Parametrizations of three-body hadronic B- and D-decay amplitudes in terms of analytic and unitary meson-meson form factors

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We introduce parametrizations of hadronic three-body B and D weak decay amplitudes that can be readily implemented in experimental analyses and are a sound alternative to the simplistic and widely used sum of Breit-Wigner type amplitudes, also known as the isobar model. These parametrizations can be particularly useful in the interpretation of CP asymmetries in the Dalitz plots. They are derived from previous calculations based on a quasi-two-body factorization approach in which two-body hadronic final-state interactions are fully taken into account in terms of unitary S- and P-wave $\pi\pi$, πK, and $K\bar{K}$ form factors. These form factors can be determined rigorously, fulfilling fundamental properties of quantum field-theory amplitudes such as analyticity and unitarity, and are in agreement with the low-energy behavior predicted by effective theories of QCD. They are derived from sets of coupled-channel equations using T-matrix elements constrained by experimental meson-meson phase shifts and inelasticities, chiral symmetry, and asymptotic QCD. We provide explicit amplitude expressions for the decays $B^\pm \to \pi^+\pi^-\pi^\pm$, $B \to K\pi^+\pi^-$, $B^\pm \to K^+K^-K^\pm$, $D^+ \to \pi^-\pi^+\pi^+$, $D^+ \to K^-\pi^+\pi^+$, and $D^0 \to K_S^0\pi^+\pi^-$, for which we have shown in previous studies that this approach is phenomenologically successful; in addition, we provide expressions for the $D^0 \to K_S^0 K^+K^-$ decay. Other three-body hadronic channels can be parametrized likewise.

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I. INTRODUCTION

Three-body hadronic decays of B and D mesons are a rich field for searches on CP violation, for tests of the Standard Model and of QCD in particular [1–5]. Furthermore, they provide an interesting ground to study hadron physics, as strong interaction effects, through the presence of two-body resonances and their interferences, have an impact on weak-decay observables. In order to extract these observables most reliably, the meson-meson final-state interactions must be addressed using theoretical constraints such as unitarity, analyticity and chiral symmetry, as well as constraints from experimental data from processes other than B and D decays. However, in Dalitz-plot analyses, the event distributions are often studied using the isobar model in which the decay amplitudes are parametrized by coherent sums of Breit-Wigner amplitudes with a background contribution, in disagreement with the fundamental principles listed above. In this work, we suggest replacing these sums by parametrizations in terms of unitary two-meson form factors, without losing contact with the description of the weak-interaction dynamics that governs the underlying flavor-changing process. These parametrizations are constructed, in part, from results published previously [6–14] and are motivated by the forthcoming analyses of high-statistics data sets for many three-body decay channels of B and D decays, in particular by the LHCb Collaboration [15].

The theoretical amplitude expressions in Refs. [6–14] from which we derive the present parametrizations are based on models of QCD factorization. The factorization beyond the leading approximation can be expressed as an expansion in the strong coupling, α_s, and inverse powers of the bottom quark mass, m_b, and has been applied with success to charmless nonleptonic two-body B decays (see, e.g., Ref. [16]). Parallel analyses of three-body B decays in the contexts of QCD factorization and perturbative QCD can be found in Refs. [17,18] and [19,20], respectively. In D decays, this factorization approach is less predictive inasmuch as it does not allow for a systematic improvement owing to the charm quark mass, $m_c \simeq m_b/3$, which enhances significant corrections to the factorized results. It is, therefore, downgraded from an effective theory that can be systematically improved, in the case of B decays, to a phenomenological procedure, in the case of D decays. Nevertheless, as a purely phenomenological approach, based on the seminal work by Bauer, Stech and Wirbel [21], the factorization hypothesis has been applied successfully to D decays, provided one treats Wilson coefficients as phenomenological parameters to account for nonfactorizable corrections [22].

Besides a recent extension of the QCD factorization framework to nonleptonic B decays into three light mesons [23], no rigorous factorization theorem valid for the entire three-body phase space and full three light-meson Dalitz plot exists. On the other hand, three-body decays of B and D mesons clearly receive important contributions from intermediate resonances—such as the $\rho(770)$, $K^*(892)$, and $\phi(1020)$—and can therefore be considered as *quasi-two-body decays*. One then assumes that two of the three final-state mesons form a single state originating from a quark-antiquark pair, which is interpreted as an intermediate quasi-two-body final state in which case the factorization can be applied. Then, the three-body final state is reconstructed with the use of two-body mesonic form factors to account for the important hadronic final-state interactions. For instance, in the $D^0 \to K_S^0\pi^-\pi^+$ decay, the three-meson final state $K_S^0\pi^+\pi^-$ is initially preceded by the quasi-two-body pairs, $[K_S^0\pi^+]_L\pi^-$, $[K_S^0\pi^-]_L\pi^+$, and $K_S^0[\pi^+\pi^-]_L$, where two of the three mesons form a state in an $L = S$ or P wave. This framework has been successfully applied to several hadronic three-body B and D decays [6–14,24,25].

The factorization of a nonleptonic weak B decay into a quasi-two-body state can be schematically described as follows. The decays are mediated by local dimension-6 four-quark operators $O_i(\mu)$ that form the weak effective nonrenormalizable Hamiltonian. However, depending on flavor content, spin, charge, and parity symmetry of the final states, only specific operators will contribute to a given decay. The B-decay amplitude into two mesons, M_1 and M_2^* with four momenta p_1 and p_2, respectively, can be written as

$$\langle M_1(p_1)M_2^*(p_2)|\mathcal{H}_{\rm eff}|B(p_B)\rangle = \frac{G_F}{\sqrt{2}}V_{\rm CKM}\sum_i C_i(\mu)\langle M_1(p_1)M_2^*(p_2)|O_i(\mu)|B(p_B)\rangle, \tag{1}$$

where $p_B = p_1 + p_2$, G_F is the Fermi constant, $V_{\rm CKM}$ is a product of Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, $C_i(\mu)$ are Wilson coefficients renormalized at the scale μ [26], and $M_2^*(p_2)$ is the resonant quasi-two-body state which decays into two lighter mesons. The hadronic amplitude $\langle M_1(p_1)M_2^*(p_2)|O_i(\mu)|B(p_B)\rangle$ describes long-distance physics. In the factorization approach we henceforth employ, this amplitude is the sum of two matrix-element products,

$$\begin{aligned}\langle M_1(p_1)&M_2^*(p_2)|O_i(\mu)|B(p_B)\rangle \\ &= (\langle M_1(p_1)|J_1^\nu|B(p_B)\rangle\langle M_2^*(p_2)|J_{2\nu}|0\rangle \\ &\quad + \langle M_1(p_1)|J_3^\nu|0\rangle\langle M_2^*(p_2)|J_{4\nu}|B(p_B)\rangle) \\ &\quad \times\left[1+\sum_n r_n\alpha_s^n(\mu)+\mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{m_b}\right)\right],\end{aligned} \tag{2}$$

where the strong coupling is evaluated at a scale μ, r_n is a combination of constant strong-interaction factors, and $|0\rangle$ is the vacuum state. Thus, at leading order, the decay amplitudes factorize into two matrix elements with either the weak quark currents J_1 and J_2 or J_3 and J_4. Radiative corrections can be systematically taken into account to a given order $\alpha_s^n(\mu)$, whereas corrections to the heavy-quark limit are of nonperturbative nature and therefore much less controlled. This is in particular true for the charm quark, which is neither a light nor heavy enough quark [27–30]. This fact makes the systematic improvements of Eq. (2), enclosed in square brackets, less reliable for D decays. One should keep this limitation in mind, but for lack of a better theoretical framework, the phenomenological approach to Eq. (2) remains a good starting point to organize the description of D decays and can be used to provide a first step beyond the isobar model.

The weak effective Hamiltonian, $\mathcal{H}_{\rm eff}$, in Eq. (1) is given by the sum of local operators $O_i(\mu)$ multiplied by Wilson coefficients $C_i(\mu)$ which encode the short-distance effects above the renormalization scale μ. For a $\Delta B = 1$ transition, for example, the Hamiltonian is given by [31,32]

$$\begin{aligned}\mathcal{H}_{\rm eff}^{\Delta B=1} = \frac{G_F}{\sqrt{2}}\sum_{p=u,c}V_{pq}^*V_{pb}\Big[&C_1(\mu)O_1^p(\mu)+C_2(\mu)O_2^p(\mu) \\ &+\sum_{i=3}^{10}C_i(\mu)O_i(\mu)+C_{7\gamma}(\mu)O_{7\gamma}(\mu) \\ &+C_{8g}(\mu)O_{8g}(\mu)\Big]+\text{H.c.},\end{aligned} \tag{3}$$

where the quark flavor can be $q = d$, s and V_{ij} are CKM matrix elements. In the decays, the weak-interaction W-boson exchange diagram gives rise to two current-current operators with different color structure owing to QCD corrections and $SU(3)$ color algebra:

$$O_1^p(\mu) = \bar{q}_i\gamma^\mu(1-\gamma_5)p_i\bar{p}_j\gamma_\mu(1-\gamma_5)b_j \tag{4}$$

$$O_2^p(\mu) = \bar{q}_i\gamma^\mu(1-\gamma_5)p_j\bar{p}_j\gamma_\mu(1-\gamma_5)b_i. \tag{5}$$

In Eqs. (4) and (5), i, j are color indices, and for the corresponding Wilson coefficients, one has $C_1(\mu) \simeq 1 + \mathcal{O}(\alpha_s(\mu))$ and $C_2(\mu) \simeq \mathcal{O}(\alpha_s(\mu))$. The operators O_i, $i =$ 3–10 stem from QCD and electroweak penguin diagrams, while $O_{7\gamma}$ and O_{8g} are electromagnetic and chromomagnetic dipole operators. The explicit tensor structure of these operators as well as their Wilson coefficients at next-to-leading logarithms can be found, for example, in Ref. [33]. With the use of an appropriate Fierz transformation and the $SU(N_c)$ identity,

$$(\bar{q}_i p_j)(\bar{p}_j b_i) = 2(\bar{q}_i T^a_{ik} p_k)(\bar{p}_j T^a_{jl} b_l) + \frac{1}{N_c}(\bar{q}_i p_i)(\bar{p}_j b_j), \tag{6}$$

where T^a_{ij} are the $SU(N_c)$ generators, the quark bilinears can be rearranged to match the flavor and color structure of the final mesons. In this transformation, the color-octet contribution in Eq. (6) is commonly neglected. The two resulting combinations of $C_1(\mu)$ and $C_2(\mu)$,

$$a_1(\mu) = C_1(\mu) + \frac{1}{N_c} C_2(\mu), \quad a_2(\mu) = C_2(\mu) + \frac{1}{N_c} C_1(\mu), \tag{7}$$

lead to "color allowed" and "color suppressed" amplitudes, respectively, which are topologically different. Typically, the Wilson coefficients are evaluated at a renormalization scale of the heavy quark, i.e. $\mu \simeq m_c, m_b$.

On the right-hand side of Eq. (2), the two matrix-element products describe different physical processes. Namely, the creation of a final two-meson state from a $\bar{q}q$ pair is described by the form factors $\langle M_2^*(p_2)|J_{2\nu}|0\rangle$, where $M_2^* \to M_3 M_4$ denotes resonant intermediate states in the different two-meson coupled channels that lead to the final three-body state. As mentioned, these form factors can be constructed so as to preserve two-body unitarity and reproduce asymptotic QCD and are constrained by chiral symmetry at low energies. We discuss them in Appendix A. In Eq. (2), the matrix element $\langle M_1(p_1)|J_3^\nu|0\rangle$ defines the weak-decay constant of a scalar, pseudoscalar, or vector meson, which is either well known from experiment, for instance, f_π and f_K, or has been evaluated with lattice-regularized QCD and other nonperturbative approaches. The transition $\langle M_2^*(p_2)|J_{4\nu}|B\rangle$ of a B meson to a strongly interacting two-meson pair via a resonance is a complicated process and the biggest source of uncertainty in our approach. It could be extracted experimentally from semileptonic processes such as $B^0 \to K^+\pi^-\mu^+\mu^-$ [34] or $D^0 \to K^-\pi^+\mu^+\mu^-$ [35]. It has also been conjectured within soft-collinear effective theory that the amplitude can be factorized in terms of a generalized B-to-two-body form factor and two-hadron light-cone distribution amplitudes [36]. In the derivation of the amplitude expressions presented here, we employ a model approximation which relates this matrix element $\langle M_2^*(p_2)|J_{4\nu}|B\rangle$ to the two-body meson form factor $\langle M_2^*(p_2)[\to M_3 M_4]|J_{2\nu}|0\rangle$.

