Accelerated cosmos in a nonextensive setup

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Here we consider a flat FRW universe whose horizon entropy meets the Rényi entropy of nonextensive systems. In our model, the ordinary energy-momentum conservation law is not always valid. By applying the Clausius relation as well as the Cai-Kim temperature to the apparent horizon of a flat FRW universe, we obtain modified Friedmann equations. Fitting the model to the observational data on the current accelerated universe, some values for the model parameters are also addressed. Our study shows that the current accelerating phase of universe expansion may be described by a geometrical fluid, originated from the nonextensive aspects of geometry, which models a varying dark energy source interacting with the matter field in the Rastall way. Moreover, our results indicate that the probable nonextensive features of spacetime may also be used to model a varying dark energy source which does not interact with the matter field and is compatible with the current accelerated phase of the Universe.

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I. INTRODUCTION

The violation of the energy-momentum conservation law in curved spacetime was first proposed by P. Rastall to modify the general relativity theory of Einstein (GR) [1]. After his pioneering work, various types of modified gravity in which matter fields are nonminimally coupled to geometry have been proposed [2–8]. It has been shown that the Rastall correction term to the Einstein field equations cannot describe dark energy, meaning that a dark energylike source is needed to model the current phase of universe expansion in this framework [9]. But, if one generalizes this theory in a suitable manner, then the mutual nonminimal coupling between geometry and matter field may be considered as the origin of the current accelerating phase and the inflation era [5]. More studies on Rastall theory can be found in Refs. [10–30].

In Einstein gravity, horizons may meet the Bekenstein-Hawking entropy-area law which is a nonextensive entropy [31–46]. Moreover, it has recently been argued that a deep connection between dark energy and horizon entropy may exist in gravitational theories [47–58]. Indeed, although extensive statistical mechanics and its corresponding thermodynamics lead to interesting results about the expansion history of the Universe [59], the mentioned points encourage physicists to use nonextensive statistical mechanics [60,61] in order to study the thermodynamic properties of spacetime and its related subjects [62](14) [63–74].

Recently, applying Rényi entropy to the horizon of FRW universe and considering a varying dark energy source interacting with the matter field, N. Komatsu found modified Friedmann equations in agreement with observational data on the current phase of universe expansion [73]. Therefore, in his model, the total energy-momentum tensor, including the matter field and varying dark-energy-like sources, is conserved. Combining this entropy with entropic force scenario, one can also obtain a theoretical basis for the MOND theory [74]. In fact, the probable nonextensive features of spacetime may be considered as an origin for both the MOND theory and the current accelerated expansion phase in a universe filled by a pressureless source satisfying ordinary conservation law [74]. Finally, it is useful to note here that both mentioned attempts [73,74] used the Padmanabhan holographic approach [75] in getting their models of universe expansion.

Here, we are interested in obtaining a model for the dynamics of the Universe by applying the Clausius relation as well as the Rényi entropy to the horizon of the FRW universe which has nonminimally been coupled to the matter field. Therefore, the total energy-momentum tensor does not necessarily satisfy the ordinary conservation law and, in fact, follows the Rastall hypothesis in our setup.

This paper is organized as follows. In the next section, introducing our approach, we present a thermodynamic description for Friedmann equations in Rastall theory. Using the Rényi entropy, our model of the Universe is obtained in Sec. III. In the fourth section, we consider a universe filled by a pressureless source, and show that, in our formalism, it can experience an accelerated expansion. Section V includes the observational constraints of

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model. The last section is devoted to a summary and concluding remarks. We use the unit of $c = \hbar = k_B = 1$ in our calculations.

II. THERMODYNAMIC DESCRIPTION OF FRIEDMANN EQUATIONS IN RASTALL THEORY

Based on the Rastall hypothesis [1],

$$T^{\mu}{}_{\nu;\mu} = \lambda R_{,\nu}, \tag{1}$$

where λ denotes the Rastall constant parameter, and $T^{\mu\nu}$ is the energy-momentum tensor of the source which fills the background. Moreover, R is the Ricci scalar of spacetime. This equation says that there is an energy exchange between spacetime and cosmic fluids due to the tendency of geometry to couple with matter fields in a nonminimal way [5,29]. For example, the $\lambda = \frac{1}{4k}$ case, where k is called the Rastall gravitational coupling constant, can support the primary inflationary era in an empty FRW universe [5]. In this manner, the ability of geometry to couple with matter fields in a nonminimal way generates a constant energy density equal to λR [5]. Some features of this nonminimal mutual interaction and its corresponding energy flux as well as its applications in cosmic eras have been studied in Ref. [5].

Bearing the Bianchi identity in mind and integrating Eq. (1), one can reach [1]

$$G_{\mu\nu} + k\lambda g_{\mu\nu} R = kT_{\mu\nu}. \tag{2}$$

The Newtonian limit of the Rastall theory implies [26,28]

$$k = \frac{\gamma}{\lambda} = \frac{4\gamma - 1}{6\gamma - 1} 8\pi G, \tag{3}$$

where $\gamma \equiv k\lambda$. Applying the thermodynamic laws to the horizon of spacetime and using Eqs. (2) and (3), it is shown that the horizon entropy is achieved as [25–27]

$$S_A^R = \frac{2\pi A}{k}, \tag{4}$$

in which A is the horizon area. Combining this result with Eq. (3), one can easily find that entropy is positive whenever γ meets either the $\gamma < \frac{1}{6}$ or $\gamma > \frac{1}{4}$ condition [26]. This equation can also be written as $S_A^R = \frac{6\gamma - 1}{4\gamma - 1} S_B$, where $S_B = \frac{A}{4G}$ is the Bekenstein-Hawking entropy [26], meaning that the second law of thermodynamics is obtained for the mentioned values of γ if it is satisfied by the Bekenstein-Hawking entropy. In addition, simple calculation leads to

$$\lambda = \frac{\gamma}{k} = \frac{\gamma(6\gamma - 1)}{(4\gamma - 1) 8\pi G} \tag{5}$$

for the Rastall constant parameter [26]. Indeed, Eqs. (3) and (5) are the direct results of imposing the Newtonian limit on Eq. (2), indicating that only for $\lambda = \gamma = 0$ do we have $k = 8\pi G$ [1,28]. In Ref. [30], assuming $k = 8\pi G$ and studying Neutron stars in Rastall gravity, authors found out that λ is close to zero. Therefore, we see that since they consider $k = 8\pi G$, their result does not reject the Rastall hypothesis. Finally, it is worth remembering here that the Bekenstein-Hawking entropy ($S_B = \frac{A}{4G}$) is also obtainable at the appropriate limit of $\gamma \to 0$.

