PHYSICAL REVIEW D **96,** 106022 (2017)

Generalized diffeomorphisms for E_9

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(Received 28 September 2017; published 29 November 2017)

We construct generalized diffeomorphisms for E_9 exceptional field theory. The transformations, which like in the E_8 case contain constrained local transformations, close when acting on fields. This is the first example of a generalized diffeomorphism algebra based on an infinite-dimensional Lie algebra and an infinite-dimensional coordinate module. As a byproduct, we give a simple generic expression for the invariant tensors used in any extended geometry. We perform a generalized Scherk–Schwarz reduction and verify that our transformations reproduce the structure of gauged supergravity in two dimensions. The results are valid also for other affine algebras.

DOI: 10.1103/PhysRevD.96.106022

I. INTRODUCTION

Exceptional symmetries are one of the deepest features of ungauged maximal supergravity, and symmetry groups of split real form $E_n(\mathbb{R})$ have been established for $n \le 9$, corresponding to supergravity in $D = 11 - n$ space-time dimensions [1–5]. These symmetries are not only important for constructing gauged supergravity models with interesting vacuum structures, but also play an important role for understanding the string theory effective action that conjecturally exhibits a discrete U-duality symmetry $E_n(\mathbb{Z})$ [6], at least for $n \le 7$.

Many papers have been devoted to understanding the origin of the $E_n(\mathbb{R})$ hidden symmetries, and recently there has been considerable progress on "geometrizing" the $E_n(\mathbb{R})$ symmetries for $n \le 8$. This geometrization requires first of all constructing an extended geometry that has $E_n(\mathbb{R})$ symmetry and then, secondly, constructing a model based on this so-called exceptional geometry. A crucial role in both steps is played by a constraint on the geometry called the (strong) section constraint that is necessary for defining a consistent algebra of generalized diffeomorphisms and for making sure that the resulting exceptional field theory reduces consistently to just standard supergravity for a particular choice of exceptional geometric background. All these steps have been carried out for finite-dimensional $E_n(\mathbb{R})$ for $n \le 8$ in a series of papers [7–24]. More generally, one can consider generalized Scherk–Schwarz reductions of these theories [25–29] to obtain gauged supergravity theories.

In the present paper, we will begin the construction of E_9 exceptional field theory, where E_9 denotes the affine extension of the largest finite-dimensional exceptional Lie group E_8. This infinite-dimensional group is known to be a symmetry of two-dimensional maximal ungauged supergravity [2,3], and gaugings of this symmetry have been considered in [30]. The first step in the construction of E_9 exceptional field theory is to establish a consistent gauge algebra of generalized diffeomorphisms similar to [14,23,24]. This requires identifying an appropriate set of coordinates that transform under E_9 together with section constraints. They allow the definition of a generalized Lie derivative that forms a closed algebraic structure. A construction of a model based on the E_9 exceptional geometry will be left to future work. In this sense, we are providing the kinematical background for the construction of a dynamical model.

The main result of this paper will be to provide a consistent algebra of generalized diffeomorphisms based on E_9 together with consistency checks using a generalized Scherk–Schwarz reduction. The coordinates lie in the simplest $\mathfrak{e}_9$ highest weight representation, sometimes called the "basic" or "fundamental" representation [30–32], that can be identified with the Hilbert space of a CFT on the E_8 lattice [33] and whose construction will be reviewed in algebraic terms below. Due to the Hilbert space structure, it will prove very convenient to employ Dirac notation to write elements in this representation, its dual and tensor products.

We will show in this paper that the full Lie derivative can be put in a remarkably compact form

$$\mathcal{L}_{\xi,\Sigma}|V\rangle = \langle\partial_V|\xi\rangle|V\rangle + \langle\partial_\xi|(C_0 - 1)|\xi\rangle \otimes |V\rangle + \langle\pi_\Sigma|C_{-1}|\Sigma\rangle \otimes |V\rangle, \tag{1.1}$$

acting on a fundamental vector $|V\rangle$ with the rescaled coset Virasoro generators $C_n \equiv 32L_n^{\text{coset}}$ [34], acting on tensor products of fundamental representations. The gauge parameters combine a fundamental vector $|\xi\rangle$ as the generic diffeomorphism parameter together with a two-index tensor which we denote as $\Sigma \equiv |\Sigma\rangle\langle\pi_\Sigma|$ and which is constrained in its second index as we specify below. The latter parameter is required for closure of the algebra, in analogy to a similar term in the E_8 exceptional field theory with three external dimensions [23,24]. This additional gauge transformation in (1.1) does not absorb the standard diffeomorphism acting on the highest weights components of the vector field $|V\rangle$, and will therefore only gauge away unphysical components of the generalized vielbein in the exceptional field theory. Generalized diffeomorphisms based on the infinite-dimensional Kac–Moody algebra $\mathfrak{e}_{11}$ have been proposed in [35] up to an unknown connection. The section constraint and the extra constrained gauge parameter Σ that we crucially need for the closure do not feature in the proposal of [35], whereas we believe that they will be needed for the closure of the algebra.

The transformations (1.1) close into an algebra, provided we impose the section constraint

$$\begin{aligned} \langle\partial_1|\otimes\langle\partial_2|(C_0-1+\sigma)&=0,\\ \langle\partial_1|\otimes\langle\partial_2|C_{-n}&=0, \qquad \forall\ n>0,\\ (\langle\partial_1|\otimes\langle\partial_2|+\langle\partial_2|\otimes\langle\partial_1|)C_1&=0. \end{aligned} \tag{1.2}$$

where σ is the operator that exchanges the two factors of the tensor product $\langle\partial_1|\otimes\langle\partial_2|$. This is a special case of a general expression for the section condition that applies in all extended geometries,

$$\langle\partial_1|\otimes\langle\partial_2|[-\eta_{AB}T^A\otimes T^B+(\lambda,\lambda)+\sigma-1]=0. \tag{1.3}$$

After a generalized Scherk–Schwarz reduction with an appropriate Ansatz for the gauge parameters $|\xi\rangle$ and Σ and the vector $|V\rangle$, the generalized diffeomorphisms (1.1) reduce to an algebraic action which precisely reproduces the gauge structure of two-dimensional gauged supergravity [30]. The section constraints above then imply the quadratic constraints on the two-dimensional embedding tensor.

Remarkably, the entire construction appears to make little use of the explicit structure of E_8 and its specific tensor identities, in marked contrast to the analogous constructions for the finite dimensional groups [14,23]. Rather, most of the consistency of the diffeomorphism algebra is a consequence of the underlying coset Virasoro symmetry. It is thus natural to expect that the present construction is not limited to the case of E_8 and its affine extension but naturally generalizes to other affine algebras. We show that this is indeed the case.

Section II reviews some basic facts about $\mathfrak{e}_9$ and its representations, including in particular some tensor products, and the construction of coset Virasoro generators. In Sec. III, we introduce coordinates and derivatives and deduce the form of the section constraint using the coset Virasoro generators. Generalized diffeomorphisms, including "extra" local $\mathfrak{e}_9$-transformations, are introduced in Sec. IV, and are shown to close when acting on vectors. Section V deals with the generalized Scherk–Schwarz reduction, and shows that the diffeomorphisms reproduce the correct structures, both for standard and non-Lagrangian gaugings, of two-dimensional gauged supergravity. In Sec. VI, it is first shown how our results are generalized to other affine algebras, and then how a completely general expression, valid for arbitrary Kac–Moody algebras and highest weight coordinate representations, for the generalized Lie derivative and the section constraint can be derived. We conclude with a summary and discussion of our results and indicate some questions for future research in Sec. VII.

II. E_9: ALGEBRA AND REPRESENTATIONS

Here we review the structure of the affine algebra $\mathfrak{e}_9$ and some useful facts about its representations. We denote by $\mathfrak{e}_9$ the centrally extended loop algebra over $\mathfrak{e}_8$, together with the derivation generator d. The generators are

$$\mathfrak{e}_9 = \langle T_m^A : A=1,\ldots,248, m\in\mathbb{Z}\rangle \oplus \mathbb{R}\mathsf{K}\oplus\mathbb{R}\mathsf{d}. \tag{2.1}$$

The first part is the loop algebra, K is the central element and d the derivation acting by $[\mathsf{d}, T_m^A] = -mT_m^A$. The remaining commutators are

$$[T_m^A, T_n^B] = f^{AB}{}_C T_{m+n}^C + \eta^{AB} m\delta_{m+n,0}\mathsf{K}, \tag{2.2}$$

with $\mathfrak{e}_8$ structure constants $f^{AB}{}_C$ and Killing metric η^{AB}, and where the standard normalization is used, so that $f^{AC}{}_D f^{BD}{}_C = 2g^\vee\eta^{AB} = 60\eta^{AB}$. The horizontal $\mathfrak{e}_8$ subalgebra of $\mathfrak{e}_9$ is generated by the T_0^A as usual.

The algebra $\mathfrak{e}_9$ admits highest and lowest weight representations. Highest weight representations $R(\Lambda)$ are labeled by a dominant integral weight $\sum_{i=0}^8 \ell^i\Lambda_i$ where the labels are those of figure 1 and are distinguished by their "level" k which is the eigenvalue of the generator K acting on the module. The level of $R(\Lambda)$ is $k=\sum_{i=0}^8 a_i\ell^i$, where a_i are the Coxeter labels $(a_0,\ldots,a_8)=(1,2,3,4,5,6,4,2,3)$. The leading states of $R(\Lambda)$ form the $\mathfrak{e}_8$ representation

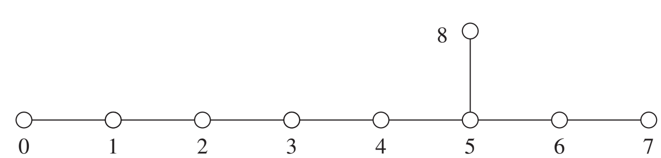

FIG. 1. The Dynkin diagram of $\mathfrak{e}_9$.

$r(\lambda)$ with highest weight $\lambda = \sum_{i=1}^{8} \ell^i \lambda_i$. Any dominant integral highest weight can be shifted by an arbitrary real amount $-h\delta$, where δ is the lowest positive null root of the affine algebra and dual to the derivation d. This means that the d eigenvalue on a weight $\Lambda = \sum_{i=0}^{8} \ell^i \Lambda_i - h\delta$ is h. The lowest weight module conjugate to the highest weight module $R(\Lambda)$ will be denoted by $\overline{R(\Lambda)}$.

At $k = 1$ there is (up to δ shifts) only one dominant weight $\Lambda_0 = (100000000)$ and the corresponding module is called the "basic" representation of $\mathfrak{e}_9$. By extrapolation of the coordinate representations of E_n exceptional geometries (see e.g. [14]), one would expect this to be the right representation for the E_9 coordinates and we will show from different angles that this is indeed the case. When one constructs the relevant invariant tensors used in the generalized diffeomorphisms and appearing in the section condition, it is important to have control over tensor products of highest weight states, especially $R(\Lambda_0)$'s.[1]

Using the affine grading of (2.2), the module $R(\Lambda_0)$ is generated by acting with the generators on an $\mathfrak{e}_8$ invariant scalar highest weight state $|0\rangle$ satisfying

$$T^A_n|0\rangle = 0, \qquad n \geq 0,$$
$$\mathsf{d}|0\rangle = 0, \qquad (\mathsf{K}-1)|0\rangle = 0. \tag{2.3}$$

The basic null states in the module appear as

$$\mathbb{P}_{(\mathbf{27000})}{}^{AB}{}_{CD} T^C_{-1} T^D_{-1}|0\rangle, \tag{2.4}$$

which is straightforward to verify using (2.2) and (2.3) together with the projection operators on the tensor product of two $\mathfrak{e}_8$ adjoint representations given in Appendix A, where also the dimensionalities of some $\mathfrak{e}_8$ representations are listed.

The first few levels of $R(\Lambda_0)$ are

$$\begin{aligned} R(\Lambda_0) = \mathbf{1}_0 &\oplus \mathbf{248}_{-1} \oplus (\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875})_{-2} \oplus (\mathbf{1} \oplus 2\cdot\mathbf{248} \oplus \mathbf{3875} \oplus \mathbf{30380})_{-3} \\ &\oplus (2\cdot\mathbf{1} \oplus 3\cdot\mathbf{248} \oplus 2\cdot\mathbf{3875} \oplus \mathbf{30380} \oplus \mathbf{27000} \oplus \mathbf{147250})_{-4} \\ &\oplus (2\cdot\mathbf{1} \oplus 5\cdot\mathbf{248} \oplus 3\cdot\mathbf{3875} \oplus 3\cdot\mathbf{30380} \oplus \mathbf{27000} \oplus \mathbf{147250} \oplus \mathbf{779247})_{-5} \oplus \ldots . \end{aligned} \tag{2.5}$$

The subscripts in the above equation refer to minus the number of times the lowering generator of node 0 where used. Equivalently, it is minus the eigenvalue of the operator d, and we refer to the subscript as "affine level." We shall sometimes denote isomorphic modules with shifted affine level h for the vacuum by $R(\Lambda_0)_{-h}$. They satisfy $\mathsf{d}|0\rangle = h|0\rangle$. The character for $R(\Lambda_0)$ that counts only affine level (where a term $c_n q^n$ corresponds to c_n states at level $-n$) has a remarkable form [38,39] in terms of the modular invariant function j:

$$\chi_{R(\Lambda_0)}(q) = (qj(q))^{1/3}, \tag{2.6}$$

and we discuss this Hilbert space in some more detail in appendix B.

In a grading with respect to the exceptional root (the simple root corresponding to node 8), the fundamental representation has the following expansion in terms of $\mathfrak{sl}(9)$ representations,

$$\begin{aligned} R(\Lambda_0) = (10000000)_0 &\oplus (00000010)_{-1} \oplus (00010000)_{-2} \oplus [(10000000) \oplus (01000001)]_{-3} \\ &\oplus [(00000010) \oplus (00000002) \oplus (10000100)]_{-4} \oplus [(00010000) \oplus (10100000) \oplus (00001001)]_{-5} \\ &\oplus [2(10000000) \oplus 2(01000001) \oplus (20000001) \oplus (00100010)]_{-6} \oplus \ldots, \end{aligned} \tag{2.7}$$

while the adjoint is

$$\begin{aligned} \mathrm{adj} = \bigoplus_{n\in\mathbb{Z}} [(00100000)_{3n+1} &\oplus (10000001)_{3n} \\ &\oplus (00000100)_{3n-1}] \oplus 2(00000000)_0. \end{aligned} \tag{2.8}$$

In these equations, the subscript is now given by minus the number of times the lowering generator of node 8 was used. Such a grading is suitable for analysing explicit solutions of the section condition. We will however mostly use the affine grading, mainly because it is better adapted to the Virasoro generators.

When dealing with representations of affine algebras, it is convenient to use their CFT or current algebra interpretation. The Sugawara construction [40] implies the presence of a Virasoro algebra, with generators

[1]Highest weight modules of affine Kac–Moody algebras are closed under the tensor product operation, since they belong to "category $\mathcal{O}$" [36], and tensor products are completely reducible but infinitely so, see also [37].

$$L_m^{(k)} = \frac{1}{2(k+g^\vee)} \sum_{n\in\mathbb{Z}} \eta_{AB} : T_n^A T_{m-n}^B : , \qquad (2.9)$$

and central charge $c_k = \frac{k \dim \mathfrak{g}}{k+g^\vee}$. The dual Coxeter number $g^\vee$ for $\mathfrak{e}_8$ is $g^\vee = 30$ and the colons refer to standard normal ordering moving positive mode numbers to the right. The Sugawara–Virasoro generators satisfy the commutation relations

$$[L_m^{(k)}, L_n^{(k)}] = (m-n)L_{m+n}^{(k)} + \frac{c_k}{12}(m^3 - m)\delta_{m+n,0}, \qquad (2.10)$$

and

$$[L_m^{(k)}, T_n^A] = -nT_{m+n}^A, \qquad (2.11)$$

with the loop algebra. Often, we will not write the level k superscript when the module is clear from the context in order to keep the notation light.

In an irreducible highest weight representation one can relate L_0 to the derivation operator d. The eigenvalue of L_0 is given by the Sugawara construction (2.9), where L_0 reduces to $\frac{1}{k+g^\vee}$ times the $\mathfrak{e}_8$ quadratic Casimir, whereas the eigenvalue of d on the highest weight state can be shifted regardless of the weight of the centrally extended loop algebra. (However, in the non-highest weight representation on the centrally extended loop algebra itself, L_0 and d act in the same way, and can be identified which each other in the further extension to the affine algebra).