Finally, the transition amplitudes $\langle M_1(p_1)|J_1^\nu|B\rangle$ ($= \langle M_1(p_1)\bar{B}|J_1^\nu|0\rangle$) are parametrized by heavy-to-light transition form factors, which are discussed in Appendix A 4.

As a definite example of the procedure outlined above, let us consider the $D^+ \to [K^-\pi^+]_{S,P}\pi^+$ decay, where the $K^-\pi^+$ pairs are in the S- or P-wave state. The matrix element given by $\langle [K^-\pi^+]_{S,P}\pi^+|\mathcal{H}_{\text{eff}}|D^+\rangle$ receives contributions from the two amplitudes $a_1(\mu)$ and $a_2(\mu)$ and factorizes as

$$\begin{aligned}
&\langle [K^-\pi^+]_{S,P}\pi^+|\mathcal{H}_{\text{eff}}|D^+\rangle \\
&\quad = \frac{G_F}{\sqrt{2}} \cos^2\theta_C [a_1 \langle [K^-\pi_1^+]_{S,P}|\bar{s}\gamma^\nu(1-\gamma_5)c|D^+\rangle \\
&\qquad \times \langle \pi_2^+|\bar{u}\gamma_\nu(1-\gamma_5)d|0\rangle \\
&\qquad + a_2 \langle [K^-\pi_1^+]_{S,P}|\bar{s}\gamma^\nu(1-\gamma_5)d|0\rangle \\
&\qquad \times \langle \pi_2^+|\bar{u}\gamma_\nu(1-\gamma_5)c|D^+\rangle] + (\pi_1^+ \leftrightarrow \pi_2^+),
\end{aligned} \tag{8}$$

θ_C being the Cabbibo angle. The $K\pi$ form factors appear explicitly in the matrix element $\langle [K^-\pi_1^+]_{S,P}|\bar{s}\gamma^\nu(1-\gamma_5)d|0\rangle$. The evaluation of $\langle [K^-\pi_1^+]_{S,P}|\bar{s}\gamma^\nu(1-\gamma_5)c|D^+\rangle$ is less straightforward. However, assuming this transition to proceed through the dominant intermediate resonances, this matrix element can also be written in terms of the $K\pi$ form factors as shown in Refs. [13,25]. This feature is of crucial importance to the parametrizations that we propose in this work. It is interesting to note that the calculation of a generalized three-body form factor using light-cone sum rules, in the spirit of Ref. [36], also leads to the appearance of the two-body meson form factors [37–41]. The other matrix elements of Eq. (8) can be written in terms of decay constants or transitions form factors that can be extracted from semileptonic decays, as outlined above. Strong phases in the mesonic final-state interactions are accounted for by the hadronic form factors, which makes this type of description particularly suitable for the interpretation of CP asymmetries that have been observed in B decays [8–11,42]. Amplitude expressions, such as in Eq. (8), are used throughout this paper as a starting point to build parametrizations based on unitary two-body hadronic form factors. Within this approach, explicit forms of parametrizations for $D^+ \to K^-\pi^+\pi^+$ and $D^0 \to K_S^0\pi^-\pi^+$ amplitudes have already been presented in Ref. [15] (see p. 27 therein).

The paper is structured as follows. In Sec. II, we introduce the parametrizations for three-body hadronic B-decay amplitudes based on the quasi-two-body factorization approaches of Ref. [9] for $B^\pm \to \pi^+\pi^-\pi^\pm$, of Refs. [6–8] for $B \to K\pi^+\pi^-$, and of Ref. [10] for $B^\pm \to K^+K^-K^\pm$. Section III applies the same procedure to D-decay amplitudes, viz. $D^+ \to \pi^+\pi^-\pi^+$ [12], $D^+ \to K^-\pi^+\pi^+$ [13], $D^0 \to K_S^0\pi^+\pi^-$ [14], and $D^0 \to K_S^0 K^+K^-$ [24]. The meson-meson and heavy-to-light meson form factors which have been used can be found in the original papers. Nevertheless, a short reminder about the derivations of unitary S- and P-waves $\pi\pi$-, πK-, and $K\bar{K}$-meson form factors entering these parametrizations is given in Appendix A together with a short review on heavy-to-light meson form factors. We wrap up with some concluding remarks about the merits of the proposed parametrizations in Sec. IV. The relations between the free parameters of the different proposed parametrizations and the theoretical decay amplitudes are presented explicitly in Appendix B.

II. PARAMETRIZATIONS OF THREE-BODY HADRONIC B-DECAY AMPLITUDES

A. Amplitudes for $B^\pm \to \pi^\pm\pi^+\pi^-$

The contributions of pion-pion interactions to CP-violating phases in $B^\pm \to \pi^\pm\pi^\mp\pi^\pm$ decays have been studied [9] within the quasi-two-body factorization approach discussed in the Introduction.[1] The amplitudes were derived as matrix elements of the weak effective Hamiltonian given by Eq. (3) with $q \equiv d$. The $\pi\pi$ effective mass distributions of the $B^\pm \to \pi^\pm\pi^+\pi^-$ data [43] are well reproduced for an invariant mass, $m_{\pi^+\pi^-} \lesssim 1.64$ GeV [9]. To parametrize the amplitudes of $B^\pm \to \pi^\pm[\pi^+\pi^-]_{S,P}$, we label the momenta of the decay as $B^\pm(p_B) \to \pi^\pm(p_1)\pi^+(p_2)\pi^-(p_3)$, where p_B, the $B^\pm$ meson momentum, satisfies $p_B = p_1 + p_2 + p_3$. The amplitudes must be symmetrized by exchanging the $\pi^+(p_2)\pi^-(p_3)$ and $\pi^-(p_1)\pi^+(p_2)$ pairs in case of a B^- decay or equivalently the $\pi^+(p_2)\pi^-(p_3)$ and $\pi^+(p_1)\pi^-(p_3)$ pairs in case of a B^+ decay. Defining the invariants, $s_{ij} = (p_i + p_j)^2$ (for $i \neq j$), with $s_{12} + s_{13} + s_{23} = m_B^2 + 3m_\pi^2$, the interacting pairs of pions in a relative S or P wave are described by s_{12} or s_{23} in the case of a B^- decay and by s_{13} and s_{23} in the case of a B^+ decay.

The symmetrized amplitude (see also Eq. (21) in Ref. [9]) for the $B^- \to \pi^-[\pi^+\pi^-]_{S,P}$ decay reads

$$\begin{aligned}\mathcal{A}^-_{\rm sym}(s_{12}, s_{23}) = \frac{1}{\sqrt{2}}[&\mathcal{A}^-_S(s_{12}) + \mathcal{A}^-_S(s_{23}) \\ &+ (s_{13} - s_{23})\mathcal{A}^-_P(s_{12}) \\ &+ (s_{13} - s_{12})\mathcal{A}^-_P(s_{23})],\end{aligned} \tag{9}$$

and an analogous amplitude holds for the $B^+ \to \pi^+[\pi^-\pi^+]_{S,P}$ decay. The amplitudes $\mathcal{A}^-_{S,P}(s_{ij})$, $ij = 12$ or 23, given by Eqs. (22) and (23) of Ref. [9], can be parametrized in terms of four complex parameters, $a^{S,P}_{1,2}$, as[2]

$$\mathcal{A}^-_S(s_{ij}) = [a^S_1(M_B^2 - s_{ij}) + a^S_2 F_0^{B\pi}(s_{ij})]F^{\pi\pi}_{0n}(s_{ij}), \tag{10}$$

$$\mathcal{A}^-_P(s_{ij}) = [a^P_1 + a^P_2 F_1^{B\pi}(s_{ij})]F^{\pi\pi}_1(s_{ij}), \tag{11}$$

where M_B is the charged B-meson mass. As done in the *BABAR* Collaboration analysis [43] and in Ref. [9], a contribution from the $f_2(1270)$ resonance can be accounted for by a Breit-Wigner line shape in a D-wave amplitude of the $\pi^+\pi^-$ pair. The $B \to \pi$ scalar and vector transition form factors $F^{B\pi}_{0,1}(s)$ in Eqs. (10) and (11) are discussed in Appendix A 4. The $\pi\pi S$-wave amplitude $\mathcal{A}^-_S(s_{ij})$ includes via the nonstrange scalar form factor $F^{\pi\pi}_{0n}(s_{ij})$ the contributions of the scalar $f_0(500)$, $f_0(980)$, and $f_0(1400)$ resonances. In a Dalitz-plot analysis, one can use, for example, the pion scalar form factor derived in Refs. [9,44]. More details are given in Appendix A 1.

The P-wave amplitude $\mathcal{A}^-_P(s_{ij})$, proportional to the pion vector form factor $F^{\pi\pi}_1(s_{ij})$, contains the $\rho(770)^0$, $\rho(1450)$, and $\rho(1700)$ contributions. In Ref. [9], the $(\pi\pi)_P$ form factor was extracted from the Belle Collaboration analysis of $\tau^- \to \pi^-\pi^0\nu_\tau$ decay data [45]. Alternatively, one can employ the unitary parametrization of Ref. [46], which fits simultaneously the $(\pi\pi)_P$-wave phase shifts and inelasticities, the $e^+e^- \to \pi^+\pi^-$ data, and the $\tau^- \to \pi^-\pi^0\nu_\tau$-decay data, as done in the $D^0 \to K^0_S\pi^+\pi^-$ Dalitz-plot fit of Ref. [14]; see Appendix A 1.

Setting the phase of a^P_1 in $\mathcal{A}^-_P(s_{ij})$ to zero yields a total of seven real parameters to be fitted. The fully symmetrized CP-conjugate $B^+ \to \pi^+\pi^-\pi^+$-decay amplitude is given by expressions similar to Eqs. (9)–(11) with again seven free real parameters. The reproduction of the Dalitz-plot data over the full phase space, in particular for the high invariant-mass regions, might require some adjustment of the $\pi\pi$ form factors. The addition of further phenomenological amplitudes that represent contributions of higher $\pi\pi$-interacting waves and possible three-body rescattering terms may be necessary.

B. Amplitudes for $B \to K\pi^+\pi^-$

The amplitude is based on the weak effective Hamiltonian in Eq. (3) with $q \equiv s$. The momenta are labeled as $B(p_B) \to K(p_1)\pi^+(p_2)\pi^-(p_3)$, with $s_{12} = (p_1 + p_2)^2$, $s_{13} = (p_1 + p_3)^2$, $s_{23} = (p_2 + p_3)^2$, and $s_{12} + s_{13} + s_{23} = m_B^2 + m_K^2 + 2m_\pi^2$.

1. Parametrization of the $B \to K[\pi^\pm\pi^\mp]_{S,P}$ amplitude

The isoscalar S-wave $\pi^+\pi^-$ final-state interactions in $B \to K\pi^+\pi^-$ decays were studied in Ref. [6] in the quasi-two-body factorization approach with an extension in Ref. [7] to include the $\pi^+\pi^-$ isovector P wave. These studies reproduce very well the Belle and *BABAR* data in an effective $\pi\pi$ mass range up to about 1.2 GeV. Following Eq. (1) of Ref. [6], the $B \to K[\pi^+\pi^-]_S$ decay amplitude can be parametrized in terms of three complex parameters, b^S_i, $i = 1, 2, 3$, for the different charges $B = B^\pm$, $K = K^\pm$ and $B = B^0(\bar{B}^0)$, $K = K^0(\bar{K}^0)$ or K^0_S,

[1] During the preparation of this manuscript, Ref. [42] appeared. Their treatment is very similar to the one we describe here.

[2] In a fit to a Dalitz plot, there is always a global phase that cannot be observed. Therefore, the phase of one of the complex parameters can be set to zero. This is also valid for the other channels discussed in the remainder of this paper.

$$\mathcal{A}_S(s_{23}) \equiv \langle K[\pi^+\pi^-]_S|\mathcal{H}_{\rm eff}|B\rangle = b_1^S(M_B^2 - s_{23})F_{0n}^{\pi\pi}(s_{23}) + (b_2^S F_0^{BK}(s_{23}) + b_3^S)F_{0s}^{\pi\pi}(s_{23}). \quad (12)$$

For the scalar-isoscalar strange form factor $F_{0s}^{\pi\pi}(s)$ in Eq. (12), one can employ its numerical expression given in Ref. [9] or that in Ref. [44] (see Appendix A 1 a). Parametrizations for the $B \to K$ scalar transition form factors $F_0^{BK}(s)$ in Eq. (12) are reviewed in Appendix A 4.

