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S-duality for holographic p-wave superconductors

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We consider the generalization of the S-duality transformation previously investigated in the context of the fractional quantum Hall effect (FQHE) and s-wave superconductivity to p-wave superconductivity in $2+1$ dimensions in the framework of the AdS/CFT correspondence. The vector Cooper condensate transforms under the S-duality action to the pseudovector condensate at the dual side. The $3+1$-dimensional Einstein-Yang-Mills theory, the holographic dual to p-wave superconductivity, is used to investigate the S-duality action via the AdS/CFT correspondence. It is shown that, in order to implement the duality transformation, chemical potentials on both the electric and magnetic sides of the duality have to be introduced. A relation for the product of the non-Abelian conductivities in the dual models is derived. We also conjecture a flavor S-duality transformation in the holographic dual to $3+1$-dimensional QCD low-energy QCD with non-Abelian flavor gauge groups. The conjectured S-duality interchanges isospin and baryonic chemical potentials.

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I. INTRODUCTION

The AdS/CFT correspondence opens a new route to studying strongly correlated systems with bosonic and fermionic degrees of freedoms at finite chemical potential and density. In gauge/gravity duality, the charge density ρ conjugate to the chemical potential is dual to an electric flux emanating from the boundary of a space-time which is asymptotically anti-de Sitter (AdS). Due to flux conservation, this charge has to reside behind the horizon of a black hole in the interior of the AdS space-time. A magnetic field in the dual field theory corresponds to switching on a magnetic flux component in the bulk space-time.

The S-duality transformation, which exchanges electric with magnetic field strengths, is a well-studied symmetry for the U(1) Einstein-Maxwell theory in $3+1$-dimensional asymptotically AdS space-times. It acts nontrivially on the boundary conditions of the U(1) gauge fields, exchanging Neumann to Dirichlet boundary conditions [1], in this way exchanging electrically charged with magnetically charged black holes and hence charge density with magnetic field in the dual field theory. It corresponds to particle-vortex duality in the $2+1$-dimensional dual field theory, and also acts in this way on the conductivities in the field theory.

S-duality together with the T-duality transformation generates the group of modular transformations $\mathrm{SL}(2,Z)$. The bulk S-duality acts naturally on the conserved U(1) currents of $2+1$-dimensional conformal field theories [1] and corresponds to three-dimensional mirror symmetry [2]. The most studied example of subgroups of the $\mathrm{SL}(2,Z)$ symmetry in $2+1$ dimensions is in the framework of the FQHE. It was observed long ago [3] that a combination of the conductivities $\sigma=\sigma_{xy}+i\sigma_{xx}$ transforms fractionally linear under modular transformations, as does the filling fraction $\nu=\frac{\rho}{B}$, where ρ is the density and B is the magnetic field. The modular subgroups $\Gamma_0(2)\subset\mathrm{SL}(2,Z)$ and $\Gamma_\theta(2)\subset\mathrm{SL}(2,Z)$ act as flux attachment transformations on, respectively, the odd-denominator and even-denominator filling fractions states. The full $\mathrm{SL}(2,Z)$ was argued to be relevant for certain $\mathcal{N}=2$ supersymmetric field theories [4]. The generator of the S-duality transformation $\nu\to\frac{1}{\nu}$ interchanges the density and magnetic field and can be understood as particle-hole transformation [1,5]. The nontrivial modular properties of the complex conductivity $\sigma=\sigma_{xy}+i\sigma_{xx}$ can be argued to hold along the renormalization group (RG) flow for energies sufficiently low such that higher derivative operators in the external electromagnetic field can be neglected [3] and the RG flow itself happens on the two-dimensional submanifold in coupling space parametrized by σ. The $\mathrm{SL}(2,Z)$ (subgroup) action in the FQHE was implemented as the S-duality transformation in several $3+1$-dimensional holographic dual systems [6–8], where the dual gravitational theory used was an Einstein-Maxwell-dilaton-axion model. The state parameters ρ, B are mapped to the charge and magnetic field emanating from the bulk black hole solution, and the modular transformation of $\sigma_{xy}+i\sigma_{xx}$ is related to the transformation of the bulk axiodilaton field and the boundary two-point correlators for the conductivities via the Kubo formula, where the axiodilaton encodes coupling constants σ_{xy} and σ_{xx} governing the two-dimensional low-energy RG flow.

Having understood the holographic S-duality action in the simplest case of a single conserved $U(1)$ current in the presence of charge density and magnetic field, the next natural step is to allow for charged scalar operators to condense in the context of s-wave holographic superconductivity[1] in the (μ, B) space of the grand-canonical ensemble. The new feature introduced by superconductivity is the presence of the condensate and the mass gap and, since holographic superconductors are type II, Abrikosov vortices in the presence of an external magnetic field. The natural guess for the duality mapping of the S-duality action, which exchanges finite density and zero magnetic field with finite magnetic field and zero density, is that the superconducting state with the charged order parameter condensate is mapped to the magnetic catalysis problem triggered by the finite magnetic field and which admits a neutral excitonic condensate. Indeed, such a mapping between the BCS and magnetic catalysis (MC) models has been discussed in $(1+1)$ dimensions (here, DOS stands for the density of states, h is the magnetic field strength, and μ is the chemical potential) [9].

