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Butterfly effect in 3D gravity

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We study the butterfly effect by considering shock wave solutions near the horizon of the anti–de Sitter black hole in some three-dimensional gravity models including 3D Einstein gravity, minimal massive 3D gravity, new massive gravity, generalized massive gravity, Born-Infeld 3D gravity, and new bigravity. We calculate the butterfly velocities of these models and also we consider the critical points and different limits in some of these models. By studying the butterfly effect in the generalized massive gravity, we observe a correspondence between the butterfly velocities and right-left moving degrees of freedom or the central charges of the dual 2D conformal field theories.

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I. INTRODUCTION

It was shown [1–4] that chaos in thermal conformal field theory (CFT) may be described by a shock wave near the horizon of an anti–de Sitter (AdS) black hole. In other words, holographically, the propagation of the shock wave on the horizon provides a description of the butterfly effect in the dual field theory. In field theory, the side butterfly effect may be diagnosed by an out-of-time order four point function between pairs of local operators

$$\langle V_x(0)W_y(t)V_x(0)W_y(t)\rangle_\beta, \tag{1.1}$$

where β is the inverse of the temperature. The butterfly effect may be seen by a sudden decay after the scrambling time, t_*,

$$\frac{\langle V_x(0)W_y(t)V_x(0)W_y(t)\rangle_\beta}{\langle V_x(0)V_x(0)\rangle_\beta\langle W_y(t)W_y(t)\rangle_\beta} \sim 1 - e^{\lambda_L(t-t_*-\frac{|x-y|}{v_B})}, \tag{1.2}$$

where λ_L is the Lyapunov exponent and v_B is the butterfly velocity. The Lyapunov exponent is $\lambda_L = \frac{2\pi}{\beta}$, where β is the inverse of Hawking temperature. And also the butterfly velocity should be identified by the velocity of the shock wave by which the perturbation spreads in the space. People have done some work recently on the butterfly effect and its different aspects [5–17] and also it needs more investigation and calculation to access a better understanding of this interesting natural phenomenon.

Gravity in three dimensions is special and interesting because it is a good toy model for the quantum gravity and by studying 3D gravity we can access a deeper understanding about gravity in higher dimensions. It is also is a good context for holography because we are more familiar with conformal field theories in two dimensions in comparison with other dimensions. In addition, we can construct the ghost free higher derivative gravity models in three dimensions. The 3D Einstein gravity does not have propagating degrees of freedom or gravitons in the bulk but by adding higher derivative terms to the action we can have massive propagating degrees of freedom. For example, by adding a gravitational Chern-Simons action to the 3D Einstein gravity action we have a massive graviton in the linearized level of the topologically massive gravity theory [18].

In this paper, we study the butterfly effect in some of 3D gravity models; we calculate the butterfly velocities of these models and also consider the critical points and different limits in some of them. In Sec. II we study the butterfly effect in the 3D Einstein gravity and we find that the butterfly velocity of 3D Einstein gravity is equal to the velocity of light. In Sec. III we study the butterfly effect in the minimal massive 3D gravity [19] which is proposed for resolving the bulk-boundary clash problem in the topologically massive gravity (TMG) [18] and we consider the TMG limit and the critical point in this model.

In Sec. IV we first review the butterfly effect in the new massive gravity (NMG) [20] by details. Then we study the generalized massive gravity (GMG) [20,21] and its different limits and the critical lines. Then we observe a correspondence between the butterfly velocities and the central charges of the dual 2D conformal field theory. In Sec. V we study the butterfly effect in the Born-Infeld 3D gravity [22–24] and its critical point and we see that at the critical point of the model both of the butterfly velocities vanish. In Sec. VI we study the butterfly effect in the new bigravity (NBG) [25,26] and consider the causality bound in this model and also we consider the logarithmic solutions limit of the new bigravity. The last section is devoted to conclusions.

II. 3D EINSTEIN GRAVITY

The action of the 3D Einstein gravity with a cosmological constant is

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$$\frac{1}{16\pi G}\int d^3x\sqrt{-g}(R-2\Lambda). \tag{2.1}$$

If we vary the action with respect to the metric we find the equations of motion as follows:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=\kappa T_{\mu\nu}. \tag{2.2}$$

To study the butterfly effect, we must to consider the black hole solution. The equations of motion of the 3D Einstein gravity admit this asymptotically AdS black hole solution, which is similar to a nonrotating Banados-Teitelboim-Zanelli black hole [27,28]:

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\varphi^2,$$
$$f(r)=\frac{r^2}{l^2}\left(1-\frac{r_h^2}{r^2}\right),\qquad \Lambda=-\frac{1}{l^2}, \tag{2.3}$$

where r_h is the radius of horizon and l is the AdS radius. The φ coordinate is dimensionless and compact; $0\le\varphi\le 2\pi$. Now let us introduce a coordinate with length dimension $x=l\varphi$; then we have $d\varphi=\frac{dx}{l}$. Also, x coordinate is compact, $0\le x\le 2\pi l$; therefore, the AdS black hole metric is

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+\frac{r^2}{l^2}dx^2. \tag{2.4}$$

Now the aim is to study the shock wave of this model when the above black hole solution of this theory is perturbed by an injection of a small amount of energy. For this aim, it is better to rewrite the solution in the Kruskal coordinate [1],

$$u=\exp\left[\frac{2\pi}{\beta}(r_*-t)\right],$$
$$v=-\exp\left[\frac{2\pi}{\beta}(r_*+t)\right], \tag{2.5}$$

where $\beta=\frac{4\pi}{f'(r)}$ is the inverse of temperature and $dr_*=\frac{dr}{f(r)}$ is the tortoise coordinate.