The amplitude for $B \to K[\pi^+\pi^-]_P$ decays can be written in terms of the complex parameter b_1^P as

$$\mathcal{A}_P(s_{12}, s_{13}, s_{23}) \equiv \langle K[\pi^+\pi^-]_P|\mathcal{H}_{\rm eff}|B\rangle = b_1^P(s_{13} - s_{12})F_1^{\pi\pi}(s_{23}). \quad (13)$$

In Ref. [7], the pion vector form factor, $F_1^{\pi\pi}(s)$, is approximated by a Breit-Wigner form. However, we recommend the use of the unitary vector form factor derived in Ref. [46] described in Appendix A 1 b. We stress that the b_i^S in Eq. (12) and b_1^P in Eq. (13) represent different parameters for each charge state.

As in the $B^\pm \to \pi^+\pi^-\pi^\pm$ case (see Sec. II A), the addition of the $[\pi^+\pi^-]$ D-wave contribution, parametrized in terms of the $f_2(1270)$ resonance, is required; higher invariant-mass phenomenological amplitudes may also be necessary.

2. Parametrization of the $B \to [K\pi^\pm]_{S,P}\pi^\mp$ amplitude

A parametrization of the $B \to [K\pi^\pm]_S\pi^\mp$ channel was introduced in Ref. [8] [see Eq. (68) therein], where in the center of mass of the $K\pi$ pair the S-wave amplitude in case of the $B^- \to [K^-\pi^+]_S\pi^-$ decay can be represented by

$$\mathcal{A}_S(s_{12}) \equiv \langle \pi^-[K^-\pi^+]_S|\mathcal{H}_{\rm eff}|B^-\rangle = (c_1^S + c_2^S s_{12})\frac{F_0^{B\pi}(s_{12})F_0^{K\pi}(s_{12})}{s_{12}}, \quad (14)$$

which follows from Eq. (10) of Ref. [8]. In Eq. (14), s_{12} is the invariant mass squared of the interacting $K^-\pi^+$ pair, whereas for B^+ and $\bar{B}^0$ decays, the kinematic variable is s_{13}. The complex parameters, c_1^S and c_2^S, can be determined through the Dalitz-plot analysis for each given charge state. We note that the isolated $K_0^*(1430)$ resonance contribution can be obtained by replacing, once the parameters c_1^S and c_2^S are determined, $F_0^{K\pi}(s)$ by its pole part $F_0^{\rm pole}(s)$ given in Eqs. (45)–(47) of Ref. [8].

Following the momentum conventions of the S-wave above, the $K\pi P$-wave amplitude of the $B^- \to [K^-\pi^+]_P\pi^-$ decays can be parametrized as

$$\mathcal{A}_P(s_{12}, s_{23}) \equiv \langle \pi^-[K^-\pi^+]_P|\mathcal{H}_{\rm eff}|B^-\rangle = c_1^P\left(s_{13} - s_{23} - (M_B^2 - m_\pi^2)\frac{m_K^2 - m_\pi^2}{s_{12}}\right) \times F_1^{B\pi}(s_{12})F_1^{K\pi}(s_{12}). \quad (15)$$

The parametrizations for the transition form factors, $F_{0(1)}^{B\pi}(s)$, are discussed in Appendix A 4, and those for the $K\pi$ scalar and vector form factors, $F_{0(1)}^{K\pi}(s)$, are discussed in Appendix A 2.

The then available Belle and *BABAR* data were well reproduced in the $\mathcal{A}_{S,P}$ amplitude analysis of Ref. [8] in a $m_{K\pi}$ range from threshold up to 1.8 GeV. Within the factorization approximation, there is no contribution from $[K\pi]$ partial waves, $l \geq 2$; thus, one expects the $[K\pi]$ D-wave contribution to be small. However, in order to analyze the Dalitz-plot data over the full energy ranges, additional phenomenological amplitudes are required.

C. Amplitudes for $B^\pm \to K^\pm K^+ K^-$

The weak effective Hamiltonian that describes this decay channel is given by Eq. (3) with $q \equiv s$. Explicit factorized expressions of $B^- \to K^-K^+K^-$ amplitudes can be found in Appendix A of Ref. [47]. The effective invariant K^+K^- mass distributions of the decays $B^\pm \to K^+K^-K^\pm$ [48,49] up to 1.8 GeV were shown to be well reproduced in the factorization approach of Ref. [10], where the $B^\pm(p_B) \to K^\pm(p_1)K^+(p_2)K^-(p_3)$ amplitudes were derived for interacting K^+K^- pairs in a relative S or P state. The symmetrized term for a B^- decay is obtained by exchange of the $K^+(p_2)K^-(p_3)$ pair with the $K^-(p_1)K^+(p_2)$ one [or exchanging the $K^+(p_2)K^-(p_3)$ and $K^+(p_1)K^-(p_3)$ pairs in case of a B^+ meson] and is added to the amplitude. The totally symmetrized amplitude using the Lorentz invariants $s_{ij} = (p_i + p_j)^2$ for $i \neq j$ is given by

$$\mathcal{A}^-(s_{12}, s_{13}, s_{23}) = \frac{1}{\sqrt{2}}[\mathcal{A}_S^-(s_{12}) + \mathcal{A}_S^-(s_{23}) + \mathcal{A}_P^-(s_{12})(s_{13} - s_{23}) + \mathcal{A}_P^-(s_{23})(s_{13} - s_{12})], \quad (16)$$

with $s_{12} + s_{13} + s_{23} = m_B^2 + 3m_K^2$.

Following Eqs. (2) and (3) of Ref. [10], the amplitudes $\mathcal{A}_{S,P}^-(s_{ij})$, $ij = 12$ or 23, can be also written in terms of six complex parameters $d_{1,2}^S$ and $d_{1,2,3,4}^P$,

$$\mathcal{A}_S^-(s_{ij}) = d_1^S(M_B^2 - s_{ij})F_{0n}^{KK}(s_{ij}) + d_2^S F_0^{BK}(s_{ij})F_{0s}^{KK}(s_{ij}), \quad (17)$$

$$\mathcal{A}_P^-(s_{ij}) = d_1^P F_{1u}^{KK}(s_{ij}) + F_1^{BK}(s_{ij})[d_2^P F_{1u}^{KK}(s_{ij}) + d_3^P F_{1d}^{KK}(s_{ij}) + d_4^P F_{1s}^{KK}(s_{ij})], \quad (18)$$

where the $B \to K$ scalar and vector transition form factors $F_{0,1}^{BK}(s)$ in Eqs. (17) and (18) are discussed in Appendix A 4 and the scalar and vector form factors $F_{0n}^{KK}, F_{0s}^{KK}, F_{1u}^{KK}$, F_{1d}^{KK}, and F_{1s}^{KK} are introduced in Appendix A 3.

Because of its small branching fraction to $K\bar{K}$ (4.6%), the $f_2(1270)$ contribution was not introduced in Ref. [10]. However, in their study of the Dalitz-plot dependence of CP asymmetry, the authors of Ref. [11] have included it, and an amplitude analysis of the full Dalitz plot should also add it together with a phenomenological term representing the high invariant-mass contributions.

III. PARAMETRIZATIONS OF THREE-BODY HADRONIC D-DECAY AMPLITUDES

A. Amplitudes for $D^+ \to \pi^+\pi^-\pi^+$

The decay $D^+ \to \pi^-\pi^+\pi^+$ is a Cabibbo suppressed mode governed by the quark-level transition $c \to du\bar{d}$. The leading contribution to the amplitude arises from the current-current operators and is proportional to $V_{cd}V_{ud}^*$, which is $\mathcal{O}(\lambda)$ in Wolfenstein parametrization, with $\lambda = 0.2257$. At next-to-leading order (NLO) in QCD, penguin operators contribute to the decay amplitude. However, CKM unitarity implies that those come with a coefficient $V_{cb}V_{ub}^*$, which is $\mathcal{O}(\lambda^5)$. It is therefore safe to neglect the penguin contributions. The dominant contributions to the effective Hamiltonian therefore are

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V_{cd}V_{ud}^*[C_1(\mu)O_1 + C_2(\mu)O_2] + \text{H.c.}, \quad (19)$$

where the two four-quark operators read

$$O_1 = [\bar{d}_i\gamma^\nu(1-\gamma_5)c_i][\bar{u}_j\gamma_\nu(1-\gamma_5)d_j], \quad (20)$$

$$O_2 = [\bar{u}_i\gamma^\nu(1-\gamma_5)c_j][\bar{d}_j\gamma_\nu(1-\gamma_5)d_i]. \quad (21)$$

In the description of D decays, however, because of the limitations discussed in the Introduction, the Wilson coefficients depart from their calculated values due to nonfactorizable corrections. In the spirit of Ref. [22], it is safe to assume they can be complex numbers and that the corrections will depend on whether the $\pi\pi$ pair is in the S- or P-wave state (the same applies to the $K\pi$ pairs in the next section). Our parametrizations below encompass this assumption.

The parametrization we propose here is chiefly based on the work of Ref. [12]. The crucial dynamical ingredient to describe the two-body hadronic final-state interactions are the scalar and vector $\pi\pi$ form factors. The description of the decay proceeds in full analogy to that outlined in the Introduction for $D^+ \to K^-\pi^+\pi^+$. Within this framework, the main difference between these two decays is that the relevant two-pion matrix element, namely, $\langle\pi^-\pi^+|\bar{d}\gamma^\nu(1-\gamma_5)d|0\rangle$ is proportional to the $\pi\pi$ vector form factor only—no scalar contribution appears because the pseudoscalars involved have the same mass. This implies that the $\pi^+\pi^-$ S wave in the decay $D^+ \to \pi^-\pi^+\pi^+$ receives no contribution from the diagram proportional to $a_2(\mu)$. The decay amplitude for S- and P-wave $\pi^+\pi^-$ pairs can be written as

$$\langle\pi^+[\pi^-\pi^+]_{S,P}|\mathcal{H}_{\rm eff}|D^+\rangle = \mathcal{A}_S^+ + \mathcal{A}_P^+, \quad (22)$$

where $\mathcal{A}_S^+$ and $\mathcal{A}_P^+$ are, respectively, the S- and P-wave $\pi^+\pi^-$ amplitudes. The S wave is dominated by the intermediate scalar-isoscalar resonances $f_0(500)$ and $f_0(980)$, while the P wave is largely dominated by the $\rho(770)^0$.

We label the four-momenta as $D^+(p_D) \to \pi^+(p_1)\pi^-(p_2)\pi^+(p_3)$ and define the invariant masses squared $s_{12} = (p_1+p_2)^2$, $s_{23} = (p_2+p_3)^2$, and $s_{13} = (p_1+p_3)^2$, with $s_{12}+s_{13}+s_{23} = m_D^2 + 3m_\pi^2$. Resonances occur in the $\pi^+\pi^-$ states described in terms of s_{12} and s_{23} invariants. With these definitions, the amplitudes $\mathcal{A}_{S,P}^+$ of Eq. (22) can be parametrized with three complex parameters, e_1^S and $e_{1,2}^P$, as

$$\mathcal{A}_S^+(s_{12},s_{23}) = e_1^S(m_D^2 - s_{12})F_{0n}^{\pi\pi}(s_{12}) + (s_{12} \leftrightarrow s_{23}), \quad (23)$$

$$\mathcal{A}_P^+(s_{12},s_{13},s_{23}) = [e_1^P + e_2^P F_1^{D\pi}(s_{12})](s_{23}-s_{13})F_1^{\pi\pi}(s_{12}) + (s_{12} \leftrightarrow s_{23}). \quad (24)$$

We are implicitly assuming that nonfactorizable corrections depend on the spin of the $\pi^+\pi^-$ pair and can be absorbed in the parameters e_i^L. In this parametrization, the two-body $\pi^+\pi^-$ interactions are fully taken into account by the scalar and vector $\pi\pi$ form factors, $F_{0n}^{\pi\pi}$ and $F_1^{\pi\pi}(s)$, respectively, which are detailed in Appendix A 1. The vector $D \to \pi$ transition form factor, $F_1^{D\pi}(s)$, in Eq. (24) is discussed in Appendix A 4. We observe that the D-wave resonance contribution, arising from the $f_2(1270)$, is sizeable (with fit fractions of about 20% [50,51]) and could be included in data analyses through usual isobar model expressions. Finally, in one of the models employed by the CLEO Collaboration [52], some evidence for a contribution from isospin-2 $\pi^+\pi^+$ interactions is presented, which may have to be included in a realistic analysis.