At $k=1$, we have $c_1 = 8$, and the highest weight state has $h = 0$, where h is the L_0 eigenvalue. At $k = 2$, there are three irreducible highest weight representations, namely $R(2\Lambda_0)$, $R(\Lambda_7)$ and $R(\Lambda_1)$. The leading $\mathfrak{e}_8$ levels are

$$\begin{aligned} R(2\Lambda_0) &= \mathbf{1}_0 \oplus \mathbf{248}_{-1} \oplus (\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875} \oplus \mathbf{27000})_{-2} \oplus (\mathbf{1} \oplus 3\cdot\mathbf{248} \oplus \mathbf{3875} \oplus \mathbf{27000} \oplus 2\cdot\mathbf{30380} \oplus \mathbf{779247})_{-3} \oplus \ldots, \\ R(\Lambda_7) &= \mathbf{3875}_0 \oplus (\mathbf{248} \oplus \mathbf{3875} \oplus \mathbf{30380} \oplus \mathbf{147250})_{-1} \oplus \ldots, \\ R(\Lambda_1) &= \mathbf{248}_0 \oplus (\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875} \oplus \mathbf{30380})_{-1} \oplus (\mathbf{1} \oplus 3\cdot\mathbf{248} \oplus 2\cdot\mathbf{3875} \oplus \mathbf{27000} \oplus 2\cdot\mathbf{30380} \oplus \mathbf{147250} \oplus \mathbf{779247})_{-2} \\ &\quad \oplus (2\cdot\mathbf{1} \oplus 6\cdot\mathbf{248} \oplus 5\cdot\mathbf{3875} \oplus 3\cdot\mathbf{27000} \oplus 5\cdot\mathbf{30380} \oplus 3\cdot\mathbf{147250} \\ &\quad \oplus 3\cdot\mathbf{779247} \oplus \mathbf{2450240} \oplus \mathbf{4096000} \oplus \mathbf{6696000})_{-3} \oplus \ldots. \end{aligned} \qquad (2.12)$$

The value of the Virasoro central charge at $k=2$ is $c_2 = \frac{31}{2}$, so tensor products of two $R(\Lambda_0)$'s must also contain "compensating" Virasoro modules with $c = 2c_1 - c_2 = \frac{1}{2}$. This is within the minimal series [41] (with $m = 3$, the Ising model), where

$$c = 1 - \frac{6}{m(m+1)}, \qquad m = 3, 4, \ldots,$$
$$h_{r,s}^m = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \qquad r = 1, \ldots m-1, \qquad s = 1, \ldots r. \qquad (2.13)$$

We can easily read off the eigenvalue h of L_0 on the highest weight states of the three representations, since the values of the $\mathfrak{e}_8$ quadratic Casimir $C_2(r(\lambda))$, normalized to $g^\vee$ in the adjoint representation, can be calculated as $C_2(r(\lambda)) = \frac{1}{2}(\lambda, \lambda + 2\varrho)$. They are 0, 48, and 30 in the three representations $r(0) = \mathbf{1}$, $r(\lambda_7) = \mathbf{3875}$ and $r(\lambda_1) = \mathbf{248}$, leading to $h = 0, \frac{3}{2}$, and $\frac{15}{16}$, respectively. These values must be matched (see e.g. [42]) by the possible values of $h_{r,s}^3$, which are $h_{1,1}^3 = 0$, $h_{2,1}^3 = \frac{1}{2}$, $h_{2,2}^3 = \frac{1}{16}$. There is the possibility of shifting with an integer, since the eigenvalue of d on a highest weight state can be shifted. Conservation of h leads to possible matchings $0 = 0 + h_{1,1}^3$, $2 = \frac{3}{2} + h_{2,1}^3$, $1 = \frac{15}{16} + h_{1,1}^3$. This shows that the first appearances of $R(2\Lambda_0)$, $R(\Lambda_7)$ and $R(\Lambda_1)$ in $R(\Lambda_0) \otimes R(\Lambda_0)$ may be (with some integer multiplicity) at affine levels 0, -2 and -1, respectively. It thus suffices to check the tensor product to affine level -2 in order to establish the (integer) coefficients, which all turn out to be 1, such that [43]

$$\begin{aligned} R(\Lambda_0) \otimes R(\Lambda_0) = \mathrm{Vir}_{1,1}^3 \otimes R(2\Lambda_0)_0 \oplus \mathrm{Vir}_{2,1}^3 \otimes R(\Lambda_7)_{-3/2} \\ \oplus \mathrm{Vir}_{2,2}^3 \otimes R(\Lambda_1)_{-15/16}, \end{aligned} \qquad (2.14)$$

where $\mathrm{Vir}_{r,s}^m$ are Virasoro modules, keeping track of the repeated occurrence of the three representations in $R(\Lambda_0) \otimes R(\Lambda_0)$. It is also easily checked that the first two terms in (2.14) represent the symmetric product and the last one the antisymmetric product.

The corresponding Virasoro characters are

$$\begin{aligned} \chi_{1,1}^3 &= \frac{1}{2}\left(\frac{\phi(q)^2}{\phi(\sqrt{q})\phi(q^2)} + \frac{\phi(\sqrt{q})}{\phi(q)}\right) \\ &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8 + 5q^9 + 7q^{10} + 8q^{11} + 11q^{12} + 12q^{13} + 16q^{14} + 18q^{15} + 23q^{16} + O(q^{17}), \end{aligned}$$

$$\chi^3_{2,1} = \frac{1}{2}\left(\frac{\phi(q)^2}{\phi(\sqrt{q})\phi(q^2)} - \frac{\phi(\sqrt{q})}{\phi(q)}\right)$$
$$= \sqrt{q}(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 4q^7 + 5q^8 + 6q^9 + 8q^{10} + 9q^{11} + 12q^{12} + 14q^{13} + 17q^{14} + 20q^{15} + 25q^{16} + O(q^{17})),$$
$$\chi^3_{2,2} = \frac{q^{1/16}\phi(q^2)}{\phi(q)}$$
$$= q^{1/16}(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 + 10q^{10} + 12q^{11} + 15q^{12} + 18q^{13} + 22q^{14} + 27q^{15} + 32q^{16} + O(q^{17})), \tag{2.15}$$

where $\phi(q) = \prod_{n=1}^{\infty}(1 - q^n)$. Note the absence of states at level -1 in the first of these representations, which of course derives from the SL(2)-invariance of the highest weight state. This property will become important later. The characters satisfy

$$((\chi^3_{1,1})^2 - (\chi^3_{2,1})^2)\chi^3_{2,2} = q^{1/16}. \tag{2.16}$$

The coset Virasoro generators acting on (2.15) are given by

$$L_n^{\text{coset}} \equiv \mathbb{1} \otimes L_n^{(1)} + L_n^{(1)} \otimes \mathbb{1} - L_n^{(2)}, \tag{2.17}$$

in terms of the level 1 and level 2 Virasoro–Sugawara operators (2.9), as a particular case of the coset construction [34]. We will in the following often make use of the following rescaled coset Virasoro generators:

$$C_n \equiv 32 L_n^{\text{coset}} = 32(\mathbb{1} \otimes L_n^{(1)} + L_n^{(1)} \otimes \mathbb{1} - L_n^{(2)})$$
$$= \mathbb{1} \otimes L_n^{(1)} + L_n^{(1)} \otimes \mathbb{1} - \sum_{p\in\mathbb{Z}} \eta_{AB} T_p^A \otimes T_{n-p}^B. \tag{2.18}$$

A general coset Virasoro generator, acting on a tensor product of states at $k = k_1$ and $k = k_2$, is

$$L_n^{\text{coset}} = L_n^{(k_1)} \otimes \mathbb{1} + \mathbb{1} \otimes L_n^{(k_2)} - L_n^{(k_1+k_2)}$$
$$= \frac{1}{k_1 + k_2 + g^\vee} \times \left(L_n^{(k_1)} \otimes \mathsf{K} + \mathsf{K} \otimes L_n^{(k_2)} - \sum_{p\in\mathbb{Z}} \eta_{AB} T_p^A \otimes T_{n-p}^B\right)$$
$$\equiv -\frac{1}{k_1 + k_2 + g^\vee} \eta_{(n)\mathcal{AB}} T^{\mathcal{A}} \otimes T^{\mathcal{B}}, \tag{2.19}$$

where the indices $\mathcal{A}$, $\mathcal{B}$ in the last expression run over the semidirect sum of the centrally extended loop algebra with the full Virasoro algebra (although in this case the expression is zero whenever $\mathcal{A}$ or $\mathcal{B}$ corresponds to a Virasoro generator different from L_n), and the (noninvertible) bilinear forms $\eta_{(n)\mathcal{AB}}$ are defined by this equation and invariant under the loop algebra of $\mathfrak{e}_9$. For $n = 0$ we get the standard invariant form on $\mathfrak{e}_9$ (if we identify L_0 with d).

By construction the operators (2.18) satisfy the Virasoro algebra up to a factor of 32 and for central charge $\frac{1}{2}$. What will be important in the following is the algebra they satisfy when acting on different $R(\Lambda_0) \otimes R(\Lambda_0)$ subspaces of the level 3 tensor product $R(\Lambda_0) \otimes R(\Lambda_0) \otimes R(\Lambda_0)$. Let us consider the action of (2.18) on two factors of this triple tensor product, where we use the notation

$$\overset{12}{C}_n \equiv -\eta_{(n)\mathcal{AB}} T^{\mathcal{A}} \otimes T^{\mathcal{B}} \otimes \mathbb{1},$$
$$\overset{13}{C}_n \equiv -\eta_{(n)\mathcal{AB}} T^{\mathcal{A}} \otimes \mathbb{1} \otimes T^{\mathcal{B}}, \quad \text{etc.}. \tag{2.20}$$

Straightforward computation then shows the following structure

$$[\overset{13}{C}_m, \overset{23}{C}_n] = \frac{m-n}{2}(\overset{13}{C}_{m+n} + \overset{23}{C}_{m+n} - \overset{12}{C}_{m+n}) + \frac{2}{3}m(m^2-1)\delta_{m+n,0} + \overset{123}{C}_{m+n}, \tag{2.21}$$

where the last operator is defined as

$$\overset{123}{C}_m = \sum_{p,q\in\mathbb{Z}} f_{ABC} T_p^A \otimes T_q^B \otimes T_{m-p-q}^C + \sum_{p\in\mathbb{Z}}\left(\frac{m}{2} - p\right)\eta_{AB}(\mathbb{1} \otimes T_p^A \otimes T_{m-p}^B + T_{m-p}^A \otimes \mathbb{1} \otimes T_p^B + T_p^A \otimes T_{m-p}^B \otimes \mathbb{1}), \tag{2.22}$$

and completely antisymmetric under the exchange of the three spaces. It can be written in compact form as

$$\overset{123}{C}_m = f^{\mathcal{AB}}{}_{\mathcal{C}} \eta_{(m)\mathcal{A}[\mathcal{D}} \eta_{(0)\mathcal{E}]\mathcal{B}} T^{\mathcal{D}} \otimes T^{\mathcal{E}} \otimes T^{\mathcal{C}} \tag{2.23}$$

with the bilinear forms $\eta_{(n)\mathcal{AB}}$ from (2.19) and structure constants $f^{\mathcal{AB}}{}_{\mathcal{C}}$ combining (2.2), (2.10), (2.11). Using its antisymmetry one can show the following relations between commutators

$$[\overset{13}{C}_m, \overset{23}{C}_n] - [\overset{12}{C}_m, \overset{13}{C}_n] = (m-n)(\overset{23}{C}_{m+n} - \overset{12}{C}_{m+n}), \tag{2.24}$$

$$[\overset{23}{C}_m, \overset{12}{C}_n + \overset{13}{C}_n] = (m-n)\overset{23}{C}_{m+n} + \frac{4}{3}m(m^2-1)\delta_{m+n,0}, \tag{2.25}$$

$$[\overset{13}{C}_n, \overset{23}{C}_{m-n}] - [\overset{13}{C}_p, \overset{23}{C}_{m-p}] = (n-p)(\overset{13}{C}_m + \overset{23}{C}_m - \overset{12}{C}_m) + \frac{2}{3}(n(n^2-1) - p(p^2-1))\delta_{m,0}, \tag{2.26}$$

which will be useful in the following.

III. COORDINATES AND SECTION CONSTRAINT

By extrapolation from the systematics of the coordinate representation for E_n one expects that the internal coordinates of E_9 exceptional field theory should transform in the fundamental representation $R(\Lambda_0)$ of $\mathfrak{e}_9$ [30–32]. We proceed with this assumption and denote the coordinates as Y^M. As for the finite-dimensional groups, consistency of the theory should require a section constraint that eliminates the dependence of fields on all but the physical coordinates. Derivatives ∂_M transform in the dual $\overline{R(\Lambda_0)}$ of the fundamental representation that decomposes in analogy with (2.5) according to

$$\overline{R(\Lambda_0)} = \mathbf{1}_0 \oplus \mathbf{248}_1 \oplus (\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875})_2 \oplus \ldots \tag{3.1}$$

under $\mathfrak{e}_8 \subset \mathfrak{e}_9$.

The section constraint is expected to be bilinear in derivatives, i.e., to lie in the tensor product of $\overline{R(\Lambda_0)} \otimes \overline{R(\Lambda_0)}$ which can be decomposed in analogy to (2.14). The possible projectors onto $\mathfrak{e}_9$ representations within this tensor product are naturally expressed in terms of the coset Virasoro generators defined in (2.17). Our Ansatz for the (strong) $\mathfrak{e}_9$ section constraint is

$$\langle\partial_1| \otimes \langle\partial_2|(C_0 - 1 + \sigma) = 0. \tag{3.2}$$

Here and in the following we use a notation in which the fundamental representation and its dual are represented by ket- and bra-vectors, respectively. In particular, derivatives ∂_M are seen as bra-states in lowest-weight modules at $k=-1$. Subscripts $_{1,2}$ on the derivatives indicate that these derivatives may act on different objects. The operator C_0 is the rescaled coset Virasoro generator from (2.18), and σ denotes the permutation operator on a tensor product

$$\langle\partial_1| \otimes \langle\partial_2|\sigma \equiv \langle\partial_2| \otimes \langle\partial_1|. \tag{3.3}$$

We will show below that the Ansatz (3.2) is compatible with the expected solutions of the section constraint. As a first check, let us verify that (3.2) indeed reproduces the section constraints from three-dimensional E_8 exceptional field theory upon proper embedding. Comparing the coordinates to E_8 exceptional field theory with three external dimensions, we expect the lowest singlet $\mathbf{1}_0$ in the level decomposition (2.5) to correspond to the singlet in the $3 \to 2+1$ decomposition of external dimensions while the adjoint $\mathbf{248}_{-1}$ on the first level should correspond to the internal coordinates of E_8 exceptional field theory. Restricting coordinates to these two lowest levels, i.e., assuming

$$\langle\partial| = \langle 0|(\partial_0 + T_1^A\partial_A), \tag{3.4}$$

we can then evaluate the constraint (3.2) as

$$\begin{aligned} 0 &= \langle\partial_1| \otimes \langle\partial_2|(C_0 - 1 + \sigma) \\ &= \langle 0| \otimes \langle 0|\partial_{1A}\partial_{2B}(\Pi^{AB}{}_{CD}T_1^C \otimes T_1^D \\ &\quad - T_1^A T_1^B \otimes \mathbb{1} - \mathbb{1} \otimes T_1^B T_1^A), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \Pi^{AB}{}_{CD} &\equiv 2\delta_C^{(A}\delta_D^{B)} - f^A{}_{CE}f^{EB}{}_D \\ &= 14(\mathbb{P}_{\mathbf{3875}})^{AB}{}_{CD} + 4\eta^{AB}\eta_{CD} - 2f^{AB}{}_E f^E{}_{CD} \end{aligned} \tag{3.6}$$

is given as a linear combination of projectors onto the $\mathbf{1}, \mathbf{248}$ and $\mathbf{3875}$, cf. (A2). Using the property that $\langle 0|T_1^A T_1^B$ is only nonzero for (AB) in the $\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875}$, cf. (2.4), one recovers the E_8 section constraint [23]

$$\partial_A \otimes \partial_B(\mathbb{P}_{\mathbf{1}} + \mathbb{P}_{\mathbf{248}} + \mathbb{P}_{\mathbf{3875}})^{AB}{}_{CD} = 0. \tag{3.7}$$

In turn, one observes that with derivatives ∂_A constrained by (3.7), the tensor product of two derivatives (3.4) is exclusively contained in the leading $\overline{R(2\Lambda_0)}_0$ and the leading $\overline{R(\Lambda_1)}_1$ in the expansion (dual to) (2.14). The full $\mathfrak{e}_9$ section condition is then expected to be equivalent to the vanishing of the remaining (infinite number of) irreducible representations in $\overline{R(\Lambda_0)} \otimes \overline{R(\Lambda_0)}$, among them all $\overline{R(\Lambda_7)}$'s. As a simple consequence of the grading, all L_m^{coset}, $m<0$ vanish when acting on products of (3.4), so they may be included in the (conjugate) section condition "for free". Moreover, the absence of level -1 states in $\text{Vir}^3_{1,1}$, cf. (2.15), then implies that also C_1 annihilates these products. Together, we arrive at the following proposal for the $\mathfrak{e}_9$ section constraints

$$\langle\partial_1| \otimes \langle\partial_2|(C_0 - 1 + \sigma) = 0, \tag{3.8a}$$

$$\langle\partial_1| \otimes \langle\partial_2|C_{-n} = 0, \quad \forall\ n > 0, \tag{3.8b}$$

$$(\langle\partial_1| \otimes \langle\partial_2| + \langle\partial_2| \otimes \langle\partial_1|)C_1 = 0, \tag{3.8c}$$

which correctly reproduces the $D = 3$, E_8 section constraint. Moreover, we show in Sec. VI B that (3.4) satisfying (3.7) is the unique solution to (3.8) up to conjugation in E_9.

