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Galilei group with multiple central extension, vorticity, and entropy generation: Exotic fluid in 3+1 dimensions

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A noncommutative extension of an ideal (Hamiltonian) fluid model in $3 + 1$ dimensions is proposed. The model enjoys several interesting features: it allows a multiparameter central extension in Galilean boost algebra (which is significant being contrary to the existing belief that a similar feature can appear only in $2 + 1$-dimensions); noncommutativity generates vorticity in a canonically irrotational fluid; it induces a non-barotropic pressure leading to a nonisentropic system. (Barotropic fluids are entropy preserving as the pressure depends only on the matter density.) Our fluid model is termed "exotic" since it has a close resemblance with the extensively studied planar ($2 + 1$ dimensions) exotic models and exotic (noncommutative) field theories.

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I. INTRODUCTION

The stage for nonrelativistic particles and their wave equations was set much earlier by Levy-Leblond [1] when he put through the case of Galilean invariant theories being independent entities and not just as nonrelativistic limits of relativistic Poincaré invariant theories. His equation for a spin $1/2$ particle (same as the Pauli equation but in which, unlike the latter, the spinor features were inherent) clearly showed that the spin (as well as a correct Lande g factor) was not an offshoot of the relativistic effect, whereas the spin-orbit interaction and Thomas precession were. Nonrelativistic equations for arbitrary spin particles were also derived in [1], and the role of the mass parameter leading to the superselection rule of Bargemann [2] was revealed explicitly. The present work deals with a generalized form of nonrelativistic fluid dynamics and is connected to works of Levy.

In this paper, we propose a generalization of nonrelativistic fluid theory in $3 + 1$ dimensions, where spatial noncommutativity (NC) gives rise to a number of striking features: (i) the NC model admits a multiparameter central extension (CE) in Galilean boost algebra. (ii) NC induces vorticity in an otherwise irrotational fluid. (iii) NC generates nonbarotropy in the fluid effective pressure that can lead to nonisentropic dynamics. In a barotropic fluid, the pressure depends on matter density alone and is associated with entropy preserving dynamics. Let us elaborate briefly on the significance of each of the above themes.

(i) In classical physics, CEs naturally arise in Hamiltonian classical mechanics [2,3] from the nonunique nature of canonical generators for a given (Hamiltonian) phase space vector field. In quantum physics, a CE can appear from singularities related to operator ordering anomaly terms [4]. CEs commute with all the generators and can consist of purely c numbers (in general, nonremovable) or canonical variables as Casimir operators (that can be shifted or removed by redefining generators). We will comment later on the nontriviality of a CE in the latter case.

It was argued long ago and accepted till date [5] that *only* $2 + 1$-dimensional Galilean algebra allows a two-parameter central extension (in boost algebra), the reason being the abelian nature of planar rotations. The present NC fluid model goes against the common lore. The second (set of) parameter appears in noncommuting Galilean boost generators, a hallmark of "exotic" physics. We have borrowed the term exotic from the series of works by Duval and Horvathy [6] and Horvathy *et al.* [7], who first constructed exotic planar nonrelativistic particle and field theory models. It was further put in firm footing by Jackiw and Nair, who identified the exotic parameter with a particle spin in a nonrelativistic limit of their relativistic spinning particle model [8] (for an alternative point of view, see [9]). The topic generated excitement as these planar models are directly connected to anyons [8,10], planar excitations of arbitrary spin and statistics. We have termed our $3 + 1$ dimensions NC fluid as an exotic fluid since it has a lot of similarities with the $2 + 1$-dimensions exotic models [6,7].

CEs can impact both theoretical as well as experimental physics. Thus, Bargmann's research (see also [1,4]) on projective representations of continuous groups [specifically, the Galilean group in $(3 + 1)$ dimensions] showed how the concept of mass and its related superselection rule appears through the central extension of a Galilean group. On the experimental side, this is evident from the recent works: the CE in Ward identities in $2 + 1$ dimensions momentum algebra leads to a direct relation between thermal Hall conductivity and topological charge density; a gapped insulating phase, the so-called Haldane insulator, appears between the Mott and density wave phases where phase boundaries were determined from the central charge; in black hole physics (see, e.g. [11] for relevant works).

(ii) and (iii) In conventional fluid dynamics, a frictionless barotropic fluid is an extremely common and useful approximation of a realistic fluid. Here, Kelvin's theorem that circulation along a closed fluid line stays constant for all

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times (or equivalently Lagrange's statement that an irrotational fluid particle will stay irrotational) is valid [12]. Although it is applicable in a variety of physical situations, a barotropic fluid can not explain topical areas of interest, e.g. high velocity aerodynamics, supersonic phenomena giving rise to shock waves [13], among others. Furthermore, a change of circulation (appearance or disappearance of vortices) is due to viscosity or a nonbarotropic equation of state [14]. An interesting astrophysical example of nonbarotropy is given in modeling nonbarotropic multifluid neutron stars [15]. A barotropic nature leads to isentropy whereas nonbarotropy, i.e., dependence of the pressure on density and other variables signals nonisentropic behavior. Further theoretical works in these contexts are [16].