Effectively one-dimensional dynamics in both cases leads to similarities in formulas for the pairing gap Δ and the gain in thermodynamic potential $\delta\Omega$ as compared to the normal unpaired state. The parameters in the two systems at a nonzero density and at a nonzero magnetic field are mapped onto each other as follows: The last line expresses a

$$\mathrm{MC} \leftrightarrow \mathrm{BCS}$$
$$\langle\bar{\psi}\psi\rangle \neq 0 \leftrightarrow \langle\psi\psi\rangle \neq 0$$
$$\text{finite}\, h \leftrightarrow \text{finite}\, \rho$$
$$\text{small}\, \mu \leftrightarrow \text{small}\, \delta\mu$$
$$h \gg \mu \leftrightarrow \mu \gg \delta\mu$$

hierarchy of scales. A similar mapping has been obtained in case of the Gross-Neveu and the BCS model [10], where the magnetic field h maps to the chemical potential mismatch $\delta\mu$ and is relevant for the inhomogeneous superconductors in the incommensurate phase. In the Sakai-Sugimoto model of holographic QCD [9], analytical formulas for the free energy difference between condensed and normal states have been obtained, proving the stability of both condensed states. In the strong magnetic field regime ("direct" magnetic catalysis), the free energy difference takes a remarkably simple form [9],

$$\delta\Omega \sim -h\left(\frac{\Delta(h)^2}{2} - \mu^2\right), \tag{1}$$

which exactly maps to the result obtained in the field theory, Table I.

[1] In most holographic systems, the $U(1)$ symmetry, which gets broken in the superconducting phase, is actually a global $U(1)$ which was ungauged during the process of holographic renormalization. During this process, the UV cutoff is taken to infinity, and gauge symmetries become global. Nevertheless, the physics of the broken state more closely resembles a superconductor than a superfluid, as it e.g. admits a dynamically generated gap.

TABLE I. Mapping between the BCS and magnetic catalysis models.

MC	BCS
$(3+1)\mathrm{d} \to (1+1)\mathrm{d}$ in x − space	$(1+1)\mathrm{d}$ in p − space
LLL and $\varepsilon = 0$ surface	Fermi surface $\varepsilon = \mu$
$\varepsilon = \sqrt{p_z^2 + 2\lvert eh\rvert n}$	$\varepsilon = p - p_F,\ p = \sqrt{\vec{p}^2}$
Excitonic: $\Delta \sim G\langle\bar{\psi}\psi\rangle$	SC: $\Delta \sim G\langle\psi\psi\rangle$
$\Delta \sim \sqrt{eh}\exp(-\frac{\mathrm{const}}{G\nu_0})$	$\Delta \sim \mu\exp(-\frac{\mathrm{const}}{G\nu_F})$
ν_0 is DOS at $\varepsilon = 0$	ν_F is DOS at $\varepsilon = \mu$
h enhances, μ destroys Δ	μ enhances, h destroys Δ
$\delta\Omega \sim h(\mu^2 - \frac{\Delta^2}{2})$	$\delta\Omega \sim \mu^2(\delta\mu^2 - \frac{\Delta^2}{2})$
$h \gg \mu, \Delta$	$\mu \gg \delta\mu, \Delta$
It can have $\mu = 0$	It can have $h = 0$
T_c grows with h (MC)	T_c decreases with h
T_c decreases with μ	T_c grows with μ (SC)

In the gravity dual description, the duality mapping between the two setups, superconductivity and MC [11], is as follows. Some explicit examples for the S-duality transformation between the $2+1$-dimensional field theories at finite density are known in the literature. In Ref. [12], the mapping between the XY model and the Abelian Higgs model was studied. The superfluid phase corresponds to the condensate of the scalar, while the solid phase is generated via the condensate of the monopole operator [12]. The solid state is a kind of Abrikosov lattice of vortices. Similar considerations have been made in the context of the CP_1 model [13], in which the form of the action is identical for the S-dual theory however the global U(1) and topological U(1) currents are interchanged. Some analysis of the corresponding holographic model which involves the electrically and magnetically charged black holes can be found in Ref. [12,14]. The instabilities which yield the bulk condensates were identified. In particular, the crystalline phase at the boundary corresponds to the nontrivial magnetic condensate in the bulk. Another example concerns the mapping between the BCS model at large chemical potentials and Gross-Neveu model in strong external magnetic field [15], Table II.

In this paper, we go one step further and consider the S-duality action on a holographic p-wave superconductor. In a p-wave superconductor, the order parameter is vectorlike since Cooper pairing occurs in the $L = 1$ state. There are two different holographic models of p-wave superconductivity [16–20]: a model with the SU(2) gauge group [16–19] and an Abelian bulk model with additional vector mesons [20]. We shall discuss the bulk SU(2) approach [16] and search for the S-duality action in the bulk theory and what it implies in the dual $2+1$-dimensional boundary theory. We shall argue that, looking at the particular solutions to the bulk equation of motion for the condensate

TABLE II. Mapping between the BCS and magnetic catalysis models in the gravity dual description.