There can be different definitions of E_9, in particular for the space of functions defining the loop group. The proof of Sec. VI B uses the definition of a Kac–Moody group of [44] that corresponds in the affine case to taking the loop group of meromorphic functions in E_8 with poles at zero and infinity only. It follows by iterations that the maximal vector spaces in $R(\Lambda)$ of solutions to (3.8) are E_9 conjugate to the expected type IIB and eleven-dimensional supergravity solutions. The latter can be seen explicitly in the $\mathfrak{sl}(9)$ level decomposition (2.7) of the coordinate representation, for which a solution to the section constraints (3.8) is given by restricting the coordinate dependence to the $\mathfrak{sl}(9)$ vector on the lowest level, which corresponds to the nine coordinates that allow to embed the full eleven-dimensional supergravity in exceptional field theory.

Although the constraints in (3.8) are independent as algebraic equations, already the symmetric part of (3.8a) is sufficient to imply that they are all satisfied. There is no clear consensus in the literature about what is to be called a section constraint (except that it should be strong enough). Sometimes, the complement to $\overline{R(2\Lambda_0)}$ in the symmetric product $\overline{R(\Lambda_0)} \otimes_s \overline{R(\Lambda_0)}$ is taken as the constraint. This is suitable in the context of e.g. the tensor hierarchy algebra [32,45–47]. Here, we choose to include all representations that vanish in the section, also antisymmetric ones.

In addition to reproducing the expected physical solutions, the main and defining characteristics of the proper section constraints is the fact that they should guarantee closure of the algebra of generalized diffeomorphisms. This is what we will show in the next section.

IV. GENERALIZED DIFFEOMORPHISMS

Having identified a reasonable set of section constraints (3.8), we will now establish the algebra of generalized diffeomorphisms. For the finite-dimensional groups, the generic action of a generalized diffeomorphism on a vector field is of the form [12,14]

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M + Z^{MN}{}_{PQ} \partial_N \xi^P V^Q, \tag{4.1}$$

with an invariant tensor $Z^{MN}{}_{PQ}$ which up to a possible weight term is built from the projector onto the adjoint representation

$$Z^{MN}{}_{PQ} = -\alpha \mathbb{P}^M{}_Q{}^N{}_P + \beta \delta_P{}^N \delta_Q{}^M, \tag{4.2}$$

and is unique up to two constants α and β. With a vector field we mean a vector that could be a gauge transformation parameter ξ; we do not consider vectors of different weight. For $\mathfrak{e}_9$, the natural candidate for this tensor is thus given by

$$Z^{MN}{}_{PQ} = \alpha \left(\sum_{n\in\mathbb{Z}} \eta_{AB} (T^A_n)^M{}_Q (T^B_{-n})^N{}_P - \delta^M{}_Q (L_0)^N{}_P - (L_0)^M{}_Q \delta^N{}_P \right) + \beta \delta^M{}_Q \delta^N{}_P. \tag{4.3}$$

It is important that $Z^{MN}{}_{PQ}$ (up to a possible scaling) is $\mathfrak{e}_9$ valued in the pairs ${}^M{}_Q$ and ${}^N{}_P$. In the following we will often turn to an index-free notation in which (4.3) takes the compact form

$$Z = \sigma(-\alpha C_0 + \beta), \tag{4.4}$$

with the permutation operator σ from (3.3) and the rescaled coset Virasoro generator C_0 from (2.18). The coefficients α, β are usually determined from closure of the algebra of transformations (4.1), for which a crucial role is played by the fact that the section constraint of the theory ensures the vanishing of [14]

$$\langle \partial_1 | \otimes \langle \partial_2 | Y = 0, \tag{4.5}$$

for the tensor $Y \equiv Z + 1$, i.e., Y has to be a linear combination of projections on irreducible representations in the section condition. In the present case this will be an infinite number of representations. Comparing (4.4) and (4.5) to the section constraints (3.8) identified in the previous section, we read off the values $\alpha = \beta = -1$, for which

$$Y = \sigma(C_0 + \sigma - 1). \tag{4.6}$$

In particular, this implies that the canonical weight of a vector is $\beta = -1$. With "canonical weight" (sometimes also called "distinguished weight" in the literature) we mean the weight of the gauge parameter ξ. For E_d exceptional field theory it is $\beta = \frac{1}{9-d}$, which would diverge for $d = 9$, but we shall see in Sec. VI B that the appropriate definition for the highest weight coordinate module $R(\lambda)$ is $\beta = (\lambda, \lambda) - 1$ that gives indeed $\beta = -1$ for E_9. A canonical co-vector (like e.g. a derivative) then has weight $\beta = +1$.

The tensors Z and Y (and thus the section constraint) can also be derived from extensions of $\mathfrak{e}_9$ in the same was as in [48] for finite-dimensional $\mathfrak{e}_d$. These extensions are the Lie algebra $\mathfrak{e}_{10}$ and a Lie superalgebra of Borcherds type, giving the antisymmetric and symmetric parts of Y, respectively. In both cases the algebra is obtained by adding a node to the Dynkin diagram of $\mathfrak{e}_9$ ("white" or "gray"), and d can be identified with the Cartan generator corresponding to this additional node.

In the index-free notation, the generalized diffeomorphism (4.1) now reads

$$\mathcal{L}_\xi |V\rangle = \langle \partial_V | \xi \rangle |V\rangle + \langle \partial_\xi | (C_0 - 1) | \xi \rangle \otimes |V\rangle, \tag{4.7}$$

where the subscript on the derivatives indicate what they act on, e.g.

$$\langle \partial_V | \otimes |V\rangle \otimes |\xi\rangle = \left(\left\langle \frac{\partial}{\partial Y} \right| \otimes |V(Y)\rangle \right) \otimes |\xi(Y)\rangle. \quad (4.8)$$

Specifically, our index-free conventions are such that for a tensor product one understands the bra and the ket states to be ordered from left to right, such that for example

$$|\overset{2}{W}\rangle = \langle \overset{1}{\omega} | \overset{12}{X} | \overset{1}{\xi} \rangle \otimes |\overset{2}{V}\rangle \Leftrightarrow |W\rangle = \langle \omega | X | \xi \rangle \otimes |V\rangle, \quad (4.9)$$

corresponding to the following expression in indices

$$W^M = \omega_N X^N{}_P{}^M{}_Q \xi^P V^Q. \quad (4.10)$$

Similarly, the labels on the states will be avoided in expressions of the type

$$|\overset{3}{W}\rangle = \langle \overset{1}{\omega} | \otimes \langle \overset{2}{v} | \overset{12}{X} \overset{23}{Y} | \overset{1}{\xi} \rangle \otimes |\overset{2}{\eta}\rangle \otimes |\overset{3}{V}\rangle$$
$$\Leftrightarrow |W\rangle = \langle \omega | \otimes \langle v | \overset{12}{X} \overset{23}{Y} | \xi \rangle \otimes |\eta\rangle \otimes |V\rangle, \quad (4.11)$$

corresponding to the following expression in indices

$$W^M = \omega_N v_P X^N{}_Q{}^P{}_R Y^R{}_S{}^M{}_T \xi^Q \eta^S V^T. \quad (4.12)$$

Having set up the notation, let us come back to the generalized diffeomorphisms (4.7). It comes as no surprise that the transformations (4.7) do not close into an algebra. This is the case already for the generalized diffeomorphisms associated with the algebra $\mathfrak{e}_8$ and it can be seen as a manifestation of the fact that in three dimensions dual gravity degrees of freedom become part of the scalar sector [12,14]. Yet, in this case a consistent symmetry algebra can be defined upon enlarging (4.1) by local algebra-valued rotations with constrained gauge parameters [23,49]. The generic pattern in exceptional field theories for $\mathfrak{e}_d$ (i.e., with $11-d$ external dimensions) is the appearance of additional covariantly constrained $(9-d)$-forms in the dual fundamental representation. For E_8 exceptional field theory these are the gauge fields whose associated gauge transformations are required for closure of the diffeomorphism algebra. For E_9 exceptional field theory in contrast, one expects additional fields to appear among the scalar fields, i.e. its scalar sector should carry not only a group valued matrix $\mathcal{M}_{MN}$ but also 0-forms of type χ_M algebraically constrained by the section constraints (3.8). In the gauge sector we then expect vector fields $A_\mu{}^M$ in the fundamental representation together with two-index gauge fields $B_\mu{}^N{}_M$ algebraically constrained in its last index according to the section constraints. Their associated gauge transformations with parameter $\Sigma^N{}_M$ are then responsible for closure of the full diffeomorphism algebra. Fields of the same two-index structure appear in E_8 exceptional field theory among the two-forms and are required in order to close the algebra of gauge transformations and supersymmetry on the vector fields [23,50].

In index-free notation, we will denote the new gauge parameter as

$$\Sigma^N{}_M : |\Sigma\rangle\langle \pi_\Sigma|, \quad (4.13)$$

to keep track of its two-index nature (keeping in mind that in general this matrix is not factorized). The constrained nature of its first index is then expressed via (3.8) as

$$\langle \partial | \otimes \langle \pi_\Sigma | (C_0 - 1 + \sigma) = 0,$$
$$\langle \partial | \otimes \langle \pi_\Sigma | C_{-n} = 0, \quad \forall\ n > 0,$$
$$(\langle \partial | \otimes \langle \pi_\Sigma | + \langle \pi_\Sigma | \otimes \langle \partial |) C_1 = 0. \quad (4.14)$$

Combining (4.7) with the new gauge transformations, we arrive at the following definition for a generalized diffeomorphism,

$$\mathcal{L}_{\xi,\Sigma} |V\rangle = \langle \partial_V | \xi \rangle |V\rangle + \langle \partial_\xi | (C_0 - 1) | \xi \rangle \otimes |V\rangle$$
$$+ \langle \pi_\Sigma | C_{-1} | \Sigma \rangle \otimes |V\rangle, \quad (4.15)$$

with gauge parameters given by a vector field $|\xi\rangle$ and a constrained tensor $|\Sigma\rangle\langle\pi_\Sigma|$ constrained by (4.14). The last term in (4.15) carries the coset Virasoro generator C_{-1} from (2.18), such that it maps the $R(\Lambda_0)$ module to the isomorphic module with an L_0 spectrum shifted by 1, so that

$$\mathsf{d} |\Sigma\rangle\langle\pi_\Sigma| = (L_0 + 1) |\Sigma\rangle\langle\pi_\Sigma|, \qquad |\Sigma\rangle\langle\pi_\Sigma| \mathsf{d} = |\Sigma\rangle\langle\pi_\Sigma| L_0, \quad (4.16)$$

with L_0 being the Sugawara–Virasoro operator (2.9). The weight of the gauge parameter $|\Sigma\rangle$ is 0 in contrast to $|\xi\rangle$ that has weight 1, such that in overall $|\Sigma\rangle\langle\pi_\Sigma|$ has weight -1. More generally, one computes that the operator C_n acting on the product of two vectors $|V\rangle$ and $|W\rangle$ of canonical weight -1 shifts the weight from -2 to $n-2$:

$$C_n \mathcal{L}_\xi (|V\rangle \otimes |W\rangle) = \langle \partial_V + \partial_W | \xi \rangle C_n |V\rangle \otimes |W\rangle + \langle \partial_\xi | \overset{23}{C}_n (\overset{12}{C}_0 + \overset{13}{C}_0 - 2) | \xi \rangle \otimes |V\rangle \otimes |W\rangle$$
$$= \langle \partial_V + \partial_W | \xi \rangle C_n |V\rangle \otimes |W\rangle + \langle \partial_\xi | ([\overset{23}{C}_n, \overset{12}{C}_0 + \overset{13}{C}_0] + (\overset{12}{C}_0 + \overset{13}{C}_0 - 2) \overset{23}{C}_n) | \xi \rangle \otimes |V\rangle \otimes |W\rangle$$
$$= \langle \partial_V + \partial_W | \xi \rangle C_n |V\rangle \otimes |W\rangle + \langle \partial_\xi | (\overset{12}{C}_0 + \overset{13}{C}_0 + n - 2) | \xi \rangle \otimes (C_n |V\rangle \otimes |W\rangle), \quad (4.17)$$

where we have made use of (2.25). We recall that the weight appears in the generalized Lie derivative (4.7) as the integral shift of C_0.

Note that in order to view the extra local rotations in the last term in (4.15) as an element "in the algebra," the centrally extended loop algebra has to be supplemented by L_{-1}. This extension is (up to a sign convention) the symmetry algebra $\mathfrak{G}$ used in [30] to describe the structure of gauged supergravity in two dimensions, which we will rederive from the generalized diffeomorphisms (4.15) in Sec. V. Moreover, it agrees precisely with the level zero content of the tensor hierarchy algebra corresponding to $\mathfrak{e}_9$, as defined in [46] for general $\mathfrak{e}_d$. In general there is an additional highest weight module of generators, which reduces to the single element L_{-1} for $d = 9$.

Before we address the closure of the algebra of transformations (4.15), let us spell out the action on a covector of canonical weight

$$\mathcal{L}_{\xi,\Sigma}\langle\omega| = \langle\partial_\omega|\xi\rangle\langle\omega| - \langle\partial_\xi| \otimes \langle\omega|(C_0 - 1)|\xi\rangle - \langle\pi_\Sigma| \otimes \langle\omega|C_{-1}|\Sigma\rangle, \tag{4.18}$$

and note that if the covector $\langle\omega|$ is constrained by the section constraint, such as the gauge parameter $\langle\pi_\Sigma|$ in (4.14), it follows directly that

$$\mathcal{L}_{\xi,\Sigma}\langle\omega| = \langle\partial_\omega|\xi\rangle\langle\omega| + \langle\omega|\xi\rangle\langle\partial_\xi|, \tag{4.19}$$

i.e., also its resulting Lie derivative is constrained, and reduces to the ordinary Lie derivative.

Let us now check that the algebra of generalized diffeomorphisms (4.15) closes. As a first step we compute the obstruction to the closure of the pure Lie derivative $\mathcal{L}_{\xi,0} = \mathcal{L}_\xi$. We thus calculate $([\mathcal{L}_\xi, \mathcal{L}_\eta] - \mathcal{L}_{[\![\xi,\eta]\!]})|V\rangle$, where $[\![\xi,\eta]\!] \equiv \frac{1}{2}(\mathcal{L}_\xi\eta - \mathcal{L}_\eta\xi)$. For $\mathfrak{e}_d$ with $d \leq 7$, this difference is 0, and for $d = 8$ it gives the "extra" local $\mathfrak{e}_8$ transformation [23,24]. Let us go through the different types of terms arising. The terms with two derivatives on $|V\rangle$ vanish trivially (due to antisymmetry under $\xi \leftrightarrow \eta$). The terms with one derivative on $|V\rangle$ become (here, antisymmetry between the parameters is implicit)

$$-\langle\partial_\xi| \otimes \langle\partial_V|(\overset{12}{C}_0 + \overset{12}{\sigma} - 1)|\xi\rangle \otimes |\eta\rangle \otimes |V\rangle, \tag{4.20}$$

which vanishes thanks to the section condition (the superscripts on C_0 and σ indicate which pair of positions it acts on).