By making use of this coordinate system, the metric becomes the following [1,5]:

$$ds^2=2A(uv)dudv+B(uv)dx^2. \tag{2.6}$$

Here $A(uv)$ and $B(uv)$ are two functions, given by $f(r)$, whose near horizon expansions are

$$A(uv)=-2cl^2(1-2cuv+3c^2u^2v^2-4c^3u^3v^3+\cdots),$$
$$B(uv)=\frac{r_h^2}{l^2}(1-4cuv+8c^2u^2v^2-12c^3u^3v^3+\cdots), \tag{2.7}$$

where c is an integration constant to be fixed later. Now we must study the shock wave; for this aim let us consider an injection of a small amount of energy from the boundary toward the horizon at time $-t_w$. This will cross the $t=0$ time slice while it is red shifted. Therefore the equations of motion should be deformed as

$$\mathcal{E}_{\mu\nu}=\kappa T^s_{\mu\nu}, \tag{2.8}$$

where $\kappa=8\pi G_N$, the energy-momentum tensor has only uu component due to the energy injection:

$$T^S_{uu}=lE\left(\exp\left(\frac{2\pi t_w}{\beta}\right)\delta(u)\delta(x)\right). \tag{2.9}$$

For solving the equations of motion near the horizon to find the shock wave solution, we consider this *Ansatz* for backreacted geometry

$$ds^2=2A(UV)dUdV+B(UV)dx^2$$
$$-2A(UV)h(x)\delta(U)dU^2, \tag{2.10}$$

where the new coordinates U and V are

$$U\equiv u,\qquad V\equiv v+h(x)\Theta(u). \tag{2.11}$$

Plugging the *Ansatz* into the equations of motion (2.2) near the horizon at the leading order, one finds a second order differential equation for $h(x)$

$$(l^4\partial_x^2-r_h^2)h(x)=-\frac{r_h^2}{2c}(\kappa lEe^{2\pi t_w/\beta})\delta(x). \tag{2.12}$$

We can reduce the equation of motion into the following:

$$(\partial_x^2-a^2)h(x)=\xi\delta(x),\qquad a^2=\frac{r_h^2}{l^4},$$
$$\xi=-\frac{r_h^2}{2cl^4}(\kappa lEe^{2\pi t_w/\beta}), \tag{2.13}$$

whose solution is

$$h(x)=-\frac{\xi}{2a}e^{-a|x|}. \tag{2.14}$$

By replacing the values of a and ξ, one can see $h(x)\sim e^{\frac{2\pi}{\beta}[(t_w-t_*)-|x|/v_B]}$, where the scrambling time is $t_*=\frac{\beta}{2\pi}\log(\frac{l}{\kappa})$, with $\kappa=8\pi G_N$ and G_N is Newton's constant in $D=3$, for true value of scrambling time [3]. If we assume $lE\sim 1$, we have to fix the value of integration constant to $c=\frac{r_h}{4l}$ in the above expression. Then one can read the value of the butterfly velocity [3,5,29]:

$$v_B = \frac{2\pi}{\beta a} = 1, \qquad \frac{2\pi}{\beta} = \frac{f'(r)}{2} = \frac{r_h}{l^2}, \qquad (2.15)$$

which is in agreement with [1], where the butterfly velocity in the Einstein gravity in the D dimension is $v_B = \sqrt{\frac{D-1}{2(D-2)}}$. Note that the largest possible butterfly velocity in the Einstein gravity is in $D = 3$ which is equal to light velocity ($v_B = 1$). It is important to note, although in the 3D Einstein gravity there is no propagating degrees of freedom in the bulk, due to boundary degrees of freedom or boundary gravitons the butterfly velocity is nonzero. It is a sign of the relationship between butterfly velocities and boundary degrees of freedom or boundary gravitons. Note that in contradiction of no propagating degrees of freedom in the bulk in 3D Einstein gravity, its dual 2D conformal field theory has a nonzero central charge [30]. In the forth section we will see a correspondence between butterfly velocities and central charges of dual 2D conformal field theory.

III. MINIMAL MASSIVE 3D GRAVITY

The MMG is a model which is proposed for resolving the bulk-boundary clash problem in the TMG (we do not have the positive energy of a graviton and unitary dual 2D conformal field theory at the same time in TMG) [18] by adding a new term to the action in the vielbein formalism [19]. And also we know that the linearized equations of motion of MMG are equal to linearized equations of motion of TMG by making use a redefinition of the topological mass parameter [31–33]. Therefore, the model has a single local degree of freedom that is realized as a massive graviton in the linearization as TMG. The Lagrangian of the minimal massive 3D gravity in the vielbein formalism is

$$L_{\mathrm{MMG}} = -\sigma e.R + \frac{\Lambda_0}{6} e.e \times e + h.T(\omega) + \frac{1}{2\mu}\left(\omega.d\omega + \frac{1}{3}\omega.\omega \times \omega\right) + \frac{\alpha}{2} e.h \times h, \qquad (3.1)$$

where e is the vielbein, ω is the spin connection, and h is a Lagrange multiplier or auxiliary field. Note that the dot and cross mean internal and external product, respectively, the dot implies a contraction of Lorenz indices of two fields with each other, and the cross means a contraction of Lorenz indices of two fields with two indices of the Levi-Civita tensor. The equations of motion of the MMG in the metric formalism is

$$\bar{\sigma} G_{\mu\nu} + \bar{\Lambda}_0 g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{\gamma}{\mu^2} J_{\mu\nu} = \kappa T_{\mu\nu}, \qquad (3.2)$$

where $G_{\mu\nu}$ is the Einstein tensor, $C_{\mu\nu}$ is the 3D Cotton tensor,

$$C_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta} \nabla_\alpha \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}\right), \qquad (3.3)$$

and $J_{\mu\nu}$ is a curvature squared, symmetric tensor:

$$J_{\mu\nu} = R_\mu{}^\lambda R_{\lambda\nu} - \frac{3}{4} R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\left(R_{\rho\sigma} R^{\rho\sigma} - \frac{5}{8} R^2\right). \qquad (3.4)$$

Note that the relations between the parameters of the vielbein and metric formalisms are

$$\gamma = \frac{\alpha}{(1+\sigma\alpha)^2}, \qquad \bar{\sigma} = -\left(\sigma + \alpha + \frac{\alpha^2 \Lambda_0}{2\mu^2(1+\sigma\alpha)^2}\right),$$
$$\bar{\Lambda}_0 = -\Lambda_0\left(1 + \sigma\alpha - \frac{\alpha^3 \Lambda_0}{4\mu^2(1+\sigma\alpha)^2}\right). \qquad (3.5)$$

