &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} X^\nu Z^\xi W^{0i}{}_{\nu\xi} N d\sigma_i \\ &= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} X^\nu Z^\xi W^{\alpha\beta}{}_{\nu\xi} dS_{\alpha\beta}, \end{aligned} \quad (4.27)$$

which, at this stage, hold for all  $X$  belonging to the  $(n+1)$ -dimensional family of Killing vectors of the anti-de Sitter background which are normal to  $\{t=0\}$ .

If  $X = \partial_\varphi$ , then we have

$$\begin{aligned}\bar{\nabla}_\sigma X_\nu dx^\sigma \otimes dx^\nu &= \bar{\nabla}_{[\sigma} X_{\nu]} dx^\sigma \otimes dx^\nu = \frac{1}{2} \partial_\sigma \bar{g}_{\nu\varphi} dx^\sigma \wedge dx^\nu = \frac{1}{2} \partial_r \bar{g}_{\varphi\varphi} dr \wedge d\varphi + \frac{1}{2} \partial_\theta \bar{g}_{\varphi\varphi} d\theta \wedge d\varphi \\ &= \sqrt{V} \sin \theta \bar{\Theta}^1 \wedge \bar{\Theta}^3 + \cos \theta \bar{\Theta}^2 \wedge \bar{\Theta}^3,\end{aligned}$$

where we used the coframe of the background metric (B1) with the following cobasis:

$$\bar{\Theta}^0 = \sqrt{V} dt, \quad \bar{\Theta}^1 = \frac{1}{\sqrt{V}} dr, \quad \bar{\Theta}^2 = r d\theta, \quad \bar{\Theta}^3 = r \sin \theta d\varphi. \quad (4.28)$$

Hence, in this coframe one obtains

$$\bar{\nabla}_1 X_3 = \sqrt{V} \sin \theta = -\bar{\nabla}_3 X_1, \quad \bar{\nabla}_2 X_3 = \cos \theta = -\bar{\nabla}_3 X_2.$$

Therefore, the second term of the integrand in Eq. (4.26) vanishes for  $r \rightarrow \infty$ , since (keeping in mind that  $dS_{\hat{0}\hat{i}}$  for  $i \neq 1$  gives zero contribution to the integrals)

$$\begin{aligned}\frac{\bar{R}}{n(n+1)} \Delta^{\mu[0}(X_\mu Z^1] - Z_\mu X^1] - \Delta^{\mu[0} \bar{\nabla}_\mu X^1] &= -\frac{\lambda}{2} \Delta^{\hat{\mu}\hat{0}} X_{\hat{\mu}} Z^{\hat{1}} - \frac{1}{2} \Delta^{\hat{\mu}\hat{0}} \bar{\nabla}_{\hat{\mu}} X^{\hat{1}} = -\frac{\lambda}{2} \Delta^{\hat{3}\hat{0}} \underbrace{X_{\hat{3}}}_{=(\bar{g}_{33} + \Delta_{33})X^{\hat{3}}} Z^{\hat{1}} - \frac{1}{2} \Delta^{\hat{3}\hat{0}} \bar{\nabla}_{\hat{3}} X^{\hat{1}} \\ &= \left( -\frac{\lambda}{2} \cdot \frac{r^2}{\sqrt{V}} + \frac{1}{2} \sqrt{V} \right) \sin \theta \Delta^{\hat{3}\hat{0}} \xrightarrow{r \rightarrow \infty} 0.\end{aligned}$$

Hence, Eq. (4.27) also holds for  $X = \partial_\varphi$ . Since all Killing vectors of AdS spacetime can be obtained as linear combinations of images of these two vectors by isometries preserving  $\{t = 0\}$ , we conclude that Eq. (4.27) holds for all Killing vectors of the AdS metric.

Once this work was completed, we were informed that Eq. (4.27) had already been observed in Ref. [25], following up on the pioneering definitions in Refs. [26,27]. We note that our conditions for the equality in Eq. (4.27) are quite weaker than those in Ref. [25].

## 2. $\Lambda = 0$

We pass to the case  $\Lambda = 0$ . We will impose conditions which guarantee that all terms which are quadratic or higher in  $g_{\mu\nu} - \bar{g}_{\mu\nu}$  give zero contribution to the integrals in the limit  $R \rightarrow \infty$ . Without these assumptions the final formulas become unreasonably long. Hence, we assume Eqs. (4.16), (4.17), (4.19), and (4.20).

In the current context, the calculation (4.5) is replaced by

$$\begin{aligned}&\epsilon_{\lambda\mu\nu\xi_1 \dots \xi_{d+K-3}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu_{\gamma\rho} dx^{\xi_1} \wedge \dots \wedge dx^{\xi_{d+K-3}} \wedge dx^\gamma \\ &= \frac{(d+K-3)!}{(d-3)!N!} \epsilon_{bcfa_1 \dots a_{d-3} A_1 \dots A_K} X^f g^{be} \delta\Gamma^c_{ae} \underbrace{dx^{a_1} \wedge \dots \wedge dx^{A_K}}_{\bar{\nabla}_h Z^{a_1} dx^h + \text{n.c.} = \delta_h^{a_1} dx^h + \text{n.c.}} \wedge dx^a \\ &+ \frac{(d+K-3)!}{(d-2)!(N-1)!} \epsilon_{\lambda\mu\nu a_1 \dots a_{d-2} A_1 \dots A_{K-1}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu_{A\rho} dx^{a_1} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A \\ &= \frac{(d+K-3)!}{(d-3)!N!} [\bar{\nabla}_h (\epsilon_{bcfa_1 \dots a_{d-3} A_1 \dots A_K} X^f g^{be} \delta\Gamma^c_{ae} Z^{a_1}) dx^h \wedge dx^{a_2} \wedge \dots \wedge dx^{A_K} \wedge dx^a \\ &- \underbrace{\epsilon_{bcfa_1 \dots a_{d-3} A_1 \dots A_K} \bar{\nabla}_h X^f g^{be} \delta\Gamma^c_{ae} Z^{a_1}}_{\text{n.c.}} dx^h \wedge dx^{a_2} \wedge \dots \wedge dx^{A_K} \wedge dx^a \\ &- \epsilon_{bcfa_1 \dots a_{d-3} A_1 \dots A_K} X^f g^{be} Z^{a_1} \bar{\nabla}_h \delta\Gamma^c_{ae} dx^h \wedge dx^{a_2} \wedge \dots \wedge dx^{A_K} \wedge dx^a + \text{n.c.}] \\ &+ \frac{(d+K-3)!}{(d-2)!(N-1)!} [\bar{\nabla}_h (\epsilon_{\lambda\mu\nu a_1 \dots a_{d-2} A_1 \dots A_{K-1}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu_{A\rho} Z^{a_1}) dx^h \wedge dx^{a_2} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A \\ &- \underbrace{\epsilon_{\lambda\mu\nu a_1 \dots a_{d-2} A_1 \dots A_{K-1}} \bar{\nabla}_h X^\nu g^{\lambda\rho} \delta\Gamma^\mu_{A\rho} Z^{a_1}}_{\text{n.c.}} dx^h \wedge dx^{a_2} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A \\ &- \epsilon_{\lambda\mu\nu a_1 \dots a_{d-2} A_1 \dots A_{K-1}} X^\nu g^{\lambda\rho} Z^{a_1} \bar{\nabla}_h \delta\Gamma^\mu_{A\rho} dx^h \wedge dx^{a_2} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A + \text{n.c.}]\end{aligned}$$

$$\begin{aligned}
&= \frac{(d+K-3)!}{(d-3)!N!} d(\epsilon_{bcfa_1\dots a_{d-3}A_1\dots A_K} X^f g^{be} \delta\Gamma^c_{ae} Z^{a_1} dx^{a_2} \wedge \dots \wedge dx^{A_K} \wedge dx^a) \\
&+ \frac{(d+K-3)!}{(d-2)!(N-1)!} d(\epsilon_{\lambda\mu\nu a_1\dots a_{d-2}A_1\dots A_{K-1}} X^\nu g^{\lambda\rho} \delta\Gamma^\mu_{A\rho} Z^{a_1} dx^h \wedge dx^{a_2} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A) \\
&- \frac{(d+K-3)!}{(d-3)!N!} \epsilon_{bcfa_1\dots a_{d-3}A_1\dots A_K} X^f x^{a_1} g^{be} \delta\mathbf{R}^c_{ea_{d-1}a_{d-2}} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \dots \wedge dx^{A_K} \wedge dx^{a_{d-2}} \\
&- \frac{(d+K-3)!}{2(d-3)!N!} \epsilon_{\lambda\mu\nu a_1\dots a_{d-2}A_1\dots A_{K-1}} X^\nu x^{a_1} g^{\lambda\rho} \delta\mathbf{R}^\mu_{\rho a_{d-1}A} dx^{a_{d-1}} \wedge dx^{a_2} \wedge \dots \wedge dx^{A_{K-1}} \wedge dx^A + \text{n.c.}
\end{aligned}$$

As before, in the last equality we have used the fact that the first  $\bar{\nabla}_h$  terms in the first expression in each of the square brackets can be replaced by  $\bar{\nabla}_\mu$ , because each form appearing in the first line above must already contain  $K$  factors of the  $KK$  differentials  $dx^A$ ; otherwise, it will give zero contribution to the integral.

In addition to all of the hypotheses so far, we will also assume that the Riemann tensor decays at a rate  $o(r^{-\beta_R})$ :

$$\mathbf{R}^\alpha_{\beta\gamma\delta} = o(r^{-\beta_R}), \quad \bar{\mathbf{R}}^\alpha_{\beta\gamma\delta} = o(r^{-\beta_R}), \quad (4.29)$$

with  $\beta_R$  chosen so that

terms  $|X||Z|o(r^{-\alpha-\beta_R})$  give no contribution to the integral in the limit  $R \rightarrow \infty$ .