The terms without derivatives on $|V\rangle$ come in two groups, either the two derivatives act on different parameters or on the same. When the two derivatives act on different gauge parameters, one obtains

$$\begin{aligned}&\frac{1}{2}\langle\partial_\xi| \otimes \langle\partial_\eta|(2(\overset{13}{C}_0 - 1)(\overset{23}{C}_0 - 1) - (\overset{23}{C}_0 - 1)(\overset{12}{C}_0 - 1) \\ &\quad - \overset{12}{\sigma}(\overset{23}{C}_0 - 1))|\xi\rangle \otimes |\eta\rangle \otimes |V\rangle - (\eta \leftrightarrow \xi) \\ &= \frac{1}{2}\langle\partial_\xi| \otimes \langle\partial_\eta|[\overset{12}{C}_0 + \overset{13}{C}_0, \overset{23}{C}_0]|\xi\rangle \otimes |\eta\rangle \otimes |V\rangle - (\eta \leftrightarrow \xi) \\ &= 0,\end{aligned} \tag{4.21}$$

where we have used the section constraint (3.8a) to reexpress $\overset{12}{\sigma}$ and used (2.25). The terms with both derivatives on the same gauge parameter are the only nonvanishing ones and can be arranged as

$$\begin{aligned}\Delta_{\xi,\eta}|V\rangle &\equiv ([\mathcal{L}_\xi, \mathcal{L}_\eta] - \mathcal{L}_{[\![\xi,\eta]\!]})|V\rangle \\ &= \frac{1}{2}\langle\partial_\eta| \otimes \langle\partial_\eta|(-\overset{13}{C}_0 + \overset{23}{C}_0 - \overset{123}{C}_0)|\xi\rangle \otimes |\eta\rangle \otimes |V\rangle \\ &\quad + \frac{1}{2}\langle\partial_\xi| \otimes \langle\partial_\xi|(-\overset{13}{C}_0 + \overset{23}{C}_0 - \overset{123}{C}_0)|\xi\rangle \otimes |\eta\rangle \otimes |V\rangle.\end{aligned} \tag{4.22}$$

Now we shall show that this variation can be absorbed in a transformation of the type (4.15) with a constrained gauge parameter $|\Sigma\rangle\langle\pi_\Sigma|$. Using the identity (2.21) one shows that

$$-\overset{13}{C}_0 + \overset{23}{C}_0 - \overset{123}{C}_0 = \frac{1}{2}[\overset{13}{C}_{-1} - \overset{23}{C}_{-1}, \overset{12}{C}_1]. \tag{4.23}$$

Substituting this into $\Delta_{\xi,\eta}|V\rangle$ one finds that the term of the commutator with $\overset{12}{C}_1$ on the left vanishes according to the section constraint (3.8c), such that the result can be written as

$$\begin{aligned}\Delta_{\xi,\eta}|V\rangle &= \frac{1}{4}\langle\partial_\eta|C_{-1}(\langle\partial_\eta|C_1|\eta\rangle \otimes |\xi\rangle \\ &\quad - \langle\partial_\eta|C_1|\xi\rangle \otimes |\eta\rangle) \otimes |V\rangle \\ &\quad + \frac{1}{4}\langle\partial_\xi|C_{-1}(\langle\partial_\xi|C_1|\eta\rangle \otimes |\xi\rangle \\ &\quad - \langle\partial_\xi|C_1|\xi\rangle \otimes |\eta\rangle) \otimes |V\rangle.\end{aligned} \tag{4.24}$$

We thus obtain closure of pure Lie derivatives into full generalized diffeomorphisms (4.15) with the additional gauge parameter given by

$$\begin{aligned}|\Sigma\rangle\langle\pi_\Sigma| &\equiv \frac{1}{4}\langle\partial_\eta|C_1|(|\eta\rangle \otimes |\xi\rangle - |\xi\rangle \otimes |\eta\rangle)\langle\partial_\eta| \\ &\quad + \frac{1}{4}\langle\partial_\xi|C_1|(|\eta\rangle \otimes |\xi\rangle - |\xi\rangle \otimes |\eta\rangle)\langle\partial_\xi|.\end{aligned} \tag{4.25}$$

Note that this is manifestly constrained in its last index since the bra components are all partial derivatives.

Next, we need to check that also the commutator of both kinds of transformations in (4.15) closes into a gauge transformation

$$\begin{aligned}[\mathcal{L}_{\xi,0},\mathcal{L}_{0,\Sigma}]|V\rangle &= \langle\partial_\Sigma+\partial_V|\xi\rangle\langle\pi_\Sigma|C_{-1}|\Sigma\rangle\otimes|V\rangle+\langle\partial_\xi|\otimes\langle\pi_\Sigma|(\overset{13}{C}_0-1)\overset{23}{C}_{-1}|\xi\rangle\otimes|\Sigma\rangle\otimes|V\rangle\\ &\quad-\langle\partial_V|\xi\rangle\langle\pi_\Sigma|C_{-1}|\Sigma\rangle\otimes|V\rangle-\langle\partial_\xi|\otimes\langle\pi_\Sigma|\overset{23}{C}_{-1}(\overset{13}{C}_0-1)|\xi\rangle\otimes|\Sigma\rangle\otimes|V\rangle\\ &=\langle\pi_\Sigma|C_{-1}(\langle\partial_\Sigma|\xi\rangle|\Sigma\rangle)\otimes|V\rangle+\langle\partial_\xi|\otimes\langle\pi_\Sigma|[\overset{13}{C}_0,\overset{23}{C}_{-1}]|\xi\rangle\otimes|\Sigma\rangle\otimes|V\rangle. \end{aligned}\tag{4.26}$$

We then use (2.25) on the last term

$$\begin{aligned}\langle\partial_\xi|\otimes\langle\pi_\Sigma|[\overset{13}{C}_0,\overset{23}{C}_{-1}]|\xi\rangle\otimes|\Sigma\rangle\otimes|V\rangle &= \langle\partial_\xi|\otimes\langle\pi_\Sigma|\left([\overset{23}{C}_{-1},\overset{12}{C}_0]+\overset{23}{C}_{-1}\right)|\xi\rangle\otimes|\Sigma\rangle\otimes|V\rangle\\ &=\langle\partial_\xi|\otimes\langle\pi_\Sigma|\left(\overset{23}{C}_{-1}\overset{12}{C}_0-(\overset{12}{C}_0-1+\overset{12}{\sigma})\overset{23}{C}_{-1}+\overset{12}{\sigma}\overset{23}{C}_{-1}\right)|\xi\rangle\otimes|\Sigma\rangle\otimes|V\rangle\\ &=\langle\partial_\xi|\otimes\langle\pi_\Sigma|\left(\overset{23}{C}_{-1}\overset{12}{C}_0+\overset{12}{\sigma}\overset{23}{C}_{-1}\right)|\xi\rangle\otimes|\Sigma\rangle\otimes|V\rangle,\end{aligned}\tag{4.27}$$

where we used the section constraint (3.8a) in the last step. Together, we obtain

$$\begin{aligned}[\mathcal{L}_{\xi,0},\mathcal{L}_{0,\Sigma}]|V\rangle &= \langle\pi_\Sigma|C_{-1}(\langle\partial_\Sigma|\xi\rangle|\Sigma\rangle\\ &\quad+\langle\partial_\xi|C_0|\xi\rangle\otimes|\Sigma\rangle)\otimes|V\rangle\\ &\quad+\langle\partial_\xi|C_{-1}(\langle\pi_\Sigma|\xi\rangle|\Sigma\rangle)\otimes|V\rangle,\end{aligned}\tag{4.28}$$

which indeed gives a gauge transformation with parameter equal to the Lie derivative of the gauge parameter $|\Sigma\rangle\langle\pi_\Sigma|$:

$$\begin{aligned}\mathcal{L}_\xi(|\Sigma\rangle\langle\pi_\Sigma|) &= \langle\partial_\Sigma|\xi\rangle|\Sigma\rangle\langle\pi_\Sigma|+\langle\partial_\xi|C_0(|\xi\rangle\otimes|\Sigma\rangle)\langle\pi_\Sigma|\\ &\quad+|\Sigma\rangle\langle\pi_\Sigma|\xi\rangle\langle\partial_\xi|,\end{aligned}\tag{4.29}$$

cf., (4.19). We recall that the weight of $|\Sigma\rangle\langle\pi_\Sigma|$ is shifted due to (4.16), explaining the absence of the -1 in the C_0 term in the Lie derivative.

As a last step we consider the commutator of two Σ gauge transformations. Two successive Σ transformations give

$$\mathcal{L}_{0,\Sigma_1}\mathcal{L}_{0,\Sigma_2}|V\rangle=\langle\pi_{\Sigma_1}|\otimes\langle\pi_{\Sigma_2}|\overset{13}{C}_{-1}\overset{23}{C}_{-1}|\Sigma_1\rangle\otimes|\Sigma_2\rangle\otimes|V\rangle,\tag{4.30}$$

so that their commutator is

$$\begin{aligned}[\mathcal{L}_{0,\Sigma_1},\mathcal{L}_{0,\Sigma_2}]|V\rangle &= \langle\pi_{\Sigma_1}|\otimes\langle\pi_{\Sigma_2}|[\overset{13}{C}_{-1},\overset{23}{C}_{-1}]|\Sigma_1\rangle\otimes|\Sigma_2\rangle\otimes|V\rangle\\ &=\langle\pi_{\Sigma_1}|\otimes\langle\pi_{\Sigma_2}|[\overset{12}{C}_{-1},\overset{13}{C}_{-1}]|\Sigma_1\rangle\otimes|\Sigma_2\rangle\otimes|V\rangle\\ &=-\langle\pi_{\Sigma_1}|\otimes\langle\pi_{\Sigma_2}|\overset{13}{C}_{-1}\overset{12}{C}_{-1}|\Sigma_1\rangle\otimes|\Sigma_2\rangle\otimes|V\rangle\\ &=\langle\pi_{\Sigma_1}|C_{-1}(-\langle\pi_{\Sigma_2}|C_{-1}|\Sigma_2\rangle\otimes|\Sigma_1\rangle)\otimes|V\rangle\\ &=\mathcal{L}_{0,\frac{1}{2}(\langle\pi_{\Sigma_1}|C_{-1}|\Sigma_1\rangle\otimes|\Sigma_2\rangle\langle\pi_{\Sigma_2}|-\langle\pi_{\Sigma_2}|C_{-1}|\Sigma_2\rangle\otimes|\Sigma_1\rangle\langle\pi_{\Sigma_1}|)}|V\rangle,\end{aligned}\tag{4.31}$$

where we used the identity (2.25) in the first step, the section constraint (3.8b) in the second, and finally that the result is antisymmetric, modulo the same section constraint. This concludes the proof of closure of the gauge algebra.

To summarize, we have shown the closure of transformations (4.15) into an "algebra"[2]

$$[\mathcal{L}_{\xi_1,\Sigma_1},\mathcal{L}_{\xi_2,\Sigma_2}]=\mathcal{L}_{\xi_{12},\Sigma_{12}},\tag{4.32}$$

defined by

$$\begin{aligned}\xi_{12}&\equiv[\![\xi_1,\xi_2]\!]\equiv\frac{1}{2}(\mathcal{L}_{\xi_1}\xi_2-\mathcal{L}_{\xi_2}\xi_1),\\ |\Sigma_{12}\rangle\langle\pi_{\Sigma_{12}}|&\equiv\mathcal{L}_{\xi_1}(|\Sigma_2\rangle\langle\pi_{\Sigma_2}|)+\frac{1}{2}\langle\pi_{\Sigma_1}|C_{-1}|\Sigma_2\rangle\otimes|\Sigma_1\rangle\langle\pi_{\Sigma_2}|\\ &\quad+\frac{1}{4}\langle\partial_{\xi_2}|C_1|(|\xi_2\rangle\otimes|\xi_1\rangle-|\xi_1\rangle\otimes|\xi_2\rangle)\langle\partial_{\xi_2}|\\ &\quad-(1\leftrightarrow2).\end{aligned}\tag{4.33}$$

[2]As in the lower-dimensional cases, this will not be a Lie algebra, since the corresponding brackets do not satisfy Jacobi identities. The proper structure, which in the double field theory situation is a Courant algebroid, is maybe best described in an L_∞ framework [51,52].

Finally, it is instructive to decompose the generalized diffeomorphisms (4.15) under E_8, and to recover the structure of E_8 exceptional field theory. Expanding the gauge parameter $|\Sigma\rangle\langle\pi_\Sigma|$ according to (2.5) yields

$$\begin{aligned} |\Sigma\rangle\langle\pi_\Sigma| = &(\sigma_1 + \sigma_{2A}T^A_{-1} + \sigma_{3AB}T^A_{-1}T^B_{-1} + \cdots)|0\rangle \\ &\times\langle 0| - (\Sigma_{0A} + \Sigma_{1A,B}T^B_{-1} + \Sigma_{2A,BC}T^B_{-1}T^C_{-1} + \cdots)|0\rangle \\ &\times\langle 0|T^A_1, \end{aligned} \tag{4.34}$$

where the indices AB of σ_{3AB} and the indices BC of $\Sigma_{2A,BC}$ are restricted to $\mathbf{1}\oplus\mathbf{248}\oplus\mathbf{3875}$, and similar terms are hidden in the ellipses for all higher L_0 weights. The section constraint implies no constraint on the parameters $\sigma_{n,\Xi}$, and the parameters $\Sigma_{n,A,\Xi}$ are constrained on their first index according to the E_8 section constraints (3.7). Similarly, we expand the diffeomorphism parameter ξ as

$$|\xi\rangle = (\xi^0 + \eta_{AB}\xi^A_1 T^B_{-1} + \xi_{2AB}T^A_{-1}T^B_{-1} + \cdots)|0\rangle. \tag{4.35}$$

Assuming partial derivatives of the form (3.4), one then obtains for the Lie derivative

$$\begin{aligned} \mathcal{L}_{\xi,\Sigma} = &\xi^0\partial_0 + \xi^A_1\partial_A - \partial_A\xi^0 T^A_1 + \partial_0\xi^0(L_0 - 1) + \partial_A\xi^A_1 L_0 \\ &+ (f^B{}_{CA}\partial_B\xi^C_1 + \Sigma_{0A})T^A_0 + \sigma_1 L_{-1} \\ &- (\partial_0\xi^A_1 + \Pi^{AB,CD}\partial_B\xi_{2CD} - f^{ABC}\Sigma_{1B,C})\eta_{AE}T^E_{-1} \\ &+ \sum_{n>1}\omega_{nA}T^A_{-n}, \end{aligned} \tag{4.36}$$

for some linear combinations ω_{nA} of $\partial_0\xi_{n\Xi}, \partial_A\xi_{n+1\Xi}, \sigma_{n\Xi}$, $\Sigma_{nA,\Xi}$. It is important to note that, although $\sigma_{n\Xi}$ is defined in the L_0 weight $n-1$ component of $R(\Lambda_0)$, and $\Sigma_{nA,\Xi}$ in the tensor product of the L_0 weight n component of $R(\Lambda_0)$ with the 248 of E_8, they only appear in ω_{nA} through an appropriate projection to the 248 of E_8. One understands indeed that Σ belongs to the tensor product $R(\Lambda_0)_{-1}\otimes\overline{R(\Lambda_0)}$, but it only appears in the generalized diffeomorphism through a projection to $\mathfrak{e}_9$.

Decomposing the vector $|V\rangle$ accordingly,

$$|V\rangle = (V^0 + \eta_{AB}V^A_1 T^B_{-1} + V_{2AB}T^A_{-1}T^B_{-1} + \cdots)|0\rangle, \tag{4.37}$$

one obtains for the action on its lowest components

$$\begin{aligned} \mathcal{L}_{\xi,\Sigma}V^0 &= \xi^0\partial_0 V^0 - V^0\partial_0\xi^0 + \xi^A_1\partial_A V^0 - V^A_1\partial_A\xi^0, \\ \mathcal{L}_{\xi,\Sigma}V^A_1 &= \xi^0\partial_0 V^A_1 - V^0\partial_0\xi^A_1 \\ &\quad + \xi^B_1\partial_B V^A_1 + V^A_1\partial_B\xi^B_1 \\ &\quad - (f^{EA}{}_B f^C{}_{DE}\partial_C\xi^D_1 + f^{CA}{}_B\Sigma_{0C})V^B_1 \\ &\quad - \Pi^{BA,CD}V_{2CD}\partial_B\xi^0 - \Pi^{AB,CD}V^0\partial_B\xi_{2CD} \\ &\quad + f^{ABC}\Sigma_{1B,C}V^0, \end{aligned} \tag{4.38}$$

with $\Pi^{AB,CD}$ from (3.6). The weight of the covariant derivative indicates that in three dimensions, V^0 is a vector field, V^A_1 a scalar and V_{2AB} a 1-form. The second line in the Lie derivative of V^A_1 reproduces precisely the E_8 internal Lie derivative with respect to the vector field ξ^A_1 and the constrained parameter Σ_{0A} [23]. We know from E_8 geometry [23,24] that such a transformation only removes unphysical parts of the vielbein. In particular, this decomposition illustrates that the additional gauge transformations in (4.15) cannot absorb the standard diffeomorphisms of the first term (which ultimately is a consequence of the shift of the L_0 charge by the operator C_{-1}). The latter thus survive as physical gauge symmetries of the theory as expected. Note that the parameters in Σ enter in a way that does not disturb the above interpretation of the transformations of the lowest components of $|V\rangle$. This is essential, so that it will not affect the physical components of a generalized vielbein.