To study the butterfly effect, we must consider the black hole solution. The equations of motion of the minimal massive 3D gravity (MMG) admit this asymptotically AdS black hole solution equation (2.4) with

$$\bar{\Lambda}_0 = -\frac{\gamma + 4l^2\mu^2\bar{\sigma}}{4l^4\mu^2}. \qquad (3.6)$$

Plugging the *Ansatz* of the Kruskal coordinate equation (2.10) into equations of motion (3.2) near the horizon at the leading order, one finds a third order differential equation for $h(x)$

$$\frac{d^3 h(x)}{dx^3} + \frac{r_h}{2\mu l^3}(\gamma + 2\mu^2 l^2 \bar{\sigma}) h''(x) - \frac{r_h^2}{l^4} h'(x) - \frac{r_h^3}{2\mu l^7}(\gamma + 2\mu^2 l^2\bar{\sigma}) h(x) = -\frac{r_h^3 \mu}{2cl^5}(\kappa l E e^{2\pi t_w/\beta})\delta(x). \qquad (3.7)$$

We can reduce the differential equation to

$$(\partial_x + a)(\partial_x^2 - b^2) h(x) = \xi\delta(x), \qquad a = \frac{r_h}{2\mu l^3}(\gamma + 2\mu^2 l^2 \bar{\sigma}),$$
$$b = \frac{r_h}{l^2}, \qquad \xi = -\frac{r_h^3\mu}{2cl^5}(\kappa l E e^{2\pi t_w/\beta}). \qquad (3.8)$$

We can decompose the above differential equation into two differential equation as follows:

$$q'(x) + a q(x) = \xi\delta(x),$$
$$h''(x) - b^2 h(x) = q(x). \qquad (3.9)$$

The solution of first equation is

$$q(x) = \xi\Theta(x)e^{-ax}. \tag{3.10}$$

If we solve the second equation by making use the above $q(x)$ we find.

$$h(x) = -\frac{\xi}{2b}\left(\frac{e^{-bx}}{a-b} - \frac{2be^{-ax}}{a^2-b^2}\right). \tag{3.11}$$

By replacing the values of a, b, and ξ we can read the scrambling time and the butterfly velocities [5,34] as follows:

$$t_* = \frac{\beta}{2\pi}\log\frac{l}{\kappa}, \qquad v_B^{(1)} = \frac{2\pi}{\beta b} = 1,$$
$$v_B^{(2)} = \frac{2\pi}{\beta a} = \frac{2\mu l}{\gamma + 2\mu^2 l^2 \bar{\sigma}}, \qquad \frac{2\pi}{\beta} = \frac{f'(r)}{2} = \frac{r_h}{l^2}. \tag{3.12}$$

The butterfly effect in the TMG has been studied in [5] and they found the butterfly velocities as follows:

$$v_B^{(1)} = 1, \qquad v_B^{(2)} = \frac{1}{\mu l}. \tag{3.13}$$

One can see that the butterfly velocities of MMG in the TMG limit ($\gamma = 0$, $\bar{\sigma} = 1$) are equal to the butterfly velocities of the TMG equation (3.12).

Now let us consider the critical point of MMG $\gamma = -2\mu l(\mu l\bar{\sigma} - 1)$ where massive and massless modes degenerate and the model has logarithmic solutions [32,33]. At this point, one can see at the critical point that both velocities degenerate and are equal to the butterfly velocity of the 3D Einstein gravity which is equal to the velocity of light:

$$v_B^{(1)} = v_B^{(2)} = 1. \tag{3.14}$$

IV. NEW MASSIVE GRAVITY AND GENERALIZED MASSIVE GRAVITY

The butterfly velocities of the NMG have been obtained in [5]; here we review it with more details. The action of NMG is [20]

$$S_{\rm NMG} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)\right]. \tag{4.1}$$

One can obtain the equations of motion by varying the action with respect to the metric:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.2}$$

where

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}(\nabla_\mu\nabla_\nu R + g_{\mu\nu}\Box R) - 8R_\mu{}^\sigma R_{\sigma\nu} + \frac{9}{2}RR_{\mu\nu} + \left(3R_{\alpha\beta}R^{\alpha\beta} - \frac{13}{8}R^2\right)g_{\mu\nu}. \tag{4.3}$$

The equations of motion of the new massive gravity admit this asymptotically AdS black hole solution equation (2.4) with

$$\Lambda = -\left(\frac{1}{l^2} + \frac{1}{4m^2l^4}\right). \tag{4.4}$$

Plugging the *Ansatz* of the Kruskal coordinate equation (2.10) into the equations of motion, Eq. (4.2), near the horizon at the leading order, one finds a forth order differential equation for $h(x)$[1]:

$$\frac{d^4h(x)}{dx_4} - \frac{r_h^2}{2l^4}(3 + 2m^2l^2)h''(x) + \frac{r_h^2}{2l^8}(1 + 2m^2l^2)h(x) = \frac{r_h^4m^2}{2cl^6}(\kappa lEe^{2\pi t_w/\beta})\delta(x). \tag{4.5}$$

We can reduce the above differential equation to

$$(\partial_x^2 - b_1^2)(\partial_x^2 - b_2^2)h(x) = \xi\delta(x), \qquad b_1^2 = \frac{r_h^2}{l^4},$$
$$b_2^2 = \frac{r_h^2}{2l^4}(1 + 2m^2l^2),$$
$$\xi = \frac{r_h^4m^2}{2cl^6}(\kappa lEe^{2\pi t_w/\beta}). \tag{4.6}$$

We can decompose the above differential equation into two differential equation as follows:

$$q''(x) - b_1^2q(x) = \xi\delta(x),$$
$$h''(x) - b_2^2h(x) = q(x). \tag{4.7}$$

By solving the first equation we have

$$q(x) = -\frac{\xi}{2b_1}e^{-b_1|x|}. \tag{4.8}$$

By replacing the above $q(x)$ in the second equation and solving the equation we find

$$h(x) = \frac{\xi}{2b_1b_2(b_1^2 - b_2^2)}(b_1e^{-b_2x} - b_2e^{-b_1x}). \tag{4.9}$$