All of these conditions are satisfied by the five-dimensional Rasheed metrics, with  $\alpha > 0$  as close to one as one wishes,  $\beta = 1 + \alpha$ ,  $\beta_R = 3$ , and with  $\gamma$  as large as desired.

In line with our previous notation, we will write  $\mathbf{R}^\alpha_{\beta\gamma\delta} - \bar{\mathbf{R}}^\alpha_{\beta\gamma\delta}$  for the difference of Riemann tensors of the  $(d+K)$ -dimensional metrics  $g_{\mu\nu} dx^\mu dx^\nu$  and  $\bar{g}_{\mu\nu} dx^\mu dx^\nu$ ,  $R^a_{bcd} - \bar{R}^a_{bcd}$  for that of the  $d$ -dimensional metrics  $g_{ab} dx^a dx^b$  and  $\bar{g}_{ab} dx^a dx^b$ ,  $\mathbf{R}^I_{JKL} - \bar{\mathbf{R}}^I_{JKL}$  for that of the  $(n+K)$ -dimensional metrics  $g_{IJ} dx^I dx^J$  and  $\bar{g}_{IJ} dx^I dx^J$ , and  $\mathcal{R}^i_{jk\ell} - \bar{\mathcal{R}}^i_{jk\ell}$  for that of the  $n$ -dimensional metrics  $g_{ij} dx^i dx^j$  and  $\bar{g}_{ij} dx^i dx^j$ .

With the above hypotheses, the derivation of the key formula (4.10) follows closely the remaining calculations in Sec. IV A, and leads to

$$\begin{aligned}
H_b(X, \mathcal{S}) &= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \left\{ \int_{\partial E(R)} \left( X^a Z^b (\mathbf{R}_{ab}{}^{ef} - \bar{\mathbf{R}}_{ab}{}^{ef}) + 4Z^{[e} X^{a]} (\mathbf{R}^f{}_a - \bar{\mathbf{R}}^f{}_a) \right. \right. \\
&- Z^e X^f (\mathbf{R}^c{}_c - \bar{\mathbf{R}}^c{}_c) - \frac{1}{n-1} [(n-3)Z^e X^f (\mathbf{R}^{bA}{}_{bA} - \bar{\mathbf{R}}^{bA}{}_{bA}) + 4Z^{[e} X^{a]} (\mathbf{R}^{fA}{}_{aA} - \bar{\mathbf{R}}^{fA}{}_{aA}) \\
&\left. \left. + 3(n-2)X^A Z^e (\mathbf{R}^{fb}{}_{bA} - \bar{\mathbf{R}}^{fb}{}_{bA}) \right] \right\} dS_{ef} - (n-2) \int_{\partial E(R)} (g^{\mu[a} - e^{-1} \bar{g}^{\mu[a}]) \bar{\nabla}_\mu X^{b]} dS_{ab}. \quad (4.31)
\end{aligned}$$

For Rasheed solutions, or more generally for solutions which asymptote to the Rasheed backgrounds  $\bar{g}$  given by Eq. (A34) with the usual decay  $o(r^{-(n-2)/2})$ , with  $T_{\mu\nu} = o(r^{-3})$ , one has [cf. Eqs. (A43)–(A44)]  $\bar{\mathbf{R}}^0_{\lambda\mu\nu} = 0$ ,  $\bar{\mathbf{R}}^{\hat{\mu}}_{\hat{4}\hat{\alpha}\hat{4}} = O(r^{-4})$ , and  $\bar{\mathbf{R}}_{\mu\nu} = O(r^{-4})$ . Thus, for  $X = \partial_\mu$  and after passing to the limit  $R \rightarrow \infty$ , we obtain an integrand which is formally identical to that for metrics which are  $KK$ -asymptotically flat:

$$\begin{aligned}
H_b(X, \mathcal{S}) &= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \int_{S(R) \times S^1} X^\nu Z^\mu \mathbf{R}^{\alpha\beta}_{\nu\mu} dS_{\alpha\beta} \\
&= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \int_{S(R) \times S^1} X^\nu x^j \mathbf{R}^{0i}_{\nu j} N d\sigma_i. \quad (4.32)
\end{aligned}$$

Some special cases, without necessarily assuming that  $g$  asymptotes to the Rasheed background, are of interest:

- (1) Suppose that  $X^\mu = \delta_0^\mu$ ; thus,  $X$  has only a time component. Keeping in mind that  $Z^0 = 0$  and  $\partial E(R) \subset \{x^0 = 0\}$ , we have

$$\begin{aligned}
 H_b(\partial_0, \mathcal{S}) &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \left\{ \int_{\partial E(R)} \left( Z^j(\mathbf{R}_{0j}{}^{0i} - \bar{\mathbf{R}}_{0j}{}^{0i}) + \frac{1}{2} Z^i(\mathbf{R}^j{}_j - \mathbf{R}^0{}_0 + \bar{\mathbf{R}}^j{}_j - \bar{\mathbf{R}}^0{}_0) - Z^j(\mathbf{R}^i{}_j - \bar{\mathbf{R}}^i{}_j) \right. \right. \\
 &\quad \left. \left. - \frac{1}{2(n-1)} [Z^i(\mathbf{R}^{0A}{}_{0A} - \bar{\mathbf{R}}^{0A}{}_{0A}) - 2Z^j(\mathbf{R}^{iA}{}_{jA} - \bar{\mathbf{R}}^{iA}{}_{jA})] \right) dS_{0i} \right. \\
 &\quad \left. - (n-2) \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\} \\
 &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \left\{ \int_{\partial E(R)} \left( \frac{1}{2} Z^i(\mathbf{R}^j{}_j - \bar{\mathbf{R}}^j{}_j) - Z^j(\mathbf{R}_{Ij}{}^{Ii} - \bar{\mathbf{R}}_{Ij}{}^{Ii}) \right. \right. \\
 &\quad \left. \left. - \frac{1}{2(n-1)} [Z^i(\mathbf{R}^{0A}{}_{0A} - \bar{\mathbf{R}}^{0A}{}_{0A}) - 2Z^j(\mathbf{R}^{iA}{}_{jA} - \bar{\mathbf{R}}^{iA}{}_{jA})] \right) dS_{0i} \right. \\
 &\quad \left. - (n-2) \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\}. \tag{4.33}
 \end{aligned}$$

- (2) Suppose that  $X^A = 0$ ; thus,  $X$  has only spacetime components. Then

$$\begin{aligned}
 H_b(X, \mathcal{S}) &= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \left\{ \int_{\partial E(R)} \left( X^a Z^b(\mathbf{R}_{ab}{}^{ef} - \bar{\mathbf{R}}_{ab}{}^{ef}) + 4Z^{[e} X^a](\mathbf{R}^f{}_a - \bar{\mathbf{R}}^f{}_a) \right. \right. \\
 &\quad \left. \left. - Z^e X^f(\mathbf{R}^c{}_c - \bar{\mathbf{R}}^c{}_c) - \frac{1}{n-1} [(n-3)Z^e X^f(\mathbf{R}^{bA}{}_{bA} - \bar{\mathbf{R}}^{bA}{}_{bA}) + 4Z^{[e} X^a](\mathbf{R}^{fA}{}_{aA} - \bar{\mathbf{R}}^{fA}{}_{aA})] \right) dS_{ef} \right. \\
 &\quad \left. - 2(n-2) \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\}. \tag{4.34}
 \end{aligned}$$

We will see below that the first term on the right-hand side is related to the Komar integral. It is not clear whether or not the remaining terms vanish in general. However, when  $X^0 = 0$ , at  $t = 0$  the third and fourth terms in the integrand in Eq. (4.34) give a vanishing contribution so that the generators of space translations read

$$\begin{aligned}
 H_b(X, \mathcal{S}) &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \left\{ \int_{\partial E(R)} \left( X^i Z^k(\mathbf{R}_{ik}{}^{0j} - \bar{\mathbf{R}}_{ik}{}^{0j}) + Z^i X^j \left[ \frac{2}{n-1} (\mathbf{R}^{0A}{}_{iA} - \bar{\mathbf{R}}^{0A}{}_{iA}) + \mathbf{R}^0{}_i - \bar{\mathbf{R}}^0{}_i \right] \right) dS_{0j} \right. \\
 &\quad \left. - (n-2) \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\}. \tag{4.35}
 \end{aligned}$$

- (3) Suppose instead that  $X^a = 0$ ; thus,  $X$  has only components tangential to the Kaluza-Klein fibers. Then, again at  $x^0 = 0$ ,

$$H_b(X, \mathcal{S}) = \lim_{R \rightarrow \infty} \left\{ \frac{3}{16(n-1)\pi} \int_{\partial E(R)} X^A Z^e (\mathbf{R}_{AB}{}^{fb} - \bar{\mathbf{R}}_{Ab}{}^{fb}) dS_{ef} - \frac{1}{8\pi} \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0}) \bar{\nabla}_\mu X^{i]} dS_{0i} \right\}. \tag{4.36}$$

### C. $(n+K)+1$ -decomposition

In a Cauchy-data context it is convenient to express the global charges explicitly in terms of Cauchy data. Here one can use the Gauss-Codazzi-Mainardi embedding equations to reexpress our spacetime-Riemann-tensor integrals in terms of the Riemann tensor of the initial-data

metric and of the extrinsic curvature tensor. For this we consider  $X^\mu = \delta_0^\mu$  and  $x^0 = 0$ , i.e., we consider Eq. (4.33).