V. GENERALIZED SCHERK–SCHWARZ REDUCTION

We will now perform another consistency check on the proposed form of E_9 generalized diffeomorphisms (4.15). We will study the behaviour of these transformations under a suitably generalized Scherk–Schwarz Ansatz [53] for vectors and gauge parameters. With the internal coordinate dependence of all fields carried by a Scherk–Schwarz twist matrix U we will show that under certain assumptions on this twist matrix, all Y^M dependence in the transformations (4.15) consistently factors out such that the generalized diffeomorphisms translate into an algebraic action on the two-dimensional fields. We find that this precisely reproduces the gauge structures identified in two-dimensional gauged supergravities [30].

Before writing down the Scherk–Schwarz Ansatz in the Dirac formalism we introduce a few definitions. First of all, we need to define the so-called twist matrix U. The group of symmetries of the theory includes not only E_9, but also the Virasoro group Vir [54]. In two dimensions, the metric scaling factor $e^{2\sigma}$ in the conformal gauge $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$ scales under the action of the central operator K in $\mathfrak{e}_9$ [55]. The scalar fields in $E_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$ and their infinite tower of dual scalar fields, together with the scaling factor $e^{2\sigma}$, parametrize a coset element of the central extension of the loop group [55–57]. On the other hand, the two-dimensional dilaton ρ is a free field. This field and its (single) dual $\tilde\rho$ transform non-trivially under the Virasoro reparametrizations of the loop group spectral parameter w. To see this one observes that an affine redefinition of the spectral parameter $w\to aw+b$ can be compensated by the affine transformation of $(\rho,\tilde\rho)\to(a\rho, a\tilde\rho - b)$ [54]. These affine transformations define the parabolic subgroup $\mathbb{R}_+\ltimes\mathbb{R}\subset \mathrm{SL}(2,\mathbb{R})\subset$ Vir generated by L_0 and L_{-1}. We therefore expect that a general Scherk–Schwarz Ansatz will be described by a twist matrix in the product of this parabolic subgroup and the central extension of the E_8 loop group. We decompose the twist matrix U accordingly as the product of a Virasoro parabolic subgroup element $U_{\rm Vir}(Y)$ and a loop group element $U_{\rm loop}(Y)$, which includes both the generators T^A_n and the central charge generator,

$$U(Y) = U_{\rm Vir}(Y)U_{\rm loop}(Y). \tag{5.1}$$

The definition of the exceptional E_9 theory is beyond the scope of this paper. Nonetheless, we expect that the Scherk–Schwarz Ansatz for the scalar fields $\mathcal{M}(x,Y)$ and the metric conformal factor $\boldsymbol{\sigma}(x,Y)$ should be determined in terms of $U_{\rm loop}(Y)$ as

$$e^{-2\boldsymbol{\sigma}(x,Y)}\mathcal{M}(x,Y) = U_{\rm loop}(Y)^T e^{-2\sigma(x)}M(x)U_{\rm loop}(Y), \tag{5.2}$$

whereas the dilaton field and its dual should be determined by[3]

$$U^T_{\rm Vir}(Y) = e^{\varsigma(Y)L_{-1}}e^{\upsilon(Y)L_0} \Rightarrow \boldsymbol{\rho}(x,Y) = e^{-\upsilon(Y)}\rho(x),$$
$$\tilde{\boldsymbol{\rho}}(x,Y) = e^{-\upsilon(Y)}(\tilde\rho(x) - \varsigma(Y)). \tag{5.3}$$

The shift of $\tilde{\boldsymbol{\rho}}(x,Y)$ in $\varsigma(Y)$ is indeed consistent with the gauging defined in [30], where the L_0 generator is not gauged and so $\upsilon(Y)=0$. Although the theory remains to be constructed, one can infer from this discussion that the Scherk–Schwarz Ansatz should involve in general both a twist matrix in the loop group and a twist matrix in the parabolic subgroup of $SL(2,\mathbb{R})$. Assuming that this is indeed the case, we shall now see that this permits to define a gauge algebra from the generalized diffeomorphisms introduced in the last section.

Note that in higher dimensions one does not only introduce a twist matrix $U(Y)\in E_d$ (for $d\le 8$), but also a scaling factor $\rho(Y)$ for the metric field Ansatz, not to be confused with the dilaton $\boldsymbol{\rho}(x,Y)$ discussed above. Since the central charge of the loop algebra acts as a Weyl rescaling of the metric in two dimensions, this scaling factor $\rho(Y)$ is already included in $U_{\rm loop}(Y)$ by construction.

It will be convenient to write the Maurer–Cartan form[4]

$$\mathcal{J} = U^{-T}dU^T = U^{-T}_{\rm Vir}dU^T_{\rm Vir} + U^{-T}_{\rm Vir}(U^{-T}_{\rm loop}dU^T_{\rm loop})U^T_{\rm Vir}$$
$$= \mathcal{J}_{\rm Vir} + \mathcal{J}_{\rm loop}, \tag{5.4}$$

in Dirac notation as

$$|\underline{J}\rangle\langle\underline{J}|\otimes\langle\partial_J| = \underline{L}_0\otimes\langle\partial\upsilon| + \underline{L}_{-1}\otimes e^{-\upsilon}\langle\partial\varsigma| + \sum_n \underline{T}^A_n\otimes\langle j_{nA}| + \underline{\mathbb{1}}\otimes\langle j_c|, \tag{5.5}$$

where we understand that the $\langle\partial_J|$ bra defines the derivative index and the $\mathfrak{vir}\oplus\mathfrak{e}_9$ matrix is written as $|\underline{J}\rangle\langle\underline{J}|$. The notation is such that

$$U^T(Y)\otimes\langle\overleftarrow{\partial}_Y| = U^T\underline{L}_0\otimes\langle\partial\upsilon| + U^T\underline{L}_{-1}\otimes e^{-\upsilon}\langle\partial\varsigma| + \sum_n U^T\underline{T}^A_n\otimes\langle j_{nA}| + U^T\otimes\langle j_c|. \tag{5.6}$$

To distinguish the ket vectors that are acted on the left by U^T and U^{-T}, we use the underlined notation such that in practice, U^T acts on an underlined ket to give a not underlined ket. The same convention applies to the bra. It follows for instance that the Maurer–Cartan form (5.5) acts on an underlined ket vector to give another underlined ket vector, which justifies that we use the notation $\underline{J}$. The underlined operators are identical to the nonunderlined ones, but are simply understood to act on underlined kets.

Before spelling out the Scherk–Schwarz Ansatz, it is important to understand the covariance under the parabolic subgroup $\mathbb{R}_+\ltimes\mathbb{R}\subset$ Vir. The algebraic part of the generalized Lie derivative (4.15) involves the derivative of the vector field $|\xi\rangle\langle\partial_\xi|$ through the operator C_0, and the constrained gauge parameter $|\Sigma\rangle\langle\pi_\Sigma|$ through the operator C_{-1}. The action of $\mathbb{R}_+\ltimes\mathbb{R}$ on these two operators is determined by the commutation relation

$$[\mathbb{1}\otimes L_m + L_m\otimes\mathbb{1}, C_n] = (m-n)C_{m+n} + \frac{4}{3}m(m^2-1)\delta_{m+n,0}, \tag{5.7}$$

to be such that a twist matrix parametrized as in (5.1) acts on C_{-1} and C_0 in the adjoint representation,

$$(U^T\otimes U^T)C_{-1}(U^{-T}\otimes U^{-T}) = e^{\upsilon}\underline{C}_{-1},$$
$$(U^T\otimes U^T)C_0(U^{-T}\otimes U^{-T}) = \underline{C}_0 - \varsigma\underline{C}_{-1}, \tag{5.8}$$

where $\underline{C}_n$ is C_n acting on flattened (underlined) vectors. Because $|\xi\rangle\langle\partial_\xi|$ and $|\Sigma\rangle\langle\pi_\Sigma|$ are naturally paired with C_{-1} and C_0, they transform in the coadjoint representation of the parabolic subgroup $\mathbb{R}_+\ltimes\mathbb{R}$,

$$|\Sigma\rangle\langle\pi_\Sigma| \to e^{-\upsilon}(|\Sigma\rangle\langle\pi_\Sigma| + \varsigma|\xi\rangle\langle\partial_\xi|), \qquad |\xi\rangle\langle\partial_\xi| \to |\xi\rangle\langle\partial_\xi|. \tag{5.9}$$

The Scherk–Schwarz Ansatz for vectors and gauge parameters written in Dirac notation now takes the form

[3] On the spectral parameter $L_0 = -w\partial_w$ and $L_{-1} = -\partial_w$.
[4] Here, we use the notation $U^{-T}\equiv(U^{-1})^T$ to denote the transpose of the inverse.

$$|V\rangle = U^{-T}|\underline{V}\rangle, \qquad |\xi\rangle = U^{-T}|\underline{\xi}\rangle,$$
$$|\Sigma\rangle\langle\pi_\Sigma| = e^{-\upsilon}U^{-T}\left(\sum_n \underline{T}^A_{1+n}|\underline{\xi}\rangle\langle j_{nA}| + \underline{L}_1|\underline{\xi}\rangle\langle\partial\upsilon| + \underline{L}_0|\underline{\xi}\rangle e^{-\upsilon}\langle\partial\varsigma|\right) + e^{-\upsilon}\varsigma|\xi\rangle\langle\partial_\xi|, \tag{5.10}$$

where the flat (underlined) ket vectors only depend on external coordinates. The Ansatz for the vectors $|V\rangle$, $|\xi\rangle$ is of the standard form, while the Ansatz for the gauge parameter Σ matches the $\mathbb{R}_+ \ltimes \mathbb{R}$ covariance (5.9) and is explicitly compatible with the constraints (4.14) that this parameter satisfies. Its expression can be written formally for constant ς in terms of a properly renormalized trace (see Appendix B)

$$|\Sigma\rangle\langle\pi_\Sigma| = e^{-\upsilon}\frac{1}{\mathcal{N}}U^{-T}(\langle\underline{J}|\underline{C}_1|\underline{J}\rangle\otimes|\underline{\xi}\rangle)\langle\partial_J| + e^{-\upsilon}\varsigma|\xi\rangle\langle\partial_\xi|, \tag{5.11}$$

which exhibits that this Ansatz preserves E$_9$ covariance.

Let us now consider the action of such a generalized diffeomorphism,

$$\begin{aligned}\mathcal{L}_{\xi,\Sigma}|V\rangle &= \langle\partial_V|\xi\rangle(U^{-T}|\underline{V}\rangle) + \langle\partial_\xi|(C_0-1)(U^{-T}|\underline{\xi}\rangle\otimes|V\rangle)\\ &\quad + e^{-\upsilon}\varsigma\langle\partial_\xi|C_{-1}(U^{-T}|\underline{\xi}\rangle\otimes|V\rangle)\\ &\quad + e^{-\upsilon}\Big(\langle\partial\upsilon|C_{-1}U^{-T}\underline{L}_1|\underline{\xi}\rangle\\ &\quad + e^{-\upsilon}\langle\partial\varsigma|C_{-1}U^{-T}\underline{L}_0|\underline{\xi}\rangle\\ &\quad + \sum_n\langle j_{nA}|C_{-1}U^{-T}\underline{T}^A_{1+n}|\underline{\xi}\rangle\Big)\otimes|V\rangle,\end{aligned} \tag{5.12}$$

where $\langle\partial_V|$ and $\langle\partial_\xi|$ are understood to derive the twist matrix U^{-T} multiplying respectively the constant vectors $|\underline{V}\rangle$ and $|\underline{\xi}\rangle$ using (5.6). Using (5.8) to write everything in terms of flat vectors, one obtains that the explicit dependence in υ and ς drops out (such that they only appear through their derivatives $\langle\partial\upsilon|$ and $e^{-\upsilon}\langle\partial\varsigma|$). For example

$$\begin{aligned}&-\langle\underline{j}_{nA}|U^T\otimes U^TC_0U^{-T}\otimes U^{-T}\underline{T}^A_n|\underline{\xi}\rangle + e^{-\upsilon}\langle\underline{j}_{nA}|U^T\otimes U^TC_{-1}U^{-T}\otimes U^{-T}(\underline{T}^A_{n+1}-\varsigma\underline{T}^A_n)|\underline{\xi}\rangle\\ &\quad = -\langle\underline{j}_{nA}|(\underline{C}_0-\varsigma\underline{C}_{-1})\underline{T}^A_n|\underline{\xi}\rangle + e^{-\upsilon}\langle\underline{j}_{nA}|e^{\upsilon}\underline{C}_{-1}(\underline{T}^A_{n+1}-\varsigma\underline{T}^A_n)|\underline{\xi}\rangle\\ &\quad = -\langle\underline{j}_{nA}|\underline{C}_0\underline{T}^A_n|\underline{\xi}\rangle + \langle\underline{j}_{nA}|\underline{C}_{-1}\underline{T}^A_{n+1}|\underline{\xi}\rangle.\end{aligned} \tag{5.13}$$

This exhibits that the Ansatz (5.10) is indeed covariant with respect to $\mathbb{R}_+ \ltimes \mathbb{R}$, as advocated above. For convenience, we introduce the flat derivative bra $\langle\underline{\partial}| = \langle\partial|U^{-T}$. One then obtains

$$\begin{aligned}U^T\mathcal{L}_{\xi,\Sigma}|V\rangle &= -\Big(\langle\underline{\partial}\upsilon|\underline{\xi}\rangle\underline{L}_0 + e^{-\upsilon}\langle\underline{\partial}\varsigma|\underline{\xi}\rangle\underline{L}_{-1} + \sum_n\langle\underline{j}_{nA}|\underline{\xi}\rangle\underline{T}^A_n + \langle\underline{j}_c|\underline{\xi}\rangle\Big)|\underline{V}\rangle\\ &\quad - \Big(\langle\underline{\partial}\upsilon|(\underline{C}_0-1)\underline{L}_0|\underline{\xi}\rangle + e^{-\upsilon}\langle\underline{\partial}\varsigma|(\underline{C}_0-1)\underline{L}_{-1}|\underline{\xi}\rangle + \sum_n\langle\underline{j}_{nA}|(\underline{C}_0-1)\underline{T}^A_n|\underline{\xi}\rangle + \langle\underline{j}_c|(\underline{C}_0-1)|\underline{\xi}\rangle\Big)|\underline{V}\rangle\\ &\quad + \Big(\langle\underline{\partial}\upsilon|\underline{C}_{-1}\underline{L}_1|\underline{\xi}\rangle + e^{-\upsilon}\langle\underline{\partial}\varsigma|\underline{C}_{-1}\underline{L}_0|\underline{\xi}\rangle + \sum_n\langle\underline{j}_{nA}|\underline{C}_{-1}\underline{T}^A_{n+1}|\underline{\xi}\rangle\Big)|\underline{V}\rangle\\ &= \langle\underline{\partial}\upsilon|((1-\underline{C}_0)\underline{L}_0\otimes\mathbb{1} - \mathbb{1}\otimes\underline{L}_0 + \underline{C}_{-1}\underline{L}_1\otimes\mathbb{1})|\underline{\xi}\rangle\otimes|\underline{V}\rangle - \langle\underline{j}_c|\underline{C}_0|\underline{\xi}\rangle|\underline{V}\rangle\\ &\quad + e^{-\upsilon}\langle\underline{\partial}\varsigma|((1-\underline{C}_0)\underline{L}_{-1}\otimes\mathbb{1} - \mathbb{1}\otimes\underline{L}_{-1} + \underline{C}_{-1}\underline{L}_0\otimes\mathbb{1})|\underline{\xi}\rangle\otimes|\underline{V}\rangle\\ &\quad + \sum_n\langle\underline{j}_{nA}|((1-\underline{C}_0)\underline{T}^A_n\otimes\mathbb{1} - \mathbb{1}\otimes\underline{T}^A_n + \underline{C}_{-1}\underline{T}^A_{n+1}\otimes\mathbb{1})|\underline{\xi}\rangle\otimes|\underline{V}\rangle\\ &= -\Big(\langle\underline{\partial}\upsilon|(\underline{L}_0+1) + e^{-\upsilon}\langle\underline{\partial}\varsigma|\underline{L}_{-1} + \sum_n\langle\underline{j}_{nA}|\underline{T}^A_n + \langle\underline{j}_c|\Big)\underline{C}_0|\underline{\xi}\rangle\otimes|\underline{V}\rangle\\ &\quad + \Big(\langle\underline{\partial}\upsilon|\underline{L}_1 + e^{-\upsilon}\langle\underline{\partial}\varsigma|(\underline{L}_0-1) + \sum_n\langle\underline{j}_{nA}|\underline{T}^A_{n+1}\Big)\underline{C}_{-1}|\underline{\xi}\rangle\otimes|\underline{V}\rangle,\end{aligned} \tag{5.14}$$

where in the last step we have used

$$\begin{aligned}&[C_0, L_n\otimes\mathbb{1}] - L_n\otimes\mathbb{1} + \mathbb{1}\otimes L_n = [C_{-1}, L_{n+1}\otimes\mathbb{1}] + C_n,\\ &[C_0, T^A_n\otimes\mathbb{1}] - T^A_n\otimes\mathbb{1} + \mathbb{1}\otimes T^A_n = [C_{-1}, T^A_{n+1}\otimes\mathbb{1}],\end{aligned} \tag{5.15}$$

which one computes straightforwardly using the definition of C_n. Defining

$$\langle \underline{\theta}| \equiv \langle \underline{\partial} v|\underline{L}_1 + e^{-v}\langle \underline{\partial} \varsigma|(\underline{L}_0 - 1) + \sum_n \langle \underline{j}_{nA}|\underline{T}^A_{n+1},$$
$$\langle \underline{\vartheta}| \equiv -\langle \underline{\partial} v|(\underline{L}_0 + 1) - e^{-v}\langle \underline{\partial} \varsigma|\underline{L}_{-1} - \sum_n \langle \underline{j}_{nA}|\underline{T}^A_{n} - \langle \underline{j}_c|, \tag{5.16}$$

the action (5.14) can be put in the compact form

$$\delta_{\underline{\xi}}|\underline{V}\rangle \equiv U^T \mathcal{L}_{\xi,\Sigma}|V\rangle = \langle \underline{\theta}|\underline{C}_{-1}|\underline{\xi}\rangle \otimes |\underline{V}\rangle + \langle \underline{\vartheta}|\underline{C}_0|\underline{\xi}\rangle \otimes |\underline{V}\rangle. \tag{5.17}$$