For a flat FRW universe with scale factor $a(t)$ and line element

$$ds^2 = -dt^2 + a(t)^2 [dr^2 + r^2 (d\theta^2 + \sin(\theta)^2 d\phi^2)], \tag{6}$$

the apparent horizon, equal to the Hubble horizon, is located at

$$\tilde{r}_A = a(t) r_A = \frac{1}{H}, \tag{7}$$

and therefore $A = \frac{4\pi}{H^2}$. Now, if the geometry is filled by a prefect fluid with energy density ρ and pressure p ($T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, p, p)$), then Eq. (1) leads to

$$\dot{\rho} + 3H(\rho + p) = -\lambda \dot{R}, \tag{8}$$

where dot denotes derivative with respect to time. It is also useful to remember here that, for the flat FRW universe,

$$R = 6\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right]. \tag{9}$$

Moreover, the use of Eq. (2) yields [25]

$$\begin{aligned} (12\gamma - 3)H^2 + 6\gamma \dot{H} &= -\frac{4\gamma - 1}{6\gamma - 1} 8\pi G \rho, \\ (12\gamma - 3)H^2 + (6\gamma - 2)\dot{H} &= \frac{4\gamma - 1}{6\gamma - 1} 8\pi G p, \end{aligned} \tag{10}$$

The evolution of density perturbation in this model has been studied in Refs. [9,16,18]. It has been shown that the story is the same as those of the standard cosmology at the background and linear perturbation level [9]. Finally, one can use Eq. (10) to get [25]

$$\dot{H} = -\frac{k}{2}(\rho + p). \tag{11}$$

From the standpoint of tensor calculus, Eq. (2) is a solution for Eq. (1) leading to the above results. But, does thermodynamics lead to the same solutions? Indeed, since entropy is the backbone of thermodynamic approach, it is expected that Eq. (2) and, thus, the above results are available only whenever the horizon entropy meets Eq. (4). In order to find the Friedmann equations corresponding on Eq. (1) from the thermodynamics point of view, we consider the general form of entropy as $S_A = S(\frac{2\pi A}{k})$. Additionally, one can use

$$\delta Q^m = (T_a^b \partial_b \tilde{r} + W \partial_a \tilde{r}) dx^a, \tag{12}$$

to evaluate the energy flux crossing the apparent horizon [76,77]. Here, we also focus on an energy-momentum source as $T_\mu^\nu = \mathrm{diag}(-\rho, p, p, p)$ which yields $W = \frac{\rho - p}{2}$ for the work density. Finally, we see [25,27]

$$\delta Q^m = -A(\rho + p) dt. \tag{13}$$

Now, applying the Clausius relation ($T dS_A = -\delta Q^m$) to the horizon [78] and using the Cai-Kim temperature ($T = \frac{H}{2\pi}$) [79], one can easily find

$$\dot{H} = -\frac{\pi}{S'}(\rho + p), \tag{14}$$

where $S' = \frac{dS_A}{dA}$ [80]. Now, combining this result with Eq. (8), we obtain

$$S' dH^2 = \frac{2\pi}{3} d(\rho + \lambda R), \tag{15}$$

as the differential form of the first Friedmann equation. The result of this equation can be combined with Eq. (14) to get the second Friedmann equation.

Now, if the system entropy meets Eq. (4), then Eq. (14) leads to $2\dot{H} = -k(\rho + p)$. It is also easy to insert Eq. (4) into Eq. (15) to get

$$(12\gamma - 3)H^2 + 6\gamma \dot{H} + \mathcal{C} = -\frac{4\gamma - 1}{6\gamma - 1} 8\pi G \rho, \tag{16}$$

where $\mathcal{C}$ is the integration constant of Eq. (15). Now, adding and subtracting $2\dot{H}$ from the lhs of this equation, and using the $2\dot{H} = -k(\rho + p)$ relation, we can reach

$$(12\gamma - 3)H^2 + (6\gamma - 2)\dot{H} + \mathcal{C} = \frac{4\gamma - 1}{6\gamma - 1} 8\pi G p. \tag{17}$$

It is apparent that, for the $\mathcal{C} = 0$ case, the original Friedmann equations in the Rastall framework are reproduced (16). Indeed, since divergence of $\mathcal{C} g_{\mu\nu}$ is zero, one may add the $\mathcal{C} g_{\mu\nu}$ term to the rhs of Rastall field equations (2) to directly get Eq. (16) instead of Eq. (10). Anyway, we know that this term represents an unusual fluid in the context of ordinary physics leading to dark energy concept and, thus, its problems.

Now, Let us consider a flat FRW universe filled by a fluid with constant state parameter defined as $w = \frac{p}{\rho}$. In this manner, calculations lead to [25]

$$\rho = \rho_0 a^{\frac{-3(1+\omega)(4\gamma - 1)}{3\gamma(1+\omega) - 1}}, \tag{18}$$

where ρ_0 is constant, for the energy density profile. For a universe filled by a pressureless component, where $\rho = \rho_0 a^{\frac{-3(4\gamma - 1)}{3\gamma - 1}}$, combining this equation with Eq. (16), one reaches at

$$\dot{H} = -\frac{\rho_0 (4\gamma - 1) 4\pi G}{6\gamma - 1} a^{\frac{-3(4\gamma - 1)}{3\gamma - 1}}, \tag{19}$$

and

$$H^2 = \frac{\rho_0 (3\gamma - 1) 8\pi G}{3(6\gamma - 1)} a^{\frac{-3(4\gamma - 1)}{3\gamma - 1}} + \mathcal{C}. \tag{20}$$

Now, it is natural expectation that the matter density should be diluted during the expansion of the Universe. This limits us to the Rastall theories with $\gamma < \frac{1}{4}$ and $\frac{1}{3} < \gamma$. Applying this result to Eq. (19), we find out that, at the long run limit ($a(t) \gg 1$), we have $\dot{H} \to 0$, and therefore, the Universe may experience an accelerating phase. In this situation, Eq. (20) implies $H = H_0 \approx \mathcal{C}$ meaning that a nonminimal coupling between geometry and energy-momentum source cannot describe the current accelerating phase of the Universe in the Rastall framework. In fact, as it is apparent, we should have $\mathcal{C} \neq 0$ to get $H_0 \neq 0$, and, thus, an accelerating universe. Therefore, the same as the standard Friedmann equations, a dark energy-like source is needed to model the accelerating universe in the Rastall theory, a result in agreement with recent study by Batista *et al.* [9].

In summary, our thermodynamic based study shows the dark energy problem is also valid in this formalism [9], unless one generalizes the Rastall theory in a suitable manner [5].

III. RÉNYI ENTROPY AND FRIEDMANN EQUATIONS

Recently, Rényi entropy has been used in order to study the effects of probable nonextensive aspects of spacetime which led to interesting results in both cosmological and gravitational setups [31,73,74,81–84]. For a nonextensive system including W discrete states, the Rényi entropy is defined as [60]

$$\mathcal{S} = \frac{1}{1-q} \ln \sum_{i=1}^{W} P_i^q, \tag{21}$$

in which P_i and q denote the probability of i^{th} state and the nonextensive parameter, respectively. Moreover, the Tsallis entropy of this system is as follows [61]

$$S_T = \frac{1}{1-q} \sum_{i=1}^{W} (P_i^q - P_i). \tag{22}$$

The linear relation between entropy and area is the key point of the Bekenstein-Hawking entropy ($S_B \sim A$), which can also be obtained from the Tsallis' nonadditive entropy definition [62]. As it is apparent from Eq. (4), the functionality of S_A^R with respect to the horizon area (A) is the same as that of the Bekenstein-Hawking entropy ($S_A^R \sim A$) meaning that S_A^R may be considered as a special case of Eq. (22). More detailed studies on gravitational and cosmological implications of Tsallis entropy (22) can be found in Refs. [62–71] and references therein. Eq. (22) can be combined with Eq. (21) to show that

$$\mathcal{S} = \frac{1}{\delta} \ln(1 + \delta S_T), \tag{23}$$

where $\delta \equiv 1 - q$, and we used $\sum_{i=1}^{W} P_i = 1$ to obtain this equation [74,81]. It has frequently been argued that the Bekenstein-Hawking entropy is not an extensive entropy [31–46], and in fact, the Bekenstein-Hawking entropy can be considered as a proper candidate for S_T in gravitational and cosmological setups [31,73,74,81], a choice in full agreement with Ref. [62]. Here, following the above arguments and recent studies [31,62,73,74,81], we use Eq. (4) as the Tsallis entropy candidate in our calculations leading to