B. Amplitudes for $D^+ \to K^-\pi^+\pi^+$

The $D^+ \to K^-\pi^+\pi^+$ decay is Cabibbo allowed, governed by the quark-level transition $c \to su\bar{d}$. Since four different quark flavors intervene, the effective Hamiltonian for this processes does not include penguin-type operators. At NLO in QCD, there are only two operators to be considered,

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V_{cs}V_{ud}^*[C_1(\mu)O_1 + C_2(\mu)O_2] + \text{H.c.}, \quad (25)$$

where the relevant four-quark operators are

$$O_1 = [\bar{c}_i \gamma^\nu (1-\gamma_5) s_i][\bar{d}_j \gamma_\nu (1-\gamma_5) u_j], \tag{26}$$

$$O_2 = [\bar{c}_i \gamma^\nu (1-\gamma_5) s_j][\bar{d}_j \gamma_\nu (1-\gamma_5) u_i]. \tag{27}$$

In Ref. [13], the $K\pi$ S- and P-wave amplitudes in this decay were written in terms of the scalar and vector $K\pi$ form factors, $F_{0,1}^{K\pi}(s)$. We use these results as the basis for our suggested parametrization. We label the momenta as $D^+(p_D) \to \pi^+(p_1)\pi^+(p_2)K^-(p_3)$ and define the following invariant masses squared of the final state: $s_{13} = (p_1+p_3)^2$, $s_{23} = (p_2+p_3)^2$, and $s_{12} = (p_1+p_2)^2$, with $s_{12}+s_{13}+s_{23} = m_D^2 + m_K^2 + 2m_\pi^2$. Thus, the S- and P-wave amplitudes for $D^+ \to [K^-\pi^+]_{S,P}\pi^+$ are

$$\langle [K^-\pi^+]_{S,P}\pi^+ | \mathcal{H}_{\rm eff} | D^+ \rangle = \mathcal{A}_S^+ + \mathcal{A}_P^+. \tag{28}$$

Contributions from D-wave resonances are known to be rather small in this decay [51]. The S- and P-wave amplitudes can be parametrized with complex parameters, $f_{1,2}^S$ and $f_{1,2}^P$, as follows,

$$\mathcal{A}_S^+(s_{13}, s_{23}) = \left[f_1^S (m_D^2 - s_{13}) + f_2^S \frac{F_0^{D\pi}(s_{13})}{s_{13}} \right] \times F_0^{K\pi}(s_{13}) + (s_{13} \leftrightarrow s_{23}), \tag{29}$$

$$\mathcal{A}_P^+(s_{12}, s_{13}, s_{23}) = \left[f_1^P \Omega(s_{13}, s_{23}) + f_2^P \left(\frac{s_{23}-s_{12}}{\Delta^2} - \frac{1}{s_{13}} \right) \times F_1^{D\pi}(s_{13}) \right] F_1^{K\pi}(s_{13}) + (s_{13} \leftrightarrow s_{23}), \tag{30}$$

with $\Omega(s_{13}, s_{23}) = s_{23} - s_{12} - \Delta^2/s_{13}$ and $\Delta^2 = ({m_K}^2 - m_\pi^2)(m_{D^+}^2 - m_\pi^2)$. We assume that nonfactorizable corrections are absorbed in the complex parameters, $f_{1,2}^S$ and $f_{1,2}^P$, to be fitted to the data.

The parametrization we introduce in Eqs. (29) and (30) makes the emergence of the scalar and vector $K\pi$ form factors, $F_{0,1}^{K\pi}(s)$, explicit. They are discussed in more detail in Appendix A 2. The scalar and vector $D\pi$ transition form factors also appear in Eqs. (29) and (30). Their variation with energy for the physical values of s is not significant, but they affect the shape of the amplitudes close to the edges of the Dalitz plot. Possible parametrizations are discussed in Appendix A 4.

A version of the above description put forward here has been employed successfully in Ref. [13]. Additional contributions, e.g., with higher angular momentum or the isospin-2 $\pi^+\pi^+$ interactions,[3] are small in this process. In a realistic high-statistics Dalitz-plot analysis, however, they may be required and would have to be included in the signal function through usual isobar model expressions, for instance.

[3]Most experimental analyses agree this contribution is negligible [53,54] with the exception of the CLEO Collaboration analysis where a fit fraction of ~20% is attributed to the $\pi^+\pi^+$ interactions [55].

A final ingredient that is not present in our parametrizations is the genuine three-body hadronic final-state interactions, which are most often neglected in experimental analyses. Their treatment is somewhat involved, and only recently did this problem start to be dealt with. An approach based on Feynman diagrams from effective Lagrangians was introduced in Refs. [56,57] precisely in the case of $D^+ \to K^-\pi^+\pi^+$. Alternatively, a description based on a dispersive treatment introduced in Ref. [58] for $(\omega/\phi) \to \pi\pi\pi$ decays has been applied to $D \to K\pi\pi$ decays in Refs. [59,60]. Finally, a coupled-channel description including three-body scattering was performed in Ref. [61]. These treatments do not allow for a simple parametrization of the type we advocate here with the goal of replacing isobar model expressions. These three-body effects, if important, should show as deviations from our description and represent a refinement to the amplitudes discussed here that should be addressed in the future.

C. Amplitudes for $D^0 \to K_S^0\pi^+\pi^-$

The decay $D^0 \to K_S^0\pi^-\pi^+$ was treated within the framework of quasi-two-body factorization in Ref. [14]. A good reproduction of the Belle Dalitz-plot density distributions [62] was obtained, and so were the distributions produced by the *BABAR* model.[4] The parametrizations that follow in the next subsections are based on the quasi-two-body amplitudes derived in this study. The Hamiltonian that describes this decay channel is similar to that of Eq. (25), but besides a Cabibbo favored term proportional to $V_{cs}^* V_{ud}$, there is also a doubly Cabibbo suppressed contribution proportional to $V_{cd}^* V_{us}$. The momenta are labeled as $D^0(p_D) \to K_S^0(p_1)\pi^-(p_2)\pi^+(p_3)$ where the kinematic configuration is defined by $s_{12} = (p_1+p_2)^2$, $s_{13} = (p_1+p_3)^2$, and $s_{23} = (p_2+p_3)^2$, with $s_{12}+s_{13}+s_{23} = m_{D^0}^2 + m_{K^0}^2 + 2m_\pi^2$. We start with the parametrization of the amplitude for the interacting $K_S^0\pi^-$ in an S- or P-wave state.

1. Parametrization of the $D^0 \to [K_S^0\pi^-]_{S,P}\pi^+$ amplitudes

The following parametrizations are derived from Eqs. (66) and (68) of Ref. [14]. In terms of three complex parameters $g_{1,2}^S$ and g_1^P, the parametrized amplitudes read

$$\mathcal{A}_{S,-}^0(s_{12}) = (g_1^S + g_2^S s_{12}) F_0^{K\pi}(s_{12}), \tag{31}$$

$$\mathcal{A}_{P,-}^0(s_{12}, s_{13}, s_{23}) = g_1^P \left(s_{23} - s_{13} + \frac{\Delta_0^2}{s_{12}} \right) F_1^{K\pi}(s_{12}), \tag{32}$$

[4]The model is built from a fit to the *BABAR* Dalitz-plot data; see Ref. [14].

with $\Delta_0{}^2 = (m_{K^0}{}^2 - m_\pi^2)(m_{D^0}^2 - m_\pi^2)$. The πK S-wave $\mathcal{A}^0_{S,-}$ amplitude includes the contribution of the scalar $K_0^*(800)^-$ and $K_0^*(1430)^-$ resonances, and the P-wave $\mathcal{A}^0_{P,-}$ amplitude includes that of the vector $K^*(892)^-$. Despite its small fit fraction, the πK D-wave $D^0 \to [K_S^0\pi^-]_D\pi^+$ amplitude plays an important role through interference. The contribution of the tensor $K_2^{*-}(1430)$ resonance can be parametrized by a relativistic Breit-Wigner formula, with a magnitude and phase that should be obtained through a fit to the data, as done successfully in Ref. [14]. This component should be added to the S- and P-wave amplitudes parametrized above.

2. Parametrization of the $D^0 \to [K_S^0\pi^+]_{S,P}\pi^-$ amplitudes

Likewise, the decay amplitudes for $D^0 \to [K_S^0\pi^+]_{S,P}\pi^-$ are given in Ref. [14] [see Eqs. (84) and (85)] and can be parametrized as

$$\mathcal{A}^0_{S,+}(s_{13}) = \left[g_3^S(m_\pi^2 - s_{13}) + g_4^S\frac{\Delta_0^2}{s_{13}}F_0^{D\pi}(s_{13})\right]F_0^{K\pi}(s_{13}), \tag{33}$$

$$\mathcal{A}^0_{P,+}(s_{12},s_{13},s_{23}) = [g_2^P + g_3^P F_1^{D\pi}(s_{13})]\left(s_{23} - s_{12} + \frac{\Delta_0^2}{s_{13}}\right) \times F_1^{K\pi}(s_{13}). \tag{34}$$

The πK S-wave $\mathcal{A}^0_{S,+}$ amplitude includes the contribution of the scalar $K_0^*(800)^+$ and $K_0^*(1430)^+$ resonances, and the P-wave $\mathcal{A}^0_{P,+}$ amplitude includes that of the vector $K^*(892)^+$. The contribution from the D wave, that stems mainly from the $K_2^*(1430)^+$, could be parametrized by the usual Breit-Wigner expressions.

3. Parametrization of the $D^0 \to K_S^0[\pi^+\pi^-]_{S,P}$ amplitudes

The weak $D^0 \to K_S^0[\pi^+\pi^-]_{S,P}$ decay amplitudes, following Ref. [14], can be parametrized, using the same momentum definition as before, as

$$\mathcal{A}^0_{S,0}(s_{23}) = (g_5^S + g_6^S s_{23})F_{0n}^{\pi\pi}(s_{23}), \tag{35}$$

$$\mathcal{A}^0_{P,0}(s_{12},s_{13},s_{23}) = (s_{12} - s_{13})[g_4^P F_1^{\pi\pi}(s_{23}) + g_5^P F_1^{\omega}(s_{23})]. \tag{36}$$

The $\pi\pi$ S-wave $\mathcal{A}^0_{S,0}$ amplitude includes the contributions of the scalar $f_0(500)$ (or σ), $f_0(980)$, and $f_0(1400)$ resonances. The $[\pi^+\pi^-]_P$ pair can originate from the ω resonance through isospin violation. This introduces a term proportional to the vector form factor $F_1^\omega(s_{23})$ in Eq. (36).[5] The effects of the vector $\rho(770)^0$ and $\omega(782)$ resonances are included in the P-wave amplitudes, $\mathcal{A}^0_{P,0}$, which also contain the contribution of the $\rho(1450)^0$ and $\rho(1700)^0$; see the details in Appendix A 1. The D wave is dominated by the $f_2(1270)$ tensor meson and must be included in a realistic amplitude. In Ref. [14], it was parametrized by the usual relativistic Breit-Wigner line shape.

[5]In Eq. (71) of Ref. [14], this term was explicitly written as $F_1^\omega(s_{23}) = m_\omega^2/(m_\omega^2 - s_{23} - im_\omega\Gamma_\omega)$.

The full decay amplitude is thus given by

$$\mathcal{A}^0 = \mathcal{A}^0_{S,-} + \mathcal{A}^0_{P,-} + \mathcal{A}^0_{S,+} + \mathcal{A}_{P,+} + \mathcal{A}^0_{S,0} + \mathcal{A}^0_{P,0} + \cdots, \tag{37}$$

where the ellipsis denotes D- and higher-wave contributions and possible high invariant-mass contribution.