We start with the case of KK-asymptotically flat initial data sets. Keeping in mind our convention that  $(x^I) = (x^i, x^A)$ , we can replace  $\mathbf{R}^i{}_{JKL}$  with the  $(n+K)$ -dimensional

Riemann tensor, which we denote by  $\mathbf{R}^I_{JKL}$ , by means of the Gauss-Codazzi relation

$$\mathbf{R}^I_{JKL} = \mathbf{R}^I_{JKL} + o(r^{-2\alpha-2}). \quad (4.37)$$

Hence, from Eq. (4.11) we obtain

$$p_0 = -\frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{\mathbb{T}^N} x^j \left( \mathbf{R}^i_j + \frac{1}{2(n-1)} \mathbf{R}^k_k \delta^i_j + \frac{1}{n-1} \mathbf{R}^{iA}_{jA} \right) N d\sigma_i. \quad (4.38)$$

We note that in the usual asymptotically flat case,  $K = 0$ , the last integral is not present. Further,  $\mathbf{R}^k_k$  then becomes the Ricci scalar of the initial data metric, with  $\mathbf{R}^k_k = o(r^{-2\alpha-2})$  because of the scalar constraint equation, and hence does not contribute to the integral. Thus, the above reproduces the well-known-by-now formula for the ADM mass in terms of the Ricci tensor of the initial data metric [28–31] when the Ricci scalar decays fast enough, as we assumed here.

We pass now to the case covered in Sec. IV B 1, namely,  $K = 0$  but  $\Lambda < 0$ , with the background metric  $\bar{g}$  as in

Eqs. (B1)–(B3). Let  $k_{IJ}$  be the extrinsic curvature tensor of the slices  $\{x^0 = \text{const}\}$ . If we assume that  $k_{IJ}$  satisfies

$$|k| := \sqrt{g^{IJ} g^{LM} k_{IL} k_{JM}} = o(r^{-n/2}), \quad (4.39)$$

from Eq. (4.27) we obtain a formula first observed in Ref. [30]:

$$H_b(X, \mathcal{S}) = -\frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} X^0 Z^j \left( \mathbf{R}^i_j - \frac{\mathbf{R}}{n} \delta^i_j \right) \times N d\sigma_i, \quad (4.40)$$

where in Eq. (4.40) we have assumed that  $X$  is a Killing vector of the anti-de Sitter background which is normal to the hypersurface  $\{t = 0\}$ .

Finally, consider general configurations as in Sec. IV B 2. Under the hypothesis that

$$|k|^2 |Z| |\partial E(R)| \rightarrow_{R \rightarrow \infty} 0, \quad (4.41)$$

from Eq. (4.33) we find

$$\begin{aligned} H_b(\partial_0, \mathcal{S}) &= \frac{1}{8(n-2)\pi} \lim_{R \rightarrow \infty} \left\{ \int_{\partial E(R)} \left( \left[ \frac{1}{2} Z^i (\mathbf{R}^j_j - \bar{\mathbf{R}}^j_j) - Z^j (\mathbf{R}^i_j - \bar{\mathbf{R}}^i_j) \right] \right. \right. \\ &\quad \left. \left. - \frac{1}{2(n-1)} [Z^i ((\mathbf{R}^A_A - \bar{\mathbf{R}}^A_A) - (\mathbf{R}^A_A - \bar{\mathbf{R}}^A_A)) - 2Z^j ((\mathbf{R}^i_j - \bar{\mathbf{R}}^i_j) - (\mathbf{R}^{ik}_{jk} - \bar{\mathbf{R}}^{ik}_{jk}))] \right) N d\sigma_i \right. \\ &\quad \left. - (n-2) \lim_{R \rightarrow \infty} \int_{\partial E(R)} (g^{\mu[0} - e^{-1} \bar{g}^{\mu[0} \bar{\nabla}_\mu X^i] N d\sigma_i \right\}. \end{aligned} \quad (4.42)$$

## V. KOMAR INTEGRALS

If  $X^\alpha$  is a Killing vector field of both  $g$  and  $\bar{g}$ , we have

$$X^\mu \mathbf{R}_{\mu bcd} = \nabla_b \nabla_c X_d, \quad \text{and} \quad X^\mu \bar{\mathbf{R}}_{\mu bcd} = \bar{\nabla}_b \bar{\nabla}_c X_d. \quad (5.1)$$

This allows us to express some of the integrals above as Komar-type integrals.

We start with the setup of Sec. IV B 2; the KK-asymptotically flat case can be obtained directly from the calculations here by setting  $\bar{\mathbf{R}}_{\alpha\beta\gamma\delta} = 0$ . To make things clear, we assume Eqs. (4.16)–(4.17) and (4.19)–(4.20), together with Eqs. (4.29)–(4.30), and recall that all these hypotheses are satisfied under the corresponding hypotheses made in the KK-asymptotically flat case.

The contribution from the first integrand in Eq. (4.31) can be manipulated as [32]

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\partial E(R)} X^a Z^b (\mathbf{R}_{ab}{}^{ef} - \bar{\mathbf{R}}_{ab}{}^{ef}) dS_{ef} &= \lim_{R \rightarrow \infty} \int_{\partial E(R)} [(X^{[f:e]:b} - X^{[f|e|}_{||b}) Z^b - X^A Z^b (\mathbf{R}_{Ab}{}^{ef} - \bar{\mathbf{R}}_{Ab}{}^{ef})] dS_{ef} \\ &= \lim_{R \rightarrow \infty} \int_{\partial E(R)} \{ (n-1) (X^{[e:f]} - X^{[e||f]}) - 3(X^{[e:f} Z^b]_{;b} + 3(X^{[e||f} Z^b]_{||b} \\ &\quad + 2(\mathbf{R}_{\mu b}{}^{b[f} Z^{e]} - \bar{\mathbf{R}}_{\mu b}{}^{b[f} Z^{e]}) X^\mu dS_{ef} - X^A Z^b (\mathbf{R}_{Ab}{}^{ef} - \bar{\mathbf{R}}_{Ab}{}^{ef}) \} dS_{ef} \\ &= \lim_{R \rightarrow \infty} \left\{ (n-1) \int_{\partial E(R)} (X^{[\alpha\beta]} - X^{[\alpha||\beta]}) dS_{\alpha\beta} + \int_{\partial E(R)} [2X^\mu Z^e (\mathbf{R}^{fb}{}_{b\mu} - \bar{\mathbf{R}}^{fb}{}_{b\mu}) \right. \\ &\quad \left. - X^A Z^b (\mathbf{R}_{Ab}{}^{ef} - \bar{\mathbf{R}}_{Ab}{}^{ef})] dS_{ef} \right\}, \end{aligned} \quad (5.2)$$

where the semicolon (;) denotes the covariant derivative of the metric  $g$  and the double bar ( $\parallel$ ) denotes the covariant derivative of the background metric  $\bar{g}$ . Moreover, we used Gauss' theorem, e.g.,

$$\lim_{R \rightarrow \infty} \int_{\partial E(R)} (X^{[e\parallel f} Z^{b]})_{\parallel b} \sqrt{|\det g|} d\Sigma_{ef} = \lim_{R \rightarrow \infty} \int_{\partial E(R)} (X^{[e\parallel f} Z^{b]})_{\parallel b} \sqrt{|\det \bar{g}|} d\Sigma_{ef} = 0. \quad (5.3)$$

Hence, under the hypotheses used in the derivation of Eq. (4.31), we can rewrite Eq. (4.31) as

$$\begin{aligned} H_b(X, \mathcal{S}) &= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \left\{ \int_{\partial E(R)} (n-1)(X^{[\alpha\beta]} - X^{[\alpha\parallel\beta]}) dS_{\alpha\beta} \right. \\ &\quad + \int_{\partial E(R)} \left( 2X^\mu Z^e (\mathbf{R}^{fb}_{b\mu} - \bar{\mathbf{R}}^{fb}_{b\mu}) - X^A Z^b (\mathbf{R}_{Ab}{}^{ef} - \bar{\mathbf{R}}_{Ab}{}^{ef}) + 4Z^{[e} X^a] (\mathbf{R}^f{}_a - \bar{\mathbf{R}}^f{}_a) \right. \\ &\quad - Z^e X^f (\mathbf{R}^c{}_c - \bar{\mathbf{R}}^c{}_c) - \frac{1}{n-1} [(n-3)Z^e X^f (\mathbf{R}^{bA}_{bA} - \bar{\mathbf{R}}^{bA}_{bA}) + 4Z^{[e} X^a] (\mathbf{R}^f{}_{aA} - \bar{\mathbf{R}}^f{}_{aA}) \\ &\quad \left. \left. + 3(n-2)X^A Z^e (\mathbf{R}^{fb}_{bA} - \bar{\mathbf{R}}^{fb}_{bA}) \right] \right) dS_{ef} - (n-2) \lim_{R \rightarrow \infty} \int_{\partial E(R)} (g^{\mu[a} - e^{-1} \bar{g}^{\mu[a}]) \bar{\nabla}_\mu X^{b]} dS_{ab} \left. \right\}. \quad (5.4) \end{aligned}$$

The first integrand is the difference of the Komar integrands of  $g$  and  $\bar{g}$ .