Holographic MC	Holographic SC
dyonic AdS RN BH, Schwarzschild BH	AdS RN BH
$\lvert H\rvert > \lvert Q\rvert$	$\lvert Q\rvert > \lvert H\rvert$
it can be $Q = 0$	it can be $H = 0$
Z_2(chiral SB) broken	$U(1)$ broken
magnetic field enhances it	magnetic field destroys it
electric field destroys it	electric field enhances it
Callan – Rubakov effect	dual Callan – Rubakov effect

and the U(1) part of the SU(2) gauge field in the background of the charged black hole (BH) in AdS_4, some general claims concerning the S-duality for the p-wave superconductor can be made. In particular, we will show that in order to define the S-duality a dual "magnetic" chemical potential has to be introduced. It will also be demonstrated that there is a relation between the conductivities in the S-dual theories similar to the one found in holographic models with Abelian bulk fields [6–8]. The vector order parameter is mapped onto the pseudovector of the dual SU(2) field strength under the S-duality transform.

In low-energy QCD, the analog of p-wave superconductivity occurs at nonvanishing isotopic chemical potential [21–24] where the vector mesons condense. Holographic QCD models like [25] include a non-Abelian flavor gauge theory in the dual $4+1$ bulk geometry, and hence it is natural to ask whether it is possible to define a kind of S-duality transform in the dual bulk theory which then induces an S-duality action in the dual low-energy QCD at the boundary. To this aim, we conjecture that the proper S-dual pair in $4+1$ bulk is the isotopic U(1) and the topological U(1) symmetries generated by the topologically conserved current. At the $3+1$ boundary, this pair corresponds to the isotopic and baryonic global charges. Therefore, it is natural to conjecture that the flavor S-duality interchanges the low-energy QCD with isotopic and baryonic chemical potentials correspondingly. We give several consistency arguments in favor of this conjecture.

The paper is organized as follows. In Sec. II, we consider the holographic p-wave superconductor and construct its S-dual in the dual asymptotically AdS_4 space-time. In Sec. III, we solve the equations of motion numerically and check that the solutions satisfy the duality relations close to the phase transition. In Sec. IV, we speculate on a possible S-duality relation for low-energy QCD with isotopic and baryonic chemical potentials. We conclude and discuss our findings in Sec. V. In the Appendixes, we prove $SL(2,Z)$ invariance of the SU(2) symmetric axiodilaton-Yang-Mills action and review the p-wave superconductor in the five-dimensional setting.

II. p-WAVE SUPERCONDUCTOR AND ITS S-DUAL

A. Equations of motion

We base our consideration of the p-wave superconductor on the electric-magnetic duality for the non-Abelian gauge fields in four dimensions [18]. In full analogy to the U(1) gauge field case, we show the SL(2,Z) invariance of the Einstein-Maxwell action with the SU(2) gauge fields which are coupled to an axion and a dilaton field in Appendix A. The gauge-gravity action coupled to an axiodilaton is

$$S_{\phi,\chi} = -\int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}\left[R-2\Lambda+\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi+e^{2\phi}\partial_\mu\chi\partial^\mu\chi)\right]+\frac{1}{4}e^{-\phi}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}\chi F_{\mu\nu}*F^{\mu\nu}\right), \tag{2}$$

where the scalar fields are dilaton ϕ and axion χ and the field strength $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$ with SU(2) gauge field $A_\mu=A^a_\mu\tau_a$. The dual field strength is obtained by applying the Hodge star operation $*F_{\mu\nu}\coloneqq\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$, where a completely antisymmetric Levi-Cività tensor $\epsilon_{\mu\nu\rho\lambda}$ has a factor of $\sqrt{-g}$ with $g=\det g_{\mu\nu}$ extracted and transforms as a tensor and not as a tensor density, and indices are freely raised and lowered using the metric $g_{\mu\nu}$ of which the signature is Lorentzian $(-+++)$. The constant $\Lambda=3/L^2$ is the AdS cosmological constant, and $\kappa^2=8\pi G$ is Newton's constant. The weak curvature means $\kappa^2/L^2\ll 1$. The relation to the gauge coupling g and the θ-angle is

$$e^{-\phi}=\frac{1}{g_E^2}=\frac{1}{g^2},\qquad \chi=\frac{1}{g_B^2}=\theta, \tag{3}$$

where subscripts E and B stand for the electric and magnetic parts. Therefore, weak coupling corresponds to $e^\phi\ll 1$. The S-operator acts on the axiodilaton as

$$\tau\to\tilde{\tau}=-\frac{1}{\tau}. \tag{4}$$

We restrict ourselves to the vanishing axion field

$$\chi=0. \tag{5}$$

According to Eq. (4) and the definition of $\tau=\chi+ie^{-\phi}$, the axion field is not generated by the S-duality transformation. In this case, the S-operator acts on the gauge field strength as

$$F^a_{\mu\nu}\to\tilde{F}^a_{\mu\nu}=-e^{-\phi}*F^a_{\mu\nu}. \tag{6}$$

Equations (4), (6) express a familiar electric-magnetic duality where the field strength transforms into a Hodge-dual one and the coupling transformation is $g^2 \to \frac{1}{g^2}$ and therefore the weak-strong coupling regimes are interchanged. Electric-magnetic duality exists only in $(3+1)$ dimensions where a 2-form $F_{\mu\nu}$ is dual to a two form again $\star F_{\mu\nu}$, as opposed for example to $(4+1)$ dimensions where a 2-form is dual to a 3-form. In $(3+1)$ dimensions, the Hodge-dual is defined as $\star F = \frac{\sqrt{-g}}{4}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}dx^\mu \wedge dx^\nu$ [26,27].