After the generalities, let us discuss our NC fluid model and its connection to exotic systems more closely. Anyonic excitations emerge as charged vortex solutions in planar Chern-Simons quantum gauge field theories [17] where boost generators commute. However, the second extension is recovered in its generalization to NC theory via a Moyal star product [7]. We will see that in our NC fluid, the second central extension is structurally identical to the above form [7], although our NC approach is totally different from (and does not introduce) a Moyal star product framework [18].

An intriguing but well-known property of Galilean boosts for massive nonrelativistic quantum systems is that their action is characterized up to a phase [2,5], leading to a one-parameter CE. CEs are associated with nontrivial Lie algebra cohomology [19], and Bargmann [2] has proved that in three or higher spatial dimensions there can exist only one CE proportional to the mass parameter. However, the important question whether there can exist other CEs was settled in [5], who recognized that *the abelian nature of planar rotations admits a second central extension*, the new exotic parameter being spin. In fact, the planar Poincaré group reduces to the exotic Galilei group following the Jackiw-Nair prescription [8]. Explicit physical models pertaining to this feature appeared in [6,7,20], which were endowed with NC planar coordinates. We comment later that in $3+1$ dimensions, NC in fluid generates a vorticity similar to exotic parameter-spin mapping in $2+1$ dimensions.

Another area of recent excitement is a generalization of quantum mechanics in NC space to include a coordinate-coordinate NC algebra together with the conventional coordinate-momentum (Heisenberg) NC algebra unchanged. Generally, the purely momentum sector is kept commutative. NC spacetime, although introduced long ago by Snyder [21] to weaken the short distance singularity in quantum field theory (which incidentally did not meet success), has captured recent interest after the work of Seiberg and Witten [18]. It was shown [18,22] that in certain low energy limits, an open string ending on D-branes can be represented by the NC generalization of conventional field theory. Different aspects of NC quantum mechanics and quantum field theories have been studied extensively [23]. The NC extension of quantum mechanics has a natural echo in classical mechanics since Poisson brackets in the latter are elevated to quantum commutators in the former. The perfect setting to generate classical noncanonical brackets is the symplectic framework [24] or equivalently Hamiltonian (Dirac) constraint analysis [25]. In the present work, we follow the Dirac formalism, where the NC generalized brackets are naturally identified with Dirac brackets.

Finally, we note that hydrodynamics [12], one of the earliest developed disciplines in applied sciences, provides a universal description of long wavelength physics that deals with low energy effective excitations of a classical or quantum field theory. It is applicable both at microscopic and macroscopic scales, from liquid drop model (nuclear physics), quark-gluon-plasma produced at RHIC/LHC to generic fluid models in cosmology. Quite interestingly, in recent times, fluid dynamics is enjoying a renewed interest from theoretical high energy physics perspective [26] (for an exhaustive review, see [27]).

II. CANONICAL FLUID DYNAMICS

In the present work, we extend the conventional barotropic fluid dynamics [27] to NC space. We work in the Hamiltonian field theoretic framework known as the Euler fluid model. In a previous work [28], we constructed the NC fluid system from a Lagrangian (fluid) approach with NC coordinates in the latter (see, e.g. [29] for connection between the Lagrange and Euler approach of fluid dynamics). The canonical model consists of density $\rho(x)$ and velocity fields $v^i = \partial_i\theta(x)$, endowed with a Poisson algebra and a Hamiltonian, $\{\theta(x), \theta(y)\} = \{\rho(x), \rho(y)\} = 0$, $\{\rho(x), \theta(y)\} = \delta(x-y)$, $H = \int d^3x[1/2(\rho(\partial_i\theta)^2) + U(\rho)]$. The continuity and Euler equations for fluid are recovered as Hamiltonian equations of motion $\dot{\rho} = \{\rho, H\} = -\partial_i(\rho v^i)$, $\dot{v}^i = -\partial_i(\frac{v^2}{2} + U')$, where $U' = dU/d\rho$. The fluid is irrotational with no vorticity and barotropic pressure $P = \rho U - U'$ depends only on density ρ. An action formulation for the conventional model also exists [30] (reviewed in [27]).

The NC fluid model was initiated in [31] (see also [27] for the Lagrangian fluid point of view). This was pursued further to completion in [28]. An NC generalization of the above canonical fluid algebra is derived in these works, and it is seen that the same NC density-density bracket,

$$\{\rho(x), \rho(y)\} = -\theta^{ij}\partial_i\rho(x)\partial_j^x\delta(x-y), \tag{1}$$

with $\theta^{ij} = -\theta^{ji}$ being the NC parameter, is reproduced by introducing NC Lagrangian particle coordinates [28]. However, the rest of the NC fluid algebra appears to be model dependent. This new NC structure will alter the fluid dynamics in a nontrivial way.