Specializing to the KK-asymptotically flat case for background-covariantly constant Killing vectors, this reads

$$\begin{aligned} p_\mu X^\mu_\infty &= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \left\{ (n-1) \int_{S(R)} \int_{\mathbb{T}^N} X^{\alpha\beta} dS_{\alpha\beta} \right. \\ &\quad + \int_{S(R)} \int_{\mathbb{T}^N} \left( 2X^\mu x^e \mathbf{R}^{fb}_{b\mu} - X^A x^b \mathbf{R}_{Ab}{}^{ef} - \frac{1}{n-1} [(n-3)x^e X^f \mathbf{R}^{bA}_{bA} + 4x^{[e} X^a] \mathbf{R}^f{}_{aA} \right. \\ &\quad \left. \left. + 3(n-2)X^A x^e \mathbf{R}^{fb}_{bA}] \right) dS_{ef} \right\}. \quad (5.5) \end{aligned}$$

Thus, it appears that in general Komar-type integrals do *not* coincide with the Hamiltonian generators. This is really the case, as can be seen for the Rasheed solutions. Using Eq. (A33), one readily finds for  $X = \partial_t$  (keeping in mind that  $n = 3$ )

$$\frac{1}{8\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{S^1} X^{\alpha\beta} dS_{\alpha\beta} = \begin{cases} 2\pi(M + \frac{\Sigma}{\sqrt{3}}), & P = 0, \\ 8\pi P(M + \frac{\Sigma}{\sqrt{3}}), & P \neq 0, \end{cases} \quad (5.6)$$

which does *neither* coincide with  $p_0$  [cf. Eq. (3.23)] *nor* with the ADM mass of the space metric  $g_{ij} dx^i dx^j$ . Note that the Komar integral of the spacetime metric  $g_{ab} dx^a dx^b$  will equal  $M + \frac{\Sigma}{\sqrt{3}}$  regardless of the value of  $P$ .

Next, for  $X = \partial_4$  we obtain

$$\frac{1}{8\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{S^1} X^{\alpha\beta} dS_{\alpha\beta} = \begin{cases} 4\pi Q, & P = 0, \\ 16\pi P Q, & P \neq 0, \end{cases} \quad (5.7)$$

which is twice the Hamiltonian charge  $p_4$ .

As a simple application of Eq. (5.6), suppose that there exists a Rasheed metric without a black-hole region. Since

the divergence of the Komar integrand is zero, we obtain  $M = -\Sigma/\sqrt{3}$ . But this is precisely one of the parameter values excluded in the Rasheed metrics, cf. Eq. (A4) below. We conclude that the regular metrics in the Rasheed family must be black-hole solutions.

For the case of metrics which asymptote to a maximally symmetric background  $\bar{g}$  with  $\Lambda \neq 0$ , as in Sec. IV B 1, the Komar integral resulting from Eq. (4.27) reads

$$\begin{aligned} H_b(X, \mathcal{S}) &= \frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \int_{\partial E(R)} X^\nu Z^\xi W^{\alpha\beta}_{\nu\xi} dS_{\alpha\beta} \\ &= \lim_{R \rightarrow \infty} \left\{ \frac{n-1}{16(n-2)\pi} \int_{\partial E(R)} X^{[\alpha\beta]} dS_{\alpha\beta} \right. \\ &\quad \left. - \frac{\Lambda}{4(n-2)(n-1)n\pi} \int_{\partial E(R)} X^\alpha Z^\beta dS_{\alpha\beta} \right\}. \quad (5.8) \end{aligned}$$

## VI. WITTEN'S POSITIVITY ARGUMENT

The Witten positive-energy argument [33,34] (cf. Ref. [35]) generalizes in an obvious manner to KK-asymptotically flat metrics. Assuming that the initial data hypersurface  $\mathcal{S}$  is spin, we consider the Witten boundary integral  $\mathcal{W}$  defined as

$$\mathcal{W}(\phi_\infty) := \lim_{R \rightarrow \infty} \int_{S(R) \times \mathbb{T}^K} \mathcal{U}^i d\sigma_i, \quad (6.1)$$

$$\mathcal{U}^I = \langle \phi, D^I \phi + \gamma^I \mathcal{D} \phi \rangle, \quad (6.2)$$

where  $\phi$  is a spinor field which asymptotes to a constant spinor  $\phi_\infty$  at an appropriate rate as one recedes to infinity in the asymptotic end, and  $\mathcal{D} := \gamma^J D_J$  is the Dirac operator on  $\mathcal{S}$ . (Note that the asymptotic spinors  $\phi_\infty$  might be incompatible with the spin structure of  $\mathcal{S}$ , in which case the argument below of course does not apply; cf. Refs. [36–38].) It is standard to show that in the natural spin frame we have

$$\mathcal{U}^I = \frac{1}{4} \sum_{L=1}^{n+K} (\partial_L g_{IL} - \partial_I g_{LL}) |\phi_\infty|^2 + o(r^{-2\alpha-1}). \quad (6.3)$$

Assuming a positive and suitably decaying energy density on a maximal (i.e.,  $g^{IJ} K_{IJ} = 0$ ) initial data hypersurface, such that

$\mathcal{S}$  is metrically complete and either is boundaryless or has a trapped compact boundary, (6.4)

the proof of the existence of the desired solutions of the Witten equation  $\mathcal{D}\phi = 0$  can be carried out along lines identical to the usual asymptotically flat case, cf., e.g., Refs. [39,40]. Comparing with Eq. (3.19), we conclude that the positivity of  $\mathcal{W}$  is equivalent to positivity of the Hamiltonian mass:

$$p_0 \geq 0.$$

It should be emphasized that  $p_0$  does *not* necessarily coincide with the ADM mass of  $g_{IJ} dx^I dx^J$ .

The above argument required the positivity of the scalar curvature of  $g_{IJ} dx^I dx^J$ . This is not needed if one replaces the usual spinor covariant derivative in Eq. (6.2) by

$$D_I \rightarrow D_I + \frac{1}{2} K_I^J \gamma_J \gamma_0. \quad (6.5)$$

The Witten quadratic form  $\mathcal{W}$  instead becomes

$$\lim_{R \rightarrow \infty} \oint_{S(R) \times \mathbb{T}^K} \mathcal{U}^i d\sigma_i = 4\pi p_\mu \langle \phi_\infty, \gamma^\mu \gamma^0 \phi_\infty \rangle, \quad (6.6)$$

and is non-negative for all  $\phi_\infty$  when the dominant energy condition is assumed on initial data hypersurfaces as in Eq. (6.4). The positivity of  $\mathcal{W}$  is equivalent to the timelikeness of the  $(n+K+1)$ -vector  $p_\mu$ . Equivalently,

$$p_0^2 - \sum_{i=1}^n p_i^2 \geq \sum_{A=n+1}^{n+K} p_A^2 \geq 0. \quad (6.7)$$

The first inequality is saturated if and only if the initial data set can be isometrically embedded in  $\mathbb{R} \times \mathbb{R}^n \times \mathbb{T}^K$  equipped with the flat Lorentzian metric (cf. Ref. [41]).

As an example, consider the Rasheed metrics with  $P = 0$ . The corresponding domains of outer communications have the topology  $\mathbb{R} \times S^1 \times (\mathbb{R}^3 \setminus B(R))$ , where the  $\mathbb{R}$  factor corresponds to the time variable,  $S^1$  is the Kaluza-Klein factor, and the  $\mathbb{R}^3 \setminus B(R)$  factor describes the space topology of the black hole. It thus has the obvious spin structure inherited from a flat  $\mathbb{R} \times S^1 \times \mathbb{R}^3$ , together with the obvious associated parallel spinors. Therefore the Witten-type argument just described applies, leading to

$$M^2 \geq Q^2, \quad (6.8)$$

where the inequality is strict for black-hole solutions. If we denote by  $M_{\text{ADM}}$  the ADM mass of the three-dimensional-space part of the Rasheed metric, this can be equivalently rewritten as

$$\left( M_{\text{ADM}} + \frac{\Sigma}{\sqrt{3}} \right)^2 \geq Q^2, \quad (6.9)$$

cf. Ref. [42].

Note that Eq. (6.9) does not exclude the possibility of a negative or vanishing  $M_{\text{ADM}}$  (cf. Refs. [36,43,44]). We have not attempted a systematic analysis of this issue, and only checked that all Rasheed solutions with  $a = 0$  and  $M = 0$  have naked singularities outside of the horizon.

## VII. SUMMARY

In this work we have considered families of metrics asymptotic to various background metrics, and studied the Hamiltonians associated with the flow of Killing vectors of the background. We have derived several new formulas for these Hamiltonians, generalizing previous work by allowing a cosmological constant, or nonstandard backgrounds, and allowing higher dimensions. In particular:

We have derived an ADM-type formula for Hamiltonians generating time translations for a wide class of background metrics, cf. Eq. (3.39).

We have provided a formula for Hamiltonians generating translations for KK-asymptotically flat metrics in terms of the spacetime curvature tensor [Eq. (4.10)].

We have derived a formula for Hamiltonians associated with generators of all background Killing fields for asymptotically anti-de Sitter spacetimes in terms of the spacetime curvature tensor [Eq. (4.27)].

Equation (4.31) provides a similar formula for a wide class of backgrounds with  $\Lambda = 0$ .

Equations (4.40) and (4.42) provide space-and-time decomposed versions of the last two Hamiltonians.

In Sec. V we have derived several Komar-type formulas for the Hamiltonians above for vector fields  $X$  which are

Killing vectors for both the background and the physical metric.

In Sec. VI we have pointed out the consequences of a Witten-type positivity argument for *KK*-asymptotically flat spacetimes: instead of proving the positivity of the ADM energy, the argument provides an inequality involving the Kaluza-Klein charges and the energy. An explicit version of the inequality has been established for *KK*-asymptotically flat Rasheed metrics.

In addition to the above, we have carried out a careful study of Rasheed metrics (see Appendix A) to obtain a nontrivial family of metrics with singularity-free domains of outer communications to which our formulas apply. We have pointed out the restrictions (A20) and (A22) on the parameters needed to guarantee the absence of naked singularities in the metric. We have shown that all metrics satisfying these conditions together with  $P = 0$  have a stably causal domain of outer communications, and we have given sufficient conditions for stable causality when  $P \neq 0$  in Eq. (A24). In Appendix A 3 we point out that the Rasheed metrics with  $P \neq 0$  are not *KK*-asymptotically flat, and describe their asymptotics. We have determined their global charges, which are significantly different according to whether or not  $P$  vanishes.