$$\mathcal{S}_A = \frac{1}{\delta} \ln(1 + \delta S_A^R) = \frac{1}{\delta} \ln\left(1 + \frac{2\pi\delta}{k} A\right), \tag{24}$$

$$\mathcal{S}'_A = \frac{d\mathcal{S}_A}{dA} = \frac{2\pi H^2}{k[H^2 + \Delta]}, \tag{25}$$

for the Rényi entropy of horizon ($\mathcal{S}_A$) and its derivative with respect to A ($\mathcal{S}'_A$), respectively. Here, $\Delta \equiv \frac{(6\gamma-1)\delta\pi}{(4\gamma-1)G}$, and it is easy to check that whenever the nonextensive features of system approach zero (or equally $\delta \to 0$), the $\mathcal{S}_A$ relation recovers Eq. (4). Now, inserting Eq. (25) into Eqs. (14) and (15), one can obtain

$$\dot{H} = -\frac{k[H^2 + \Delta]}{2H^2} (\rho + p), \tag{26}$$

and

$$H^2 - \Delta \ln(\Delta + H^2) + C_1 = \frac{k}{3}\rho + \frac{\gamma}{3} R, \tag{27}$$

where C_1 is the integration constant, respectively. Defining a new constant $C = C_1 - \Delta \ln \Delta$, one can rewrite the last equation as

$$H^2 - \Delta \ln\left(1 + \frac{H^2}{\Delta}\right) + C = \frac{k}{3}\rho + \frac{\gamma}{3} R. \tag{28}$$

Since $H = \frac{\dot{a}}{a}$, Eq. (9) can be rewritten as

$$R = 6(\dot{H} + 2H^2). \tag{29}$$

Finally, combining this equation with Eq. (28), and inserting the result into Eq. (26), we find

$$H^2 = \frac{k}{3}(\rho + \rho_e^\gamma), \tag{30}$$

and

$$H^2 + \frac{2}{3}\dot{H} = -\frac{k}{3}(p + p_e^\gamma), \tag{31}$$

for the first and second Friedmann equations in our model, respectively. In fact, one should combine Eqs. (30) and (26) with each other, and then add and subtract the $\frac{2}{3}\dot{H}$ term to the result to find the last equation. In the above equations,

$$\begin{aligned}
\rho_e^\gamma &= \frac{3}{k}\left(4\gamma H^2 + \Delta \ln\left(1 + \frac{H^2}{\Delta}\right) + 2\gamma\dot{H} - C\right), \\
p_e^\gamma &= -\frac{3}{k}\left(4\gamma H^2 + \Delta \ln\left(1 + \frac{H^2}{\Delta}\right)\right. \\
&\quad \left. + 2\dot{H}\left(\gamma + \frac{1}{3(\frac{H^2}{\Delta} + 1)}\right) - C\right),
\end{aligned} \tag{32}$$

denote effective energy density and effective pressure in this framework, respectively. It is easy to see that, at the $\delta \to 0$ limit (or equally $\Delta \to 0$), the results of Rastall theory are recovered. Additionally, we have $p_e \to -\rho_e$ whenever $\dot{H} \to 0$. Now, since $\frac{\ddot{a}}{a} = H^2 + \dot{H}$, one can rewrite Eq. (31) as

$$2\frac{\ddot{a}}{a} + H^2 = -k(p + p_e^\gamma), \tag{33}$$

where its lhs has the same form as that of the standard Friedmann equation [85]. Besides, simple calculations reach

$$\frac{\ddot{a}}{a} = -\frac{k}{6}[\rho + \rho_e^\gamma + 3(p + p_e^\gamma)], \tag{34}$$

for the acceleration equation. Therefore, we deal with two fluids. The first fluid is the ordinary energy-momentum tensor T_μ^ν corresponding to the real fluid with energy density ρ and pressure p. The second fluid, which has geometrical origin, is called the effective energy-momentum tensor, and it is defined as

$$\Theta_\mu^\nu = \mathrm{diag}(-\rho_e^\gamma, p_e^\gamma, p_e^\gamma, p_e^\gamma). \tag{35}$$

In fact, the obtained effective fluid consists of two parts: (i) the nonextensive aspects of spacetime, and (ii) the nonminimal coupling between geometry and matter fields which follows the Rastall hypothesis. Therefore, the $\gamma \to 0$ limit of Θ_μ^ν only includes the nonextensive effects. We show it by $\mathcal{T}_\mu^\nu = \mathrm{diag}(-\rho_e, p_e, p_e, p_e)$ in which

$$\rho_e = \frac{3}{8\pi G}\left(\Delta_{(\gamma=0)} \ln\left(1 + \frac{H^2}{\Delta_{(\gamma=0)}}\right) - C_{(\gamma=0)}\right),$$
$$p_e = -\rho_e - \frac{3}{8\pi G}\left(\frac{2\dot{H}}{3(\frac{H^2}{\Delta_{(\gamma=0)}} + 1)}\right), \tag{36}$$

where $\Delta_{(\gamma=0)} = \frac{\delta\pi}{G}$ and $C_{(\gamma=0)} = C_1 - \Delta_{(\gamma=0)} \ln \Delta_{(\gamma=0)}$. In fact, it is a geometrical fluid originated from the nonextensive aspects of spacetime, and recovers the ordinary cosmological constant model of dark energy at the appropriate limit of $\gamma \to 0$. It is also easy to check that this source satisfies the conservation law, i.e.,

$$\dot{\rho}_e + 3H(\rho_e + p_e) = 0, \tag{37}$$

Therefore, Θ_μ^ν acts as a time-varying dark energy model which satisfies the conservation law only for $\gamma = 0$.

Bearing the Bianchi identity in mind, since the lhs of Eqs. (30) and (33) are compatible with the Einstein tensor, we should have $(\Theta_\mu^\nu + T_\mu^\nu)^{;\mu} = 0$ leading to

$$\dot{\rho} + 3H(\rho + p) = -[\dot{\rho}_e^\gamma + 3H(\rho_e^\gamma + p_e^\gamma)], \tag{38}$$

and, thus,

$$\lambda\dot{R} = \dot{\rho}_e^\gamma + 3H(\rho_e^\gamma + p_e^\gamma). \tag{39}$$

In fact, the above results would also be obtained by writing Einstein field equations as $G_{\mu\nu} = k(\Theta_{\mu\nu} + T_{\mu\nu})$. In addition, Eqs. (37) and (39) tell us that there is no energy flux between geometry and matter fields at the appropriate limit of $\lambda \to 0$ (or equally $\gamma \to 0$), a result in full agreement with Eq. (1). Indeed, although $\Theta_\mu^\nu \to \mathcal{T}_\mu^\nu$ at the $\gamma \to 0$ limit, since $\mathcal{T}_\mu^\nu$ is a divergence-less tensor (37), we have $\lambda\dot{R} = 0$ and, thus, the ordinary energy-momentum conservation law is met by the T_μ^ν source. Applying the $\gamma \to 0$ and $\Delta \to 0$ limits to the above equations, one can easily reach the standard Friedmann equations compatible with the Bekenstein-Hawking entropy of horizon [50,56,71]. Therefore, the $\gamma \to 0$ limit helps us in obtaining the modification of considering Rényi entropy to the standard Friedmann equations as