A consistent reduction thus corresponds to a twist matrix constructed such that the combinations (5.16) are constant, corresponding to two different types of gaugings. The first one, parametrized by a constant embedding tensor $\langle \underline{\theta}|$, precisely reproduces the standard gauge structure of two-dimensional gauged supergravity [30]. The second type of gauging, parametrized by a constant $\langle \underline{\vartheta}|$, is slightly less standard. As follows from (5.17), it gauges the generator $\mathsf{d} \in \mathfrak{e}_9$ that is represented as L_0, which is not a symmetry of the ungauged Lagrangian. The resulting gaugings thus do not admit an action but are defined only on the level of their field equations. In this sense they are the analogues of the trombone gaugings [58] that gauge the trombone scaling symmetry [59] of higher-dimensional supergravities. Note that in the two-dimensional case the trombone symmetry as defined in [59] is an ordinary (and off-shell) Weyl symmetry of the two-dimensional theory that is generated by the central charge K of E_9. It is gauged by both parameters $\langle \underline{\theta}|$ and $\langle \underline{\vartheta}|$ and thus part of a generic gauging in two dimensions.

Explicitly, one has

$$\langle \underline{\theta}|\underline{C}_{-1}|\underline{\xi}\rangle + \langle \underline{\vartheta}|\underline{C}_0|\underline{\xi}\rangle = (\langle \underline{\theta}|\underline{L}_{-1}|\underline{\xi}\rangle + \langle \underline{\vartheta}|\underline{L}_0|\underline{\xi}\rangle) - \eta_{AB}\sum_{n\in\mathbb{Z}}(\langle \underline{\theta}|\underline{T}^A_{-n-1}|\underline{\xi}\rangle + \langle \underline{\vartheta}|\underline{T}^A_{-n}|\underline{\xi}\rangle)\underline{T}^B_n + \langle \underline{\theta}|\underline{\xi}\rangle \underline{L}_{-1} + \langle \underline{\vartheta}|\underline{\xi}\rangle \underline{L}_0. \tag{5.18}$$

A straightforward computation shows that the algebra of gauge transformations (5.17) closes according to

$$[\delta_{\xi_1}, \delta_{\xi_2}]|\underline{V}\rangle = \delta_{\xi_{12}}|\underline{V}\rangle, \tag{5.19}$$

with gauge parameter

$$|\xi_{12}\rangle \equiv \frac{1}{2}(\langle \underline{\theta}|\underline{C}_{-1} + \langle \underline{\vartheta}|\underline{C}_0)(|\underline{\xi}_1\rangle \otimes |\underline{\xi}_2\rangle - |\underline{\xi}_2\rangle \otimes |\underline{\xi}_1\rangle), \tag{5.20}$$

provided that the components of the embedding tensor satisfy the constraints

$$\langle \underline{\theta}| \otimes \langle \underline{\theta}|\underline{C}_{-1} + \langle \underline{\vartheta}| \otimes \langle \underline{\theta}|(\underline{C}_0 + \sigma - 1) = 0,$$
$$\langle \underline{\vartheta}| \otimes \langle \underline{\vartheta}|\underline{C}_0 + \langle \underline{\theta}| \otimes \langle \underline{\vartheta}|\underline{C}_{-1} = 0. \tag{5.21}$$

If the twist matrix from which this embedding tensor is obtained satisfies the section constraint, these constraints must be automatically satisfied since closure of the algebra is guaranteed by construction by the closure of the generalized diffeomorphism algebra. In the absence of an L_0-gauging ($\langle \underline{\vartheta}| = 0$), we recover the condition

$$\langle \underline{\theta}| \otimes \langle \underline{\theta}|\underline{C}_{-1} = 0, \tag{5.22}$$

which had been identified as the quadratic constraint on the embedding tensor in [30]. For pure L_0-gaugings ($\langle \underline{\theta}| = 0$) on the other hand, we precisely recover the section constraint

$$\langle \underline{\vartheta}| \otimes \langle \underline{\vartheta}|\underline{C}_0 = 0, \tag{5.23}$$

as for pure trombone gaugings in higher dimensions.

VI. GENERALIZATION TO OTHER GROUPS

In this section, we discuss two generalizations of our formulas for the generalized diffeomorphisms (1.1) and section constraint (1.2). The first generalization is to arbitrary affine algebras and the second one to arbitrary Kac–Moody algebras. In the most general case, the generalizations we present only give the generalized form of the section constraint and generalized Lie derivative, but we have not checked directly closure of the gauge algebra which also requires the introduction of extra constrained transformation parameters Σ. For the generalization to other affine algebras with coordinates in the basic representation, the parameter Σ can be defined in analogy with the $\mathfrak{e}_9$ case considered in detail above and the gauge algebra closes in exactly the same way. For general Kac–Moody algebras, a systematic introduction of Σ most probably requires the language and properties of tensor hierarchy algebras that we shall not attempt here. We also note that even if a consistent gauge algebra is established, this does not guarantee the existence of a nontrivial physical model for any Kac–Moody algebra.

A. Extension to other affine groups

In this section, we discuss how much of the structure of the E_9 exceptional geometry will carry over to affine extensions of other “exceptional” field theories based on simple symmetry groups in $D = 3$ space-time dimensions [60,61].[5] An example of a double field theory with $SO(8, n)$ symmetry with three external dimensions was

[5]The case of semisimple symmetries and their affine and further extensions was discussed in [62].

recently constructed in [63], the duality covariant theory based on the Ehlers group SL(2) was constructed in [49], and the picture for higher SL(n) was given in [24].

The important steps in the construction of the E_9 exceptional geometry performed in this paper were (i) the identification of an appropriate representation $R(\Lambda_0)$ for the coordinates, (ii) the identification of an appropriate section constraint in $\overline{R(\Lambda_0)} \otimes \overline{R(\Lambda_0)}$ and (iii) verification of the closure of the generalized diffeomorphisms up to section constraint. It is noticeable that in the definition of the generalized diffeomorphism (4.15) and section constraint (3.8) only the coset Virasoro generators appear. Little use of the structure of E_8 itself is made.

Let us consider an arbitrary simple finite-dimensional algebra $\mathfrak{g}$ (replacing $\mathfrak{e}_8$) and its associated (nontwisted) affine extension $\mathfrak{g}^+$ (replacing $\mathfrak{e}_9$). The associated groups will be denoted by G and G^+, respectively. The known structure of exceptional field theory with G symmetry have internal coordinates in the adjoint representation $\mathbf{adj}$ of G satisfying a section constraint in the representation $\mathbf{sec}$ of G that lies in the tensor product of two adjoint representations. The pieces of the section constraint that lie in the symmetric part of the tensor product correspond to the three-dimensional embedding tensor (as a consequence of the duality between level 2 and level -1 in the tensor hierarchy algebra for compactifications to three dimensions [32]). There is also an antisymmetric contribution to the section constraint [23,24,49,63]. In addition, the generalized Lie derivatives with three external dimensions contain also constrained parameters Σ besides the standard parameters ξ. The standard physical solution to the $D = 3$ section constraint is given by taking from $\mathbf{adj}$ a d-dimensional subspace that corresponds to the maximal number of dimensions that can be oxidized [61,64].

All affine algebras afford a "basic" representation $R(\Lambda_0)$ at level $k = 1$ [33,36]. Its distinguishing property is that it is an irreducible highest weight module of $\mathfrak{g}^+$ that decomposes under $\mathfrak{g}$ as

$$R(\Lambda_0) = \mathbf{1}_0 \oplus \mathbf{adj}_{-1} \oplus \mathbf{sec}_{-2} \oplus \ldots, \tag{6.1}$$

where the antisymmetric part of $\mathbf{sec}$ (in the tensor product of two adjoints) is $\mathbf{sec}_a = \mathbf{adj}$. This is the generalization of (2.5). It is a generic property of $R(\Lambda_0)$ that there are null states at affine level -2. They are a consequence of $f_0 f_0|0\rangle = 0$, where f_0 is the generator corresponding to the root $-\alpha_0$. It is easily shown that this state is annihilated by e_0. In terms of $\mathfrak{g}$, the state $f_0 f_0|0\rangle$ would carry the weight 2θ, where θ is the highest root of $\mathfrak{g}$. Therefore, the "big" representation in the symmetric product of two $\mathfrak{g}$-adjoints is always absent at affine level -2 in $R(\Lambda_0)$, and the symmetric part of $\mathbf{sec}$ is some smaller representation: $\mathbf{adj} \otimes_s \mathbf{adj} = r(2\theta) \oplus \mathbf{sec}_s$.

We can then work out the general tensor product of two elements in $R(\Lambda_0)$ at low $\mathfrak{g}$ levels,

$$R(\Lambda_0) \otimes R(\Lambda_0) = \mathbf{1}_0 \oplus (2 \cdot \mathbf{adj})_{-1} \oplus (2 \cdot \mathbf{sec} \oplus \mathbf{adj} \otimes \mathbf{adj})_{-2} \oplus \ldots. \tag{6.2}$$

A physically expected solution to the $D = 2$ section constraint is taking the singlet at level 0 and the d-dimensional subspace in $\mathbf{adj}$ that corresponds to the solution of the $D = 3$ section constraint. These $d + 1$ coordinates together correspond to the oxidation from $D = 2$ external space to the same maximal oxidation endpoint in $3 + d$ dimensions. We would therefore like the $D = 2$ section constraint to be strong enough to remove everything but a solution of this type.

From the representation theory of affine algebras we know that the tensor product $R(\Lambda_0) \otimes R(\Lambda_0)$ decomposes into representations at level $k = 2$. More precisely, there is again a coset construction similar to (2.14) where the standard modules at $k = 2$ appear multiplied with coset Virasoro characters that are q-series. What types of $k = 2$ modules exist does depend on the structure of $\mathfrak{g}^+$. Two $k = 2$ modules that always exist are

$$R(2\Lambda_0) = \mathbf{1}_0 \oplus \mathbf{adj}_{-1} \oplus (\mathbf{sec} \oplus r(2\theta))_{-2} \oplus \ldots \tag{6.3}$$

that is the leading term in the symmetric part of the tensor product and

$$R(\Lambda_1)_{-1} = \mathbf{adj}_{-1} \oplus (\mathbf{adj} \wedge \mathbf{adj} \oplus \mathbf{sec}_s)_{-2} \oplus \ldots \tag{6.4}$$

by which we denote the leading term in the antisymmetric part of the tensor product[6] The first null states in $R(2\Lambda_0)$ appear at level -3. The null states in $r(2\theta)$ at affine level -1 in $R(\Lambda_1)$ come from the observation that $f_0|\Lambda_1\rangle$ is a null state.

Comparing these two leading expansions with the full tensor product (6.2) we conclude that any other $k = 2$ module can only start contributing from level -2 onwards. Writing the levels as a q-series this means that

$$R(\Lambda_0) \otimes_s R(\Lambda_0) = (1 + q^2) R(2\Lambda_0) \oplus q^2 \mathbf{sec}'_s \oplus \ldots$$
$$R(\Lambda_0) \wedge R(\Lambda_0) = (q + q^2) R(\Lambda_1) \oplus \ldots \tag{6.5}$$

where the ellipses denote terms at affine level -3 and lower, and where $\mathbf{sec}_s = \mathbf{1} \oplus \mathbf{sec}'_s$. Note that the identification of an irreducible highest weight affine representation from its leading irreducible $\mathfrak{g}$ representation is unique at a given k. What is noteworthy is the absence of a term at level -1 in the $R(2\Lambda_0)$ piece.

[6]The notation may seem to indicate that there is a unique simple root α_1 connected with a single line to α_0. This is not necessarily the case (e.g., in A_n^+); then Λ_1 has to be reinterpreted as the weight $\sum_{i=1}^{\mathrm{rank}\,\mathfrak{g}} a_i \Lambda_i$, where the highest root of $\mathfrak{g}$ is $\theta = \sum_{i=1}^{\mathrm{rank}\,\mathfrak{g}} a_i \lambda_i$.

There is a coset Virasoro construction associated with the tensor product of two $k = 1$ modules. The q-series (after an appropriate shift of the conformal weight of the affine representations) are characters of this coset Virasoro algebra. Unlike the case for E_9, it is not true in general that they are characters in the minimal series since the central charge can be $c \geq 1$ [65].[7] Nevertheless, the q-series always represent characters of (possibly reducible) unitary representations of the Virasoro algebra.

The contribution to h from the $\mathfrak{g}$ quadratic Casimir is 0 for $R(2\Lambda_0)$ and $\frac{g^\vee}{2+g^\vee}$ for $R(\Lambda_1)$. The generalization of (2.14) is

$$R(\Lambda_0) \otimes R(\Lambda_0) = \mathrm{Vir}_0 \otimes R(2\Lambda_0)_0 \oplus \mathrm{Vir}_{-\frac{2}{2+g^\vee}} \otimes R(\Lambda_1)_{-\frac{g^\vee}{2+g^\vee}} \oplus \dots \tag{6.6}$$

where the subscript on the coset Virasoro modules is $-h$. If we define the rescaled coset Virasoro operators $C_n = (2 + g^\vee) L_n^{\mathrm{coset}}$, we can conclude that the appropriate section constraints remain of the precise form (3.8):

$$\begin{aligned} \langle \partial_1 | \otimes \langle \partial_2 | (C_0 - 1 + \sigma) &= 0, \\ \langle \partial_1 | \otimes \langle \partial_2 | C_{-n} &= 0, \quad \forall\ n > 0, \\ (\langle \partial_1 | \otimes \langle \partial_2 | + \langle \partial_2 | \otimes \langle \partial_1 |) C_1 &= 0, \end{aligned} \tag{6.7}$$

since C_0 then takes the value 0 and 2 in the leading symmetric and antisymmetric states, respectively. Then, the only modules remaining in the product of two derivatives are the leading ones, corresponding to the highest weights in the (conjugate) Virasoro modules corresponding to $\overline{R(2\Lambda_0)}$ and $\overline{R(\Lambda_1)}$.

Since all the remaining steps in the calculation only depend on the coset Virasoro algebra, we conclude that the form of the generalized diffeomorphism and the closure of the gauge algebra proceed in the same way for all affine symmetries G^+.

B. Strong section constraint for an arbitrary Kac–Moody algebra

The section constraint is an important starting point for the construction of any "extended geometry," be it double or exceptional field theory, or some other model with enhanced symmetry algebra $\mathfrak{g}$. The actual form of the Y tensor defining this constraint has normally been determined on a case-by-case basis. This applies in particular to exceptional field theory, where it is notoriously difficult to find tensorial identities applying to every member of the series of exceptional algebras. However, in [48] a general construction of the Y tensor was given, based on bosonic and fermionic extensions of the algebra $\mathfrak{g}$. The identities needed for closure and covariance of the generalized Lie derivative are then automatically satisfied, except for one of them (whose failure is the reason for introducing an extra constrained transformation parameter in the $\mathfrak{e}_8$ case).

The construction of the Y tensor in [48] was given explicitly for exceptional field theory, but can easily be generalized to any highest (or lowest) weight representation $R(\lambda)$ of any Kac–Moody algebra $\mathfrak{g}$, except for cases where $\mathfrak{g}$ or its fermionic extension has a degenerate Cartan matrix. In this section we will obtain a general formula for the Y tensor which includes also the degenerate cases, and thus encompasses all the known finite-dimensional examples and the affine algebra examples described in this paper. We will restrict to simply laced $\mathfrak{g}$. In general we consider the Lie group G defined in [44] for an arbitrary Kac–Moody algebra $\mathfrak{g}$.