Last but perhaps not least, Eq. (C3) provides a useful identity—which we have not seen in the literature—that is satisfied by the Riemann tensor in any dimensions and generalizing the usual double-dual identity valid in four dimensions.

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### APPENDIX A: AN EXAMPLE: RASHEED'S SOLUTIONS

Rasheed [45] has constructed a family of stationary axisymmetric solutions of the five-dimensional vacuum Einstein equations which take the form

$$ds_{(5)}^2 = \frac{B}{A}(dx^4 + 2A_\mu dx^\mu)^2 + \sqrt{\frac{A}{B}} ds_{(4)}^2, \quad (\text{A1})$$

where  $a$ ,  $M$ ,  $P$ ,  $Q$ , and  $\Sigma$  are real numbers satisfying

$$\frac{Q^2}{\Sigma + M\sqrt{3}} + \frac{P^2}{\Sigma - M\sqrt{3}} = \frac{2\Sigma}{3}, \quad (\text{A2})$$

$$\begin{aligned} M^2 + \Sigma^2 - P^2 - Q^2 &\neq 0, & (M + \Sigma/\sqrt{3})^2 - Q^2 &\neq 0, \\ (M - \Sigma/\sqrt{3})^2 - P^2 &\neq 0, \end{aligned} \quad (\text{A3})$$

$$M \pm \frac{\Sigma}{\sqrt{3}} \neq 0,$$

$$F^2 := \frac{[(M + \Sigma/\sqrt{3})^2 - Q^2][(M - \Sigma/\sqrt{3})^2 - P^2]}{M^2 + \Sigma^2 - P^2 - Q^2} > 0, \quad (\text{A4})$$

and where

$$\begin{aligned} ds_{(4)}^2 = & -\frac{G}{\sqrt{AB}}(dt + \omega^0_\phi d\phi)^2 + \frac{\sqrt{AB}}{\Delta} dr^2 \\ & + \sqrt{AB} d\theta^2 + \frac{\Delta\sqrt{AB}}{G} \sin^2(\theta) d\phi^2, \end{aligned} \quad (\text{A5})$$

with

$$\begin{aligned} A = & (r - \Sigma/\sqrt{3})^2 - \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + a^2 \cos^2(\theta) \\ & + \frac{2JPQ \cos(\theta)}{(M + \Sigma/\sqrt{3})^2 - Q^2}, \\ B = & (r + \Sigma/\sqrt{3})^2 - \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + a^2 \cos^2(\theta) \\ & - \frac{2JPQ \cos(\theta)}{(M - \Sigma/\sqrt{3})^2 - P^2}, \\ G = & r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2 \cos^2(\theta), \\ \Delta = & r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2, \\ \omega^0_\phi = & \frac{2J \sin^2(\theta)}{G} [r + E], \\ J^2 = & a^2 F^2, \end{aligned} \quad (\text{A6})$$

whereas  $E$  is given by

$$E = -M + \frac{(M^2 + \Sigma^2 - P^2 - Q^2)(M + \Sigma/\sqrt{3})}{(M + \Sigma/\sqrt{3})^2 - Q^2}. \quad (\text{A7})$$

The physical-space Maxwell potential is given by

$$2A_\mu dx^\mu = \frac{C}{B} dt + \left( \omega^5_\phi + \frac{C}{B} \omega^0_\phi \right) d\phi, \quad (\text{A8})$$

where

$$C = 2Q(r - \Sigma/\sqrt{3}) - \frac{2PJ \cos(\theta)(M + \Sigma/\sqrt{3})}{(M - \Sigma/\sqrt{3})^2 - P^2}, \quad (\text{A9})$$

$$\omega^5_\phi = \frac{H}{G}, \quad (\text{A10})$$

and

$$H := 2P\Delta \cos(\theta) - \frac{2QJ \sin^2(\theta) [r(M - \Sigma/\sqrt{3}) + M\Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2]}{[(M + \Sigma/\sqrt{3})^2 - Q^2]} \quad (\text{A11})$$

The Rasheed metrics (A1) have been obtained by applying a solution-generating technique [45] (cf. Ref. [46]) to the Kerr metrics. This guarantees that these metrics solve the five-dimensional vacuum Einstein equations when the constraint (A3) is satisfied. As the procedure is somewhat involved, it appears useful to cross-check the vanishing of the Ricci tensor using computer algebra. We have been able to verify this in the  $P = 0$  case with SAGE (which required a week-long computation on a personal computer), as well as for a set of samples for the parameters  $(M, a, P, Q, \Sigma)$  in the  $P \neq 0$  case with MATHEMATICA. We have, however, not been able to do it for the full set of parameters.

Let us address the question of the global structure of the metrics above. We have

$$\det g = -A^2 \sin^2(\theta),$$

which shows that the metrics are smooth and Lorentzian except possibly at the zeros of  $A$ ,  $B$ ,  $G$ ,  $\Delta$ , and  $\sin(\theta)$ .

After a suitable periodicity of  $\phi$  (as in Sec. A 3 below) has been imposed, regularity at the axes of rotation away from the zeros of denominators follows from the factorizations

$$\left(\frac{\Delta}{G} - 1\right) = \frac{a^2 \sin^2(\theta)}{a^2 \cos^2(\theta) - 2Mr + P^2 + Q^2 + r^2 - \Sigma^2}, \quad (\text{A12})$$

$$2A_\phi - 2P \frac{\Delta}{G} \cos(\theta) = \frac{\sin^2(\theta)}{G} \left( \mathcal{H} + \frac{2JC}{B} [r + E] \right), \quad (\text{A13})$$

where

$$\mathcal{H} := - \frac{2QJ [r(M - \Sigma/\sqrt{3}) + M\Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2]}{[(M + \Sigma/\sqrt{3})^2 - Q^2]}. \quad (\text{A14})$$

It will be seen below that, after restricting the parameter ranges as in Eqs. (A20) and (A22), the location of Killing horizons is determined by the zeros of

$$\begin{vmatrix} g_{tt} & g_{t\phi} & g_{t4} \\ g_{\phi t} & g_{\phi\phi} & g_{\phi 4} \\ g_{4t} & g_{4\phi} & g_{44} \end{vmatrix} = -\Delta \sin^2(\theta), \quad (\text{A15})$$

and thus by the real roots  $r_+ \geq r_-$  of  $\Delta$ , if any:

$$r_{\pm} = M \pm \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2}. \quad (\text{A16})$$

## 1. Zeros of the denominators

The norms

$$g_{tt} = \frac{W}{AB} \quad \text{and} \quad g_{44} = \frac{B}{A}$$

of the Killing vectors  $\partial_t$  and  $\partial_4$  are geometric invariants, where  $W = -GA + C^2$ . So zeros of  $A$  and  $AB$  correspond to singularities in the five-dimensional geometry except if

- (1) a zero of  $A$  is a joint zero of  $A$ ,  $B$ , and  $W$ , or if
- (2) a zero of  $B$  which is not a zero of  $A$  is also a zero of  $W$ .

Setting

$$\mathcal{A} := \frac{2JPQ}{a^2((M + \Sigma/\sqrt{3})^2 - Q^2)}, \quad (\text{A17})$$

one checks that if

$$\begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathcal{A}|) = 0, & \text{when } |\mathcal{A}| > 2 \quad \text{or} \\ \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4} = 0, & \text{when } |\mathcal{A}| \leq 2, \end{cases} \quad (\text{A18})$$

then  $A$  vanishes exactly at one point. Otherwise the set of zeros of  $A$  forms a curve in the  $(r, \theta)$  plane. Let  $\theta \mapsto r_A^+(\theta)$  denote the curve (say,  $\gamma$ ) corresponding to the set of largest zeros of  $A$ .

Note that  $W$  and  $A$  are polynomials in  $r$ , with  $A$  of second order. If  $W/A$  is smooth, the remainder of the polynomial division of  $W$  by  $r - r_A^+$  must vanish on the part of  $\gamma$  that lies outside the horizon. One can calculate this remainder with MATHEMATICA, obtaining a function of  $\theta$  which vanishes at most at isolated points, if at all. It follows that the division of  $W$  by  $A$  is singular on the closure of the domain of outer communications (d.o.c.), i.e., the region  $\{r \geq r_+\}$ , if  $A$  has zeros there, except perhaps when Eq. (A18) holds.

One can likewise exclude a joint zero of  $W$  and  $B$  in the closure of the d.o.c. without a zero of  $A$ , except possibly for the case where this zero is isolated for  $B$  as well, which happens if

$$\begin{cases} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1 - |\mathcal{B}|) = 0, & \text{if } |\mathcal{B}| > 2 \quad \text{or} \\ \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4} = 0, & \text{if } |\mathcal{B}| \leq 2. \end{cases} \quad (\text{A19})$$

See Ref. [47] for a more detailed analysis of the borderline cases.

To summarize, a necessary condition for a black hole without obvious singularities in the closure of the domain of outer communications is that all zeros of  $A$  lie under the outermost Killing horizon  $r = r_+$ . One finds that this will be the case if and only if

$$\begin{aligned}
 |\mathcal{A}| > 2 \quad \text{and} \quad & \begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathcal{A}|) < 0, \quad \text{or} \\ M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} - a^2(1 - |\mathcal{A}|)}, \end{cases} \\
 \text{or} \\
 |\mathcal{A}| \leq 2 \quad \text{and} \quad & \begin{cases} \frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4} < 0, \quad \text{or} \\ M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > \frac{\Sigma}{3} + \sqrt{\frac{2P^2\Sigma}{\Sigma - M\sqrt{3}} + \frac{a^2\mathcal{A}^2}{4}}, \end{cases} \tag{A20}
 \end{aligned}$$

except perhaps when Eq. (A18) holds.