In this paper, for the first time, we propose a field theoretic NC extended action from which the NC fluid algebra is derived as Dirac brackets. Indeed, the NC extension of the action is not unique. The present construction reproduces the NC density bracket (1) but there is a mismatch with [28] for the rest of the algebra. Quite obviously, the action formulation has many advantages: spacetime symmetry generators and conserved quantities can be derived. In previous studies, validity of the Jacobi identity for the NC brackets was an issue, whereas in the Dirac bracket formalism, it is guaranteed since Dirac brackets preserve Jacobi identity.

III. NONCOMMUTATIVE FLUID DYNAMICS

We posit a candidate for the NC generalization of a fluid Lagrangian (which is also our primary result)

$$L = -\dot{\theta}\left(\rho - \frac{1}{2}\theta^{ij}\partial_i\rho\partial_j\theta\right) - \left(\frac{1}{2}\rho(\partial_i\theta)^2 + U(\rho)\right). \quad (2)$$

Indeed, as mentioned above, this is not a unique choice. We have based our model on the correct form of $\{\rho(x),\rho(y)\}$ bracket [27,28,31], which this Lagrangian reproduces as Dirac brackets (to be explained later). Other inequivalent forms of NC fluid models can be found in [32].

Let us first derive the equations of motion by varying ρ and θ in the action,

$$\dot{\rho} = -\partial_i((\rho\partial_i\theta) + \frac{\theta^{ij}}{2}[\partial_j\theta\partial_k(\rho\partial_k\theta) + \rho\partial_j(\partial_k\theta)^2]), \quad (3)$$

$$\dot{\theta} = -\left(\left(\frac{(\partial\theta)^2}{2} + U'\right) - \frac{\theta^{jk}}{2}\partial_k\theta\partial_j\left(\frac{(\partial\theta)^2}{2} + U'\right)\right). \quad (4)$$

Clearly the mass conservation in (3) is not violated since the new NC θ^{ij} term is a total divergence.

Noether prescription yields the canonical energy momentum tensor,

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu\theta)}\partial^\nu\theta + \frac{\partial L}{\partial(\partial_\mu\rho)}\partial^\nu\rho - \eta^{\mu\nu}L, \quad (5)$$

where $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the flat metric. Explicit expressions for energy and momentum densities are $T^{00} = \frac{1}{2}\rho(\partial_i\theta)^2 + U(\rho), T^{0i} = \rho\partial_i\theta - \frac{1}{2}\theta^{jk}\partial_j\rho\partial_k\theta\partial_i\theta$. Notice that T^{00} does not receive any NC correction but $T^{ij} \neq T^{ji}, T^{0i} \neq T^{i0}$ indicating that rotational and Lorentz symmetries are lost due to a constant θ^{ij} parameter. However, to demonstrate that T^{00}, T^{0i} properly generate time and space translations, respectively, we need the full NC brackets which we now provide.

IV. NONCOMMUTATIVE BRACKETS

It is clear from the Lagrangian (2), being first order in the time derivative, θ and the combination $\rho - \frac{1}{2}\theta^{ij}\partial_i\rho\partial_j\theta$ are a canonical pair but it is problematic to isolate the NC brackets between the basic variables θ and ρ. Instead, we exploit the Dirac bracket formalism to obtain, to a first nontrivial order in θ^{ij}, the NC fluid algebra

$$\{\theta(x),\theta(y)\} = 0, \;\; \{\rho(x),\rho(y)\} = -\theta^{ij}\partial_i\rho(x)\partial_j^x\delta(x-y),$$

$$\{\rho(x),\theta(y)\} = \delta(x-y) + \frac{1}{2}\theta^{ij}\partial_j\theta(x)\partial_i^x\delta(x-y). \quad (6)$$

We point out that the NC model of [32] will not generate any NC extended fluid algebra.

With the Hamiltonian $H = \int d^3xT^{00}$ and the NC algebra (6), we compute $\dot{\rho} = \{\rho, H\}, \dot{v}^i = \{v^i, H\}$ and ensure that these equations are identical to the previously derived (Lagrangian) dynamical equations [(3), (4)]. From the expression of momentum $P^i = \int d^3xT^{0i}$, we find $\{\theta(x), P^i\} = -\partial_i\theta, \{\rho(x), P^i\} = -\partial_i\rho$, showing that P^i is the correct momentum since it generates spatial translations for ρ and θ. Just before, we have demonstrated that H provides the correct time translation for ρ and θ. Note that the total mass operator $M = \int d^3x\rho(x)$ satisfies $\{M,\rho(x)\} = \{M,\partial_i\theta(x)\} = 0$ indicating that M will lie at the center of the Galilean algebra and will act as the central extension.