A vector $|p\rangle$ in a highest weight representation $R(\lambda)$ satisfies the (weak) section constraint if $|p\rangle \otimes |p\rangle \in R(2\lambda)$. This is equivalent to the statement that $|p\rangle$ is in a minimal $R(\lambda)$-orbit under $\mathfrak{g}$. This is discussed e.g. in [14], and a direct connection between minimal orbits and Borcherds superalgebras (the fermionic extensions of $\mathfrak{g}$) was made in [66].

The quadratic Casimir,

$$\begin{aligned} C_2 &= \frac{1}{2}\eta_{AB} : T^A T^B \\ &\coloneqq \sum_{\alpha \in \Delta_+} E_{-\alpha} E_\alpha + \frac{1}{2}(H, H) + (\varrho, H), \end{aligned} \tag{6.8}$$

is defined for finite- and infinite-dimensional Kac–Moody algebras on a highest weight module, where the Weyl vector ϱ is the sum of the fundamental weights (instead of half the sum of the possibly infinitely many positive roots in Δ_+). It is normalized by $C_2(R(\lambda)) = \frac{1}{2}(\lambda, \lambda + 2\varrho)$, so that $C_2(adj) = g^\vee$ for finite-dimensional $\mathfrak{g}$. Here T^A are the generators of $\mathfrak{g}$ and η_{AB} is the invariant symmetric bilinear form. The last term is a normal ordering term, which for finite-dimensional $\mathfrak{g}$ can be absorbed into a symmetrically ordered product of generators. We observe that $C_2(R(2\lambda)) = 2C_2(R(\lambda)) + (\lambda, \lambda)$. Also, there is no other irreducible highest weight representation in the symmetric product of $R(\lambda)$ with itself with this maximal value of C_2.

The weak section constraint on $|p\rangle$ is equivalent to the equation

$$\begin{aligned} 0 &= [C_2(R(2\lambda)) - 2C_2(R(\lambda)) - (\lambda, \lambda)] |p\rangle \otimes |p\rangle \\ &= \frac{1}{2}\eta_{AB} : T^A T^B : (|p\rangle \otimes |p\rangle) - \left(\frac{1}{2}\eta_{AB} : T^A T^B : |p\rangle\right) \otimes |p\rangle \\ &\quad - |p\rangle \otimes \left(\frac{1}{2}\eta_{AB} : T^A T^B : |p\rangle\right) - (\lambda, \lambda)|p\rangle \otimes |p\rangle \\ &= [\eta_{AB} T^A \otimes T^B - (\lambda, \lambda)] |p\rangle \otimes |p\rangle. \end{aligned} \tag{6.9}$$

[7] Another case where one has $c = \frac{1}{2}$ as for E_9 is the affine extension A_1^+ of $SL_2(\mathbb{R})$ (the Geroch group [55–57]) corresponding to pure four-dimensional Einstein gravity. Also the coset constructions based on A_2 or any finite-dimensional exceptional algebra fall in the minimal series.

Any vector satisfying $|p\rangle\otimes|p\rangle\in R(2\lambda)$ satisfies this equation by construction and it was proven in [44] that all the solutions to this equation are in the $\mathbb{R}^\times\times G$-orbit of the highest weight vector $|\lambda\rangle$ of $R(\lambda)$.[8] This equation determines therefore the unique minimal nontrivial G-orbit in $R(\lambda)$, where the $\mathbb{R}^\times$ is related to rescalings in the one-dimensional highest weight space.

In order to define the strong section constraint we now consider a second vector $|q\rangle$ such that all $|p\rangle$, $|q\rangle$ and $|p\rangle+|q\rangle$ satisfy the section constraint. Since (6.9) is by construction G invariant, one can assume without loss of generality that $|p\rangle=|\lambda\rangle$, the highest weight vector. Then it is convenient to decompose

$$|q\rangle=\sum_{k=0}^{n}|q\rangle_k,\qquad (\lambda,H)|q\rangle_k=((\lambda,\lambda)-k)|q\rangle_k,\qquad(6.10)$$

and the positive roots as $\Delta_+=\sum_{k\geq 0}\Delta_k$ such that $\alpha_k\in\Delta_k$ satisfies $(\lambda,\alpha_k)=k$, and n is the lowest weight for which $|q\rangle_n$ is nonzero. Because the weight is preserved by the operator $\eta_{AB}T^A\otimes T^B$, one obtains that the lowest weight component of the constraint on $|p\rangle\otimes|q\rangle+|q\rangle\otimes|p\rangle$ reduces to

$$\begin{aligned}0&=([\eta_{AB}T^A\otimes T^B-(\lambda,\lambda)](|\lambda\rangle\otimes|q\rangle+|q\rangle\otimes|\lambda\rangle))_n\\&=[\eta_{AB}T^A\otimes T^B-(\lambda,\lambda)](|\lambda\rangle\otimes|q\rangle_n+|q\rangle_n\otimes|\lambda\rangle)\\&=-n|\lambda\rangle\otimes|q\rangle_n-n|q\rangle_n\otimes|\lambda\rangle\\&\quad+\sum_{k=1}^{n}\sum_{\alpha_k\in\Delta_k}(E_{-\alpha_k}|\lambda\rangle\otimes E_{\alpha_k}|q\rangle_n+E_{\alpha_k}|q\rangle_n\otimes E_{-\alpha_k}|\lambda\rangle),\end{aligned}\qquad(6.11)$$

which in turn can only be satisfied if

$$n|\lambda\rangle\otimes|q\rangle_n=\sum_{\alpha_n\in\Delta_n}E_{\alpha_n}|q\rangle_n\otimes E_{-\alpha_n}|\lambda\rangle.\qquad(6.12)$$

The only solution is

$$|q\rangle_n=\sum_{\alpha_n\in\Delta_n}v_{\alpha_n}E_{-\alpha_n}|\lambda\rangle.\qquad(6.13)$$

Recalling from (6.10) that n is the maximal value for which $|q\rangle_n$ is nonzero, we now consider the lowest weight component of the constraint on $|q\rangle\otimes|q\rangle$, i.e.

$$\begin{aligned}&([\eta_{AB}T^A\otimes T^B-(\lambda,\lambda)]|q\rangle\otimes|q\rangle)_{2n}\\&=\sum_{\alpha_n,\beta_n\in\Delta_n}v_{\alpha_n}v_{\beta_n}[\eta_{AB}T^A\otimes T^B-(\lambda,\lambda)]E_{-\alpha_n}|\lambda\rangle\otimes E_{-\beta_n}|\lambda\rangle\\&=\sum_{\alpha_n,\beta_n\in\Delta_n}v_{\alpha_n}v_{\beta_n}\Big(((\alpha_n,\beta_n)-2n)E_{-\alpha_n}|\lambda\rangle\otimes E_{-\beta_n}|\lambda\rangle\\&\quad+\sum_{\gamma\in\Delta}[E_\gamma,E_{-\alpha_n}]|\lambda\rangle\otimes[E_{-\gamma},E_{-\beta_n}]|\lambda\rangle\Big).\end{aligned}\qquad(6.14)$$

There is a lowest weight $\lambda-\alpha_n$ such that $v_{\alpha_n}\neq 0$, i.e., $v_{\alpha_n+\gamma_0}=0$ for all positive γ_0 on level 0. This implies that there is no contribution to the term in $((\alpha_n,\alpha_n)-2n)E_{-\alpha_n}|\lambda\rangle\otimes E_{-\alpha_n}|\lambda\rangle$ from $[E_{\gamma_0},E_{-\alpha_n-\gamma_0}]|\lambda\rangle\otimes[E_{-\gamma_0},E_{-\alpha_n+\gamma_0}]|\lambda\rangle$ and we must therefore have $(\alpha_n,\alpha_n)=2n$. Since in general $(\alpha_n,\alpha_n)\leq 2$ for any Kac–Moody algebra, the constraint on $|q\rangle$ can only have solutions with $n=1$.

We thus have

$$|q\rangle=\Big(v_0+\sum_{\alpha_1\in\Delta_1}v_{\alpha_1}E_{-\alpha_1}\Big)|\lambda\rangle,\qquad(6.15)$$

and the weak section constraint reduces to

$$\begin{aligned}&[\eta_{AB}T^A\otimes T^B-(\lambda,\lambda)]|q\rangle\otimes|q\rangle\\&=\sum_{\alpha_1,\beta_1\in\Delta_1}v_{\alpha_1}v_{\beta_1}\Big(((\alpha_1,\beta_1)-2)E_{-\alpha_1}|\lambda\rangle\otimes E_{-\beta_1}|\lambda\rangle\\&\quad+\sum_{\pm\gamma_0\in\Delta_0}[E_{\gamma_0},E_{-\alpha_n}]|\lambda\rangle\otimes[E_{-\gamma_0},E_{-\beta_n}]|\lambda\rangle\\&\quad+E_{-\alpha_1}E_{-\beta_1}|\lambda\rangle\otimes|\lambda\rangle+|\lambda\rangle\otimes E_{-\alpha_1}E_{-\beta_1}|\lambda\rangle\Big).\end{aligned}\qquad(6.16)$$

The vector $|q\rangle$ automatically solves the section constraint if v_{α_1} is only nonzero for the simple root dual to λ, and by construction for any v_{α_1} obtained from the latter by the action of the stabilizer G_0 of (λ,H). The same theorem from [44] implies then moreover that all the solutions are G_0-conjugate to this one.

Now we can compute for the orbit representative $|q\rangle\otimes|\lambda\rangle$ [with $|q\rangle$ as in (6.15)] that

$$\begin{aligned}&[\eta_{AB}T^A\otimes T^B-(\lambda,\lambda)]|q\rangle\otimes|\lambda\rangle\\&=\sum_{\alpha_1\in\Delta_1}v_{\alpha_1}\Big(-E_{-\alpha_1}|\lambda\rangle\otimes|\lambda\rangle+\sum_{\beta_1\in\Delta_1}E_{\beta_1}E_{-\alpha_1}|\lambda\rangle\otimes E_{-\beta_1}|\lambda\rangle\Big)\\&=-|q\rangle\otimes|\lambda\rangle+|\lambda\rangle\otimes|q\rangle.\end{aligned}\qquad(6.17)$$

By G-covariance we therefore have that the strong section constraint on any pair of vectors $|p\rangle$ and $|q\rangle$ therefore implies in general that $Y|p\rangle\otimes|q\rangle=0$ for the tensor

[8]The rescaling factor $\mathbb{R}^\times$ is not included in G when $(\lambda,\lambda)=0$, unless $\mathfrak{g}$ includes a central charge. This would be the case for E_{10} for example.

$$\sigma Y = -\eta_{AB} T^A \otimes T^B + (\lambda, \lambda) + \sigma - 1. \tag{6.18}$$

This tensor permits to define the generalized diffeomorphisms uniquely and uniformly for any group G and highest weight representation $R(\lambda)$ as

$$\begin{aligned} \mathcal{L}_\xi |V\rangle &= \langle \partial_V|\xi\rangle |V\rangle - \langle \partial_\xi |V\rangle |\xi\rangle + \langle \partial_\xi |\sigma Y|\xi\rangle \otimes |V\rangle \\ &= \langle \partial_V|\xi\rangle |V\rangle - \eta_{AB}\langle \partial_\xi |T^A|\xi\rangle T^B|V\rangle \\ &\quad + ((\lambda,\lambda) - 1)\langle \partial_\xi|\xi\rangle |V\rangle, \end{aligned} \tag{6.19}$$

such that they reduce to standard diffeomorphisms if $|V\rangle$ and $|\xi\rangle$ satisfy the strong section constraint, and the connection term is valued in $\mathbb{R} \oplus \mathfrak{g}$. The Z tensor is then as usual $Z = Y - 1$. The overall normalization of the Y tensor is of course not determined by the homogeneous condition $Y|p\rangle \otimes |q\rangle = 0$, but follows from demanding that[9] $\sigma Z \in (\mathfrak{g} \oplus \mathbb{R}) \otimes (\mathfrak{g} \oplus \mathbb{R})$, i.e., that the term σ in (6.18) cancels in $\sigma Z = \sigma Y - \sigma$. Note, however, that the closure of these candidate generalized diffeomorphisms is not guaranteed by the construction, neither is it expected that any choice of algebra and representation will lead to a meaningful field theory.

The remarkably simple expression (6.19) turns out to reproduce (by necessity) the invariant tensors used in all previously constructed extended geometries. In particular $(\lambda, \lambda) - 1 = \frac{1}{9-d}$ for E_d type groups with $3 \leq d \leq 11$, except for $d = 9$, in which case one gets instead $(\lambda, \lambda) = 0$ as described in this paper. They also generalize to arbitrary Kac–Moody algebras, and have therefore a potential to be applicable also e.g. in E_d, $d > 9$.

We also remark that the construction above can be used to recover ordinary Riemannian geometry as well by taking $\mathfrak{g} = \mathfrak{sl}(n)$ and coordinates x^a in the fundamental representation. For traceless generators $K^a{}_b$ and $K^c{}_d$ the invariant metric is $\delta^a_d \delta^c_b - \frac{1}{n}\delta^a_b \delta^c_d$ and $(\lambda, \lambda) = 1 - \frac{1}{n}$ for the fundamental representation. Evaluating (6.19) on a vector with components V^a then leads to

$$\mathcal{L}_\xi V^a = \xi^b \partial_b V^a - V^b \partial_b \xi^a, \tag{6.20}$$

the usual Lie derivative for $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{R}$. Moreover, the section constraint $Y|p\rangle \otimes |q\rangle = 0$ becomes trivial in this case so that all coordinates x^a can be used at the same time.

The construction given here agrees with the one in [48], where $\mathfrak{g}$ is extended to a Borcherds superalgebra $\mathcal{B}$. The Cartan matrix A_{ij} of $\mathfrak{g}$ ($i, j = 1, 2, \ldots, r$, where r is the rank of $\mathfrak{g}$) is then extended to a Cartan matrix B_{IJ} of $\mathcal{B}$ ($I, J = 0, 1, \ldots, r$), such that

$$B_{00} = 0, \quad B_{ij} = A_{ij}, \quad B_{0i} = B_{i0} = -(\lambda, \alpha_i). \tag{6.21}$$

We assume both A and B to be nondegenerate, which implies $(B^{-1})_{00} \neq 0$, although the construction can be generalized to arbitrary Kac–Moody algebras $\mathfrak{g}$. In the notation used here, the general expression for Y that follows from the construction in [48] is then

$$\sigma Y = -\eta_{AB} T^A \otimes T^B - \left(1 + \frac{1}{(B^{-1})_{00}}\right) + \sigma. \tag{6.22}$$

Since

$$A_{ki}(B^{-1})_{i0} = B_{ki}(B^{-1})_{i0} = -B_{k0}(B^{-1})_{00}, \tag{6.23}$$

the coefficients of the weight λ in the basis of simple roots α_i of $\mathfrak{g}$ are given by

$$\lambda = -(A^{-1})_{ij} B_{j0} \alpha_i = \frac{(B^{-1})_{i0}}{(B^{-1})_{00}} \alpha_i, \tag{6.24}$$

and its length squared by

$$\begin{aligned} ((B^{-1})_{00})^2 (\lambda, \lambda) &= (B^{-1})_{0i} B_{ij} (B^{-1})_{j0} \\ &= -(B^{-1})_{0i} B_{i0} (B^{-1})_{00} \\ &= -(B^{-1})_{0I} B_{I0} (B^{-1})_{00} \\ &= -(B^{-1})_{00}, \end{aligned} \tag{6.25}$$

from which it follows that (6.22) can be rewritten as (6.18).

In terms of the present work, and $\mathfrak{e}_9$, the Y tensor (4.6) is already manifestly of the form (6.18) with $(\lambda, \lambda) = 0$. It follows from the presentation above that a representative of solutions to the strong section condition is spanned by $\langle 0|$ and a subspace representing the M-theory or type IIB branch of the E_8 strong section condition. The procedure is general and gives a recipe for such an “oxidization” procedure, which can be continued through a series of duality groups X_n with decreasing rank by sequentially removing nodes of the Dynkin diagram corresponding to the coordinate module, with highest weight $|\lambda\rangle$, each time expressing a representative of the solutions of the strong section constraint for X_n as the linear subspace spanned by λ and a section for X_{n-1}. In general X_{n-1} is the Levi stabilizer of the representative $|\lambda\rangle$, that reduces when λ is a fundamental weight to the algebra whose Dynkin diagram is the one of X_n with the node associated to λ removed. The sequence is uniquely determined provided the module $R(\lambda_n)_n$ is irreducible for all n, but this is generally not the case. Whenever the module reduces to several irreducible components, there are as many “oxidized” algebras X_{n-1} as there are irreducible components. The “oxidization” procedure therefore generally gives rise to a tree rather than a linear sequence. For maximal supersymmetry the module becomes reducible in $D = 9$, giving rise to both

[9] In the affine case the scaling is included in $\mathfrak{g}$ through the central extension.

the type IIB and the eleven-dimensional supergravity solution. For half maximal the module becomes reducible in $D = 5$, giving rise to both type IIB on K3 and heterotic solutions.