An identical argument applies to the zeros of  $B$ , with the zeros of  $B$  lying on a curve unless Eq. (A19) holds. Ignoring this last case, the zeros of  $B$  need to be similarly hidden behind the outermost Killing horizon. Setting

$$\mathcal{B} := -\frac{2JPQ}{a^2((M - \Sigma/\sqrt{3})^2 - P^2)}, \tag{A21}$$

one finds that this will be the case if and only if

$$\begin{aligned}
 |\mathcal{B}| > 2 \quad \text{and} \quad & \begin{cases} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1 - |\mathcal{B}|) < 0, \quad \text{or} \\ M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > -\frac{\Sigma}{3} + \sqrt{\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} - a^2(1 - |\mathcal{B}|)}, \end{cases} \\
 \text{or} \\
 |\mathcal{B}| \leq 2 \quad \text{and} \quad & \begin{cases} \frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4} < 0, \quad \text{or} \\ M + \sqrt{M^2 + \Sigma^2 - P^2 - Q^2 - a^2} > -\frac{\Sigma}{3} + \sqrt{\frac{2Q^2\Sigma}{\Sigma + M\sqrt{3}} + \frac{a^2\mathcal{B}^2}{4}}, \end{cases} \tag{A22}
 \end{aligned}$$

except perhaps when Eq. (A19) holds.

While the above guarantees the lack of obvious singularities in the d.o.c.  $\{r > r_+\}$ , there could still be causality violations there. Ideally, the d.o.c. should be globally hyperbolic, a question which we have not attempted to address. Barring global hyperbolicity, a decent d.o.c. should at least admit a time function, and the function  $t$  provides an obvious candidate. In order to study the issue we note the identity

$$g^{00} = \frac{4J^2[r + E]^2 \sin^2(\theta) - AB\Delta}{A\Delta G}. \tag{A23}$$

A MATHEMATICA calculation shows that the numerator factorizes through  $G$ , so that  $g^{00}$  extends smoothly through the ergosphere. When  $P = 0$ , one can verify that  $g^{00}$  is negative on the d.o.c. For  $P \neq 0$  one can find open sets of parameters which guarantee that  $g^{00}$  is strictly negative for  $r > r_+$  when  $A$  and  $B$  have no zeros there. An example is given by the condition

$$r_+ \geq \frac{EM + q}{M + E}, \tag{A24}$$

which is sufficient but not necessary, where  $q := P^2 + Q^2 - \Sigma^2 + a^2$ . We hope to return to the question of causality violations in the future.

In Fig. 1 we show the locations of the zeros of  $A$  and  $B$  for some specific sets of parameters satisfying, or violating, the conditions above.

Another potential source of singularities of the metric (A1) could be the zeros of  $G$ . It turns out that they are irrelevant, which can be seen as follows. The relevant metric coefficient is  $g_{\phi\phi}$ , which reads

$$\begin{aligned}
 g_{\phi\phi} = & \frac{B}{A} \left( \omega^5_{\phi} + \frac{C}{B} \omega^0_{\phi} \right)^2 \\
 & + \sqrt{\frac{A}{B}} \left( -\frac{G}{\sqrt{AB}} (\omega^0_{\phi})^2 + \frac{\Delta\sqrt{AB}}{G} \sin^2(\theta) \right). \tag{A25}
 \end{aligned}$$

Taking into account a  $G^{-1}$  factor in  $\omega^0_{\phi}$ , it follows that  $g_{\phi\phi}$  can be written as a fraction  $(\dots)/ABG^2$ . A MATHEMATICA

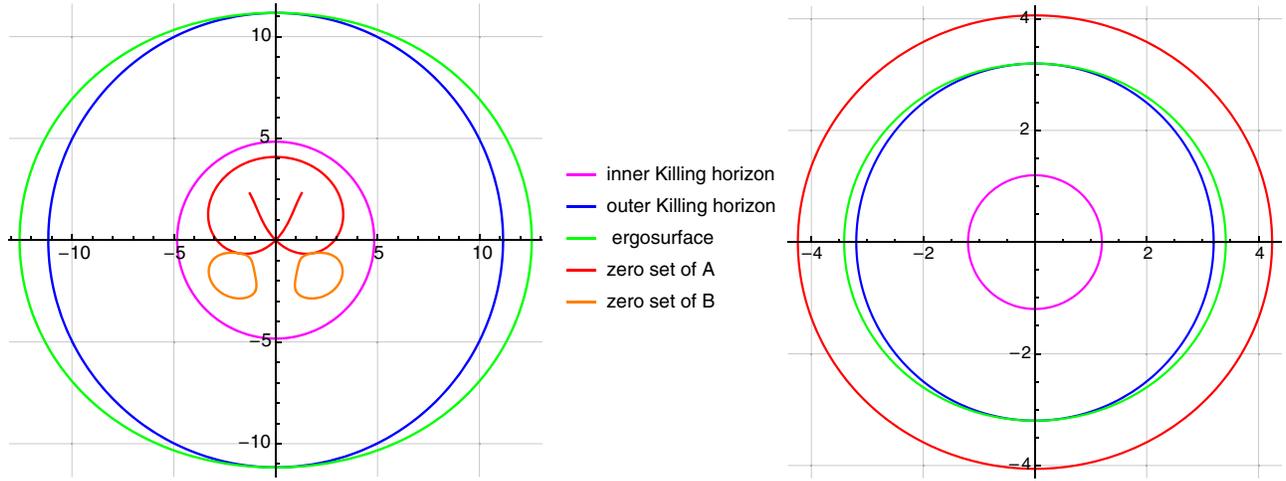


FIG. 1. Two sample plots for the location of the ergosurface (zeros of  $G$ ), the outer and inner Killing horizons (zeros of  $\Delta$ ), and the zeros of  $A$ ,  $B$ . Left plot:  $M = 8$ ,  $a = \frac{33}{10}$ ,  $Q = \frac{8}{5}$ ,  $\Sigma = -\frac{23}{5}$ ,  $P = -\frac{1}{5} \sqrt{\frac{2(4105960\sqrt{3}+2770943)}{12813}} \approx -7.86$ , with zeros of  $A$  and  $B$  under both horizons, consistently with Eq. (A20) and (A22). Right plot:  $M = 1$ ,  $a = 1$ ,  $Q = 0$ ,  $\Sigma = \sqrt{6}$ ,  $P = \sqrt{4 - 2\sqrt{2}} \approx 1.08$ ; here, Eq. (A20) is violated, while the zeros of  $B$  occur at negative  $r$ .

calculation shows that the denominator (...) factorizes through  $AG^2$ , which shows indeed that the zeros of  $G$  are innocuous for the problem at hand.

Let us write  $ds_{(4)}^2$  as  ${}^{(4)}g_{ab}dx^a dx^b$ . The factorization just described works for  $g_{\phi\phi}$  but does *not* work for  ${}^{(4)}g_{\phi\phi}$ . From what has been said we see that the quotient metric  ${}^{(4)}g_{ab}dx^a dx^b$  is always singular in the d.o.c., a fact which seems to have been ignored, and unnoticed, in the literature so far.

## 2. Regularity at the outer Killing horizon $\mathcal{H}_+$

The location of the outer Killing horizon  $\mathcal{H}_+$  of the Killing field

$$k = \partial_t + \Omega_\phi \partial_\phi + \Omega_4 \partial_{x^4} \quad (\text{A26})$$

is given by the larger root  $r_+$  of  $\Delta$ , cf. Eq. (A16). The condition that  $\mathcal{H}_+$  is a Killing horizon for  $k$  is that the pullback of  $g_{\mu\nu}k^\nu$  to  $\mathcal{H}_+$  vanishes. This, together with

$$\Delta|_{\mathcal{H}_+} = 0, \quad G|_{\mathcal{H}_+} = -a^2 \sin^2(\theta), \quad (\text{A27})$$

yields

$$\begin{aligned} \Omega_\phi &= -\frac{1}{\omega_\phi^0} \Big|_{\mathcal{H}_+} = \frac{a^2}{2J} (r_+ + E)^{-1}, \\ \Omega_4 &= -\frac{2(A_t \omega_\phi^0 - A_\phi)}{\omega_\phi^0} \Big|_{\mathcal{H}_+} \\ &= \frac{Q(-3Mr_+ - \sqrt{3}M\Sigma + 3P^2 + 3Q^2 + \sqrt{3}r\Sigma - 3\Sigma^2)}{(E + r_+)(3M^2 + 2\sqrt{3}M\Sigma - 3Q^2 + \Sigma^2)}. \end{aligned} \quad (\text{A28})$$

After the coordinate transformation

$$\bar{\phi} = \phi - \Omega_\phi dt, \quad \bar{x}^4 = x^4 - \Omega_4 dt, \quad (\text{A29})$$

the metric (A1) becomes

$$g = g_s + \frac{dr^2}{\Delta} + \Delta U dt^2, \quad (\text{A30})$$

where  $g_s$  is a smooth  $(0, 2)$  tensor, with  $U := g_{tt}/\Delta$  extending smoothly across  $\Delta = 0$ . Introducing a new time coordinate by

$$\tau = t - \sigma \ln(r - r_+) \Rightarrow d\tau = dt - \frac{\sigma}{r - r_+} dr, \quad (\text{A31})$$

where  $\sigma$  is a constant to be determined, Eq. (A30) takes the form

$$\begin{aligned} g &= g_s + \Delta U \left( d\tau + \frac{\sigma}{r - r_+} dr \right)^2 + \frac{dr^2}{\Delta} \\ &= g_s + \Delta U d\tau^2 + \frac{2\Delta U \sigma}{r - r_+} d\tau dr + \left( \frac{1}{\Delta} + \frac{\Delta U \sigma^2}{(r - r_+)^2} \right) dr^2 \\ &= g_s + \Delta U d\tau^2 + \frac{2\Delta U \sigma}{r - r_+} d\tau dr + \underbrace{\frac{(r - r_+)^2 + \Delta^2 \sigma^2 U}{\Delta (r - r_+)^2}}_V dr^2. \end{aligned} \quad (\text{A32})$$