D. Amplitudes for $D^0 \to K_S^0K^+K^-$

The $D^0 \to K_S^0K^+K^-$ decay channel was measured by the *BABAR* Collaboration with high statistics [63]. Using the quasi-two-body factorization approach [64], we parametrize [24] this decay channel with the definitions of the invariants s_{12}, s_{13}, and s_{23} similar to those introduced for $D^0 \to K_S^0\pi^+\pi^-$ in the previous section, replacing charged pions by charged kaons, the effective Hamiltonian being identical (here, the charged kaon mass is denoted m_K). The momenta are thus labeled as $D^0(p_D) \to K_S^0(p_1)K^-(p_2)K^+(p_3)$ with $s_{12} = (p_1+p_2)^2$, $s_{13} = (p_1+p_3)^2$, $s_{23} = (p_2+p_3)^2$, and $s_{12}+s_{13}+s_{23} = m_{D^0}^2 + m_{K^0}^2 + 2m_K^2$. The involved three interacting kaon pairs, $[K^+K^-]_L$, $[K_S^0K^-]_L$, and $[K_S^0K^+]_L$, can be in a scalar or vector state with $L = S$ or P, respectively. The isospin of the $[K^+K^-]_L$ pairs can be either 0 or 1, but that of the $[K_S^0K^\mp]_L$ pairs is 1.

1. Parametrization of the $D^0 \to K_S^0[K^+K^-]_S$ amplitudes

The decay amplitude in which the isoscalar $[K^+K^-]_S$ pair is associated with the $f_0(980)$ and $f_0(1370)$ resonances, and the isovector one is related to the $a_0(980)^0$ and $a_0(1450)^0$ resonances, can be parametrized as

$$\begin{aligned}\mathcal{A}^0_{S,0}(s_{23}) = {}& h_1^S(m_{D^0}^2 - s_{23})F_{0n}^{K\bar{K}}(s_{23}) \\ &+ h_2^S(m_{K^0}^2 - s_{23})F_{0s}^{K\bar{K}}(s_{23}) \\ &+ h_3^S(m_{D^0}^2 - s_{23})G_0^{K\bar{K}}(s_{23}).\end{aligned} \tag{38}$$

The amplitude with the isovector $[K^0K^-]_S$ pairs in an S-wave state, which include the $a_0(980)^-$ and $a_0(1450)^-$ resonances, can be parametrized as

$$\mathcal{A}^0_{S,-}(s_{12}) = (h_4^S + h_5^S s_{12})G_0^{K\bar{K}}(s_{12}), \tag{39}$$

and the corresponding amplitude associated with the $a_0(980)^+$ and $a_0(1450)^+$ resonances can be parametrized as

$$\mathcal{A}^0_{S,+}(s_{13}) = \left[h^S_6 \frac{F_0^{D^0K^-}(s_{13})}{s_{13}} + h^S_7(m^2_K - s_{13})\right] G_0^{K\bar{K}}(s_{13}). \tag{40}$$

The scalar-isocalar form factors, $F^{K\bar{K}}_{0n(s)}(s)$, in Eq. (38), and the scalar-isovector ones, $G_0^{K\bar{K}}(s)$, in Eqs. (39) and (40) are detailed in Appendix A 3, while the scalar D to K transition form factor $F_0^{DK}(s)$ is defined in Appendix A 4.

2. Parametrization of the $D^0 \to K^0_S[K^+K^-]_P$ amplitudes

We parametrize this decay amplitude, where the isoscalar and isovector $[K^+K^-]_P$ pairs contain contributions from the $\omega(782)$, $\omega(1420)$, $\phi(1020)$, $\rho(770)^0$, $\rho(1450)^0$, and $\rho(1700)^0$ resonances, by the expression [24]

$$\mathcal{A}^0_{P,0}(s_{12}, s_{13}, s_{23}) = (s_{12} - s_{13})(h^P_1 F^{K^+K^-}_{1u}(s_{23}) + h^P_2 F^{K^+K^-}_{1s}(s_{23})). \tag{41}$$

Likewise, one can express the amplitude in which the isovector $[K^0_S K^-]_P$ pair is associated with the three ρ^- resonances by

$$\mathcal{A}^0_{P,-}(s_{12},s_{13},s_{23}) = h^P_3\left[s_{23} - s_{13} + (m^2_{D^0} - m^2_K)\frac{m^2_{K^0} - m^2_K}{s_{12}}\right] \times F_1^{K^-K^0}(s_{12}), \tag{42}$$

while the parametrization of the amplitude associated with the ρ^+ resonances reads

$$\mathcal{A}^0_{P,+}(s_{12},s_{13},s_{23}) = [h^P_4 + h^P_5 F_1^{D^0K^-}(s_{13})] \times \left[s_{23} - s_{12} + (m^2_{D^0} - m^2_K)\frac{m^2_{K^0} - m^2_K}{s_{13}}\right] \times F_1^{K^+\bar{K}^0}(s_{13}). \tag{43}$$

The vector-isoscalar form factor, $F^{K\bar{K}}_{1u(s)}(s)$, and the vector-isovector form factors, $F_1^{K^-K^0}(s)$ and $F_1^{K^+\bar{K}^0}(s)$, appearing in Eqs. (41) to (43) are defined in Appendix A 3. The parametrization of the vector D to K transition form factor $F_1^{DK}(s)$ is discussed in Appendix A 4.

The full decay amplitude is the coherent sum of all the subamplitudes discussed above,

$$\mathcal{A}^0 = \mathcal{A}^0_{S,-} + \mathcal{A}^0_{P,-} + \mathcal{A}^0_{S,+} + \mathcal{A}^0_{P,+} + \mathcal{A}^0_{S,0} + \mathcal{A}^0_{P,0} + \cdots, \tag{44}$$

where the ellipsis denotes the omission of higher waves that could be included using Breit-Wigner line shapes.

IV. CONCLUDING REMARKS

We have introduced alternatives to the isobar-model Dalitz-plot parametrizations of weak D and B decays into exclusive final states composed of three light mesons, namely, the various charge states $\pi\pi\pi$, $K\pi\pi$, and $KK\bar{K}$. Such isobar parametrizations have been frequently employed in fits, although they do not respect unitarity, which leads, among other effects, to a sum of branching fractions that can exceed the total decay width by large amounts. As a consequence, any strong CP phases that may be extracted from these fits must be taken with caution.

Our alternative parametrizations, while not fully three-body unitary, are based on a sound theoretical application of QCD factorization to a hadronic quasi-two-body decay. We thus assume that the final three-meson state is preceded by intermediate resonant states, which is justified by ample phenomenological and experimental evidence. Analyticity, unitarity, chiral symmetry, as well as the correct asymptotic behavior of the two-meson scattering amplitude in S and P waves are implemented via analytical and unitary S- and P-wave $\pi\pi$, πK, and $K\bar{K}$ form factors which enter the hadronic final states of our amplitude parametrizations. These amplitudes can be readily used adjusting the

TABLE I. For each B-decay channel in the first column, the second column refers to the equation of the proposed amplitude parametrization, and the third column lists the dominant contributing resonances.

Quasi-two-body channel	See equation:	Dominant resonances
$B^- \to \pi^-[\pi^+\pi^-]_S$	(10)	$f_0(500)$, $f_0(980)$, $f_0(1400)$
$B^- \to \pi^-[\pi^+\pi^-]_P$	(11)	$\rho(770)^0$, $\rho(1450)^0$, $\rho(1700)^0$
$B \to K[\pi^+\pi^-]_S$	(12)	$f_0(500)$, $f_0(980)$, $f_0(1400)$
$B \to K[\pi^+\pi^-]_P$	(13)	$\rho(770)^0$, $\rho(1450)^0$, $\rho(1700)^0$
$B^{-(0)} \to [K^{-(0)}\pi^+]_S\pi^-$	(14)	$K^*_0(800)^{0(+)}$, $K^*_0(1430)^{0(+)}$
$B^{-(0)} \to [K^{-(0)}\pi^+]_P\pi^-$	(15)	$K^*(892)^{0(+)}$, $K^*(1410)^{0(+)}$
$B^- \to K^-[K^+K^-]_S$	(17)	$f_0(980)$, $f_0(1400)$
$B^- \to K^-[K^+K^-]_P$	(18)	$\rho(770)^0$, $\rho(1450)^0$, $\rho(1700)^0$, $\omega(782)$, $\omega(1420)$, $\omega(1650)$, $\phi(1020)$, $\phi(1680)$

TABLE II. As in Table I but for hadronic quasi-two-body D decays.

Quasi-two-body channel	See equation:	Dominant resonances
$D^+ \to [\pi^+\pi^-]_S\pi^+$	(23)	$f_0(500)$, $f_0(980)$, $f_0(1400)$
$D^+ \to [\pi^+\pi^-]_P\pi^+$	(24)	$\rho(770)^0$, $\rho(1450)^0$
$D^+ \to [K^-\pi^+]_S\pi^+$	(29)	$K_0^*(800)^0$, $K_0^*(1430)^0$
$D^+ \to [K^-\pi^+]_P\pi^+$	(30)	$K^*(892)^0$, $K^*(1410)^0$
$D^0 \to [K_S^0\pi^-]_S\pi^+$	(31)	$K_0^*(800)^-$, $K_0^*(1430)^-$
$D^0 \to [K_S^0\pi^-]_P\pi^+$	(32)	$K^*(892)^-$, $K^*(1410)^-$
$D^0 \to [K_S^0\pi^+]_S\pi^-$	(33)	$K_0^*(800)^+$, $K_0^*(1430)^+$
$D^0 \to [K_S^0\pi^+]_P\pi^-$	(34)	$K^*(892)^+$, $K^*(1410)^+$
$D^0 \to K_S^0[\pi^+\pi^-]_S$	(35)	$f_0(500)$, $f_0(980)$, $f_0(1400)$
$D^0 \to K_S^0[\pi^+\pi^-]_P$	(36)	$\rho(770)^0$, $\omega(782)$
$D^0 \to K_S^0[K^+K^-]_S$	(38)	$f_0(980)$, $f_0(1400)$, $a_0(980)^0$, $a_0(1450)^0$
$D^0 \to K^+[K^0K^-]_S$	(39)	$a_0(980)^-$, $a_0(1450)^-$
$D^0 \to K^-[K^0K^+]_S$	(40)	$a_0(980)^+$, $a_0(1450)^+$
$D^0 \to K_S^0[K^+K^-]_P$	(41)	$\omega(782)$, $\omega(1420)$, $\phi(1020)$, $\rho(770)^0$, $\rho(1450)^0$
$D^0 \to K^+[K^0K^-]_P$	(42)	$\rho(770)^-$, $\rho(1450)^-$
$D^0 \to K^-[K^0K^+]_P$	(43)	$\rho(770)^+$, $\rho(1450)^+$

parameters in a least-square fit to the Dalitz plot—for a given decay channel—and employing tabulated form factors as functions of momentum squared or energy. The different quasi-two-body B- and D-decay channels for which we provide explicit amplitude expressions are summarized in Tables I and II, respectively. For each channel, the relevant equation for the parametrization is cited, and the dominant contributing resonances are listed. Let us add a practical remark: in any application of the parametrized amplitudes to experimental analyses, one can set to zero one phase of the S or P wave amplitude since the Dalitz-plot density is not sensitive to its value.

With this "tool kit," we strongly hope to contribute to more sophisticated experimental extractions of three-body decay observables, in particular CP-violating phases.

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APPENDIX A: FORM FACTORS

In quantum field theory, it can be shown, using dispersion relations [65], that strong-interaction meson-meson form factors can be in principle calculated exactly by means of the coupled-channel Muskhelishvili-Omnès (MO) equations [66], provided one knows the meson-meson scattering matrices at all energies. In practice, our knowledge about scattering phases is incomplete, and one has to resort to simplifications. Eventually, different approaches to the calculation of these form factors lead to slightly different results. In the following, we briefly describe several state-of-the-art descriptions of the various form factors employed in the decay-amplitude parametrizations presented in this work. These form factors can be obtained from the authors of the original works in the form of numerical tables and be readily employed in a concrete Dalitz-plot analysis.

1. $\pi\pi$ form factors

The parametrizations of the amplitudes $B^- \to \pi^-[\pi^+\pi^-]_{S,P}$ in Eqs. (10) and (11), $B \to K[\pi^\pm\pi^\mp]_{S,P}$ in Eqs. (12) and (13), $D^+ \to \pi^+[\pi^-\pi^+]_{S,P}$ in Eqs. (23) and (24), and

$D^0 \to K_S^0[\pi^+\pi^-]_{S,P}$ in Eqs. (35) and (36) require the knowledge of the pion nonstrange scalar form factor, $F_{0n}^{\pi\pi}(s_{ij})$, and vector form factor, $F_1^{\pi^+\pi^-}(s_{ij})$. The strange pion scalar form factor $F_{0s}^{\pi\pi}(s_{ij})$ enters the parametrization of the $B \to K[\pi^\pm\pi^\mp]_S$ amplitude in Eq. (12).

a. Scalar form factors

The scalar form factors $F_{0n(s)}^{\pi\pi}(s_{ij})$ can be found, for example, in Refs. [6,9,14,44,67].[6] In Ref. [9], the form factors have been derived using a unitary relativistic coupled-channel model including $\pi\pi$, $K\bar{K}$, and effective $(2\pi)(2\pi)$ interactions together with chiral symmetry constraints (an approach put forward in Ref. [68]). The latest version of the corresponding nonstrange form factors was obtained in Ref. [14], with constraints from the high-statistics Dalitz-plot data of the $D^0 \to K_S^0\pi^+\pi^-$ from Refs. [63,69]. In this approach the $\pi\pi T$-matrix is that of the solution A of the three coupled-channel model of Ref. [70], where the effective mass is $m_{(2\pi)} = 700$ MeV.