We start with the holographic p-wave superconductor introduced by Gubser *et al.* [16]. It constitutes the electric side (E side) of the duality. We will work in the four-dimensional space, in the background of a Reissner-Nordström black hole with asymptotic AdS_4 in the UV, but for now, we write a general metric,

$$ds^2 = -g_{tt}(z)dt^2 + g_{zz}(z)dz^2 + g_{xx}(z)dx^2 + g_{yy}(z)dy^2, \tag{7}$$

without off-diagonal terms because we restrict ourselves to the probe limit. The non-Abelian SU(2) gauge field has the following components [16],

$$A(z) = A_t^3(z)\tau^3 dt + A_x^1(z)\tau^1 dx, \tag{8}$$

where A_t^3 plays the role of the chemical potential and A_x^1 is the component which condenses at T_c and leads to spontaneous symmetry breaking and superconductivity. The field strengths obtained with these two components of vector potential (15) are

$$F_{zt}^3 = \partial_z A_t^3, \tag{9}$$

$$F_{zx}^1 = \partial_z A_x^1, \tag{10}$$

$$F_{tx}^2 = A_t^3 A_x^1, \tag{11}$$

and we do not write other components such as $F_{zt}^3 = -F_{tz}^3$. Performing the S-duality transformation (4) and (6), we obtain the magnetic side (B side) of the duality,

$$\tilde{F}_{xy}^3 = -\frac{1}{g^2}\frac{\sqrt{-g}}{g_{tt}g_{zz}}F_{zt}^3, \tag{12}$$

$$\tilde{F}_{ty}^1 = -\frac{1}{g^2}\frac{\sqrt{-g}}{g_{zz}g_{xx}}F_{zx}^1, \tag{13}$$

$$\tilde{F}_{zy}^2 = -\frac{1}{g^2}\frac{\sqrt{-g}}{g_{tt}g_{xx}}F_{tx}^2. \tag{14}$$

If we add the magnetic field to the p-wave superconductor

$$A(z) = A_t^3(z)\tau^3 dt + (A_x^1(z)\tau^1 + A_x^3(y,z)\tau^3)dx, \tag{15}$$

one more duality relation can be written,

$$\tilde{F}_{zt}^3 = -\frac{1}{g^2}\frac{\sqrt{-g}}{g_{xx}g_{yy}}F_{xy}^3, \tag{16}$$

that makes the system of Eqs. (12)–(14) and (16) closed and symmetric in terms of the gauge field components on the two sides of the duality. Here, the tilde field strength on the B side (12)–(14), (16) has complimentary components to the one on the E side, $g = \frac{1}{\tilde{g}}$, and the Yang-Mills coupling g should not be confused with the determinant of the metric $\sqrt{-g}$. The metric factor arises from using the following convention for Levi-Cività tensor $\epsilon^{1234} = \frac{1}{\sqrt{-g}}$, $\epsilon_{1234} = -\sqrt{-g}$ with $g \equiv \det g_{\mu\nu}$, where the lifting/lowering of indices is done as usual, for example $\epsilon^{ijkl}g_{1i}g_{2j}g_{3k}g_{4l} = -\epsilon_{1234}$ and $\epsilon^{1234}\epsilon_{1234} = -4! = -24$; therefore, for example $\epsilon_{xyzt} \sim \sqrt{-g}/\sqrt{g_{tt}g_{zz}}$.

In order to find the field content in the S-dual frame, we need to use explicit solutions of Yang-Mills equations in the electric frame. We do not know these solutions in the condensed phase. However, we can look at two asymptotic behaviors of the fields: AdS_4 expansion in the UV and AdS_2 expansion in the IR throat of the Reissner-Nordström (RN)-AdS geometry. As a result, we obtain the following field components on the magnetic side,

$$\tilde{A}(x,z) = \tilde{A}_t^3(z)\tau^3 dt + (\tilde{A}_y^2(z)\tau^2 + \tilde{A}_y^3(x,z)\tau^3)dy, \tag{17}$$

that we explain further. Therefore, the field strengths are given by

$$\tilde{F}_{xy}^3 = \partial_x \tilde{A}_y^3, \tag{18}$$

$$\tilde{F}_{ty}^1 = -\tilde{A}_t^3\tilde{A}_y^2, \tag{19}$$

$$\tilde{F}_{zy}^2 = \partial_z \tilde{A}_y^2, \tag{20}$$

that we use in the duality relations (12)–(14). Performing the necessary substitutions, the duality relations can be written as

$$\partial_x \tilde{A}_y^3 = \frac{1}{g^2}\frac{\sqrt{-g}}{g_{tt}g_{zz}}\partial_z A_t^3, \tag{21}$$

$$\partial_z A_x^1 = \frac{1}{\tilde{g}^2}\frac{g_{zz}g_{xx}}{\sqrt{-g}}\tilde{A}_t^3\tilde{A}_y^2, \tag{22}$$

$$\partial_z \tilde{A}_y^2 = -\frac{1}{g^2}\frac{\sqrt{-g}}{g_{tt}g_{xx}}A_t^3 A_x^1, \tag{23}$$

where $g = 1/\tilde{g}$. When the magnetic field is added to the p-wave superconductor (15), an additional duality relation is written:

$$\partial_y A_x^3 = -\frac{1}{\tilde{g}^2}\frac{g_{xx}g_{yy}}{\sqrt{-g}}\partial_z\tilde{A}_t^3. \tag{24}$$

In what follows, we show that Eqs. (21) and (24) map the magnetic field to the charge density on the opposite side of duality and Eqs. (22) and ((23) relate the v.e.v.s and the sources on the electric and magnetic sides. Therefore, Eqs. (22) and (23) are the duality relations for the condensates in the electric and magnetic systems.