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_e),$$
$$H^2 + \frac{2}{3}\dot{H} = \frac{-8\pi G}{3}(p + p_e), \tag{40}$$

where ρ_e and p_e follow Eq. (36). It is worth mentioning that, independent of the values of C and Δ, we have $p_e \to -\rho_e$ whenever $\dot{H} \to 0$. As a check, one can also insert $\gamma = 0$ in Eqs. (3) and (32) to get these results. In this manner, the acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[\rho + \rho_e + 3(p + p_e)], \tag{41}$$

and the second line of Eq. (40) can also be written as

$$2\frac{\ddot{a}}{a} + H^2 = -8\pi G(p + p_e), \tag{42}$$

where its lhs is in the form of the standard Friedmann equation [85]. Bearing Eq. (37) as well as the argument after Eq. (39) in mind, it is apparent that, for $\gamma = 0$, the $T_{\mu\nu}$ source respects the continuity equation, i.e.,

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{43}$$

Although the same as Refs. [73,74], we used the Rényi entropy to get the modified Friedmann equations, which differ from those of recent studies [73,74] in the following ways: (i) While the entropy expression appears in acceleration equations obtained in Refs. [73,74], which use the Padmanabhan approach, its derivative is the backbone of getting the acceleration equation in our model based on applying thermodynamics laws to the horizon [see Eq. (14)]; (ii) The Komar mass definition has been used by authors in Refs. [73,74] only for the $T_{\mu\nu}$ source. As we saw, our results are also obtainable if one writes the Einstein field equations as $G_{\mu\nu} = k(\Theta_{\mu\nu} + T_{\mu\nu})$, meaning that the Komar mass should be written for the modified energy-momentum tensor $\Theta_{\mu\nu} + T_{\mu\nu}$ instead of $T_{\mu\nu}$; and (iii) In Ref. [73], the nonextensive features of spacetime have been introduced as an origin for a time-varying dark energy ($\Lambda(t)$) which interacts with matter fields and does not meet Eq. (37). This is while the time-varying dark energy candidate of our model interacts with matter fields only in the Rastall way and meets Eq. (37) in the absence of the Rastall hypothesis ($\gamma = 0$).

IV. A UNIVERSE FILLED BY A PRESSURELESS FLUID

In order to study a universe filled by a pressureless source, we insert $\rho = \rho_0 a^{\frac{-3(4\gamma-1)}{3\gamma-1}}$ into Eq. (26) and use Eq. (30) to reach

$$\frac{H^2(1-4\gamma)-\Delta\ln(1+\frac{H^2}{\Delta})+C}{1-3\gamma(1+\frac{\Delta}{H^2})}=\xi a^{\frac{-3(4\gamma-1)}{3\gamma-1}}, \quad (44)$$

where $\xi\equiv\frac{\rho_0(4\gamma-1)8\pi G}{3(6\gamma-1)}$. It is clear that, for a Rastall theory of $\gamma<\frac{1}{4}$ or $\frac{1}{3}<\gamma$, the rhs of this equation and, thus, its lhs are vanished at long run limit ($a\gg1$) meaning that $H\to$ constant $\equiv H_0$. Now, H_0 can be evaluated from

$$\frac{H_0^2}{\Delta}=\frac{\ln(1+\frac{H_0^2}{\Delta})}{1-4\gamma}+C_2, \quad (45)$$

where $C_2\equiv\frac{C}{(4\gamma-1)\Delta}$ is a constant. It is also apparent that, depending on the value of γ, this equation may be solvable even if we have $C_2=0$ or $C_1=0$. Additionally, since Δ is unknown parameter, this equation helps us in finding its possible values as a function of H_0. One can also use this result in order to apply the $a\gg1$ limit to Eq. (26) to see that $\dot{H}\to0$ whenever γ meets either $\gamma<\frac{1}{4}$ or $\frac{1}{3}<\gamma$. In summary, based on our results, the dark energy problem in Rastall theory can be overcame by considering the probable nonextensive features of spacetime.

Now we consider a universe filled by a pressureless source ($p=0$) whenever $\gamma=0$. In this manner, both of Eqs. (18) and (43) lead to $\rho=\rho_0a^{-3}$ for energy density. Inserting $\gamma=0$ into Eq. (40), and following the recipe which led to Eqs. (44) and (45), we reach

$$H^2-\Delta_{(\gamma=0)}\ln\left(1+\frac{H^2}{\Delta_{(\gamma=0)}}\right)+C_{(\gamma=0)}=\chi a^{-3}, \quad (46)$$

and

$$\frac{H_0^2}{\Delta_{(\gamma=0)}}=\ln\left(1+\frac{H_0^2}{\Delta_{(\gamma=0)}}\right)-C_3, \quad (47)$$

where $\chi\equiv\frac{8\pi G\rho_0}{3}$, and $C_3\equiv\frac{C_{(\gamma=0)}}{\Delta_{(\gamma=0)}}$. It also means that, whenever the divergence of T_μ^ν is zero, the probable nonextensive features of spacetime, which behave as a conserved fluid, can be considered as the nature of the current accelerating phase of the Universe if $\Delta_{(\gamma=0)}$ and H_0 meet the above equation. It is worth noting here that this equation is solvable even if $C_1=0$. This result (the $\gamma=0$ case) is in agreement with our previous results, where we found out the γ parameter should either meet $\gamma<\frac{1}{4}$ or $\frac{1}{3}<\gamma$.

Bearing in mind the results addressed after Eqs. (20) and (45), it is worth mentioning that from the viewpoint of dynamics, the $\gamma<\frac{1}{4}$ and $\frac{1}{3}<\gamma$ intervals are permissible for γ. On the other hand, thermodynamic considerations [the results of Eq. (4)], insist only the $\gamma<\frac{1}{6}$ and $\frac{1}{4}<\gamma$ intervals are admissible. Comparing these results with each other, one can easily find that $\gamma<\frac{1}{6}$ and $\frac{1}{3}<\gamma$ are common intervals. This means that the values of γ obtained from observations are allowed, if they be within in these ranges.

V. OBSERVATIONAL CONSTRAINTS

In what follows, let us discuss the observational constraints on the scenarios presented above. In order to constrain the free parameters of the models we use, the Union 2.1 sample [86], which contains 580 Supernovae type Ia (SNIa) in the redshift range $0.015\le z\le1.41$, 36 observational Hubble data ($H(z)$) in the range ($0.0708\le z\le2.36$) compiled in [87] and the baryon acoustic oscillations (BAO) distance measurements at different redshift, in order to diminish the degeneracy between the free parameters.