V. ENERGY AND MOMENTUM CONSERVATION LAWS

We start by noting that the total mass $M = \int d^3x\rho$ is conserved, $\{M,H\} = 0$. Let us now discuss the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$ which in a component form gives rise to energy and momentum conservation laws, $\partial_0T^{00} + \partial_iT^{i0} = 0, \partial_0T^{0i} + \partial_jT^{ji} = 0$.

Using (3) and (4) we find that the above local energy conservation law is satisfied identically, thus ensuring energy conservation. On the other hand, the total momentum $P^i = \int d^3xT^{0i}$ is conserved but the local conservation law receives NC corrections.

VI. EXOTIC CENTRAL EXTENSION IN GALILEAN BOOST ALGEBRA

Defining Galilean boost generators as, $B^i = tP^i - \int d^3x\rho x^i$, we find θ and ρ transform under boost as, $\{\theta(x), B^i\} = -t\partial_i\theta + x^i - \frac{1}{2}\theta^{ij}\partial_j\theta, \{\rho(x), B^i\} = -t\partial_i\rho - \theta^{ij}\partial_j\rho$. Notice that both θ and ρ behave in a non-canonical way, indicating the possibility that Galilean invariance is lost due to noncommutativity. From the behavior of θ and ρ under boost, we can compute the following relations:

$$\{B^i, P^j\} = -\delta^{ij}\int d^3x\rho = -\delta^{ij}M, \quad (7)$$

$$\{B^i, B^j\} = \theta^{ij}\int d^3x\rho = \theta^{ij}M. \quad (8)$$

This is the cherished form of multiparameter CE in 3 + 1-dimensional Galilean algebra and constitutes one of our major results. In three space dimensions, θ^{ij} introduce three additional CE parameters. The first one (7) is the well-known Bargman CE [2]. A structure, similar to the second one in (8) depending on the NC parameter θ^{ij} was discovered *only* in ($2+1$-dimensional) planar models having exotic symmetry [6,7]. Naming our model as exotic fluid is thus justified.

A comment regarding this novel form of (multiply) centrally extended Galilean algebra is in order. Notice that in the exotic models [6,20] in the plane θ^{ij}, being antisymmetric, yields one parameter, whereas in three space θ^{ij} consists of three independent parameters. In fact, from a purely algebraic point of view, after the work of Bargmann [2] on projective or ray representations of the Galilei group, it was established in [33] that, contrary to the wave functions transforming as true representations of the Galilei group, only the projective representations provide localized particle states. The works [5] showed that although in $3+1$ dimensions only one parameter CE are allowed, in $2+1$ dimensions, due to the simple structure of the planar rotation group, this restriction is relaxed, and a three parameter CE [20] is possible that reduces to two parameter (the mass and the single exotic or NC parameter) on physical grounds (since planar states classified by three CE parameters do not support nontrivial dynamics). Returning to the three (space) dimensional field theory studied here, any deformation in the algebra of the fluid variable $[\rho(x),\theta(x)]$ from the canonical one to NC one (6) is very restrictive (due to the symmetry of the algebra, Jacobi identity satisfaction among others). However, the derivative of delta function $\partial_{i(x)}\delta(x-y)$, odd under the interchange of $x\rightleftharpoons y$, provides an additional freedom (which is not enjoyed by discrete mechanical systems) that allows nontrivial modifications in the algebra. Notice that $\partial_{i(x)}\delta(x-y)$ is present in all the NC-extension terms in the NC algebra (6). Indeed, this is not a proof but a possible explanation of this novel phenomenon—a multiparameter CE in three space dimensions.

For compactness, we use vector notation for angular momentum $\mathbf{J}=\int d^3x(\mathbf{x}\times\mathbf{T})$ and with $\sigma^k=(1/2)\epsilon^{kij}\theta^{ij}$, we derive rest of the NC generalized Galilean algebra,

$$\{J^i,J^j\}=\epsilon^{ijk}J^k,\qquad \{J^i,P^j\}=\epsilon^{ijk}P^k,\tag{9}$$

$$\{J^i,B^j\}=\epsilon^{ijk}B^k+\frac{1}{2}(\boldsymbol{\sigma}.\mathbf{P}\delta^{ij}-\sigma^jP^i),\tag{10}$$

$$\{\mathbf{B},H\}=-\mathbf{P}+\int d^3x\left[\frac{1}{2}\boldsymbol{\sigma}.\left(\boldsymbol{\nabla}\frac{1}{\rho}\times\mathbf{T}\right)\mathbf{T}+\frac{1}{4}\left(\boldsymbol{\sigma}\times\boldsymbol{\nabla}\left(\frac{1}{\rho}\right)\right)\right],\tag{11}$$

$$\{\mathbf{J},H\}=\frac{1}{4}\int d^3x\mathbf{T}^2\left[(\boldsymbol{\sigma}.\mathbf{T})\boldsymbol{\nabla}\frac{1}{\rho^2}-\left(\boldsymbol{\sigma}.\boldsymbol{\nabla}\frac{1}{\rho^2}\right)\mathbf{T}\right].\tag{12}$$