Using the expressions for b_1, b_2, and ξ we can read the scrambling time and the butterfly velocities as follows:

[1] The shock wave solution in the Minkowski space background for TMG (and NMG) is studied in [35].

$$t_* = \frac{\beta}{2\pi}\log\frac{l}{\kappa}, \qquad v_B^{(1)} = \frac{2\pi}{\beta b_1} = 1,$$
$$v_B^{(2)} = \frac{2\pi}{\beta b_2} = \frac{1}{\sqrt{m^2l^2+\frac{1}{2}}}, \qquad \frac{2\pi}{\beta} = \frac{f'(r)}{2} = \frac{r_h}{l^2}. \quad (4.10)$$

One can see at the critical point of NMG, $m^2l^2 = \frac{1}{2}$ [36], the two butterfly velocities degenerate into one velocity which is the velocity of light

$$m^2l^2 = \frac{1}{2}, \qquad v_B^{(1)} = v_B^{(2)} = 1. \quad (4.11)$$

Now let us consider the GMG which is the combination of TMG and NMG [20,21]. The action of GMG is the action of NMG plus the gravitational Chern-Simons action:

$$S_{\rm GMG} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)\right] + S_{CS}, \quad (4.12)$$

where the gravitational Chern-Simons action is [37]

$$S_{CS} = \frac{1}{32\pi G\mu}\int d^3x\sqrt{-g}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left[\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right]. \quad (4.13)$$

One can obtain the equations of motion by varying the action with respect to the metric:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = \kappa T_{\mu\nu}, \quad (4.14)$$

where $K_{\mu\nu}$ is defined by Eq. (4.3) and $C_{\mu\nu}$ is the 3D Cotton tensor which is defined by Eq. (3.3). The equations of motion of the generalized massive gravity admit this asymptotically AdS black hole solution equation (2.4) with

$$\Lambda = -\left(\frac{1}{l^2} + \frac{1}{4m^2l^4}\right). \quad (4.15)$$

Plugging the *Ansatz* of the Kruskal coordinate equation (2.10) into the equations of motion, Eq. (4.14), near the horizon at the leading order, one finds a forth order differential equation for $h(x)$:

$$\frac{d^4h(x)}{dx^4} - \frac{r_hm^2}{l\mu}\frac{d^3h(x)}{dx^3} - \frac{r_h^2m^2}{2l^2}\left(2+\frac{3}{m^2l^2}\right)h''(x) + \frac{r_h^3m^2}{\mu l^5}h'(x) + \frac{r_h^4m^2}{2l^6}\left(2+\frac{1}{m^2l^2}\right)h(x) = \frac{r_h^4m^2}{2cl^6}(\kappa lEe^{2\pi t_w/\beta})\delta(x). \quad (4.16)$$

We can write the above differential equation in this form:

$$(\partial_x^2 - b_2^2)(\partial_x^2 - a\partial_x - b_1^2)h(x) = \xi\delta(x),$$
$$b_1^2 = \frac{r_h^2}{2l^4}(1+2m^2l^2),$$
$$a = \frac{r_hm^2}{l\mu}, \qquad b_2^2 = \frac{r_h^2}{l^4},$$
$$\xi = \frac{r_h^4m^2}{2cl^6}(\kappa lEe^{2\pi t_w/\beta}). \quad (4.17)$$

Now let us decompose the above differential equation:

$$q''(x) - b_2^2q(x) = \xi\delta(x),$$
$$h''(x) - ah'(x) - b_1^2h(x) = q(x). \quad (4.18)$$

The first equation is similar to the first equation of Eq. (4.7); therefore, we have $q(x) = -\frac{\xi}{2b_2}e^{-b_2|x|}$, and if we put $q(x)$ in the above second equation we find

$$h(x) = \frac{\xi}{4b_2\sqrt{a^2+4b_1^2}(b_1^2-b_2(a+b_2))}\Big[-2\sqrt{a^2+4b_1^2}e^{-b_2x} + \left(\sqrt{a^2+4b_1^2}+2b_2+a\right)e^{-\frac{1}{2}(-a+\sqrt{a^2+4b_1^2})x} + \left(\sqrt{a^2+4b_1^2}-2b_2-a\right)e^{-\frac{1}{2}(-a-\sqrt{a^2+4b_1^2})x}\Big]. \quad (4.19)$$

Using the expressions for a, b_1, b_2, and ξ we can read the scrambling time and the butterfly velocities as follows:

$$t_* = \frac{\beta}{2\pi}\log\frac{l}{\kappa}, \qquad v_B^{(1)} = \frac{2\pi}{\beta b_2} = 1,$$
$$v_B^{(2)} = \frac{2\pi}{\beta(\frac{1}{2}(-a+\sqrt{a^2+4b_1^2}))} = \frac{2\mu}{m^2l}\left(\frac{1}{-1+\sqrt{1+\frac{2\mu^2}{m^2}(2+\frac{1}{m^2l^2})}}\right),$$
$$v_B^{(3)} = \frac{2\pi}{\beta(\frac{1}{2}(-a-\sqrt{a^2+4b_1^2}))} = \frac{2\mu}{m^2l}\left(\frac{-1}{1+\sqrt{1+\frac{2\mu^2}{m^2}(2+\frac{1}{m^2l^2})}}\right), \qquad \frac{2\pi}{\beta} = \frac{r_h}{l^2}. \quad (4.20)$$

Note that for $\mu^2 > 0$ and $m^2 > 0$ in a ghost free regime $v_B^{(3)}$ is negative, which implies moving in the backward direction. At the NMG limit of the model, $\mu \to \infty$, we have

$$v_B^{(2)} = \frac{1}{\sqrt{m^2l^2 + \frac{1}{2}}}, \qquad v_B^{(3)} = -\frac{1}{\sqrt{m^2l^2 + \frac{1}{2}}}. \qquad (4.21)$$