In order to obtain a smooth metric in the domain of outer communications the constant  $\sigma$  has to be chosen

so that the numerator of  $V$  has a triple zero at  $r = r_+$ . A MATHEMATICA computation gives an explicit formula for the desired constant  $\sigma$ , which is too lengthy to be explicitly presented here. This establishes the smooth extendibility of the metric in suitable coordinates across  $r = r_+$ .

$$\begin{pmatrix} \frac{2M}{r} + \frac{2\Sigma}{\sqrt{3}r} - 1 & 0 & 0 & 0 & \frac{2Q}{r} \\ 0 & \frac{2Mx^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & \frac{2Mxy}{r^3} & \frac{2Mxz}{r^3} & 0 \\ 0 & \frac{2Mxy}{r^3} & \frac{2My^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & \frac{2Myz}{r^3} & 0 \\ 0 & \frac{2Mxz}{r^3} & \frac{2Myz}{r^3} & \frac{2Mz^2}{r^3} - \frac{2\Sigma}{\sqrt{3}r} + 1 & 0 \\ \frac{2Q}{r} & 0 & 0 & 0 & \frac{4\Sigma}{\sqrt{3}r} + 1 \end{pmatrix} + O(r^{-2}). \quad (\text{A33})$$

It turns out that when  $P \neq 0$ , the Rasheed metrics do *not* satisfy the KK-asymptotic flatness requirements anymore; indeed, the phase space decomposes into sectors, labeled by  $P \in \mathbb{R}$ , in which the metrics  $g$  asymptote to the background metric

$$\bar{g} := (dx^4 + 2P \cos(\theta)d\varphi)^2 - dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta)d\varphi^2. \quad (\text{A34})$$

The metrics (A1) and (A34) are singular at  $\sin(\theta) = 0$ . This can be resolved by replacing  $x^4$  by  $\bar{x}^4$  (respectively, by  $\tilde{x}^4$ ) on the following coordinate patches:

$$\begin{cases} \bar{x}^4 := x^4 + 2P\varphi, & \theta \in [0, \pi), \\ \tilde{x}^4 := x^4 - 2P\varphi, & \theta \in (0, \pi]. \end{cases} \quad (\text{A35})$$

Indeed, the one-form

$$\begin{aligned} dx^4 + 2P \cos(\theta)d\varphi &= d\bar{x}^4 + 2P(\cos(\theta) - 1)d\varphi \\ &= d\bar{x}^4 - \frac{2P}{r(r+z)}(xdy - ydx) \end{aligned}$$

is smooth for  $r > 0$  on  $\{\theta \in [0, \pi)\}$ . Similarly the one-form

$$\begin{aligned} dx^4 + 2P \cos(\theta)d\varphi &= d\tilde{x}^4 + 2P(\cos(\theta) + 1)d\varphi \\ &= d\tilde{x}^4 + \frac{2P}{r(r-z)}(xdy - ydx) \end{aligned}$$

is smooth on  $\{\theta \in (0, \pi], r > 0\}$ . The smoothness of both  $g$  and  $\bar{g}$  in the d.o.c., under the constraints discussed above, readily follows.

We note the relation

$$\bar{x}^4 = \tilde{x}^4 + 4P\varphi, \quad (\text{A36})$$

### 3. Asymptotic behavior

When  $P = 0$  the Rasheed metrics satisfy the KK-asymptotic flatness conditions. This can be seen by introducing manifestly asymptotically-flat coordinates  $(t, x, y, z)$  in the usual way. With some work one finds that the metric takes the form

which implies a smooth geometry with periodic coordinates  $\bar{x}^4$  and  $\tilde{x}^4$  if and only if

$$\text{both } \bar{x}^4 \text{ and } \tilde{x}^4 \text{ are periodic with period } 8P\pi. \quad (\text{A37})$$

From this perspective  $x^4$  is *not a coordinate* anymore; instead, the basic coordinates are  $\bar{x}^4$  for  $\theta \in [0, \pi)$  and  $\tilde{x}^4$  for  $\theta \in (0, \pi]$ , with  $dx^4$  (but *not*  $x^4$ ) well defined away from the axes of rotation  $\{\sin(\theta) = 0\}$  as

$$dx^4 = \begin{cases} d\bar{x}^4 - 2Pd\varphi, & \theta \in [0, \pi), \\ d\tilde{x}^4 + 2Pd\varphi, & \theta \in (0, \pi]. \end{cases} \quad (\text{A38})$$

#### a. Curvature of the asymptotic background

We continue with a calculation of the curvature tensor of the asymptotic background. It is convenient to work in the coframe

$$\begin{aligned} \bar{\Theta}^0 &= dt, & \bar{\Theta}^1 &= dx, & \bar{\Theta}^2 &= dy, \\ \bar{\Theta}^3 &= dz, & \bar{\Theta}^4 &= dx^4 + 2P \cos(\theta)d\varphi, \end{aligned} \quad (\text{A39})$$

which is manifestly smooth after replacing  $dx^4$  as in Eq. (A38). Using

$$\begin{aligned} d\bar{\Theta}^4 &= -2P \sin(\theta)d\theta \wedge d\varphi = -2P \frac{x^i}{r^3} \partial_i ](dx \wedge dy \wedge dz) \\ &= -\frac{P}{r^3} \overset{\circ}{\epsilon}_{\hat{i}\hat{j}\hat{k}} x^i dx^{\hat{j}} \wedge dx^{\hat{k}}, \end{aligned} \quad (\text{A40})$$

where  $\overset{\circ}{\epsilon}_{\hat{i}\hat{j}\hat{k}} \in \{0, \pm 1\}$  denotes the usual epsilon symbol, one finds the following nonvanishing connection coefficients:

$$\bar{\omega}^{\hat{4}}_{\hat{i}} = \frac{P}{r^3} \mathring{e}_{\hat{i}\hat{j}\hat{k}} x^{\hat{j}} \bar{\Theta}^{\hat{k}}, \quad \bar{\omega}^{\hat{i}}_{\hat{j}} = \frac{P}{r^3} \mathring{e}_{\hat{i}\hat{j}\hat{k}} x^{\hat{k}} \bar{\Theta}^{\hat{4}}, \quad (\text{A41})$$

where  $x^{\hat{i}} \equiv x^i$ . This leads to the curvature forms

$$\begin{aligned} \bar{\Omega}^{\hat{i}}_{\hat{j}} &= \frac{P}{r^3} \mathring{e}_{\hat{i}\hat{j}\hat{k}} \left( -\frac{3}{r^2} x^{\hat{k}} x^{\hat{\ell}} + \delta^{\hat{k}}_{\hat{\ell}} \right) \bar{\Theta}^{\hat{\ell}} \wedge \bar{\Theta}^{\hat{4}} \\ &\quad - \frac{2P^2}{r^6} \mathring{e}_{\hat{i}\hat{m}(\hat{k}\hat{j})\hat{n}} \hat{\ell} x^{\hat{m}} x^{\hat{n}} \bar{\Theta}^{\hat{k}} \wedge \bar{\Theta}^{\hat{\ell}}, \\ \bar{\Omega}^{\hat{4}}_{\hat{i}} &= \frac{P}{r^3} \mathring{e}_{\hat{i}\hat{j}\hat{k}} \left( -\frac{3}{r^2} x^{\hat{j}} x^{\hat{\ell}} + \delta^{\hat{j}}_{\hat{\ell}} \right) \bar{\Theta}^{\hat{\ell}} \wedge \bar{\Theta}^{\hat{k}} \\ &\quad + \frac{P^2}{r^6} \mathring{e}_{\hat{k}\hat{m}\hat{j}} \mathring{e}_{\hat{k}\hat{i}\hat{\ell}} x^{\hat{m}} x^{\hat{\ell}} \bar{\Theta}^{\hat{j}} \wedge \bar{\Theta}^{\hat{4}}, \end{aligned} \quad (\text{A42})$$

and hence the following nonvanishing curvature tensor components:

$$\begin{aligned} \bar{\mathbf{R}}^{\hat{i}}_{\hat{j}\hat{k}\hat{4}} &= \frac{P}{r^3} \mathring{e}_{\hat{i}\hat{j}\hat{\ell}} \left( -\frac{3}{r^2} x^{\hat{\ell}} x^{\hat{k}} + \delta^{\hat{\ell}}_{\hat{k}} \right), \\ \bar{\mathbf{R}}^{\hat{4}}_{\hat{i}\hat{j}\hat{4}} &= \frac{P^2}{r^6} \mathring{e}_{\hat{k}\hat{m}\hat{j}} \mathring{e}_{\hat{k}\hat{i}\hat{\ell}} x^{\hat{m}} x^{\hat{\ell}}, \\ \bar{\mathbf{R}}^{\hat{i}}_{\hat{j}\hat{k}\hat{\ell}} &= -\frac{2P^2}{r^6} (\mathring{e}_{\hat{i}\hat{j}\hat{n}} \mathring{e}_{\hat{k}\hat{\ell}} \hat{m} + \mathring{e}_{\hat{i}\hat{m}(\hat{k}\hat{\ell})\hat{j}\hat{n}}) x^{\hat{m}} x^{\hat{n}}. \end{aligned} \quad (\text{A43})$$

The nonvanishing components of the Ricci tensor read

$$\begin{aligned} \bar{\mathbf{R}}_{\hat{i}\hat{j}} &= -\frac{2P^2}{r^6} \mathring{e}_{\hat{k}\hat{m}\hat{i}} \mathring{e}_{\hat{k}\hat{n}\hat{j}} x^{\hat{m}} x^{\hat{n}}, \\ \bar{\mathbf{R}}_{\hat{4}\hat{4}} &= -\frac{P^2}{r^6} \mathring{e}_{\hat{k}\hat{m}\hat{i}} \mathring{e}_{\hat{k}\hat{i}\hat{\ell}} x^{\hat{m}} x^{\hat{\ell}}. \end{aligned} \quad (\text{A44})$$

Subsequently, the Ricci scalar is  $\bar{\mathbf{R}} = -2P^2/r^4$ .