A. Supernovae type Ia

In order to study the constraints applied to a cosmological model by the SNIa data, one can use the distance modulus $\mu(z)$ defined as

$$\mu_{\rm th}(z)=5\log_{10}D_L(z)+\mu_0, \quad (48)$$

where $\mu_0=42.38-5\log_{10}h$. Moreover, $h=H_0/100$ km $\cdot$ $s^{-1}\cdot$ Mpc^{-1} is the dimensionless Hubble parameter, and $D_L(z)$ is the luminosity distance calculated as

$$D_L(z)=\frac{c(1+z)}{H_0}\int_0^z\frac{dz'}{E(z')}, \quad (49)$$

in the flat FRW universe. Here, $E^2(z)=H^2(z)/H_0^2$, and, thus, using Eqs. (30) and (40), we can easily reach

$$E^2(z)=\Omega_m(1+z)^3+4\gamma E^2(z)+\frac{\Delta}{H_0^2}\ln\left(1+\frac{H_0^2}{\Delta}E^2(z)\right)+2\gamma\frac{\dot{H}}{H_0^2}+\Omega_C \quad (50)$$

whenever $\gamma\neq0$ and $\Omega_C=-C/H_0^2$ (model I), and

$$E^2(z)=\Omega_m(1+z)^3+\frac{\Delta_{(\gamma=0)}}{H_0^2}\ln\left(1+\frac{H_0^2}{\Delta_{(\gamma=0)}}E^2(z)\right)+\Omega_{C_{(\gamma=0)}}, \quad (51)$$

while $\gamma=0$ and $\Omega_{C_{(\gamma=0)}}=-C_{(\gamma=0)}/H_0^2$ (model II), respectively. In the above results, $\Omega_m=\rho_{m0}/\rho_{\rm cr}$ and $\rho_{\rm cr}$ is the critical density defined as $\rho_{\rm cr}=3H_0^2/8\pi G$. Now, applying the $E^2(z=0)=1$ and $\dot{H}(z=0)=0$ conditions (the usual normalization conditions at $z=0$) to Eqs. (50) and (51), we get

$$\Omega_C=1-\Omega_{m0}-4\gamma-\frac{\Delta}{H_0^2}\ln\left(1+\frac{H_0^2}{\Delta}\right), \quad (52)$$

and

$$\Omega_{C_{(\gamma=0)}} = 1 - \Omega_{m0} - \frac{\Delta_{(\gamma=0)}}{H_0^2} \ln\left(1 + \frac{H_0^2}{\Delta_{(\gamma=0)}}\right), \quad (53)$$

in model I and model II, respectively. Therefore, model I has three free parameters as $\{\Omega_{m0}, \Delta, \gamma\}$, and model II has two free parameters including $\{\Omega_{m0}, \Delta_{(\gamma=0)}\}$. It is worthwhile remembering here that if $\Delta \to 0$ and $\gamma \to 0$, then the ΛCDM model is recovered for $C = C_1 \equiv \Lambda$.

Observational constraints on cosmological model can be obtained by minimizing χ^2 given by [88,89]

$$\chi^2_{\mathrm{SNIa}} = \mathsf{A} - \frac{\mathsf{B}^2}{\mathsf{C}}, \quad (54)$$

where

$$\mathsf{A} = \sum_{i=1}^{580} \frac{[\mu_{\mathrm{th}}(z_i, p_i) - \mu_{\mathrm{obs}}(z_i)]^2}{\sigma^2_{\mu_i}},$$

$$\mathsf{B} = \sum_{i=1}^{580} \frac{\mu_{\mathrm{th}}(z_i, p_i) - \mu_{\mathrm{obs}}(z_i)}{\sigma^2_{\mu_i}},$$

$$\mathsf{C} = \sum_{i=1}^{580} \frac{1}{\sigma^2_{\mu_i}}, \quad (55)$$

and we have marginalized over the nuisance parameter μ_0 and μ_{obs}.

B. Baryon acoustic oscillations (BAO)

The expanding spherical wave of baryonic perturbations, which comes from acoustic oscillations at recombination and comoving scale of about 150 Mpc, helps us in identifying the peak of large scale correlation function measured from SDSS (Sloan Digital Sky Survey). It is worth noting that the BAO scale depends on (i) the scale of sound horizon at recombination and (ii) the transverse and radial scales at the mean redshift of galaxies in the survey. In order to obtain the corresponding constraints on the cosmological models, we begin with χ^2 for the WiggleZ BAO data [90] given as

$$\chi^2_{\mathrm{WiggleZ}} = (\bar{A}_{\mathrm{obs}} - \bar{A}_{\mathrm{th}}) C^{-1}_{\mathrm{WiggleZ}} (\bar{A}_{\mathrm{obs}} - \bar{A}_{\mathrm{th}})^T, \quad (56)$$

where $\bar{A}_{\mathrm{obs}} = (0.447, 0.442, 0.424)$ is data vector at $z = (0.44, 0.60, 0.73)$, T denotes the ordinary transpose, and $\bar{A}_{\mathrm{th}}(z, p_i)$ is [91]

$$\bar{A}_{\mathrm{th}} = D_V(z) \frac{\sqrt{\Omega_m H_0^2}}{cz}, \quad (57)$$

in which

$$D_V(z) = \frac{1}{H_0}\left[(1+z)^2 D_A(z)^2 \frac{cz}{E(z)}\right]^{1/3} \quad (58)$$

is the distance scale. Here, $D_A(z)$ denotes the angular diameter distance defined as $D_A(z) = \frac{D_L(z)}{(1+z)^2}$. Additionally, $C^{-1}_{\mathrm{WiggleZ}}$ is the inverse covariance matrix for the WiggleZ data set given by

$$C^{-1}_{\mathrm{WiggleZ}} = \begin{pmatrix} 1040.3 & -807.5 & 336.8 \\ -807.5 & 3720.3 & -1551.9 \\ 336.8 & -1551.9 & 2914.9 \end{pmatrix}. \quad (59)$$

For the SDSS DR7 BAO distance measurements, χ^2 can similarly be expressed as [92]

$$\chi^2_{\mathrm{SDSS}} = (\bar{d}_{\mathrm{obs}} - \bar{d}_{\mathrm{th}}) C^{-1}_{\mathrm{SDSS}} (\bar{d}_{\mathrm{obs}} - \bar{d}_{\mathrm{th}})^T, \quad (60)$$

where $\bar{d}_{\mathrm{obs}} = (0.1905, 0.1097)$ is the data point at $z = 0.2$ and $z = 0.35$. $\bar{d}_{\mathrm{th}}(z_d, p_i)$ is also defined as

$$\bar{d}_{\mathrm{th}} = \frac{r_s(z_d)}{D_V(z)}, \quad (61)$$

in which $r_s(z)$ is the radius of the comoving sound horizon given by

$$r_s(z) = c\int_z^\infty \frac{c_s(z')}{H(z')} dz' \quad (62)$$

and

$$c_s(z) = \frac{1}{\sqrt{3(1 + \bar{R}_b/(1+z)}}, \quad (63)$$

is the sound speed. Here, $\bar{R}_b = 31500\Omega_b h^2 (T_{\mathrm{CMB}}/2.7\ \mathrm{K})^{-4}$ and $T_{\mathrm{CMB}} = 2.726$ K. z_{drag} at the baryon drag epoch fitted with the formula, proposed in [93],

$$z_{\mathrm{drag}} = \frac{1291(\Omega_m h^2)^{0.251}}{1 + 0.659(\Omega_m h^2)^{0.828}} [1 + b_1(\Omega_b h^2)^{b_2}], \quad (64)$$

where

$$b_1 = 0.313(\Omega_m h^2)^{-0.419}[1 + 0.607(\Omega_m h^2)^{0.674}] \quad (65)$$

and

$$b_2 = 0.238(\Omega_m h^2)^{0.223}. \quad (66)$$

Here,

$$C^{-1}_{\mathrm{SDSS}} = \begin{pmatrix} 30124 & -17227 \\ -17227 & 86977 \end{pmatrix}, \quad (67)$$

TABLE I. Baryon acoustic oscillations (BAO) data measurements used in our statistical analysis.