VII. CONCLUSIONS

We have performed the first and critical step towards an exceptional field theory based on E_9 or other affine groups, which consists in the construction of a closed algebra of gauge transformations. Like in the case of E_8, extra local and constrained rotations are part of the gauge transformations. This is connected to the presence of dual gravity and other (in the present case an infinite number of) mixed tensors. These extra transformations are shown to be such that they do not interfere with the dynamics of the physical part of a vielbein. The precise covariant form of this dynamics remains to be constructed. Our construction makes heavy use of Virasoro generators in order to form and use invariant tensors. We also provide a generalized Scherk–Schwarz reduction, which shows that our gauge transformations reduce to the ones expected from two-dimensional gauged supergravity, and predicted by the tensor hierarchy algebra. A completely generic form of the Y tensor, and thereby of candidate generalized diffeomorphisms based on any Kac–Moody algebra was presented.

One important implication from our construction is that the generalized vielbein should parametrize an element of the coset $G/K(G)$, where the group G is constructed from exponentiation of an extended algebra $\mathfrak{e}_9 \oplus \mathbb{R}L_{-1}$ [just like the twist matrix (5.1)]. In such a construction, the generalized vielbein would include all the fields of the theory, including the scaling factor of the metric in the conformal gauge. It is therefore not clear whether the E_9 exceptional field theory can be formulated without resorting to the conformal gauge, such as to be manifestly invariant under both exceptional and ordinary two-dimensional diffeomorphisms.

The additional gauge transformation involving the tensor Σ is highly degenerate. The parameter Σ is a section-constrained element of $R(\Lambda_0)_{-1} \otimes \overline{R(\Lambda_0)}$, whereas it only enters the generalized diffeomorphism through its projection to $\mathfrak{e}_9 \oplus \mathbb{R}L_{-1}$ defined by C_{-1}. The existence of a Courant algebroid (or generalization thereof) underlying the algebra of generalized diffeomorphisms $\mathcal{L}_{\xi,\Sigma}$ remains unclear at the moment. Another feature of the transformations in their present form is that they are noncovariant, i.e., it is not possible to introduce tensors as in [16]. The situation is in that sense identical to that of the E_8 generalized diffeomorphisms of [23]. In the E_8 case, this was remedied by the introduction of a nondynamical background vielbein and its associated Weitzenböck connection [24]. The corresponding procedure in the present case remains an open problem.

Our construction lends strong support to the relevance of the tensor hierarchy algebra [32]. We are necessarily led to a situation where the algebra consists of T^A_m, K, L_0 and L_{-1}. Also the embedding tensor representation matches the level -1 part of the tensor hierarchy algebra. This is the first instance where additional elements (in this case L_{-1}) are seen in the algebra, and the lesson should be important for the continuation to higher exceptional algebras (see [46]). In the present work, the well developed representation theory for affine algebras, relying in particular on the presence of a Virasoro algebra, was of immense help. If one wants to continue to E_{10} or E_{11} [67], the situation is quite the opposite. Still, level expansions may be helpful, and the existence of a simple generic form for the generalized diffeomorphisms looks encouraging. It would be very interesting to see if a generalized geometry for E_{10} in some way can make contact with the E_{10} emergent space proposal of [68].

ACKNOWLEDGMENTS

We would like to thank Franz Ciceri for comments on the manuscript. This work is partially supported by a PHC PROCOPE, project No. 37667ZL, and by DAAD PPP grant 57316852 (XSUGRA). The work of M. C. and J. P. is supported by the Swedish Research Council, project no. 2015-04268. The work of G. B. was partially supported by the ANR grant Black-dS-String.

APPENDIX A: SOME E_8 REPRESENTATIONS AND TENSOR PRODUCTS

In this Appendix, we collect some useful information about $\mathfrak{e}_8$ representations. First we list the highest weights of those occurring at low levels in the expansions of the various $\mathfrak{e}_9$ representations:

$$\begin{aligned} r(0) &= \mathbf{1}, \\ r(\lambda_1) &= \mathbf{248}, \\ r(\lambda_7) &= \mathbf{3875}, \\ r(2\lambda_1) &= \mathbf{27000}, \\ r(\lambda_2) &= \mathbf{30380}, \\ r(\lambda_8) &= \mathbf{147250}, \\ r(\lambda_1+\lambda_7) &= \mathbf{779247}, \\ r(\lambda_3) &= \mathbf{2450240}, \\ r(\lambda_1+\lambda_2) &= \mathbf{4096000}, \\ r(\lambda_6) &= \mathbf{6696000}. \end{aligned} \tag{A1}$$

The tensor product of two adjoints gives $\mathbf{248} \otimes \mathbf{248} = \mathbf{1} \oplus \mathbf{3875} \oplus \mathbf{27000} \oplus \mathbf{248} \oplus \mathbf{30380}$. The first three are the symmetric product and the last two the antisymmetric. The projection operators on the five irreducible representations in the tensor product are given by [69]

$$\begin{aligned}
\mathbb{P}_{(\mathbf{1})}{}^{MN}{}_{PQ} &= \frac{1}{248}\eta^{MN}\eta_{PQ},\\
\mathbb{P}_{(\mathbf{3875})}{}^{MN}{}_{PQ} &= \frac{1}{7}\delta_P^{(M}\delta_Q^{N)} - \frac{1}{14}f^{A(M}{}_P f_A{}^{N)}{}_Q - \frac{1}{56}\eta^{MN}\eta_{PQ},\\
\mathbb{P}_{(\mathbf{27000})}{}^{MN}{}_{PQ} &= \frac{6}{7}\delta_P^{(M}\delta_Q^{N)} + \frac{1}{14}f^{A(M}{}_P f_A{}^{N)}{}_Q + \frac{3}{217}\eta^{MN}\eta_{PQ},\\
\mathbb{P}_{(\mathbf{248})}{}^{MN}{}_{PQ} &= -\frac{1}{60}f_A{}^{MN}f^A{}_{PQ},\\
\mathbb{P}_{(\mathbf{30380})}{}^{MN}{}_{PQ} &= \delta_{PQ}^{MN} + \frac{1}{60}f_A{}^{MN}f^A{}_{PQ},
\end{aligned} \tag{A2}$$

where indices are lowered and raised with η_{AB} and η^{AB}. The structure constants satisfy the identity [69]

$$f^E{}_{AG}f_{BEH}f^{GIC}f_I{}^{HD} = 24\delta^C_{(A}\delta^D_{B)} + 12\eta_{AB}\eta^{CD} - 20f^E{}_A{}^C f_{EB}{}^D + 10f^E{}_A{}^D f_{EB}{}^C. \tag{A3}$$

APPENDIX B: NORMALIZATION OF THE TRACE

We would like to define a trace on operators acting on the Hilbert space (the representation $R(\Lambda_0)$). The idea is that even if infinities are encountered, they may be consistently renormalized, or even cancel in final results of calculations. It is included here as a speculation. If the trace can be defined in a more rigorous way, it may be useful, since it seems to give correct results at least in some calculations (see below), but we should stress that we have not relied on its use in the derivation of any results in the paper.

The relation of the trace to the quadratic Casimir implies that for some possibly infinite factor $\mathcal{N}$, one must have

$$\mathrm{Tr}\mathbb{1} = 0, \qquad \mathrm{Tr}L_0 = \mathcal{N},$$
$$\mathrm{Tr}T^A_n T^B_m = -\mathcal{N}\delta_{m+n,0}\eta^{AB}, \tag{B1}$$

on the representation space $R(\Lambda_0)$ of the basic module with character

$$(qj(q))^{1/3} = \frac{E_4(q)}{\prod_{n>0}(1-q^n)^8}, \tag{B2}$$

where

$$E_4(q) = 1 + 240\sum_{n>0}\sigma_3(n)q^n = \Theta_{E_8}(q) = \sum_{Q\in E_8} q^{Q^2/2}, \tag{B3}$$

is the theta function of the E_8 lattice and the full character is the partition function of eight free chiral bosons on the E_8 torus. The Hilbert space factorizes into the momentum component in the E_8 lattice and the oscillator Hilbert space $R(\Lambda_0) = E_8 \otimes \mathcal{H}^{\otimes 8}$, and the action of L_0 on $R(\Lambda_0)$ is simply the tensor product action on E_8 and $\mathcal{H}^{\otimes 8}$. The naive computation of (B1) from the Hilbert space trace gives infinite factors for all of them, and one needs to introduce some well chosen insertion to potentially regularize them. One difficulty is to find a regularization that preserves E_9 invariance. We shall simply assume that it exists in the following.

The trace satisfies

$$\mathrm{Tr}_1(\overset{1}{X}\sigma_{12}) = \overset{2}{X}. \tag{B4}$$

Say that $|J\rangle\langle J| \in \mathfrak{e}_9$, then one can decompose it in the base $\mathbb{1}, L_0, T^A_n$, and one can define a projector using the trace formula

$$\begin{aligned}
|J\rangle\langle J| &= \frac{1}{\mathcal{N}}\Big(\mathrm{Tr}|J\rangle\langle J|\cdot L_0 + \mathrm{Tr}L_0|J\rangle\langle J|\cdot\mathbb{1} - \sum_n \eta_{AB}\mathrm{Tr}T^A_n|J\rangle\langle J|\cdot T^B_{-n}\Big)\\
&= \frac{1}{\mathcal{N}}\langle J|C_0|J\rangle.
\end{aligned} \tag{B5}$$

This permits to prove the identity

$$\langle\overset{1}{J}|\sigma_{12}\overset{13}{X}|\overset{1}{J}\rangle = \frac{1}{\mathcal{N}}\mathrm{Tr}_4(\langle\overset{1}{J}|\overset{14}{C}_0|\overset{1}{J}\rangle\sigma_{42}\overset{43}{X}) = \frac{1}{\mathcal{N}}\overset{23}{X}\langle\overset{1}{J}|\overset{12}{C}_0|\overset{1}{J}\rangle \tag{B6}$$

for any operator X acting on the tensor product $R(\Lambda_0)\otimes R(\Lambda_0)$. In the same way one obtains

$$\langle\overset{1}{J}|\sigma_{12}\overset{23}{X}|\overset{1}{J}\rangle = \langle\overset{1}{J}|\overset{13}{X}\sigma_{12}|\overset{1}{J}\rangle = \frac{1}{\mathcal{N}}\langle\overset{1}{J}|\overset{12}{C}_0|\overset{1}{J}\rangle\overset{23}{X}. \tag{B7}$$

These two identities will be very useful in the following.

Based on this formal trace, an alternative computation of the Scherk–Schwarz Ansatz in the absence of L_{-1} gauging goes as follows. For a twist matrix U solely in E_9, such that $\varsigma = 0$ in (5.3), one can define the Ansatz in terms of matrices using the normalized trace

$$\frac{1}{\mathcal{N}}\mathrm{Tr}\mathcal{J} = \frac{1}{\mathcal{N}}\langle\underline{J}|\underline{J}\rangle\langle\partial_J| = \langle\partial v|. \tag{B8}$$

The Scherk–Schwarz Ansatz written in the Dirac formalism then takes the form

$$\begin{aligned}
|V\rangle &= U^{-T}|\underline{V}\rangle,\\
|\xi\rangle &= U^{-T}|\underline{\xi}\rangle,\\
|\Sigma\rangle\langle\pi_\Sigma| &= \frac{1}{\mathcal{N}}e^{-v}\overset{2}{U}{}^{-T}(\langle\underline{J}1|\overset{12}{\underline{C}}_1|\underline{J}1\rangle\otimes|\overset{2}{\underline{\xi}}\rangle)\langle\partial_J|,
\end{aligned} \tag{B9}$$

and we have

$$
\begin{aligned}
\mathcal{L}_{\xi,\Sigma}|\overset{3}{V}\rangle &= \langle\partial_V|\xi\rangle(U^{-T}|\underline{V}\rangle) + \langle\partial_\xi|(C_0-1)(U^{-T}|\underline{\xi}\rangle\otimes|V\rangle) + \frac{1}{\mathcal{N}}e^{-\upsilon}\langle\overset{2}{\partial}_J|\overset{23}{C}_{-1}(\overset{2}{U}{}^{-T}\langle\underline{J}1|\overset{12}{\underline{C}}_1|\underline{J}1\rangle\otimes|\overset{2}{\underline{\xi}}\rangle)\otimes\overset{3}{U}{}^{-T}|\overset{3}{\underline{V}}\rangle \\
&= -U^{-T}\langle\underline{\partial}_J|\underline{\xi}\rangle|\underline{J}\rangle\langle\underline{J}|\underline{V}\rangle - U^{-T}\langle\underline{\partial}\rho|(\underline{C}_0-1)|\underline{\xi}\rangle\otimes|\underline{V}\rangle - U^{-T}\langle\underline{\partial}_J|(\underline{C}_0-1)(|\underline{J}\rangle\otimes|\underline{V}\rangle)\langle\underline{J}|\underline{\xi}\rangle \\
&\quad + \frac{1}{\mathcal{N}}U^{-T}\langle\underline{J}|\otimes\langle\underline{\partial}_J|\overset{23}{\underline{C}}_{-1}\overset{12}{\underline{C}}_1|\underline{J}\rangle\otimes|\underline{\xi}\rangle\otimes|\underline{V}\rangle \\
&= U^{-T}\left[-\langle\underline{\partial}_J|\underline{J}\rangle\langle\underline{J}|\underline{C}_0|\underline{\xi}\rangle\otimes|\underline{V}\rangle + \langle\underline{J}|\otimes\langle\partial_J|\left(-\sigma_{13}+\sigma_{12}(1-\overset{13}{\underline{C}}_0+\overset{23}{\underline{C}}_0) - \frac{1}{\mathcal{N}}\overset{23}{\underline{C}}_{-1}\overset{12}{\underline{C}}_1\right)|\underline{J}\rangle\otimes|\underline{\xi}\rangle\otimes|\underline{V}\rangle\right].
\end{aligned}
\tag{B10}
$$

We can now remove all the σ_{12} and σ_{13} operators using Eq. (B6) as

$$
\begin{aligned}
&\langle\underline{J}|\otimes\langle\partial_J|\left(-\sigma_{13}+\sigma_{12}(1-\overset{13}{\underline{C}}_0+\overset{23}{\underline{C}}_0) + \frac{1}{\mathcal{N}}\overset{23}{\underline{C}}_{-1}\overset{12}{\underline{C}}_1\right)|\underline{J}\rangle\otimes|\underline{\xi}\rangle\otimes|\underline{V}\rangle \\
&= \frac{1}{\mathcal{N}}\langle\underline{J}|\otimes\langle\partial_J|(\overset{23}{\underline{C}}_{-1}\overset{12}{\underline{C}}_1 - \overset{13}{\underline{C}}_0 + \overset{12}{\underline{C}}_0 - [\overset{23}{\underline{C}}_0,\overset{12}{\underline{C}}_0])|\underline{J}\rangle\otimes|\underline{\xi}\rangle\otimes|\underline{V}\rangle \\
&= \frac{1}{\mathcal{N}}\langle\underline{J}|\otimes\langle\partial_J|(\overset{12}{\underline{C}}_1\overset{23}{\underline{C}}_{-1} - \overset{23}{\underline{C}}_0)|\underline{J}\rangle\otimes|\underline{\xi}\rangle\otimes|\underline{V}\rangle,
\end{aligned}
\tag{B11}
$$

where we used (2.21) in the last step.

The final result is

$$
\begin{aligned}
U^T\mathcal{L}_{\xi,\Sigma}|V\rangle &= \frac{1}{\mathcal{N}}[\langle\underline{J}|\otimes\langle\underline{\partial}_J|\underline{C}_1|\underline{J}\rangle]\underline{C}_{-1}(|\underline{\xi}\rangle\otimes|\underline{V}\rangle) - \left[\langle\underline{\partial}_J|\underline{J}\rangle\langle\underline{J}| + \frac{1}{\mathcal{N}}\langle\underline{J}|\underline{J}\rangle\langle\underline{\partial}_J|\right]\underline{C}_0(|\underline{\xi}\rangle\otimes|\underline{V}\rangle) \\
&= \langle\underline{\theta}|\underline{C}_{-1}|\underline{\xi}\rangle\otimes|\underline{V}\rangle + \langle\underline{\vartheta}|\underline{C}_0|\underline{\xi}\rangle\otimes|\underline{V}\rangle,
\end{aligned}
\tag{B12}
$$

corresponding to the ordinary gauging and the L_0-gauging.

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