The gauge fields which satisfy the equations of motion (EOM) should also satisfy the duality relations. EOM are invariant under the *S*-duality transformations. Next, we summarize the equations of motion on the electric and magnetic sides of the duality. The Yang-Mills equation for the non-Abelian SU(2) gauge fields is

$$\nabla_\mu F^{a\mu\nu} = -\epsilon^{abc}A_\mu^b F^{c\mu\nu}, \tag{25}$$

which becomes for the gauge field components $A_t^3(z)$ and $A_x^1(z)$ from Eq. (15) on the electric side,

$$\frac{1}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{zz}g_{tt}}F_{zt}^3\right) = -\frac{1}{g_{tt}g_{xx}}A_x^1 F_{xt}^2, \tag{26}$$

$$\frac{1}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{zz}g_{xx}}F_{zx}^1\right) = \frac{1}{g_{tt}g_{xx}}A_t^3 F_{tx}^2, \tag{27}$$

respectively, with $g \equiv \det g$, and the metric notation is given by Eq. (7). Equations of motion on the electric side, Eqs. (27), are written explicitly:

$$\partial_z^2 A_t^3 + \frac{g_{zz}g_{tt}}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{zz}g_{tt}}\right)\partial_z A_t^3 - \frac{g_{zz}}{g_{xx}}(A_x^1)^2 A_t^3 = 0, \tag{28}$$

$$\partial_z^2 A_x^1 + \frac{g_{zz}g_{xx}}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{zz}g_{xx}}\right)\partial_z A_x^1 - \frac{g_{zz}}{g_{tt}}(A_t^3)^2 A_x^1 = 0. \tag{29}$$

The Yang-Mills equations (25) are written for the gauge field components $\tilde{A}_y^3(z,x)$, $\tilde{A}_t^3(z)$, and $\tilde{A}_y^2(z)$ (17) on the magnetic side,

$$\frac{1}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{zz}g_{yy}}\tilde{F}_{zy}^3\right) = 0, \tag{30}$$

$$\frac{1}{\sqrt{-g}}\partial_x\left(\frac{\sqrt{-g}}{g_{xx}g_{yy}}\tilde{F}_{xy}^3\right) = 0, \tag{31}$$

$$\frac{1}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{tt}g_{zz}}\tilde{F}_{zt}^3\right) = \frac{1}{g_{tt}g_{yy}}\tilde{A}_y^2\tilde{F}_{yt}^1, \tag{32}$$

$$\frac{1}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{zz}g_{yy}}\tilde{F}_{zy}^2\right) = -\frac{1}{g_{tt}g_{yy}}\tilde{A}_t^3\tilde{F}_{ty}^1, \tag{33}$$

with the metric given by Eq. (7). Equations of motion on the magnetic side become

$$\partial_z^2\tilde{A}_y^3 + \frac{g_{zz}g_{yy}}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{zz}g_{yy}}\right)\partial_z\tilde{A}_y^3 = 0, \tag{34}$$

$$\partial_x^2\tilde{A}_y^3 + \frac{g_{xx}g_{yy}}{\sqrt{-g}}\partial_x\left(\frac{\sqrt{-g}}{g_{xx}g_{yy}}\right)\partial_x\tilde{A}_y^3 = 0, \tag{35}$$

$$\partial_z^2\tilde{A}_t^3 + \frac{g_{tt}g_{zz}}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{tt}g_{zz}}\right)\partial_z\tilde{A}_t^3 - \frac{g_{zz}}{g_{yy}}(\tilde{A}_y^2)^2\tilde{A}_t^3 = 0, \tag{36}$$

$$\partial_z^2\tilde{A}_y^2 + \frac{g_{zz}g_{yy}}{\sqrt{-g}}\partial_z\left(\frac{\sqrt{-g}}{g_{zz}g_{yy}}\right)\partial_z\tilde{A}_y^2 - \frac{g_{zz}}{g_{tt}}(\tilde{A}_t^3)^2\tilde{A}_y^2 = 0. \tag{37}$$

We are able to find analytic solutions of the equations of motion in the two limiting cases, near the boundary and around the horizon of the black hole at small temperatures. Then, we verify that the duality relations are satisfied by these solutions. In what follows, we outline the asymptotic behavior of the AdS-Reissner-Nordström black hole metric. The AdS-RN black hole in $(3+1)$ dimensions is

$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + dx^2 + dy^2) + \frac{R^2}{r^2}\frac{dr^2}{f}, \tag{38}$$

$$f = 1 + \frac{3r_\star^4}{r^4} - \frac{r_0^3 + 3r_\star^4/r_0}{r^3}, \tag{39}$$

where the electric charge of the black hole is $q = \sqrt{3}r_\star^2$, and $A_t = \frac{\mu}{r}$ with $\mu \sim q$, and the radius of the black hole horizon is r_0, $f(r_0) = 0$.