Survey	z	Parameter	Measurement	Reference
6dF	0.106	r_s/D_V	0.336 ± 0.015	[94]
SDSS-MGS	0.57	r_s/D_V	0.0732 ± 0.0012	[95]
BOSS-LOWZ	0.32	D_V/r_s	8.47 ± 0.17	[95]
BOSS—Ly_α	2.36	D_A/r_s	10.08 ± 0.4	[96]

is the inverse covariance matrix for the SDSS data set. Additionally, we use the Six Degree Field Galaxy Survey (6dF) measurement [94], the Main Galaxy Sample of Data Release 7 of Sloan Digital Sky Survey (SDSS-MGS) [95], the LOWZ and CMASS galaxy samples of the Baryon Oscillation Spectroscopic Survey (BOSS-LOWZ) [95], and the distribution of the LymanForest in BOSS (BOSS—Ly_α) [96]. These measurements and their corresponding effective redshifts (z) are summarized in Table I. Therefore, the total χ^2_{BAO} includes nine data points (for all the BAO data sets).

$$\chi^2_{\mathrm{BAO}} = \chi^2_{\mathrm{WiggleZ}} + \chi^2_{\mathrm{SDSS}} + \chi^2_{6dF} + \chi^2_{\mathrm{SDSS-MGS}} + \chi^2_{\mathrm{BOSS-LOWZ}} + \chi^2_{\mathrm{BOSS-}Ly_\alpha} \tag{68}$$

C. History of the Hubble parameter

The differential evolution of early type passive galaxies provides direct information about the Hubble parameter $H(z)$. We adopt 36 Observational Hubble Data (OHD) at different redshifts ($0.0708 \le z \le 2.36$) obtained from [87], where 26 data are deduced from the differential age method, and the remaining 10 data belong to the radial BAO method. Here, we use these data to constrain the cosmological free parameters of the models under consideration. The corresponding χ^2 can be defined as [88]

$$\chi^2_{H(z)}(H_0, p_i) = \sum_{i=1}^{36} \frac{[H_{\mathrm{obs}}(z_i) - H_{\mathrm{th}}(z_i, H_0, p_i)]^2}{\sigma^2_H(z_i)}, \tag{69}$$

where $H_{\mathrm{th}}(z_i, H_0, p_i)$ is the theoretical value of the Hubble parameter at the redshift z_i. This equation can be rewritten as [88]

$$\chi^2_{H(z)}(H_0, p_i) = \mathsf{A}_1 - \mathsf{B}_1 + \mathsf{C}_1, \tag{70}$$

in which

$$\mathsf{A}_1 = H_0^2 \sum_{i=1}^{36} \frac{E^2(z_i, p_i)}{\sigma_i^2},$$
$$\mathsf{B}_1 = 2H_0 \sum_{i=1}^{36} \frac{H_{\mathrm{obs}}(z_i) E^2(z_i, p_i)}{\sigma_i^2},$$
$$\mathsf{C}_1 = \frac{H^2_{\mathrm{obs}}(z_i)}{\sigma_i^2}. \tag{71}$$

The function $\chi^2_{H(z)}$ depends on the model parameters. To marginalize over H_0, we assume that the distribution of H_0 is a Gaussian function with standard deviation width σ_{H_0} and mean $\bar{H}_0$. Then we build the posterior likelihood function $\mathcal{L}_H(p)$ that depends just on the free parameters p_i, as

$$\mathcal{L}_H(p_i) = \int \pi_H(H_0) \exp[-\chi^2_H(H_0, p_i)] dH_0, \tag{72}$$

where

$$\pi_H(H_0) = \frac{1}{\sqrt{2\pi\sigma_{H_0}}} \exp\left[-\frac{1}{2}\left(\frac{H_0 - \bar{H}_0}{\sigma_{H_0}}\right)^2\right], \tag{73}$$

is a prior probability function widely used in the literature. Finally, we minimize $\chi^2_{H(z)}(p_i) = -2\ln\mathcal{L}_H(p_i)$ with respect to the free parameters p_i to obtain the best-fit parameters values.

D. Statistic analysis and results

Maximum likelihood $\mathcal{L}_{\max}$, is the procedure of finding the value of one or more parameters for a given statistic which maximizes the known likelihood distribution. The maximum likelihood estimate for the best-fit parameters p_i is

$$\mathcal{L}_{\max}(p_i) = \exp\left[-\frac{1}{2}\chi^2_{\min}(p_i)\right], \tag{74}$$

and therefore, $\chi^2_{\min}(p_i) = -2\ln\mathcal{L}_{\max}(p_i)$ [97]. In order to find the best values of the free parameters of the models, we consider

$$\chi^2_{\mathrm{total}} = \chi^2_{\mathrm{SNIa}} + \chi^2_{\mathrm{BAO}} + \chi^2_{H(z)}. \tag{75}$$

Moreover, the Fisher matrix is widely used in analyzing the constraints of cosmological parameters from different observational data sets [98,99]. Having the best-fit $\chi^2_{\min}(p_i, \sigma_i^2)$, the Fisher matrix can be calculated as

$$F_{ij} = \frac{1}{2}\frac{\partial^2 \chi^2_{\min}}{\partial p_i \partial p_j}, \tag{76}$$

where $F_{ij}(\equiv F_{ij}(p_i, \sigma_i^2))$ depends on the uncertainties σ_i^2 of the parameters p_i for a given model. The inverse of the Fisher matrix also provides an estimate of the covariance matrix through $[C_{\mathrm{cov}}] = [F]^{-1}$. Its diagonal elements are the squares of uncertainties in each parameter marginalizing over the others, while the off-diagonal terms yield the correlation coefficients between parameters. The uncertainties obtained in the propagation of errors are also given by $\sigma_i = \sqrt{\mathrm{Diag}[C_{\mathrm{cov}}]_{ij}}$. Note that the marginalized

TABLE II. Summary of the best-fit values at 68.27% C.L. for the parameters $\Delta(\equiv\frac{(6\gamma-1)\delta\pi}{(4\gamma-1)G})$, Ω_{m0} and γ to Rényi entropy. Also, we shows the summary of the best-fit values at 68.27% C.L. for the particular case where $\gamma = 0$, where $\Delta = \Delta_{(\gamma=0)}$. We also present the value of Ω_c, derived from standard error propagation.

Data	χ^2_{min}	Δ	Ω_{m0}	γ	Ω_C
SNIa	562.227	0.001 ± 1.702	0.279 ± 0.033	0.001 ± 0.031	0.77 ± 0.36
SNIa + BAO	564.724	0.034 ± 0.064	0.289 ± 0.015	-0.021 ± 0.018	0.794 ± 0.073
SNIa + BAO + $H(z)$	583.613	0.183 ± 0.053	0.274 ± 0.013	-0.012 ± 0.016	0.725 ± 0.065
Model II	Data	χ^2_{min}	$\Delta_{(\gamma=0)}$	Ω_{m0}	$\Omega_{C_{(\gamma=0)}}$
$\gamma = 0$	SNIa	562.228	0.02 ± 1.26	0.28 ± 0.14	0.72 ± 0.15
$\gamma = 0$	SNIa + BAO	564.818	0.040 ± 0.060	0.287 ± 0.013	0.713 ± 0.014
$\gamma = 0$	SNIa + BAO + $H(z)$	585.158	0.038 ± 0.051	0.270 ± 0.010	0.729 ± 0.010

uncertainty is always greater than (or at most equal to) the nonmarginalized one. In fact, marginalization cannot decrease the error, and it has no effect if all other parameters are uncorrelated with it. Previously known uncertainties of parameters, known as priors, can trivially be added to the calculated Fisher matrix.