For an alternative, one can employ the scalar pion form factors obtained from the numerical solution of a coupled-channel MO problem, as derived in Ref. [44]. This approach has been recently revisited in the context of $B^0 \to J/\psi\pi\pi$ decays in Ref. [67]. There, the system of MO equations is solved with input from chiral symmetry constrained by recent lattice data. These form factors suffer from an uncertainty that stems from the kaon form-factor normalization at zero (which enters through the coupled-channel equations). This theoretical uncertainty is more pronounced in the scalar pion form factor at energies above 800 MeV.

The modulus of the pion nonstrange scalar form factor is characterized by a dip arising from the $f_0(980)$ contribution and by two bumps of which the origins are the $f_0(500)$ and $f_0(1400)$ resonances. The strange scalar form factor is dominated by a peak around the $f_0(980)$ contribution. The form factors are depicted, for instance, in Fig. 1 of Ref. [9] for the nonstrange scalar form factor and in Fig. 6 of Ref. [67] for both the strange and nonstrange scalar cases.

b. Vector form factor

The pion vector form factor can be extracted accurately from experimental data for $\tau^- \to \pi^-\pi^0\nu_\tau$ and $e^+e^- \to \pi^+\pi^-$. However, while in the τ^- decay the current has only an isospin-1 component, the e^+e^- annihilation also implies an isoscalar component. Recent descriptions can be found, for example, in Refs. [45,46,71].

A good fit to $D^0 \to K_S^0\pi^-\pi^+$ decay data is obtained in Ref. [14] using the vector form-factor parametrization employed by the Belle Collaboration in their data analysis of $\tau^- \to \pi^-\pi^0\nu_\tau$ decays [45]. It is based on a Gounaris-Sakurai form, and the parameters used are those of Table VII of Ref. [45]. The Dalitz plot is also well described by the unitary parametrization of Ref. [46].

Another recent unitary description that can be useful in data analysis is the dispersive representation of Ref. [71]. This description of the form factor uses Belle data on the $\tau \to \pi\pi\nu$ decays to constrain a three-time subtracted dispersive representation.

Finally, care must be exercised to correctly take into account both the isosvector and isoscalar components. For instance, in $D^+ \to \pi^-\pi^+\pi^+$ decays, the current that couples to the $\pi^+\pi^-$ pair in a P wave is $\bar{d}\gamma_\mu d$, which contains both isospin 0 and 1. One therefore expects the ω contribution to be sizeable in high-statistics data sets. The inclusion of the ω contribution can be done as discussed in detail in Ref. [67] [see in particular their Eq. (3.7)]. An alternative is to take into account the contribution of the ω using the respective isobar model amplitude, described in terms of a Breit-Wigner parametrization.

2. $K\pi$ form factors

The $K\pi$ scalar form factor, $F_0^{K\pi}$, and the $K\pi$ vector form factor, $F_1^{K\pi}$, enter our parametrizations of the $B \to [K\pi^\pm]_{S,P}\pi^\mp$, $D^+ \to [K^-\pi^+]_{S,P}\pi^+$, and $D^0 \to [K_S^0\pi^\mp]_{S,P}\pi^\mp$ amplitudes. Below, we discuss the determination of these form factors.

a. $K\pi$ scalar form factor

Sophisticated computations of the scalar $F_0^{K\pi}$ form factor by means of a coupled-channel dispersive representation can be found in Refs. [72,73]. The form factor derived in Ref. [72] from two coupled-channel MO equations depends on the ratio $r_{K\pi} = f_K/f_\pi$, f_K and f_π being the kaon and pion decay constants, and was used with success in Refs. [8,14]. It contains the contributions of the $K_0^*(800)$ [or $\kappa(800)$] [74] and $K_0^*(1430)$ resonances clearly visible as bumps. Its modulus is plotted in Fig. 2 of Ref. [8].

The same form factor was derived in a coupled-channel ($K\pi$, $K\eta$, and $K\eta'$) dispersive framework imposing constraints from Chiral Perturbation Theory at low energies in Ref. [73]. The form factors are obtained from the numerical solution of the coupled-channel equations with input from the T-matrix elements previously calculated in Ref. [75]. This is the form factor that was employed in the description of $D^+ \to K^-\pi^+\pi^+$ decays in Ref. [13].

b. $K\pi$ vector form factor

The $K\pi$ vector form factor can be extracted with accuracy from the spectrum of $\tau \to K\pi\nu$ decays. These decays are largely dominated by the vector contribution,

[6] In Refs. [6,7,9,12,14], the form factor is defined as $\Gamma_1^{n*}(s_{ij}) = \sqrt{3/2}F_{0n}^{\pi\pi}(s_{ij})$ with $F_{0n}^{\pi\pi}(0) = 1$. The relation for the strange case is ambiguous as $F_{0s}^{\pi\pi}(0) = 0$ in the lowest order of chiral symmetry (see Refs. [44,68] for more details).

and the present statistics allows for a description with good precision. The unitary form factor derived in Ref. [72] from three coupled-channel equations has been used in Ref. [8]. In Refs. [76,77], the form factor is described by a dispersive relation with three subtractions and constrained by the Belle data for $\tau^- \to K_S\pi^-\nu_\tau$ [78] and information from K_{l3} decays. The $K^*(892)$ and $K^*(1410)$ resonances contribute to this vector form factor. The contribution of the $K^*(1680)$ is difficult to assess due to the larger error bars around 1600 GeV in the spectrum of $\tau^- \to K_S\pi^-\nu_\tau$. This form factor has been employed with success in the description of $D^+ \to K^-\pi^+\pi^+$ decays of Ref. [13]. It also leads to a good fit of the present high-statistics $D^0 \to K^0_S\pi^+\pi^-$ data [14].

3. *KK* form factors

a. Scalar-isoscalar case

The kaon nonstrange and strange scalar and isoscalar form factors, $F^{K\bar{K}}_{0n(s)}(s_{ij})$,[7] enter the $B^- \to K^-[K^+K^-]_S$ amplitude in Eq. (17) and the $D^0 \to K^0_S[K^+K^-]_S$ amplitude in Eq. (38). They have been calculated in Ref. [10] with the three coupled channels $\pi\pi$, $\bar{K}K$, and 4π [effective $(2\pi)(2\pi)$ or $\sigma\sigma$ or $\eta\eta$, etc.] in the approach developed in Ref. [9] to derive the pion scalar form factors (see Appendix A 1 a). Through their coupling to $K\bar{K}$, the resonances $f_0(980)$ and $f_0(1400)$ contribute to $F^{K\bar{K}}_{0n(s)}(s_{ij})$, as can be seen from the spikes present in Fig. 1 of Ref. [10]. An alternative derivation of these form factors using MO equations has been presented in Ref. [44] and represents a sound alternative.

b. Scalar-isovector case

For an isospin 1 $[K^+K^-]$ pair and assuming isospin symmetry, the scalar-isovector form factor $G^{K\bar{K}}_0(s) = G^{[K^+K^-]}_0(s) = G^{K^0K^-}_0(s) = G^{\bar{K}^0K^+}_0(s)$ is defined as [79]

$$\mathcal{B}^0 G^{K\bar{K}}_0(s) = \langle \bar{K}^0(p_{K^-})K^+(p_{K^+})|\bar{u}d|0\rangle, \qquad \text{(A1)}$$

with $\mathcal{B}^0 = m^2_\pi/(m_u+m_d)$. This form factor, entering the $D^0 \to K^0_S[K^+K^-]_S$ amplitude in Eq. (38), was calculated in Ref. [79] from coupled MO equations for $\pi\eta$ and $K\bar{K}$ channels. The above form factor includes the contributions of the $a_0(980)$ and $a_0(1450)$ seen as bumps in their moduli (see, for instance, the right panel of Fig. 7 of Ref. [79]).

c. Vector case

For the $B^- \to K^-[K^+K^-]_P$ amplitude in Eq. (18) and for the $D^0 \to K^0_S[K^+K^-]_P$ amplitude in Eq. (41), the vector form factors $F^{K^+K^-}_{1q}(s)$ with $q = u$, d and s are defined through [80]

$$\langle K^+(p_i)K^-(p_j)|\bar{q}\gamma_\nu q|0\rangle = (p_i - p_j)_\nu F^{K^+K^-}_{1q}(s_{ij}). \qquad \text{(A2)}$$

They have been calculated using vector dominance, quark model assumptions, and isospin symmetry in Ref. [80] and receive contributions from the eight vector mesons: $\rho(770)$, $\rho(1450)$, $\rho(1700)$, $\omega(782)$, $\omega(1420)$, $\omega(1650)$, $\phi(1020)$, and $\phi(1680)$. The form factor can be written in closed form using, for example, Eqs. (23) to (25) of Ref. [10]. The parameters needed can be obtained from Table 2 of Ref. [80].

The isovector $K\bar{K}$ form factors that enter the amplitudes $D^0 \to K^\mp[K^0_S K^\pm]_P$ are defined:

$$\langle K^+(p_i)\bar{K}^0(p_j)|\bar{u}\gamma_\nu d|0\rangle = (p_i - p_j)_\nu F^{K^+\bar{K}^0}_1(s_{ij}), \qquad \text{(A3)}$$

$$\langle K^-(p_i)K^0(p_j)|\bar{d}\gamma_\nu u|0\rangle = (p_i - p_j)_\nu F^{K^-K^0}_1(s_{ij}). \qquad \text{(A4)}$$

Using isospin symmetry, one can obtain the following relations [80],

$$F^{K^+\bar{K}^0}_1(s_{ij}) = -F^{K^-K^0}_1(s_{ij}) = 2F^{K^+K^-}_{1u,I=1}(s_{ij}), \qquad \text{(A5)}$$

where $F^{K^+K^-}_{1u,I=1}(s_{ij})$ is the $I = 1$ component of the charged kaon form factor. This form factor is described by Eq. (23) of Ref. [10] keeping only the ρ meson contributions.

4. Heavy-to-light transition form factors

As discussed in the Introduction, factorization theorems allow one to perturbatively integrate out energy scales and yield approximations which are exact in the infinite heavy-quark limit. To a reasonable extent, the decay amplitudes factorize in terms of products of hard and soft matrix elements. Among the latter, heavy-to-light transitions factors have been extensively studied in the past two decades, though their precise nonperturbative evaluation remains a challenge. Full *ab initio* calculations valid in any momentum-squared region are currently out of reach, and one is mostly left with modelling the heavy-to-light amplitudes with as much input from nonperturbative QCD as possible; in many cases, form factors are only obtainable for a limited range of momentum squared, q^2, values and then extrapolated to other q^2 values.

The transition amplitude of a heavy pseudoscalar meson H to a lighter pseudoscalar meson P via an electroweak current, $\langle P(p_P)|J_\mu|H(p_H)\rangle$, is described by two dimensionless form factors,

$$\langle P(p_P)|\bar{l}\gamma_\mu(1-\gamma_5)h|H(p_H)\rangle = F_+(q^2)(p_H + p_P)_\mu + F_-(q^2)(p_H - p_P)_\mu, \qquad \text{(A6)}$$

where $l = u$, d, s, $h = c$, b and where the transferred momentum is $q = p_H - p_P$. It is convenient to rewrite this amplitude in terms of another pair of form factors, namely, the scalar and vector form factors, $F_0(q^2)$ and $F_1(q^2)$, respectively, introducing the momentum $K = p_H + p_P$ [81–83]:

[7] In Refs. [6,10,11], these form factors are also defined as $\Gamma^{n*}_2(s_{ij}) = F^{KK}_{0n}(s_{ij})/\sqrt{2}$ and $\Gamma^{s*}_2(s_{ij}) = F^{KK}_{0s}(s_{ij})$ with $F^{KK}_{0n}(0) = F^{KK}_{0s}(0) = 1$ (see Refs. [44,68] for more details).

$$\langle P(p_P)|\bar{l}\gamma_\mu(1-\gamma_5)h|H(p_H)\rangle = F_1(q^2)\left[K_\mu - \frac{K\cdot q}{q^2}q_\mu\right] + F_0(q^2)\frac{K\cdot q}{q^2}q_\mu. \quad \text{(A7)}$$

The relation between the two sets of form factors is given by

$$F_1(q^2) = F_+(q^2), \quad \text{(A8)}$$

$$F_0(q^2) = F_+(q^2) + F_-(q^2)\frac{q^2}{K\cdot q}, \quad \text{(A9)}$$

where at $q^2 = 0$ the identity, $F_1(0) = F_0(0) = F_+(0)$, holds. Notice that the above definitions are identical for the $\langle S(p_S)|J_\mu|H(p_H)\rangle$ transitions, e.g., when the final state S is a scalar meson.