In the UV, $r \to \infty$, the redshift factor $f \approx 1$, and the metric becomes asymptotically AdS_4,

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx^2 + dy^2) + \frac{R^2}{r^2}dr^2, \tag{40}$$

with the radius R. Introducing $z = \frac{R^2}{r}$, we have

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dx^2 + dy^2 + dz^2). \tag{41}$$

In the IR, near the black hole horizon $r \to r_0$, and at low enough temperatures $T/\mu \ll 1$, the redshift factor has the double zero,

$$f \approx 6\frac{(r - r_\star)^2}{r_\star^2}\left(1 - \frac{(r_0 - r_\star)^2}{(r - r_\star)^2}\right), \tag{42}$$

where the expansion is carried out in two small parameters $(r - r_\star)/r_\star \ll 1$ and $(r_0 - r_\star)/r_\star \ll 1$. Due to the double zero, the metric has the AdS_2 behavior with a small correction $\sim(r_0 - r_\star)^2$,

$$ds^2 = -\frac{(r-r_\star)^2}{R_2^2}\left(1-\frac{(r_0-r_\star)^2}{(r-r_\star)^2}\right)dt^2 + \frac{R_2^2}{(r-r_\star)^2(1-\frac{(r_0-r_\star)^2}{(r-r_\star)^2})}dr^2 + \frac{r_\star^2}{R^2}(dx^2+dy^2), \tag{43}$$

with the radius $R_2 = \frac{R}{\sqrt{6}}$. In extremal case, $T=0$, the two radii coincide $r_0 = r_\star$, and the correction $\sim(r_0-r_\star)^2$ vanishes. Introducing $z = \frac{R_2^2}{r-r_\star}$ and $z_0 = \frac{R_2^2}{r_0-r_\star}$, we have the leading AdS_2 behavior near the horizon at small temperatures $T = \frac{1}{2\pi z_0} \ll 1$,

$$ds^2 = -\frac{R_2^2}{z^2}\left(1-\frac{z^2}{z_0^2}\right)dt^2 + \frac{R_2^2}{z^2(1-\frac{z^2}{z_0^2})}dz^2 + \frac{r_\star^2}{R^2}(dx^2+dy^2), \tag{44}$$

where the correction $\sim z^2/z_0^2$ to the AdS_2 geometry is due to the small but nonzero temperature. In what follows, we use the two limiting cases with the metric given by Eqs. (41) and (44) to solve EOM for the gauge fields and to test the duality conditions.

B. UV asymptotics: AdS_4

First, we consider the UV limit with pure AdS_4 metric (41) at small $z \sim 0$,

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx^2 + dy^2), \tag{45}$$

where R is the AdS radius.

A known analytic solution of Einstein and Yang-Mills equations is the AdS RN black hole with the electric charge q, where the vector field is $A_t^3 = \mu - qz$ and $A_x^1 = 0$. There is another solution, a hairy RN black hole, which describes a condensed phase and becomes preferable in some parameter range. There, the metric and the behavior of the scalar potential A_t^3 are modified mainly in the IR region by the vector potential A_x^1 which acquires a radial profile. However, an asymptotic form of the solutions near the AdS boundary remains unchanged to the leading order. Yang-Mills equations in the AdS_4 are

$$\partial_z^2 A_t^3 + (A_x^1)^2 A_t^3 = 0, \tag{46}$$

$$\partial_z^2 A_x^1 + (A_t^3)^2 A_x^1 = 0. \tag{47}$$

Therefore, we can write the following asymptotic behavior near the AdS boundary in the probe limit,

$$A_t^3 = \mu - qz + O(z^2), \tag{48}$$

$$A_x^1 = a_x^{(0)} + a_x^{(1)} z + O(z^2), \tag{49}$$

where we can read off according to the AdS/CFT dictionary the physical terms of the boundary conformal field theory (CFT). In CFT terms, μ is the $U(1)_3$ chemical potential, and q is the electric charge density. From the asymptotic expansion of the vector potential, the AdS/CFT dictionary says that in the condensed phase where $U(1)_3$ is spontaneously broken, the source term is zero $a_x^{(0)} = 0$ and the condensate is given by the v.e.v. $a_x^{(1)} \neq 0$, and in the normal phase, $a_x^{(1)} = 0$ and $a_x^{(0)} \neq 0$.

In the normal phase, $a_x^{(1)} = 0$, from Eqs. (12)–(14), the only nonvanishing component of the field strength $\tilde{F}_{\mu\nu}$ in the magnetic frame is

$$\tilde{F}_{xy}^3 = -q = \text{const}, \tag{50}$$

where we absorbed the coupling into redefining the solution A_t^3. Therefore, the S-dual of the $U(1)_3$ charged black hole is a state with $U(1)_3$ magnetic field,

$$\tilde{A}_y^3 = -qx. \tag{51}$$

This result probably holds for a backreacted solution.