Note that $v_B^{(2)}$ is exactly one of the butterfly velocities in NMG, and also in the TMG limit $m^2 \to \infty$ with finite μ one can see

$$v_B^{(2)} = \frac{1}{\mu l}, \qquad v_B^{(3)} = 0. \qquad (4.22)$$

Here $v_B^{(2)}$ is one of the butterfly velocities in TMG which is in agreement with the result of [5] for TMG equation (3.12). In addition, there is a critical line in the parameter space of GMG at $\frac{1}{2m^2l^2} + \frac{1}{\mu l} = 1$, [21] which in the TMG limit, $m^2 \to \infty$ is the critical point of TMG, $\mu l = 1$ and in the NMG limit, $\mu \to \infty$ is the critical point of NMG, $m^2l^2 = \frac{1}{2}$. One can see at the critical line of GMG we have

$$v_B^{(2)} = 1, \qquad v_B^{(3)} = -\frac{\mu l - 1}{\mu l - \frac{1}{2}}. \qquad (4.23)$$

Note that $v_B^{(2)}$ is the butterfly velocity of the 3D Einstein gravity, and also in the NMG limit, $\mu \to \infty$, $v_B^{(3)} = -1$, which is the butterfly velocity of the 3D Einstein gravity with the negative sign, and is in agreement with Eq. (4.21) at the critical point of NMG, $m^2l^2 = \frac{1}{2}$. In addition, one can see at the critical point of TMG, $\mu l = 1$, we have $v_B^{(3)} = 0$ which is in agreement with Eq. (4.22). Note that in the TMG limit equation (4.22) for $\mu l = -1$ we have $v_B^{(2)} = -1$, which is the butterfly velocity of the 3D Einstein gravity with the negative sign, it may mean moving in the backward direction, and also here there is an interesting point; maybe the negative butterfly velocities imply some instabilities in the dual theory; these instabilities might lead to a phase transition. In [7,17] the authors proposed that the butterfly velocity v_B can be used to diagnose the quantum phase transition in holographic theories. They provided evidence for this proposal with a holographic model exhibiting metal-insulator transitions, in which the derivatives of v_B, with respect to system parameters, characterize quantum critical points with local extremes in the zero temperature limit [7].

We can consider $\mu l = -1$ as the other critical point of the theory; therefore, the other critical line of GMG is $\frac{1}{2m^2l^2} - \frac{1}{\mu l} = 1$. We know from the dual 2D CFT of TMG [38]

$$c_L = \frac{3l}{2G}\left(1 - \frac{1}{\mu l}\right), \qquad c_R = \frac{3l}{2G}\left(1 + \frac{1}{\mu l}\right). \qquad (4.24)$$

One can see at two critical points of TMG, $\mu l = 1$ and $\mu l = -1$, that we have two different chiral modes, right moving and left moving, respectively, as follows:

$$\mu l = 1, \qquad c_L = 0, \qquad c_R = \frac{3l}{G},$$
$$\mu l = -1, \qquad c_L = \frac{3l}{G}, \qquad c_R = 0. \qquad (4.25)$$

And also we know that changing the sign of the topological mass in TMG, $\mu \to -\mu$, is equivalent to the acting parity operator on the theory and is going from the left-moving mode to right-moving mode and vise versa.

Now let us consider the other critical line of GMG at $\frac{1}{2m^2l^2} - \frac{1}{\mu l} = 1$, and at this line the butterfly velocities are

$$v_B^{(3)} = -1, \qquad v_B^{(2)} = \frac{\mu l + 1}{\mu l + \frac{1}{2}}. \qquad (4.26)$$

One can see in the NMG limit, $\mu \to \infty$, $v_B^{(2)} = 1$, which is the butterfly velocity of the 3D Einstein gravity and is in agreement with Eq. (4.21) at the critical point of NMG. In addition, one can see at the other critical point of TMG $\mu l = -1$ that we have $v_B^{(2)} = 0$. We can conclude that there is a correspondence between the butterfly velocities and right-left moving degrees of freedom or the central charges of the dual conformal field theories.

We observed that at both of the critical lines at the NMG limit $\mu \to \infty$ we have both right-moving and left-moving velocities:

$$v_B^{(2)} = 1 \qquad v_B^{(3)} = -1. \qquad (4.27)$$

Note that the new massive gravity is a parity-preserving or even parity model [20]. But at the TMG limits in critical lines or critical points of TMG, we have just the right-moving velocity in one branch and just the left-moving velocity in the other branch; in other words, TMG is a parity violating or odd parity theory:

$$\mu l = 1, \qquad v_B^{(3)} = 0, \qquad v_B^{(2)} = 1,$$
$$\mu l = -1, \qquad v_B^{(3)} = -1, \qquad v_B^{(2)} = 0. \qquad (4.28)$$

In other languages, the theory is chiral at the critical points, $\mu l = 1$ and $\mu l = -1$. These relations are so similar to relations for the central charges of the dual 2D CFT equation (4.25) at the critical points where the theory is chiral. Therefore, we observe a correspondence between the butterfly velocities and the central charges of dual 2D CFT at the critical points of TMG.

Recently a conjecture has been proposed about the lower bound on a diffusion coefficient by the "butterfly velocity" [6,39]:

$$D \geq \frac{\hbar v_B^2}{k_B T}, \tag{4.29}$$

where D is the diffusion coefficient, k_B is the Boltzmann constant, and T is the temperature. And also in [40], the authors studied a universality, which determines the shear viscosity η and electrical conductivity σ in terms of the corresponding "central charges" and naturally leads to a conjectured bound on conductivity in physical systems. And we know the relation between conductivity, charge susceptibility, and the diffusion coefficient:

$$D = \frac{\sigma}{\chi}, \tag{4.30}$$

where χ is charge susceptibility. These bounds on conductivity and the diffusion coefficient may be evidence of correspondence between the butterfly velocities and the central charges of the dual conformal field theories.