The advantage of the Lorentz decomposition in Eq. (A7) lies in the simplification of the decay amplitudes: if the meson, emitted via an electroweak gauge boson, is a pseudoscalar (or scalar), then only $F_0(q^2)$ enters the decay amplitude. Analogously, if the emitted meson is a vector (or axial-vector) meson, the decay amplitude only depends on $F_1(q^2)$.

The weak transition of a heavy pseudoscalar meson H to a lighter vector meson V can be decomposed into Lorentz invariants as [31]

$$\begin{aligned}\langle V(p_V,\epsilon_V)|\bar{l}\gamma_\mu(1-\gamma_5)b|H(p_H)\rangle &= \frac{-2V}{m_H+m_V}\varepsilon_{\mu\nu\alpha\beta}\epsilon_V^{*\nu}p_H^\alpha p_V^\beta - 2im_V A_0(q^2)\frac{\epsilon_V^*\cdot q}{q^2}q_\mu \\ &- i(m_H+m_V)A_1(q^2)\left[\epsilon^*_{V\mu} - \frac{\epsilon_V^*\cdot q}{q^2}q_\mu\right] \\ &+ iA_2(q^2)\frac{\epsilon_V^*\cdot q}{m_H+m_V}\left[(p_H+p_V)_\mu - \frac{m_H^2-m_V^2}{q^2}q_\mu\right],\end{aligned} \quad \text{(A10)}$$

where ϵ_V is the polarization of the final-state vector meson, $q = p_H - p_V$, $p_V^2 = m_V^2$, and $p_H^2 = m_H^2$. Other, related decompositions are possible; see, e.g., Refs. [81–87]. Their relations with the form factor decomposition in Eq. (A10) are detailed in Ref. [83] where algebraic interpolations for the transition form factors can also be found.

A variety of theoretical approaches have been applied to the transition form factors in Eqs. (A7) and (A10), among which are analyses using light-front and relativistic constituent quark models, light-cone sum rules, continuum functional QCD approaches, and lattice-QCD simulations. An experimental extraction of the transition form factors from semileptonic decays for a range of q^2 momenta is possible and has been obtained, for instance, in the case of $D^0 \to \pi^- e^+ \nu_e$ decays [88]. These decays are considerably easier to analyze than nonleptonic decays characterized by complicated final-state interactions. For a brief summary of the theoretical approaches, we refer to Ref. [89], where a numerical comparison of the theoretical transition form factor, $F_+^{B\to\pi}(q^2)$, predictions for various q^2 values is provided in Table I and which highlights pronounced variations among the approaches. A comparison of numerical results for the $B \to K^*$ form factors obtained in lattice-QCD, light-cone sum rules, and Dyson-Schwinger equation approaches is presented in Fig. 2 of Ref. [87].

APPENDIX B: RELATIONS BETWEEN THE PARAMETRIZED AND ORIGINAL AMPLITUDES

The aim of this Appendix is to relate the amplitudes introduced in Secs. II and III to those derived in quasi-two-body QCD factorizations [6–10,12–14], which represents the original motivation of the present parametrizations. The main purpose of this section is to make contact with the original works and make explicit the physical meaning behind the different parameters of the amplitudes we discussed here. The relations are presented following the order of appearance of the three-body decay amplitudes in Secs. II and III. Explanations of and details about constants and form factors that occur in the amplitudes below can be found in the original references we quote.

1. *B*-decay amplitudes

In the parameters below, when necessary, the superscripts $-, \bar{0}, +$, and 0 refer to the $B^-, \bar{B}^0, B^+$, and B^0 mesons, respectively.

a. $B^\pm \to \pi^\pm\pi^+\pi^-$

Comparing the parametrized $B^- \to \pi^-[\pi^+\pi^-]_{S,P}$ S and P amplitudes, Eqs. (10) and (11), to the corresponding amplitudes, Eqs. (22) and (23), in Ref. [9] yields

$$a_1^S = -\frac{G_F}{\sqrt{2}}\chi_S f_\pi F_0^{BR_S}(m_\pi^2)u(R_S\pi^-), \quad \text{(B1)}$$

$$a_2^S = \frac{G_F}{\sqrt{2}}B_0\frac{M_B^2 - m_\pi^2}{m_b - m_d}v(\pi^- R_S), \quad \text{(B2)}$$

$$a_1^P = \frac{G_F}{\sqrt{2}}N_P\frac{f_\pi}{f_{R_P}}A_0^{BR_P}(m_\pi^2)u(R_P\pi^-), \quad \text{(B3)}$$

$$a_2^P = \frac{G_F}{\sqrt{2}}w(\pi^- R_P). \quad \text{(B4)}$$

The definitions and numerical values of all the quantities in Eqs. (B1) to (B4) can be found in Ref. [9]. The functions $u(R_P\pi^-)$, $v(\pi^- R_S)$, and $w(\pi^- R_P)$, corresponding to the short-distance contributions, are proportional to the CKM matrix elements and to the effective Wilson coefficients. The dominant meson resonances are $R_S \equiv f_0(980)$ and $R_P \equiv \rho(770)^0$ (see Ref. [9]). Applying CP conjugation to the right-hand side of Eqs. (B1) to (B4) yields the relations between the a_i coefficients of the parametrized $B^+ \to \pi^+[\pi^-\pi^+]_{S,P}$ amplitudes to the original amplitude parameters.

b. $B \to K\pi^+\pi^-$

Comparison of the $B^- \to K^-[\pi^+\pi^-]_S$ amplitude given in Eq. (1) of Ref. [6] with the parametrized form (12) leads to

$$b_1^{-S} = \frac{G_F}{\sqrt{2}}[\chi f_K F_0^{B\to(\pi\pi)_S}(m_K^2)U - \tilde{C}], \tag{B5}$$

$$b_2^{-S} = \frac{G_F}{\sqrt{2}}\frac{2\sqrt{2}B_0}{m_b - m_s}(M_B^2 - m_K^2)V, \tag{B6}$$

$$b_3^{-S} = -\frac{G_F}{\sqrt{2}}\chi(M_B^2 - m_K^2)\tilde{C}, \tag{B7}$$

where $\tilde{C} = f_\pi F_\pi(\lambda_u P_1^{GIM} + \lambda_t P_1)$ with $\lambda_u = V_{ub}V_{us}^*$ and $\lambda_t = V_{tb}V_{ts}^*$. Furthermore, for $i = 1, 2, 3$,

$$b_i^{\bar{0}S} = \frac{b_i^{-S}}{\sqrt{2}}, \tag{B8}$$

$$b_i^{+(0)S} = b_i^{-(\bar{0})S}(\lambda_u^*, \lambda_t^*). \tag{B9}$$

The quantities entering Eqs. (B5) to (B7) are defined in Ref. [6], where their numerical values are also given.

The parameter b_1^{-P} of the $B^- \to K^-[\pi^+\pi^-]_P$ amplitude (13) is related to the parameters described in Ref. [7] in the following way,

$$b_1^{\pm P} = \frac{A^\pm}{\sqrt{2}m_\rho f_\rho}, \tag{B10}$$

$$b_1^{0(\bar{0})P} = \frac{A^0(A^{\bar{0}})}{\sqrt{2}m_\rho f_\rho}, \tag{B11}$$

with

$$A^- = G_F m_\rho[f_K A_0^{B\to\rho}(M_K^2)(U^- - C^P) + f_\rho F_1^{B\to K}(m_\rho^2)W^-], \tag{B12}$$

$$A^{\bar{0}} = G_F m_\rho[f_K A_0^{B\to\rho}(M_K^2)(U^{\bar{0}} + C^P) + f_\rho F_1^{B\to K}(m_\rho^2)W^{\bar{0}}], \tag{B13}$$

$$A^+(A^0) = -A^-(A^{\bar{0}})(\lambda_u^*, \lambda_t^*). \tag{B14}$$

Definitions and values of the parameters appearing in Eqs. (B12) to (B14) can be found in Ref. [7].

In Eqs. (B5), (B6), (B12), and (B13), the short-distance contribution functions U, V, $U^{-(\bar{0})}$, and $W^{-(\bar{0})}$ are products of CKM quark-mixing matrix elements with effective Wilson coefficients.

Comparing the $B^- \to [K^-\pi^+]_S\pi^-$ and $\bar{B}^0 \to [\bar{K}^0\pi^-]_S\pi^+$ amplitudes given by Eqs. (10) and (14) of Ref. [8] with their parametrized forms (14) leads to

$$c_1^{-S} = \frac{G_F}{\sqrt{2}}(M_B^2 - m_\pi^2)(m_K^2 - m_\pi^2)\left[\lambda_u\left(a_4^u(S) - \frac{a_{10}^u(S)}{2} + c_4^u\right) + \lambda_c\left(a_4^c(S) - \frac{a_{10}^c(S)}{2} + c_4^c\right)\right], \tag{B15}$$

$$c_2^{-S} = -\sqrt{2}G_F\frac{(M_B^2 - m_\pi^2)(m_K^2 - m_\pi^2)}{(m_b - m_d)(m_s - m_d)}\left[\lambda_u\left(a_6^u(S) - \frac{a_8^u(S)}{2} + c_6^u\right) + \lambda_c\left(a_6^c(S) - \frac{a_8^c(S)}{2} + c_6^c\right)\right], \tag{B16}$$

$$c_1^{\bar{0}S} = \frac{G_F}{\sqrt{2}}(M_{\bar{B}^0}^2 - m_\pi^2)(m_{\bar{K}^0}^2 - m_\pi^2)[\lambda_u(a_1 + a_4^u(S) + a_{10}^u(S) + c_4^u) + \lambda_c(a_4^c(S) + a_{10}^c(S) + c_4^c)], \tag{B17}$$

$$c_2^{\bar{0}S} = -\sqrt{2}G_F\frac{(M_{\bar{B}^0}^2 - m_\pi^2)(m_{\bar{K}^0}^2 - m_\pi^2)}{(m_b - m_d)(m_s - m_d)}[\lambda_u(a_6^u(S) + a_8^u(S) + c_6^u) + \lambda_c(a_6^c(S) + a_8^c(S) + c_6^c)], \tag{B18}$$

$$c_{1,2}^{+(0)S} = c_{1,2}^{-(\bar{0})S}(\lambda_u \to \lambda_u^*, \lambda_c \to \lambda_c^*), \tag{B19}$$

where $\lambda_c = V_{cb}V_{cs}^*$.