In the condensed phase, $a_x^{(1)} \neq 0$, the dual field strengths are from Eqs. (12)–(14):

$$\tilde{F}_{xy}^3 = -q + O(z), \tag{52}$$

$$\tilde{F}_{ty}^1 = \tilde{g}^2 a_x^{(1)} + O(z), \tag{53}$$

$$\tilde{F}_{zy}^2 = \tilde{g}^2 a_x^{(1)} \mu z + O(z^2). \tag{54}$$

In order to see which operators are switched on in the S-dual frame, we need to find the gauge field $\tilde{A}_\mu^a$ corresponding to the field strength given by Eqs. (52)–(54). We work in the radial gauge

$$\tilde{A}_z^a = 0, \tag{55}$$

with $a = 1, 2, 3$. In Eq. (52), the field strength $\tilde{F}_{xy}^3$,

$$\tilde{F}_{xy}^3 = \partial_x \tilde{A}_y^3 = -q\tilde{A}_y^3 = -qx, \tag{56}$$

is easily integrated to give

$$\tilde{A}_y^3 = -qx, \tag{57}$$

which yields the magnetic field perpendicular to the (x,y)-plane with CFT. In order to integrate $\tilde{F}_{ty}^1$, we assume a stationary condition, $\partial_t(\ldots) = 0$, and no breaking of homogeneity in (x,y) directions: no $1/x$, $1/y$ terms in the potential $\tilde{A}_\mu^a$. This yields

$$\tilde{F}_{ty}^1 = \partial_t \tilde{A}_y^1 - \partial_y \tilde{A}_t^1 + (\tilde{A}_t^2 \tilde{A}_y^3 - \tilde{A}_t^3 \tilde{A}_y^2) = \tilde{g}^2 a_x^{(1)} = \text{const}. \tag{58}$$

Here, the first term is forbidden by stationarity. The second term is zero; otherwise, $\tilde{A}_t^1 \sim y$ and the field strength

$\tilde{F}^2_{ty} \sim -\tilde{A}^1_t \tilde{A}^3_y \sim xy$ will be induced. Solving for the third term, it would need $\tilde{A}^2_t \sim -\tilde{g}^2 a_x^{(1)}/qx$, and hence it would break homogeneity, $\tilde{F}^2_{tx} \sim 1/x^2$. Therefore, only the fourth term is left:

$$\tilde{A}^3_t \tilde{A}^2_y = -\tilde{g}^2 a_x^{(1)}. \tag{59}$$

Equation (59) can be solved if a new chemical potential $\tilde{\mu}$ is introduced,

$$\tilde{A}^3_t = \tilde{\mu} + O(z), \tag{60}$$

$$\tilde{A}^2_y = -\frac{\tilde{g}^2 a_x^{(1)}}{\tilde{\mu}}. \tag{61}$$

We see that $\tilde{A}^2_y$ explicitly breaks $U(1)_3$ and SO(2) spacial rotations. Integrating the last field strength in Eq. (54) gives

$$\tilde{F}^2_{zy} = \partial_z \tilde{A}^2_y - \partial_y \tilde{A}^2_z - (\tilde{A}^1_z \tilde{A}^3_y - \tilde{A}^3_z \tilde{A}^1_y) = \tilde{g}^2 a_x^{(1)} \mu z. \tag{62}$$

Here, only the first term is nonzero; three other terms are zero due to the gauge condition $\tilde{A}^a_z = 0$ (55). Therefore, the duality condition is

$$\partial_z \tilde{A}^2_y = \tilde{g}^2 a_x^{(1)} \mu z, \tag{63}$$

that means the small z expansion of $\tilde{A}^2_y$ starts with $\sim z^2$ and there is no linear term $\sim z$.

Summarizing the duality conditions, we have

$$\partial_x \tilde{A}^3_y = \tilde{g}^2 \partial_z A^3_t, \tag{64}$$

$$\partial_z A^1_x = -\frac{1}{\tilde{g}^2} \tilde{A}^3_t \tilde{A}^2_y, \tag{65}$$

$$\partial_z \tilde{A}^2_y = \frac{1}{g^2} A^3_t A^1_x, \tag{66}$$

and when the magnetic field is added to the E side,

$$\partial_y A^3_x = -g^2 \partial_z \tilde{A}^3_t. \tag{67}$$

Here, Eqs. (64) and (67) relate magnetic fields and charge densities, and Eqs. (65) and (66) relate condensates and sources between the two sides of the duality. To see the latter, we write an expansion for the gauge fields:

$$A^1_x = a_x^{(0)} + a_x^{(1)} z + \cdots, \tag{68}$$

$$\tilde{A}^2_y = \tilde{a}_y^{(0)} + \tilde{a}_y^{(1)} z + \cdots. \tag{69}$$

From the duality condition in Eqs. (59) and (65), we obtain the relation between the condensates, which are the v.e.v.s,

$$\tilde{a}_y^{(0)} = -\frac{\tilde{g}^2 a_x^{(1)}}{\tilde{\mu}}, \tag{70}$$

with $\tilde{g} = 1/g$. From the duality condition in Eqs. (63) and (66), we obtain the relation for the sources

$$a_x^{(0)} = \tilde{a}_y^{(1)} = 0, \tag{71}$$

which are switched off. The duality relations (70) and (71) confirm that the source and the v.e.v. are interchanged by the S-duality transformation. In the expansion (69), the source is the leading and the v.e.v. is the subleading term on the electric side, while the source is the subleading and the v.e.v. is the leading term on the magnetic side. Note that the duality relations for the condensates and v.e.v.s, as well as for the magnetic fields generated by the charge densities, can be obtained from one another by replacing tilde variables to nontilde ones and vice versa. They are direct and inverse S-duality transformations. The minus sign reflects that the matrix $S^2 = -1$; when the S-duality is applied twice, it gives a minus sign.