V. BORN-INFELD 3D GRAVITY

In this section we study the butterfly effect in the Born-Infeld 3D gravity [22–24], which include the AdS_3 vacuum as well as solutions with the $AdS_2 \times S^1$ symmetry. The action of the Born-Infeld 3D gravity is

$$S_{BI} = -\frac{m^2}{4\pi G}\int d^3x\sqrt{-g}F(R,K,S), \tag{5.1}$$

where

$$F(R,K,S) = \sqrt{1 + \frac{1}{2m^2}\left(R - \frac{1}{2m^2}K - \frac{1}{12m^4}S\right)} - \left(1 + \frac{\Lambda}{2m^2}\right), \tag{5.2}$$

with

$$K \equiv R_{\mu\nu}R^{\mu\nu} - \frac{1}{2}R^2,$$
$$S \equiv 8R^{\mu\nu}R_{\mu\alpha}R^{\alpha}{}_{\nu} - 6RR_{\mu\nu}R^{\mu\nu} + R^3. \tag{5.3}$$

Using this form of the action, the equations of motion read [23]

$$\begin{aligned}
-\frac{\kappa}{4m^2}T_{\mu\nu} = {} & -\frac{1}{2}Fg_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F_R + F_R R_{\mu\nu} + \frac{1}{m^2}[2\nabla_\alpha\nabla_\mu(F_R R^{\alpha}{}_{\nu}) - g_{\mu\nu}\nabla_\alpha\nabla_\beta(F_R R^{\alpha\beta}) - \Box(F_R R_{\mu\nu}) \\
& - 2F_R R_\nu{}^{\alpha}R_{\mu\alpha} + g_{\mu\nu}\Box(F_R R) - \nabla_\mu\nabla_\nu(F_R R) + F_R R R_{\mu\nu}] \\
& - \frac{1}{2m^4}\Big[4F_R R^{\rho}{}_{\mu}R_{\rho\alpha}R^{\alpha}{}_{\nu} + 2g_{\mu\nu}\nabla_\alpha\nabla_\beta(F_R R^{\beta\rho}R^{\alpha}{}_{\rho}) + 2\Box(F_R R_\nu{}^{\alpha}R_{\mu\alpha}) - 4\nabla_\alpha\nabla_\mu(F_R R_\nu{}^{\rho}R^{\alpha}{}_{\rho}) \\
& + 2\nabla_\alpha\nabla_\mu(F_R R R^{\alpha}{}_{\nu}) - g_{\mu\nu}\nabla_\alpha\nabla_\beta(F_R R R^{\alpha\beta}) - \Box(F_R R R_{\mu\nu}) - 2F_R R R_\nu{}^{\rho}R_{\mu\rho} - g_{\mu\nu}\Box(F_R R_{\alpha\beta}R^{\alpha\beta}) \\
& + \nabla_\mu\nabla_\nu(F_R R_{\alpha\beta}R^{\alpha\beta}) - F_R R_{\alpha\beta}R^{\alpha\beta}R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box(F_R R^2) - \frac{1}{2}\nabla_\mu\nabla_\nu(F_R R^2) + \frac{1}{2}F_R R^2 R_{\mu\nu}\Big],
\end{aligned} \tag{5.4}$$

where

$$F_R = \frac{\partial F}{\partial R} = \frac{1}{4m^2}\left[F + \left(1 + \frac{\Lambda}{2m^2}\right)\right]^{-1}. \tag{5.5}$$

The equations of motion of the Born-Infeld 3D gravity admit this asymptotically AdS black hole solution equation (2.4) with

$$\Lambda = -2m^2\left(1 - \sqrt{1 - \frac{1}{m^2 l^2}}\right). \tag{5.6}$$

Plugging the *Ansatz* of the Kruskal coordinate equation (2.10) into the equations of motion, Eq. (5.4), near the horizon at the leading order, one finds a forth order differential equation for $h(x)$:

$$\begin{aligned}
&\frac{d^4h(x)}{dx^4} - \frac{r_h^2}{l^4}(4 + m^2l^2)h''(x) \\
&\quad + \frac{r_h^4}{l^8}\frac{(3m^2l^2+1)(m^2l^2+1)}{m_2 l^2 - 1}h(x) \\
&\quad = -\frac{r_h^4 m}{2cl^7}\sqrt{m^2l^2-1}(\kappa l E e^{2\pi t_w/\beta}).
\end{aligned} \tag{5.7}$$

One can write the above differential equation in this form:

$$(\partial_x^2 - b_1^2)(\partial_x^2 - b_2^2)h(x) = \xi\delta(x),$$
$$b_1^2 = \frac{r_h^2}{2l^4}\left[4 + m^2l^2 + \sqrt{\frac{m^6l^6 - 5m^4l^4 - 8m^2l^2 - 20}{m^2l^2 - 1}}\right],$$
$$b_2^2 = \frac{r_h^2}{2l^4}\left[4 + m^2l^2 - \sqrt{\frac{m^6l^6 - 5m^4l^4 - 8m^2l^2 - 20}{m^2l^2 - 1}}\right],$$
$$\xi = -\frac{r_h^4 m}{2cl^7}\sqrt{m^2l^2-1}(\kappa l E e^{2\pi t_w/\beta}). \tag{5.8}$$

If we decompose the above differential equations, which are similar to Eq. (4.7), the solution of $h(x)$ is exactly the same as Eq. (4.9):

$$h(x) = \frac{\xi}{2b_1 b_2(b_1^2 - b_2^2)}(b_1 e^{-b_2 x} - b_2 e^{-b_1 x}). \tag{5.9}$$

Using the expressions for b_1, b_2, and ξ, we can read the scrambling time and the butterfly velocities as follows:

$$t_* = \frac{\beta}{2\pi}\log\frac{l}{\kappa}, \qquad v_B^{(1)} = \frac{2\pi}{\beta b_1} = \sqrt{\frac{2\sqrt{m^2l^2-1}}{(4+m^2l^2)\sqrt{m^2l^2-1}+\sqrt{m^6l^6-5m^4l^4-8m^2l^2-20}}},$$
$$v_B^{(2)} = \frac{2\pi}{\beta b_2} = \sqrt{\frac{2\sqrt{m^2l^2-1}}{(4+m^2l^2)\sqrt{m^2l^2-1}-\sqrt{m^6l^6-5m^4l^4-8m^2l^2-20}}}, \qquad \frac{2\pi}{\beta} = \frac{r_h}{l^2}. \tag{5.10}$$

Now let us consider the critical point of the Born-Infeld 3D gravity, $m^2l^2 = 1$, where the model has logarithmic wave solutions [24]. One can see that at the critical point, the above two velocities degenerate and are equal to zero

$$m^2l^2 = 1, \qquad v_B^{(1)} = v_B^{(2)} = 0. \tag{5.11}$$

In [29] we observed that by adding a higher curvature correction to the Einstein gravity, the butterfly velocity decreases at the critical point. It is interesting that the butterfly velocities in the Born-Infeld 3D gravity vanish at the critical point, and it is important to note that the Born-Infeld 3D gravity has an infinite higher derivative in the level of the action because of the square root form of the action.