Comparison of the parametrized $K\pi$ P-wave amplitude (15) to the original one in Eqs. (11) and (15) of Ref. [8] gives

$$\begin{aligned} c_1^{-P} = \frac{G_F}{\sqrt{2}}\Bigg\{&\lambda_u\left(a_4^u(P) - \frac{a_{10}^u(P)}{2} + c_4^u\right) + \lambda_c\left(a_4^c(P) - \frac{a_{10}^c(P)}{2} + c_4^c\right) \\ &+ 2\frac{m_{K^*}}{m_b}\frac{f_V^\perp(\mu)}{f_V}\left[\lambda_u\left(a_6^u(P) - \frac{a_8^u(P)}{2} + c_6^u\right) + \lambda_c\left(a_6^c(P) - \frac{a_8^c(P)}{2} + c_6^c\right)\right]\Bigg\}, \end{aligned} \tag{B20}$$

$$\begin{aligned} c_1^{\bar{0}P} = \frac{G_F}{\sqrt{2}}\Bigg\{&\lambda_u(a_1 + a_4^u(P) + a_{10}^u(P) + c_4^u) + \lambda_c(a_4^c(P) + a_{10}^c(P) + c_4^c) \\ &+ 2\frac{m_{K^*}}{m_b}\frac{f_V^\perp}{f_V}[\lambda_u(a_6^u(P) + a_8^u(P) + c_6^u) + \lambda_c(a_6^c(P) + a_8^c(P) + c_6^c)]\Bigg\}, \end{aligned} \tag{B21}$$

$$c_1^{+(0)P} = c_1^{-(\bar{0})P}(\lambda_u \to \lambda_u^*, \lambda_c \to \lambda_c^*). \tag{B22}$$

The values and the definitions of the different short range parameters entering Eqs. (B15) to (B21) can be found in Ref. [8]. Let us just mention that the a_1, $a_i^{u(c)}(S/P)$, $i = 4$, 6, 8, 10, are leading order factorization (effective Wilson) coefficients to which vertex and penguin corrections are added. The $c_i^{u(c)}$, $i = 4$, 6 are free fitted parameters representing nonperturbative and higher order contributions to the penguin diagrams [8].

c. $B^\pm \to K^+K^-K^\pm$

Comparison of the original $B^- \to K^-[K^+K^-]_{S,P}$ amplitudes (see Eqs. (2) and (3) of Ref. [10]) with the parametrized forms of Eqs. (17) and (18) leads to

$$d_1^{-S} = -\frac{G_F}{\sqrt{2}}\chi f_K F_0^{B\to[K^+K^-]_S}(m_K^2)y, \tag{B23}$$

$$d_2^{-S} = \frac{2B_0 G_F}{m_b - m_s}(M_B^2 - m_K^2)v, \tag{B24}$$

$$d_1^{-P} = \frac{G_F}{\sqrt{2}}\frac{f_K}{f_\rho}A_0^{B\rho}(m_K^2)y, \tag{B25}$$

$$d_2^{-P} = -\frac{G_F}{\sqrt{2}}w_u, \tag{B26}$$

$$d_3^{-P} = -\frac{G_F}{\sqrt{2}}w_d, \tag{B27}$$

$$d_4^{-P} = -\frac{G_F}{\sqrt{2}}w_s. \tag{B28}$$

The definition and numerical values of the different parameters entering Eqs. (B23) to (B28) can be found in Ref. [10]. The parameters y, v, w_u, w_d, and w_s represent the contribution of the short range weak-decay amplitudes. For the $B^+ \to K^+[K^+K^-]_{S,P}$ amplitudes, one has

$$d_i^{+S(P)} = d_i^{-S(P)}(\lambda_u \to \lambda_u^*, \lambda_c \to \lambda_c^*). \tag{B29}$$

2. D-decay amplitudes

a. $D^+ \to \pi^+\pi^-\pi^+$

The parameters of the $D^+ \to \pi^-\pi^+\pi^+$ amplitudes given in Eq. (23) can be related to the underlying description of Ref. [12] as follows:

$$e_1^S = \frac{G_F}{\sqrt{2}}V_{cd}V_{ud}^* a_1 f_\pi \chi_S^{\mathrm{eff}}, \tag{B30}$$

$$e_1^P = \frac{G_F}{\sqrt{2}}V_{cd}V_{ud}^* a_1 f_\pi \chi_P^{\mathrm{eff}}, \tag{B31}$$

$$e_2^P = \frac{G_F}{\sqrt{2}}V_{cd}V_{ud}^* a_2. \tag{B32}$$

The parameters $\chi_{S,P}^{\mathrm{eff}}$ are related to the contribution of intermediate resonances in the matrix element of the a_1 type [12]. We use $f_\pi = \sqrt{2}F_\pi \simeq 130.5$ MeV.

b. $D^+ \to K^-\pi^+\pi^+$

The complex parameters of the $D^+ \to K^-\pi^+\pi^+$ amplitude given in Eq. (29) can be related to the description of Ref. [13] as

$$f_1^S = \frac{G_F}{\sqrt{2}}V_{cs}V_{ud}^* f_\pi \chi_S^{\mathrm{eff}} a_1, \tag{B33}$$

$$f_1^P = \frac{G_F}{\sqrt{2}}V_{cs}V_{ud}^* f_\pi \chi_V^{\mathrm{eff}} a_1, \tag{B34}$$

$$f_2^S = \frac{G_F}{\sqrt{2}}V_{cs}V_{ud}^* \Delta_+^2 a_2, \tag{B35}$$

$$f_2^P = \frac{G_F}{\sqrt{2}}V_{cs}V_{ud}^* \Delta_+^2 a_2. \tag{B36}$$

The notation and definitions are analogous to the $D^+ \to \pi^-\pi^+\pi^+$ case. We use again $\Delta_+^2 = (m_{K^-}^2 - m_\pi^2)(m_{D^+}^2 - m_\pi^2)$. The parameters $\chi_{V,S}^{\mathrm{eff}}$ are related to the contribution of intermediate resonances in the a_1-type amplitude. We refer to Ref. [13] for their precise definition.

As a final comment, experiments found an offset of about −65° between the S- and P-wave phases [53–55] that is crucial to reproduce the Dalitz plot [13]. This offset in the phases is described, in the parametrization proposed here, by the phases of the $f_{1,2}^L$ parameters. We should point out, however, that the dynamical origin of the phase difference between the S and P waves may be related to hadronic three-body rescattering that is beyond our description [56], although some controversy persists (see Ref. [61]).

c. $D^0 \to K_S^0\pi^+\pi^-$

Comparison between the different $\mathcal{A}_{S(P)}$ amplitudes, Eqs. (31) to (36), and the $\mathcal{M}_i$ amplitudes, Eqs. (66)–(69), (71), (84), and (85) of Ref. [14] yields the following relations.

For the $D^0 \to [K_S^0\pi^-]_{S,P}\pi^+$ amplitudes, one has

$$g_1^S = \alpha_1 m_{D^0}^2 + \beta_1 m_\pi^2, \tag{B37}$$

$$g_2^S = -(\alpha_1 + \beta_1), \tag{B38}$$

$$\alpha_1 = -\frac{G_F}{2}a_1\Lambda_1\chi_1 f_\pi F_0^{D^0 R_S[\bar{K}^0\pi^-]}(m_\pi^2), \tag{B39}$$

$$\beta_1 = -\frac{G_F}{2}a_2\Lambda_1\chi_1 f_{D^0} F_0^{R_S[\bar{K}^0\pi^-]\pi^+}(m_{D^0}^2), \tag{B40}$$

$$g_1^P = \frac{G_F}{2}\Lambda_1\left[a_1\frac{f_\pi}{f_{K^{*-}}}A_0^{D^0 R_P[\bar{K}^0\pi^-]}(m_\pi^2) - a_2\frac{f_{D^0}}{f_{K^{*-}}}A_0^{\pi^+ R_P[\bar{K}^0\pi^-]}(m_{D^0}^2)\right]. \tag{B41}$$

The relations for the $D^0 \to [K_S^0\pi^+]_{S,P}\pi^-$ amplitudes are

$$g_3^S = -\frac{G_F}{2}\Lambda_2 z_8 a_2 \chi_1 f_{D^0} \quad F_0^{\pi^- R_S[K^0\pi^+]}(m_{D^0}^2), \tag{B42}$$

$$g_4^S = \frac{G_F}{2}\Lambda_2 z_8 a_1, \tag{B43}$$

$$g_2^P = -\frac{G_F}{2}\Lambda_2 z_9 a_2 \frac{f_{D^0}}{f_{K^{*+}}}A_0^{R_P[K^0\pi^+]\pi^-}(m_{D^0}^2), \tag{B44}$$

$$g_3^P = -\frac{G_F}{2}\Lambda_2 z_9 a_1. \tag{B45}$$

And for the $D^0 \to K_S^0[\pi^+\pi^-]_{S,P}$ amplitudes, it reads

$$g_5^S = \alpha_2 m_{D^0}^2 + \beta_2 m_{K^0}^2, \tag{B46}$$

$$g_6^S = -(\alpha_2 + \beta_2), \tag{B47}$$

$$\alpha_2 = -\frac{G_F}{2}a_2(\Lambda_1+\Lambda_2)\chi_2 f_{K^0}F_0^{D^0 R_S[\pi^+\pi^-]}(m_{K^0}^2), \tag{B48}$$

$$\beta_2 = -\frac{G_F}{2}a_2(\Lambda_1+\Lambda_2)\chi_2 f_{D^0}F_0^{\bar{K}^0 R_S[\pi^+\pi^-]}(m_{D^0}^2), \tag{B49}$$

$$g_4^P = \frac{G_F}{2}a_2(\Lambda_1+\Lambda_2)\frac{1}{f_\rho}[f_{K^0}A_0^{D^0 R_P[\pi^+\pi^-]}(m_{K^0}^2) + f_{D^0}A_0^{\bar{K}^0 R_P[\pi^+\pi^-]}(m_{D^0}^2)], \tag{B50}$$

$$g_5^P = \frac{G_F}{2}(\Lambda_1+\Lambda_2)\frac{a_2}{\sqrt{2}}\left[f_{K^0}A_0^{D^0\omega}(m_{K^0}^2) - f_{D^0}A_0^{\bar{K}^0[\pi^+\pi^-]_\omega}(m_{D^0}^2)\right]\frac{g_{\omega\pi\pi}}{m_\omega}. \tag{B51}$$

For the definitions and numerical values of all parameters entering Eqs. (B37) to (B51), see Ref. [14].

d. $D^0 \to K_S^0 K^+ K^-$

Comparison between the parametrized $\mathcal{A}_{S(P)}$ amplitudes, Eqs. (38) to (43), and the corresponding amplitudes of Ref. [24] yields for the kaon pairs in scalar states

$$h_1^S = -\frac{G_F}{4\sqrt{2}}(\Lambda_1+\Lambda_2)a_2\chi^n f_{K^0}F_0^{D^0 f_0}(m_{K^0}^2), \tag{B52}$$

$$h_2^S = -\frac{G_F}{2\sqrt{2}}(\Lambda_1+\Lambda_2)a_2\chi^s f_{D^0}F_0^{K^0 f_0}(m_{D^0}^2), \tag{B53}$$

$$h_3^S = -\frac{G_F}{4}(\Lambda_1+\Lambda_2)a_2\chi^{(1)} f_{K^0}F_0^{D^0 a_0^0}(m_{K^0}^2), \tag{B54}$$

$$h_4^S = -\frac{G_F}{2}\Lambda_2\chi^{(1)}[a_1 f_{K^+}m_{D^0}^2 F_0^{D^0 a_0^-}(m_K^2) + a_2 f_{D^0}m_K^2 F_0^{K^+ a_0^-}(m_{D^0}^2)], \tag{B55}$$

$$h_5^S = \frac{G_F}{2}\Lambda_2\chi^{(1)}[a_1 f_{K^+}F_0^{D^0 a_0^-}(m_K^2) + a_2 f_{D^0}F_0^{K^+ a_0^-}(m_{D^0}^2)], \tag{B56}$$

$$h_6^S = -\frac{G_F}{2}\Lambda_1 a_1(m_{D^0}^2 - m_K^2)(m_K^2 - m_{K^0}^2), \tag{B57}$$

$$h_7^S = -\frac{G_F}{2}\Lambda_1 a_2\chi^{(1)} f_{D^0}F_0^{K^- a_0^+}(m_{D^0}^2). \tag{B58}$$

For the kaon pairs in vector states, one has

$$h_1^P = \frac{G_F}{2}(\Lambda_1+\Lambda_2)a_2\frac{f_{K^0}}{f_{\rho^0}}A_0^{D^0\rho^0}(m_{K^0}^2), \tag{B59}$$

$$h_2^P = \frac{G_F}{2}(\Lambda_1+\Lambda_2)a_2\frac{f_{D^0}}{f_\phi}A_0^{K^0\phi}(m_{D^0}^2)), \tag{B60}$$

$$h_3^P = \frac{G_F}{2}\Lambda_2\left[a_1\frac{f_{K^+}}{f_\rho}A_0^{D^0\rho^-}(m_K^2) - a_2\frac{f_{D^0}}{f_\rho}A_0^{K^+\rho^-}(m_{D^0}^2)\right], \tag{B61}$$

$$h_4^P = -\frac{G_F}{2}\Lambda_1 a_2\frac{f_{D^0}}{f_\rho}A_0^{K^-\rho^+}(m_{D^0}^2), \tag{B62}$$

$$h_5^P = -\frac{G_F}{2}\Lambda_1 a_1. \tag{B63}$$

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