A few comments are in order. From (9), we find that P^i transforms canonically, which is expected since (as shown before) it correctly translates both θ, ρ but the fact that $\mathbf{J}-\mathbf{J}$ angular momentum algebra is also canonical is quite unexpected, although the probable reason is again the behavior of $\mathbf{T}$. The rest of the algebra receives NC corrections.

Thus, NC generalization leads to nonconservation of boost and angular momentum which is expected and agrees with earlier results [34] (in different NC field theory models). However, notice that the NC terms in the rhs of (11), (12) are higher order in $\mathbf{T}$ and can be ignored for low kinetic energy, thus recovering a weaker form of boost and angular momentum conservation along with the cherished *exotic central extension that is independent of* $\mathbf{T}$ *and survives the low energy limit*.

VI. DARBOUX MAP, NONCOMMUTATIVITY INDUCED VORTICITY, AND NONISENTROPY

Darboux's theorem, a fundamental property of symplectic geometry, states that any symplectic manifold is locally isomorphic to some R^{2n} with its standard symplectic form or in physics language, the NC variables ρ, θ can be expressed (at least locally) in terms of a canonical set ρ_c, θ_c obeying canonical algebra $\{\rho_c(x),\rho_c(y)\}=\{\theta_c(x),\theta_c(y)\}=0$; $\{\rho_c(x),\theta_c(y)\}=\delta(x-y)$. The explicit form of a Darboux map to $O(\theta)$ [that can be read off from the comments above (6)] is given by $\rho=\rho_c-\frac{1}{2}\theta^{ij}\partial_j\rho_c\partial_i\theta_c$; $\theta=\theta_c$. Notice that exploiting the Darboux map, $\bar{B}^i=tP^i_c-\int d^3x\rho_cx^i$ which, as expected, is just the canonical form that will satisfy $\{\bar{B}^i,\bar{B}^j\}=0$ so that the exotic central extension can be removed [without affecting the (non) conservation of Boost]. An identical situation prevails in earlier planar exotic models as well [7]. However, as pointed out by Brihaye *et al.* in [20], this does not render the CE trivial, and the models with and without CE are not physically equivalent since the Darboux map is not a canonical transformation, and also it changes the interpretation of basic degrees of freedom.

From now on, we will work with ρ_c,θ_c but keep the original notation ρ, θ. The Hamiltonian, to $O(\theta)$ and to $O(v^2)$ is,

$$H=\int dr\left[T_c-\frac{1}{2}\theta^{ij}\frac{\partial_j\rho v_i}{\rho}\left(T_c+P_c\right)\right]\tag{13}$$

where $v^i=\partial_i\theta$ and $T_c=\frac{1}{2}\rho v^2+U(\rho)$, $P_c=\rho U'-U$ are canonical energy density and pressure. The continuity equation to $O(v^2)$,

$$\dot{\rho}=\{\rho,H\}=\partial_l\left[-\rho\left(v^l-\frac{1}{2}\theta^{lj}\partial_j\rho\frac{1}{\rho}\left(\frac{1}{2}v^2+U'\right)-\frac{1}{2}\theta^{ij}\frac{(\partial_j\rho)}{\rho}v^iv^l\right)\right],\tag{14}$$

is written in a suggestive form $\dot{\rho}=-\partial_l(-\rho\bar{v}^l)$, where to $O(v^2)$, $\bar{v}^l=v^l-\frac{1}{2}\theta^{lj}\partial_j\rho\frac{1}{\rho}(\frac{1}{2}v^2+U')-\frac{1}{2}\theta^{ij}\frac{(\partial_j\rho)}{\rho}v^iv^l$, so that $\bar{v}^l$ is naturally identified as the NC corrected velocity. Clearly $\bar{v}^l$ is no longer irrotational yielding the induced vorticity to $O(v^2)$:

$$\{\bar{v}^l(x), \bar{v}^k(y)\} = \frac{1}{2}\left[\theta^{lm}\partial_k^y\left(\frac{1}{\rho(y)}U'(y)\partial_m^y\delta(x-y)\right) - \theta^{km}\partial_l^x\left(\frac{1}{\rho(x)}U'(x)\partial_m^x\delta(x-y)\right)\right]. \quad (15)$$