Table II summarizes the main results of the statistical analysis carried out by using the data sets SNIa, SNIa + BAO, and SNIa + BAO + $H(z)$ for two scenarios including (i) the Rényi entropy, taking into account the Rastall framework (model I) and (ii) the particular case of $\gamma = 0$ (model II). The parameter Ω_{m0} takes into account the content of cold dark matter plus baryons to the present. It is useful to note that SNIa does not constrain Δ very well, and in fact, results are improved by introducing the other observational tests including BAO and $H(z)$. We can also see that the sign change of γ does not affect the main thermodynamic consideration obtained from Eq. (4). Indeed, since its obtained values meet the $\gamma < \frac{1}{6}$ condition, entropy is always positive and dynamics, i.e., the acceleration of the Universe, is in agreement with the observational data, an outcome in agreement with the results of previous section. For model I, the likelihood contours arisen from the fitting analysis for the set of free parameters (Δ, Ω_{m0}), and marginalized one-dimensional posterior distributions for γ (PDF), considering the best-fit values

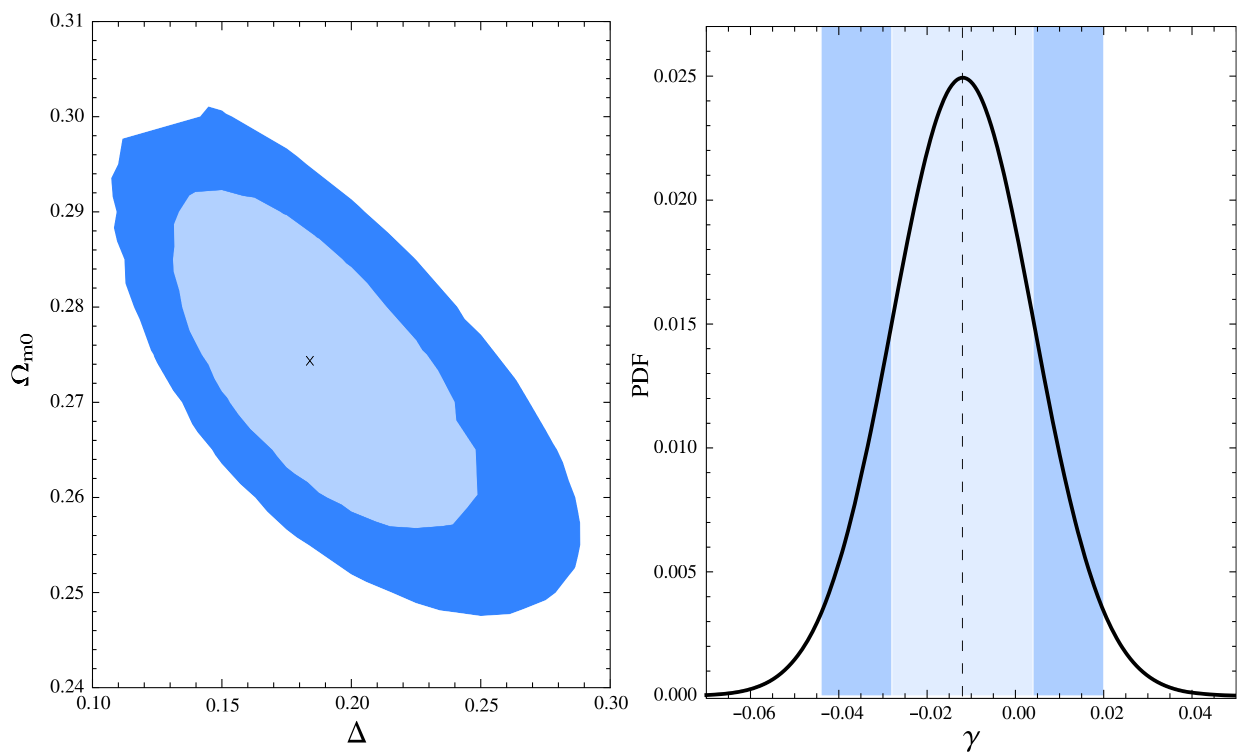

FIG. 1. Contour plots for the free parameter $\{\Delta - \Omega_{m0}\}$ at 1σ and 2σ CL for model I, from the joint analysis SNIa + BAO + $H(z)$ (left panel). Additionally, we present the corresponding marginalized one-dimensional posterior distributions for γ parameter (right panel).

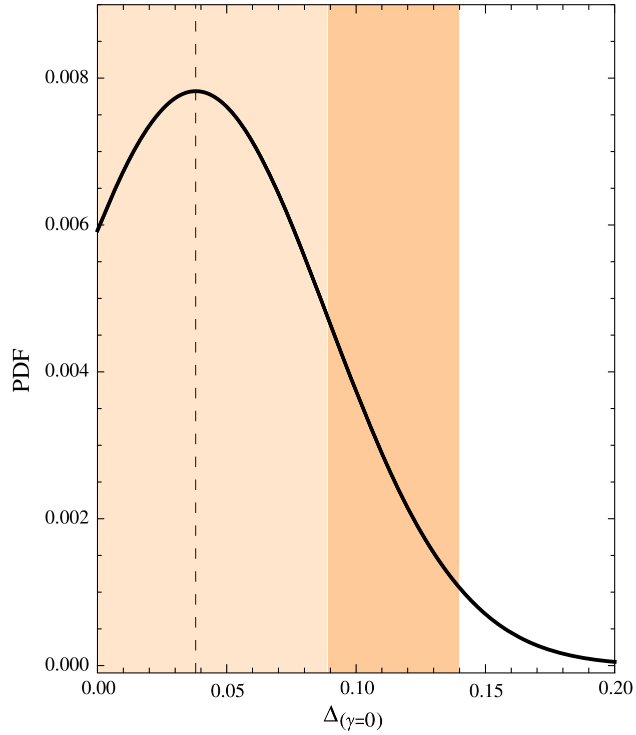

FIG. 2. Marginalized one-dimensional posterior distributions for $\Delta_{(\gamma=0)}$ parameter at 1σ and 2σ CL for model II, from the joint analysis SNIa + BAO + $H(z)$.

for used data sets SNIa + BAO + $H(z)$, are presented in Fig. 1. Moreover, Fig. 2. includes marginalized one-dimensional posterior distributions for $\Delta_{(\gamma=0)}$ parameter in model II. Here, we can appreciate slight deviations from the standard model (or equally the $\Delta_{(\gamma=0)} \to 0$ limit). This possibility does not rule out the standard model and may, in principle, be used to distinguish between the ΛCDM and our models.

The equation of state (EoS) considering a given cosmological model can be written as

$$w(z) = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = -1 + \frac{2(1+z)}{3H}\frac{dH}{dz}, \tag{77}$$

which has been derived from the combination of Eqs. (30) and (31), for the general case and Eq. (40) for the particular case of $\gamma = 0$. Fig. 3. shows the behavior of the total EoS for both cases, with error propagation at 68.27% C.L. regarding the best-fit values presented in Table II. We note that the total EoS, due to the mechanism presented in this paper, does not cross the phantom division line for the best fit of parameters. At high redshift limit, it approaches asymptotically to a value of zero, that is, behaving like a fluid without pressure. In general, we note the behavior $-1 \lesssim w \lesssim 0$ from the best-fit values. Similar behavior are found in unification models in the dark sector of the Universe.