#### 4. Global charges: A summary

For ease of future reference, we summarize the global charges of the Rasheed metrics. Let  $p_\mu$  be the Hamiltonian momentum of the level sets of  $t$ , and let  $p_{\mu,\text{ADM}}$  be the ADM four-momentum of the space metric  $g_{ij} dx^i dx^j$ . Then,

$$\begin{aligned} p_{i,\text{ADM}} = p_i &= 0, & p_{0,\text{ADM}} &= M - \frac{\Sigma}{\sqrt{3}}, \\ p_0 &= \begin{cases} 2\pi M, & P = 0, \\ 4\pi PM, & P \neq 0, \end{cases} & p_4 &= \begin{cases} 2\pi Q, & P = 0, \\ 8\pi PQ, & P \neq 0. \end{cases} \end{aligned} \quad (\text{A45})$$

The Komar integrals associated with  $X = \partial_t$  are

$$\frac{1}{8\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{S^1} X^{\alpha;\beta} dS_{\alpha\beta} = \begin{cases} 2\pi(M + \frac{\Sigma}{\sqrt{3}}), & P = 0, \\ 8\pi P(M + \frac{\Sigma}{\sqrt{3}}), & P \neq 0, \end{cases} \quad (\text{A46})$$

The Komar integrals associated with  $X = \partial_4$  are

$$\frac{1}{8\pi} \lim_{R \rightarrow \infty} \int_{S(R)} \int_{S^1} X^{\alpha;\beta} dS_{\alpha\beta} = \begin{cases} 4\pi Q, & P = 0, \\ 16\pi PQ, & P \neq 0. \end{cases} \quad (\text{A47})$$

## APPENDIX B: THE VECTOR FIELD Z

Let

$$Z = r\partial_r.$$

We wish to calculate  $\bar{\nabla}_\mu Z_\nu$  for the Kottler metrics and the Rasheed metrics.

First, let  $\bar{g}$  be the  $(n+1)$ -dimensional anti-de Sitter (Kottler) metric,

$$\bar{g} = -V dt^2 + V^{-1} dr^2 + r^2 \bar{h}, \quad (\text{B1})$$

with

$$V = \lambda r^2 + \kappa, \quad (\text{B2})$$

where  $\kappa \in \{0, \pm 1\}$  is a constant,

$$\lambda = -\frac{2\Lambda}{n(n-1)}, \quad (\text{B3})$$

and where  $\bar{h}$  is an ( $r$ -independent) Einstein metric on an  $(n-1)$ -dimensional compact manifold  $\mathcal{K}$ , with scalar curvature  $(n-1)(n-2)\kappa$ . It holds that (cf., e.g., Ref. [48])

$$\bar{\mathbf{R}} = -n(n+1)\lambda. \quad (\text{B4})$$

Further,

$$\begin{aligned}
 \bar{\nabla}_{(\mu} Z_{\nu)} dx^\mu \otimes dx^\nu &= \frac{1}{2} \mathcal{L}_Z \bar{g} = \frac{1}{2} (Z^\alpha \partial_\alpha \bar{g}_{\mu\nu} + \partial_\mu Z^\alpha \bar{g}_{\alpha\nu} + \partial_\nu Z^\alpha \bar{g}_{\alpha\mu}) dx^\mu dx^\nu \\
 &= \frac{1}{2} (r(\partial_r(-V)dt^2 + \partial_r(V^{-1})dr^2 + \partial_r(r^2)d\Omega) + 2V^{-1}dr^2) \\
 &= \frac{1}{2} \left( \frac{r\partial_r V}{V}(-Vdt^2) + (2 - rV^{-1}\partial_r V)V^{-1}dr^2 + 2r^2d\Omega^2 \right), \tag{B5}
 \end{aligned}$$

$$\bar{\nabla}_{[\mu} Z_{\nu]} dx^\mu \otimes dx^\nu = \partial_{[\mu} Z_{\nu]} dx^\mu \otimes dx^\nu = 0. \tag{B6}$$

Adding, we find

$$\bar{\nabla}_{\mu} Z_{\nu} dx^\mu \otimes dx^\nu = \bar{g} \pmod{(\delta'_\mu, \delta'_\nu)}, \tag{B7}$$

which gives Eq. (4.17).

Next, for the Rasheed background metrics (A34) one finds

$$\begin{aligned}
 \mathcal{L}_Z \bar{g} &= 2(dr^2 + r^2 d\Omega^2), \\
 d(\bar{g}_{\alpha\beta} Z^\alpha dx^\beta) &= d(rdr) = 0, \tag{B8}
 \end{aligned}$$

and Eq. (4.17) without the  $o(r^{-\gamma})$  term readily follows.

### APPENDIX C: AN IDENTITY FOR THE RIEMANN TENSOR

We write  $\delta_{\gamma\delta}^{\alpha\beta}$  for  $\delta_{\gamma}^{[\alpha} \delta_{\delta]}^{\beta]} \equiv \frac{1}{2}(\delta_{\gamma}^{\alpha} \delta_{\delta}^{\beta} - \delta_{\gamma}^{\beta} \delta_{\delta}^{\alpha})$ , etc.

For completeness, we prove the following identity satisfied by the Riemann tensor, which is valid in any dimension, is clear in dimensions two and three, implies the double-dual identity for the Weyl tensor in dimension four, and is probably well known in higher dimensions as well:

$$\delta_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} R^{\rho\sigma}{}_{\gamma\delta} = \frac{1}{3!} (R^{\alpha\beta}{}_{\mu\nu} + \delta_{\mu\nu}^{\alpha\beta} R - 4\delta_{[\mu}^{\alpha} R^{\beta]}_{\nu]). \tag{C1}$$

The above holds for any tensor field satisfying

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\beta\alpha\delta\gamma}. \tag{C2}$$

To prove Eq. (C1) one can calculate as follows:

$$\begin{aligned}
 4!\delta_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} R^{\rho\sigma}{}_{\gamma\delta} &= 2[\delta_{\mu}^{\alpha}(\delta_{\nu}^{\beta} \delta_{\rho}^{\gamma} \delta_{\sigma}^{\delta} - \delta_{\rho}^{\beta} \delta_{\nu}^{\gamma} \delta_{\sigma}^{\delta} + \delta_{\sigma}^{\beta} \delta_{\nu}^{\gamma} \delta_{\rho}^{\delta}) - \delta_{\nu}^{\alpha}(\delta_{\mu}^{\beta} \delta_{\rho}^{\gamma} \delta_{\sigma}^{\delta} - \delta_{\rho}^{\beta} \delta_{\mu}^{\gamma} \delta_{\sigma}^{\delta} + \delta_{\sigma}^{\beta} \delta_{\mu}^{\gamma} \delta_{\rho}^{\delta}) \\
 &\quad + \delta_{\rho}^{\alpha}(\delta_{\mu}^{\beta} \delta_{\nu}^{\gamma} \delta_{\sigma}^{\delta} - \delta_{\nu}^{\beta} \delta_{\mu}^{\gamma} \delta_{\sigma}^{\delta} + \delta_{\sigma}^{\beta} \delta_{\mu}^{\gamma} \delta_{\nu}^{\delta}) - \delta_{\sigma}^{\alpha}(\delta_{\mu}^{\beta} \delta_{\nu}^{\gamma} \delta_{\rho}^{\delta} - \delta_{\nu}^{\beta} \delta_{\mu}^{\gamma} \delta_{\rho}^{\delta} + \delta_{\rho}^{\beta} \delta_{\mu}^{\gamma} \delta_{\nu}^{\delta})] R^{\rho\sigma}{}_{\gamma\delta} \\
 &= 2(2\delta_{\mu\nu}^{\alpha\beta} \delta_{\rho}^{\gamma} \delta_{\sigma}^{\delta} - 4\delta_{\mu\nu}^{\alpha\gamma} \delta_{\rho}^{\beta} \delta_{\sigma}^{\delta} + 4\delta_{\mu\nu}^{\beta\gamma} \delta_{\rho}^{\alpha} \delta_{\sigma}^{\delta} + 2\delta_{\rho}^{\alpha} \delta_{\sigma}^{\beta} \delta_{\mu}^{\gamma} \delta_{\nu}^{\delta}) R^{\rho\sigma}{}_{\gamma\delta} \\
 &= 4(\delta_{\mu\nu}^{\alpha\beta} R^{\gamma\delta}{}_{\rho\sigma} - 2\delta_{\mu\nu}^{\alpha\gamma} R^{\beta\sigma}{}_{\rho\sigma} + 2\delta_{\mu\nu}^{\beta\gamma} R^{\alpha\sigma}{}_{\rho\sigma} + R^{\alpha\beta}{}_{\mu\nu}) \\
 &= 4(R^{\alpha\beta}{}_{\mu\nu} + \delta_{\mu\nu}^{\alpha\beta} R^{\gamma\delta}{}_{\rho\sigma} - 4\delta_{[\mu}^{\alpha} R^{\beta]}_{\nu]\rho\sigma}). \tag{C3}
 \end{aligned}$$

If the sums are over all indices we obtain Eq. (C1). The reader is warned, however, that in some of our calculations the sums will be only over a subset of all possible indices,

in which case the last equation remains valid but the last two terms in Eq. (C3) *cannot* be replaced by the Ricci scalar and the Ricci tensor.

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