Note that the NC induced vorticity is structurally totally different from the conventional form of vorticity ($\sim\nabla\times v$), and furthermore, the leading term (written here) is independent of $\bar{v}$ and will survive the low energy limit. Let us consider an explicit form of a conventional barotropic fluid having $U(\rho) = K\rho^\lambda$ with K, λ numerical constants, for which $P_c = (\lambda - 1)U$. For the special case of pressureless dust, ($\lambda = 1, P_c = 0$), induced NC vorticity is given by (15), $\{\bar{v}^l(x), \bar{v}^k(y)\} = \frac{K}{2}[\theta^{lm}\partial_m(\frac{1}{\rho}\partial_k\delta(x-y)) - \theta^{km}\partial_l(\frac{1}{\rho}\partial_m\delta(x-y))]$ where all arguments of fields and derivatives are on x. One immediately notices a nonabelian like feature, reminiscent of NC field theories [10,18], since $\{v^k(x), \bar{v}^k(y)\}$ even for *same* k is nonzero: $\{v^k(x), \bar{v}^k(y)\} = \theta^{km}/\rho^2(\partial_k\rho\partial_m\delta(x-y) - \partial_m\rho\partial_k\delta(x-y))$ (no sum on k).

To consider the effective pressure, we have to derive the Euler equation for $\bar{v}^i$,

$$\dot{\bar{v}}^l = -\partial_l\left(\frac{\bar{v}^2}{2}\right) - \frac{1}{\rho}\partial_l P_c + \frac{1}{2\rho}\theta^{ij}\partial_l(\bar{v}^i\partial_j U) - \frac{1}{2}\theta^{ij}U'\partial_l\left(\frac{1}{\rho}\bar{v}^i\partial_j\rho\right) + \frac{1}{2}\theta^{lj}\left[U'\partial_j\left(\frac{\partial_k(\rho\bar{v}^k)}{\rho}\right) + \frac{\bar{v}^k\partial_j\rho\partial_k U'}{\rho}\right]. \quad (16)$$

Notice that the effective pressure depends explicitly on $\bar{v}^i$ (apart from ρ) that signals a nonbarotropy in the fluid that can yield subsequent nonisentropic dynamics. Again for pressureless dust, we find an NC generated effective pressure, $\rho\dot{\bar{v}}^l = -\partial_l(\frac{\bar{v}^2}{2}) + \frac{K}{2}(\frac{1}{\rho^2}\theta^{kj}\bar{v}^k\partial_j\rho\partial_l\rho + \theta^{lj}\partial_j(\frac{\partial_k(\rho\bar{v}_k)}{\rho}))$. (15) and (16) constitute our other major results, where NC induces a vorticity and nonbarotropy (with possible entropy generation), respectively, in the simplest of ideal fluid, irrotational pressureless dust. Apart from introducing anisotropy, the signature of the NC pressure can be both positive or negative (depending on θ and the fields), which might lead to a Chaplygin fluid like behavior of negative pressure [35] that is interesting in a cosmological scenario as a dark energy candidate [36].

VIII. CONCLUSION

To summarize, we have provided, for the first time, an action for a noncommutative fluid that enjoys several interesting features: the only example till date of a multiple parameter centrally extended Galilean algebra in $3+1$-dimensions, generation of vorticity, and nonbarotropy in the fluid. All these effects vanish in the commutative limit, $\theta^{ij} = 0$. Explicit expressions of the above NC phenomena are provided for a canonical irrotational, and a barotropic fluid are derived. This "exotic" fluid has a close resemblance with popular "exotic" models studied earlier exclusively in $3+1$-dimensions.

For future work, we briefly outline a possible NC effect in the cosmological context. As we have shown earlier [37], the NC can directly modify the Friedmann equation, thereby producing an NC corrected effective curvature. Furthermore, NC necessarily generates anisotropy and inhomogeneity that can lead to structure formation effects via cosmological perturbations. (Work is in progress in these directions.) Other open problems, apart from the obvious one of extending the present work to fluids that are canonically not irrotational, are that it would be worthwhile to consider the Madelung framework to interpret the NC correction as a spin effect in the quantum fluid [38]. Also, since the NC fluid exhibits exotic features, it might be relevant in a semiclassical Bloch electron theory with a potential application in anomalous or spin Hall effects [7,39] with a novel effect for the exotic (second) central extension. Finally, it would be worthwhile to look for other nonrelativistic field theories in three (space) dimensions with multiple central extension parameters.