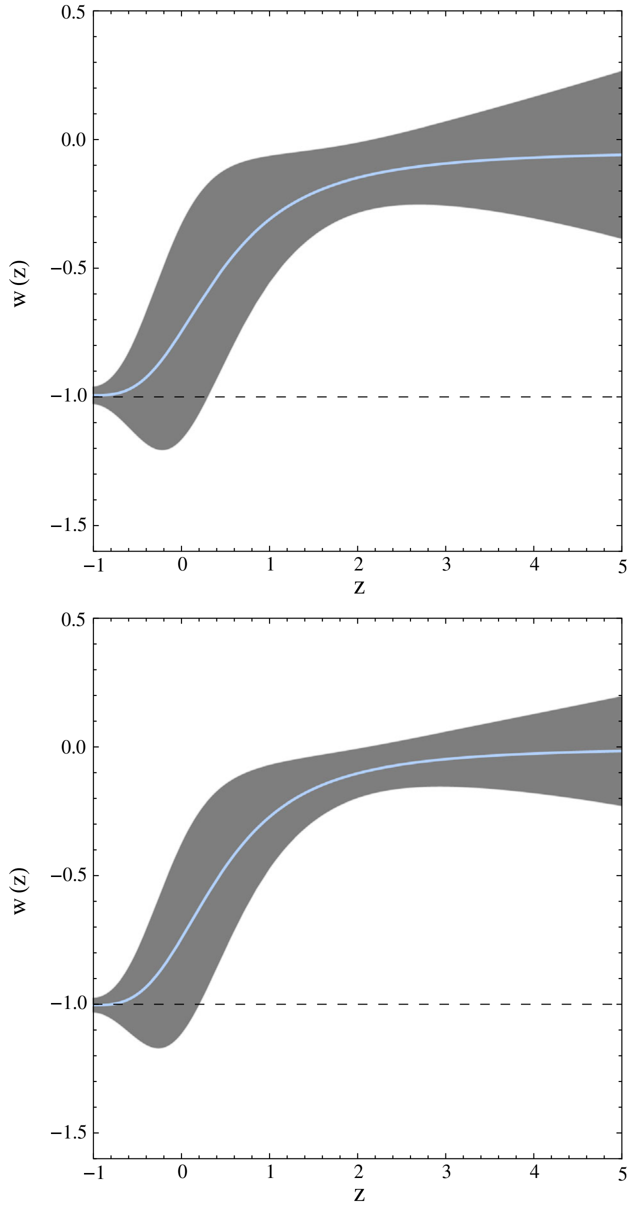

FIG. 3. Reconstruction of the EoS at 68% CL (gray region) from our joint analysis model I (top panel) and model II ($\gamma = 0$) (bottom panel). The blue line represents the best-fit value for all data set SNIa + BAO + $H(z)$.

On the other hand, it is natural to describe the kinematics of the cosmic expansion through the Hubble parameter $H(t)$, and its dependence on time, i.e., the deceleration parameter $q(z)$. The deceleration parameter is defined as $q(z) = -\ddot{a}a/\dot{a}$ combined with the $\ddot{a}/a = H^2 + \dot{H}$ relation, where $\dot{H} = dH/dt$, to get

$$q(z) = -1 + \frac{(1+z)}{H(z)}\frac{dH(z)}{dz}. \tag{78}$$

From Eq. (41) and by considering $p = 0$ along with $p_e^\gamma = -\rho_e^\gamma$, it follows that

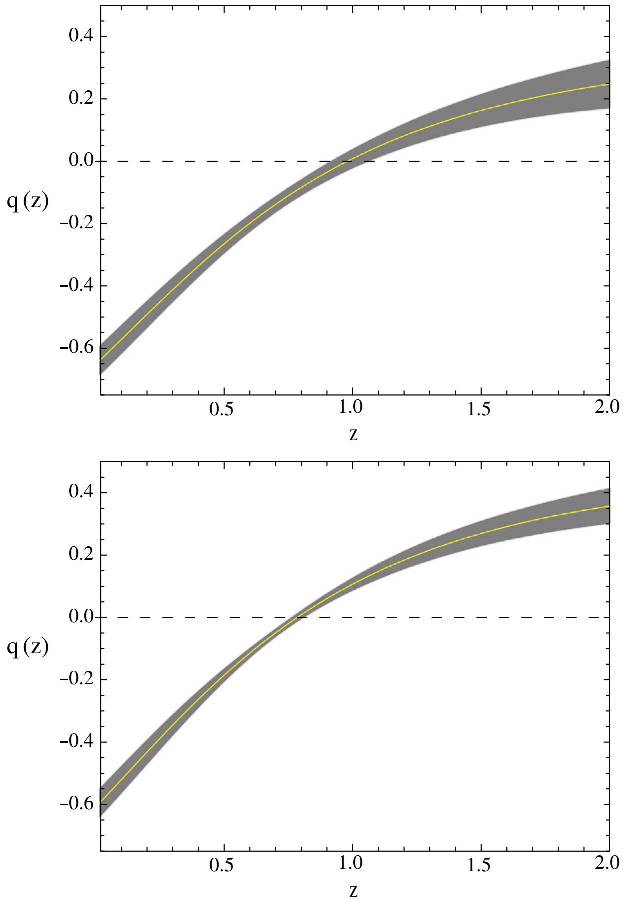

FIG. 4. Reconstruction of the $q(z)$ parameter, along with the 1σ errors (shaded region). From our joint analysis, model I (top panel) and model II ($\gamma = 0$) (bottom panel). The yellow line represents the best-fit value for all data set SNIa + BAO + $H(z)$.

$$q_0 = \frac{\Omega_{m0}}{2} - \Omega_e^\gamma, \tag{79}$$

where its rhs has to be evaluated at $z = 0$, and we have defined $\Omega_e^\gamma = \rho_e^\gamma/\rho_{\rm cr}$. In general, if Ω_e^γ is sufficiently large (i.e., $\Omega_e^\gamma > \Omega_m$), then $q(z = 0) < 0$, which corresponds to an accelerated expanding universe.

$q(z)$ has been plotted in Fig. 4. by using the best fit of parameters with all observational data SNIa + BAO + Hz (See Table II). As expected, models give $q(z) < 0$ at late times and $q(z) > 0$ at earlier epoch, which means that the expansion rate is slowed down in the past and speeded up in the present. Therefore, there is a transition between the decelerated phase ($q(z) > 0$) into an accelerated era $q(z) < 0$ at redshift z_t for these models. Our analysis admits $\{z_t = 0.98, q_0 = -0.63\}$ for model I and $\{z_t = 0.77, q_0 = -0.59\}$ for model II.

VI. SUMMARY AND CONCLUDING REMARKS

Applying the Clausius relation, the Cai-Kim temperature, and the Rényi entropy to the apparent horizon of a flat FRW universe, we arrived at a model for the dynamics of the Universe. Fitting the model to observational data, the values of model parameters were obtained. We found out that if we attribute Rényi entropy to the horizon, then the current accelerated phase of universe expansion may be described in the Rastall framework. Our study also shows that the probable nonextensive features of spacetime may play the role of a varying dark energy in a universe in which an ordinary energy-momentum conservation law is valid.

Although our model shows suitable agreement with observational data, it is very important to study the evolution of density perturbations in this model which helps us to decide about the performance of our model. We leave this subject for future work.

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