And also it is worth noting that at the critical point of the Born-Infeld gravity, both the central charges of the dual 2D CFT vanish [23,24]:

$$m^2l^2 = 1, \qquad c_L = c_R = \frac{3l}{2G}\sqrt{1-\frac{1}{m^2l^2}} = 0. \tag{5.12}$$

Maybe we can say it is another evidence for correspondence between the butterfly velocities and the central charges of the dual 2D CFT.

VI. NEW BIGRAVITY

The NBG is a recently proposed 3D gravity model for resolving the bulk-boundary clash in the new massive gravity [25,26] if we consider the NMG action by using an auxiliary field $f_{\mu\nu}$ and then promote the auxiliary field to the dynamical field. The NBG action is

$$\begin{aligned} S_{\rm NBG} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\Big(\sigma R[g] + f_{\mu\nu}\mathcal{G}^{\mu\nu}[g] \\ + \frac{1}{4}m^2(\tilde{f}^{\mu\nu}f_{\mu\nu} - \tilde{f}^2) - 2\Lambda_g\Big) \\ + \frac{1}{16\pi\tilde{G}}\int d^3x\sqrt{-f}(R[f] - 2\Lambda_f), \end{aligned} \tag{6.1}$$

where Λ_f is a new cosmological constant, $\tilde{G}$ is Newton constant of the new metric, and $R[g]$ and $R[f]$ are Ricci scalars constructed from $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. $\mathcal{G}_{\mu\nu}$ is the Einstein tensor of the metric $g_{\mu\nu}$. Note that all indices are raised by $g^{\mu\nu}$ except those in the definition of Ricci scalar $R[f]$ which are raised by the inverse metric $f^{\mu\nu}$.

By varying the above NBG action with respect to the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ one can find the equations of motion:

$$\begin{aligned} &\mathcal{G}[g]_{\mu\nu} + \Lambda_g g_{\mu\nu} + \frac{m^2}{2}\Big[\tilde{f}_\mu{}^\rho f_{\nu\rho} - \tilde{f}f_{\mu\nu} - \frac{1}{4}g_{\mu\nu}(\tilde{f}^{\rho\sigma}f_{\rho\sigma} - \tilde{f}^2)\Big] \\ &\quad + 2\tilde{f}_{(\mu}{}^\rho\mathcal{G}[g]_{\nu)\rho} + \frac{1}{2}f_{\mu\nu}R[g] - \frac{1}{2}\tilde{f}R_{\mu\nu}[g] - \frac{1}{2}g_{\mu\nu}f_{\rho\sigma}\mathcal{G}[g]^{\rho\sigma} \\ &\quad + \frac{1}{2}[\nabla^2[g]f_{\mu\nu} - 2\nabla[g]^\rho\nabla[g]_{(\mu}f_{\nu)\rho} + \nabla[g]_\mu\nabla[g]_\nu\tilde{f} \\ &\quad + (\nabla[g]^\rho\nabla[g]^\sigma f_{\rho\sigma} - \nabla^2[g]\tilde{f})g_{\mu\nu}] = \kappa T_{\mu\nu}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} &\mathcal{G}[f]_{\mu\nu} + \Lambda_f f_{\mu\nu} - \frac{1}{k}\sqrt{\frac{g}{f}}\Big[f_{\alpha\mu}f_{\beta\nu}\mathcal{G}[g]^{\alpha\beta} \\ &\quad + \frac{1}{2}m^2(g^{\sigma\alpha}g^{\tau\beta} - g^{\sigma\tau}g^{\alpha\beta})(f_{\sigma\tau}f_{\alpha\mu}f_{\beta\nu})\Big] = \kappa T_{\mu\nu}, \end{aligned} \tag{6.3}$$

where $\mathcal{G}[f]_{\mu\nu}$ is the Einstein tensor of the metric $f_{\mu\nu}$ and $k = \frac{G}{\tilde{G}}$ is the relative strength of two Newton constants associated with two metrics.

The equations of motion of new bigravity admit this asymptotically AdS black hole solution:

$$\begin{aligned} ds_g^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2}dx^2, \\ f(r) = \frac{r^2}{l^2}\Big(1 - \frac{r_h^2}{r^2}\Big), \qquad ds_f^2 = \gamma ds_g^2, \end{aligned} \tag{6.4}$$

with

$$4-\gamma^2 l_g^2 m^2+2\gamma+4l_g^2\Lambda_g=0,$$
$$1-m^2 l_f^2-k\frac{l_g}{l_f}(1+\Lambda_f l_f^2)=0. \tag{6.5}$$

For solving the equations of motion near the horizon to find the shock wave solution, we consider these *Ansätze* in the Kruskal coordinate for backreacted geometry

$$ds_g^2=2A(UV)dUdV+B(UV)dx^2 -2A(UV)h(x)\delta(U)dU^2,$$
$$ds_f^2=2A(UV)dUdV+B(UV)dx^2 -2A(UV)\rho(x)\delta(U)dU^2. \tag{6.6}$$