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[1] J.-M. Levy-Leblond, J. Math. Phys. **4**, 776 (1963); Commun. Math. Phys. **6**, 286 (1967).
[2] V. Bargmann, Ann. Math. **59**, 1 (1954).
[3] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978).
[4] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).
[5] J.-M. Souriau, *Structure des Systemes Dynamiques* (Paris, Dunod, 1969); *Structure of Dynamical Systems, a Symplectic View of Physics* (Birkauser, 1997); J.-M. Levy-Leblond in *Group Theory and Applications*, edited by E. M. Loebl (Academic Press, New York, 1972), Vol. II, p. 222; D. R. Grigore, J. Math. Phys. (N.Y.) **37**, 460 (1996); S. K. Bose, Commun. Math. Phys. **169**, 385 (1995); C. Duval, Exotic Galilei group, IQHE and ChernSimons electrodynamics (unpublished).
[6] C. Duval and P. A. Horvathy, Phys. Lett. B **479**, 284 (2000); J. Phys. A **34**, 10097 (2001); Phys. Lett. B **547**, 306 (2002);

P. A. Horvathy, Ann. Phys. (Amsterdam) **299**, 128 (2002); Nucl. Phys. **B673**, 301 (2003); for review, see P. A. Horvathy, L. Martina, and P. C. Stichel, SIGMA **6**, 060 (2010).

[7] P. A. Horvathy, L. Martina, and P. C. Stichel, Phys. Lett. B **564**, 149 (2003); Nucl. Phys. **B673**, 301 (2003).

[8] R. Jackiw and V. P. Nair, Phys. Rev. D **43**, 1933 (1991); Phys. Lett. B **480**, 237 (2000).

[9] C. R. Hagen, Phys. Lett. B **539**, 168 (2002).

[10] C. Chou, V. P. Nair, and A. P. Polychronakos, Phys. Lett. B **304**, 105 (1993); M. Chaichian, R. G. Felipe, and D. L. Martinez, Phys. Rev. Lett. **71**, 3405 (1993); **73**, 2009 (E) (1994); R. Jackiw and V. P. Nair, Phys. Rev. Lett. **73**, 2007 (1994); S. Ghosh, Phys. Lett. B **338**, 235 (1994); Phys. Rev. D **51**, 5827 (1995); P. A. Horvathy and M. S. Plyushchay, Phys. Lett. B **595**, 547 (2004); C. Duval and P. A. Horvathy, Phys. Lett. B **594**, 402 (2004); for review, see F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific Publishing Company, Singapore, 1990).

[11] J. Cardy, J. Stat. Mech. (2010) P10004; B. S. Kim and A. D. Shapere, Phys. Rev. Lett. **117**, 116805 (2016); S. Ejima and H. Fehske, J. Phys. Conf. Ser. **592**, 012134 (2015); C. Troessaert, J. High Energy Phys. 08 (2013) 044; K. Ropotenko, Mod. Phys. Lett. A **31**, 1650018 (2016).

[12] L. Landau and E. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon, Oxford, UK, 1987); H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, England, 1932).

[13] A. Vazsonyi, Q. Appl. Math. **3**, 29 (1945).

[14] L. Brekhokikh and V. Goncharov, *Springer Series on Wave Phenomena 1* (Springer-Verlag, Berlin, 1985), ISBN 3-540-13765-3.

[15] K. Glampedakis, N. Andersson, and S. K. Lander, Mon. Not. R. Astron. Soc. **420**, 1263 (2012).

[16] R. Prix, Phys. Rev. D **69**, 043001 (2004); A. Reisenegger and P. Goldreich, Astrophys. J. **395**, 240 (1992); G Ballesteros, Phys. Rev. D **94**, 025034 (2016); C. A. Stephen and A. Dar, Proc. R. Soc. A **466**, 2605 (2010).

[17] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989); S. C. Zhang, Int. J. Mod. Phys. B **06**, 25 (1992).

[18] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032; For reviews, see M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001); R. J. Szabo, Phys. Rep. **378**, 207 (2003); R. Banerjee, B. Chakraborty, S. Ghosh, P. Mukherjee, and S. Samanta, Found. Phys. **39**, 1297 (2009).

[19] D. Simms, Lectures given at Centre de Physique Theorique, CNRS, Marseille, 1969 (unpublished); V. Aldaya and J. A. de Azcarraga, Int. J. Theor. Phys. **24**, 141 (1985); G. M. Tuynman and W. A. J. J. Wiegerinck, J. Geom. Phys. **4**, 207 (1987); G. Marmo, G. Morandi, A. Simoni, and E. C. G. Sudarshan, Phys. Rev. D **37**, 2196 (1988).

[20] Y. Brihaye, C. Gonera, S. Giller, and P. Kosinski, arXiv: hep-th/9503046; J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, Ann. Phys. (N.Y.) **260**, 224 (1997); P. A. Horvathy and M. S. Plyushchay J. High Energy Phys. 06 (2002) 033.

[21] H. S. Snyder, Phys. Rev. **71**, 38 (1947); **72**, 68 (1947); See also C. N. Yang, Phys. Rev. **72**, 874 (1947).

[22] M. M. Sheikh-Jabbari, Phys. Lett. B **455**, 129 (1999); P. M. Ho and Y. S. Wu, Phys. Lett. B **398**, 52 (1997); R. Banerjee, B. Chakraborty, and S. Ghosh, Phys. Lett. B **537**, 340 (2002).

[23] J. Gamboa, M. Loewe, and J. C. Rojas, Phys. Rev. D **64**, 067901 (2001); V. P. Nair and A. P. Polychronakos, Phys. Lett. B **505**, 267 (2001); H. O. Girotti, Am. J. Phys. **72**, 608 (2004); S. Ghosh and P. Roy, Phys. Lett. B **711**, 423 (2012); O. Bertolami and P. Leal, Phys. Lett. B **750**, 6 (2015); R. Banerjee, B. D. Roy, and S. Samanta, Phys. Rev. D **74**, 045015 (2006).

[24] L. D. Faddeev and R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988).

[25] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964).

[26] V. P. Nair, arXiv:1606.06407; D. Capasso, V. P. Nair, and J. Tekel, Phys. Rev. D **88**, 085025 (2013); D. Karabali and V. P. Nair, Phys. Rev. D **90**, 105018 (2014); R. Banerjee, S. Dey, B. R. Majhi, and A. K. Mitra, Phys. Rev. D **89**, 104013 (2014); G. M. Monteiro, A. G. Abanov, and V. P. Nair, Phys. Rev. D **91**, 125033 (2015); R. Banerjee, S. Ghosh, and A. K. Mitra, Eur. Phys. J. C **75**, 207 (2015); arXiv:1604.06544.

[27] R. Jackiw, V. P. Nair, S. Y. Pi, and A. P. Polychronakos, J. Phys. A **37**, R327 (2004).

[28] P. Das and S. Ghosh, Eur. Phys. J. C **76**, 627 (2016); **77**, 64 (2017).

[29] R. Salmon, Annu. Rev. Fluid Mech. **20**, 225 (1988).

[30] A. Clebsch and J. Reine, Angew. Math. **56**, 1 (1859); C. Eckart, Phys. Rev. **54**, 920 (1938); C. C. Lin, in *International School of Physics E. Fermi (XXI)*, edited by G. Careri (Academic Press, New York, 1963).

[31] R. Jackiw, Nucl. Phys. B, Proc. Suppl. **108**, 30 (2002); Phys. Part. Nucl. **33**, S6 (2002); Lect. Notes Phys. **616**, 294 (2003); arXiv:physics/0209108; Z. Guralnik, R. Jackiw, S. Y. Pi, and A. Polychronakos, Phys. Lett. B **517**, 450 (2001).

[32] M. C. B. Abdalla, L. Holender, M. A. Santos, and I. V. Vancea, Phys. Rev. D **86**, 045019 (2012); L. Holender, M. A. Santos, M. T. D. Orlando, and I. V. Vancea, Phys. Rev. D **84**, 105024 (2011).

[33] E. Inonu and E. P. Wigner, Nuovo Cimento **9**, 705 (1952); M. Hammermesh, Ann. Phys. (N.Y.) **9**, 518 (1960).

[34] For NC nonconserved boosts, see B. Chakraborty, S. Gangopadhyay, and A. Saha, Phys. Rev. D **70**, 107707 (2004); B. Chakraborty and A. S. Majumdar, Ann. Phys. (N.Y.) **250**, 112 (1996); R. Banerjee and P. Mukherjee, Ann. Phys. (N.Y.) **264**, 30 (1998).

[35] S. Chaplygin, Sci. Mem. Moscow Univ. Math. Phys. **21**, 1 (1904).

[36] A. Kamenshchik, U. Moschella, and V. Pasquier, Phys. Lett. B **511**, 265 (2001); U. Alam, V. Sahni, T. D. Saini, and A. A. Starobinsky, Mon. Not. R. Astron. Soc. **344**, 1057 (2003); M. C. Bento, O. Bertolami, and A. A. Sen, Phys. Rev. D **66**, 043507 (2002); U. Debnath, A. Banerjee, and S. Chakraborty, Classical Quantum Gravity **21**, 5609 (2004); R. Banerjee, S. Ghosh, and S. Kulkarni, Phys. Rev. D **75**, 025008 (2007).

[37] Note that the NC $\{\rho(x), \rho(y)\}$ is the same in both here and in [28] but $\{\rho(x), \theta(y)\}$ is different (model dependence), which in the present case yields a nonzero contribution in the effective curvature.

[38] G. Salesi, Mod. Phys. Lett. A **11**, 1815 (1996).

[39] L. Goldenberg and L. Vaidman, Phys. Rev. Lett. **75**, 1239 (1995); D. Xiao, M. C. Chang, and Q. Niu, Rev. Mod. Phys. **82**, 1959 (2010).