Note that here we take $\gamma=1$. In other words, we take same background for $g_{\mu\nu}$ and $f_{\mu\nu}$ but the perturbations around the background are different [25] by the $h(x)$ and $\rho(x)$ functions. Plugging the *Ansatz* of the Kruskal coordinate equation (6.6) into the equations of motion, Eqs. (6.2) and (6.3), near the horizon at the leading order, one finds two coupled forth order differential equation for $h(x)$ and $\rho(x)$:

$$2\rho''(x)-5h''(x)-2\frac{r_h^2}{l^4}(l^2m^2-1)\rho(x) +\frac{r_h^2}{l^4}(2l^2m^2+1)h(x)=\xi\delta(x),$$
$$k\rho''(x)-h''(x)-\frac{r_h^2}{l^4}(l^2m^2+k-2)\rho(x) +\frac{r_h^2}{l^4}(l^2m^2-1)h(x)=-\frac{k}{2}\xi\delta(x), \tag{6.7}$$

where $\xi=\frac{r_h^2}{cl^4}(\kappa lEe^{2\pi t_w/\beta})$. The solutions of the above coupled differential equations are

$$h(x)=\xi\left[\frac{-(1+k)}{2r_hl^2(1+2k)}e^{-\frac{r_h}{l^2}x}+\frac{k^2+k-2}{2r_hl^2(1+2k)\sqrt{5k-2}\sqrt{l^2m^2(1+2k)+k-4}}e^{-\frac{r_h}{l^2}\sqrt{\frac{l^2m^2(1+2k)+k-4}{5k-2}}x}\right],$$
$$\rho(x)=\xi\left[\frac{-(1+k)}{2r_hl^2(1+2k)}e^{-\frac{r_h}{l^2}x}-\frac{3(k+2)}{2r_hl^2(1+2k)\sqrt{5k-2}\sqrt{l^2m^2(1+2k)+k-4}}e^{-\frac{r_h}{l^2}\sqrt{\frac{l^2m^2(1+2k)+k-4}{5k-2}}x}\right]. \tag{6.8}$$

From the above expressions, one can read the scrambling time and the butterfly velocities as follows:

$$t_*=\frac{\beta}{2\pi}\log\frac{l}{\kappa},\qquad v_B^{(1)}=\frac{2\pi}{\beta(\frac{r_h}{l^2})}=1,$$
$$v_B^{(2)}=\frac{2\pi}{\beta(\frac{r_h}{l^2}\sqrt{\frac{l^2m^2(1+2k)+k-4}{5k-2}})}=\sqrt{\frac{5k-2}{l^2m^2(1+2k)+k-4}},$$
$$\frac{2\pi}{\beta}=\frac{r_h}{l^2}. \tag{6.9}$$

Now lets consider the $k=1$ case which happens when two Newton constants are equal,

$$k=1,\qquad v_B^{(2)}=\frac{1}{\sqrt{m^2l^2-1}}. \tag{6.10}$$

For respecting the causality, the butterfly velocity must be equal or less than the velocity of light, $v_B^{(2)}\le 1$; therefore, we have

$$m^2l^2\ge 2. \tag{6.11}$$

For $m^2l^2=2$ the butterfly velocity is equal to the velocity of light, $v_B^{(2)}=1$, which is the butterfly velocity of the 3D Einstein gravity.

Finally, it is interesting to consider the logarithmic solutions limit of the new bigravity [25]:

$$1-\frac{\gamma}{2}+k\sqrt{\gamma}=0. \tag{6.12}$$

Here we take $\gamma=1$, then $k=-\frac{1}{2}$; therefore, we have

$$k=-\frac{1}{2},\qquad v_B^{(1)}=v_B^{(2)}=1. \tag{6.13}$$

This situation is similar to critical points of TMG, MMG, and NMG where the models have logarithmic solutions and the two butterfly velocities degenerate into one and it is equal to the butterfly velocity of the 3D Einstein gravity, which is the velocity of light.

VII. CONCLUSIONS

In this paper we study some of three-dimensional gravity models, we calculate the butterfly velocities of these models, and also we consider critical points and different limits in some of them. In Sec. II we study the butterfly effect in the 3D Einstein gravity by considering the shock wave in the Kruskal coordinate near the horizon of the AdS black hole, and we find that the butterfly velocity of the 3D Einstein gravity is equal to the velocity of light, which is in agreement with [1] in $D = 3$. Although in the 3D Einstein gravity there is no propagating degree of freedom or graviton in the bulk, due to boundary degrees of freedom or boundary gravitons the butterfly velocity is nonzero.

In Sec. III we study the butterfly effect of the minimal massive 3D gravity, and we consider the TMG limit of the model which was in agreement with the results of [5]. We study the critical point of the model and we observed that the two butterfly velocities degenerate at the critical point and are equal to the butterfly velocity of the 3D Einstein gravity.

In Sec. IV we first review the butterfly effect in the new massive gravity by details and consider the critical point of the model where the two butterfly velocities degenerate and are equal to the butterfly velocity of the 3D Einstein gravity, then we study the butterfly effect in the generalized massive gravity and we find three butterfly velocities for this theory. Then we consider TMG and NMG limits of the theory and critical lines and critical points of the model and we observed that there is a correspondence between the butterfly velocities and right-left moving degrees of freedom or the central charges of the dual two-dimensional conformal field theory.

In Sec. V we study the butterfly effect in the Born-Infeld 3D gravity and we find that at the critical point of the model, the two butterfly velocities degenerate and are equal to zero. It is interesting that the butterfly velocities in the Born-Infeld 3D gravity vanish at the critical point. We know that the Born-Infeld 3D gravity has an infinite higher derivative in the level of the action because of the square root form of the action and also in [29] we observed that by adding a higher curvature correction to the Einstein gravity, the butterfly velocity decreases at the critical point. And also, both of the central charges of the dual 2D CFT vanish at the critical point of the model, it may be other evidence for correspondence between the butterfly velocities and the central charges of the dual 2D CFT.

In Sec. VI we study the butterfly effect in the new bigravity model and we find a causality bound in the parameter space of the model and also we consider the logarithmic solutions limit of the new bigravity and we observed that in this limit the two butterfly velocities degenerate into one which is equal to the butterfly velocity of the 3D Einstein gravity. In following, it is so important and also interesting to study the butterfly effect in the dual 2D conformal field theories [41–43] of these models and rederive the obtained results by CFT calculations.

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