We summarize the components of the vector potential for the p-wave superconductor and its S-dual in the asymptotic AdS_4 space (small z) in Table III. The left panel represents the standard p-wave superconductor, and the right panel is for the case when a magnetic field is added.

In Table III, the $U(1)_3$ charge density in the p-wave superconductor maps to the magnetic field in the S-dual frame (left panel) and vice versa the magnetic field in the p-wave superconductor setting maps to the charge density in the dual frame (right panel).

As discussed in Refs. [1,28], performing the S-dual transformation on the Abelian gauge theory in the AdS_4 bulk corresponds to imposing two boundary conditions: Dirichlet (standard) on the electric side and Neumann (modified) on the magnetic side. There is some additional effort in extrapolating the results for the Abelian theory, where the Dirichlet and Neumann boundary conditions are simply exchanged under S-duality, to the non-Abelian case

TABLE III. Components of the gauge field for the p-wave superconductor and its S-dual in the AdS_4. The left table represents the standard holographic p-wave superconductor, and the right panel is when a magnetic field is added.

E side	B side
	$\tilde{A}^3_y = -qx$
$A^3_t = \mu - qz$	$\tilde{A}^3_t = \tilde{\mu} - \tilde{q}z$
$A^1_x = a_x^{(1)} z$	$\tilde{A}^2_y = -\frac{\tilde{g}^2 a_x^{(1)}}{\tilde{\mu}}$

E side	B side
$A^3_x = \tilde{q}y$	$\tilde{A}^3_y = -qx$
$A^3_t = \mu - qz$	$\tilde{A}^3_t = \tilde{\mu} - \tilde{q}z$
$A^1_x = a_x^{(1)} z$	$\tilde{A}^2_y = -\frac{\tilde{g}^2 a_x^{(1)}}{\tilde{\mu}}$

[28]. However, in the p-wave superconductor and in its S-dual, the SU(2) symmetry is broken down to $\mathrm{U}(1)_3$ by the chemical potential in the third color direction, and formation of condensates happens in the Abelian subgroup. The two boundary conditions are characterized in terms of the fall-off conditions of the gauge field near the boundary: when the leading/subleading term is fixed while the subleading/leading term is allowed to fluctuate gives Dirichlet/Neumann boundary conditions [28]. As is familiar from AdS/CFT, in the standard quantization (Dirichlet b.c.), the leading behavior acts as a source for the conserved current operator, and the subleading behavior gives a v.e.v. provided the source is switched off. In the modified (alternative) quantization (Neumann b.c.), the roles of the source and v.e.v. are interchanged [29]. Therefore, as the expectation value $\langle a_x^{(1)}\rangle$ is the superconducting condensate, in the S-dual frame, we associate $\langle \tilde{g}^2 a_x^{(1)}/\tilde{\mu}\rangle$ with the magnetic condensate, Table III.

We can speculate about the physical meaning of a new chemical potential $\tilde{\mu}$ in the S-dual frame. Depending on the context, it may reflect the density of magnetic monopoles. On the electric side, the conserved quantity of the boundary theory (conserved current J^μ) corresponds to electric charge in the bulk. On the S-dual side, the net magnetic charge corresponds to a conserved quantity in the boundary theory [1] (while the corresponding current vanishes at every point $\langle J^\mu\rangle = 0$). The difference also arises in CFTs that there are states charged under the global gauge group on the E side and therefore the Goldstone modes are produces as this symmetry is broken, while on the B side, there is a Gauss law instead, and no Goldstone modes arise.

There is the following pattern of breaking the non-Abelian gauge group and the spacial (x, y) rotational symmetry in the p-wave superconductor,

$$\mathrm{SU}(2)\xrightarrow{A_t^3}\mathrm{U}(1)_3\xrightarrow{A_x^1}\text{nothing}, \tag{72}$$

$$\mathrm{SO}(2)\xrightarrow{A_t^3}\mathrm{SO}(2)\xrightarrow{A_x^1}\text{nothing}, \tag{73}$$

where the chemical potential μ, which is introduced by the boundary value of the component A_t^3, breaks the SU(2) symmetry down to the diagonal subgroup U(1) which is generated by τ^3. In order to study the transition to the superconducting state, we allow solutions with nonzero $\langle J_x^1\rangle$ and therefore the nonzero dual gauge field A_x^1. This solution breaks not only color U(1) but also the spacial rotations SO(2). The symmetry breaking pattern for the S-dual of a p-wave superconductor is

$$\mathrm{SU}(2)\xrightarrow{\tilde{A}_y^3,\tilde{A}_t^3}\mathrm{U}(1)_3\xrightarrow{\tilde{A}_y^2}\text{nothing(up to discrete)}, \tag{74}$$

$$\mathrm{SO}(2)\xrightarrow{\tilde{A}_y^3,\tilde{A}_t^3}\mathrm{SO}(2)\xrightarrow{\tilde{A}_y^2}\text{nothing(up to discrete)}, \tag{75}$$

where the magnetic field present through the gauge field $\tilde{A}_y^3 \sim x$ breaks the non-Abelian gauge group SU(2) down to the diagonal subgroup U(1), but it does not break $(2+1)$-dimensional rotations SO(2). The superconducting phase with A_1^x maps to the S-dual state with a nonzero v.e.v. of an operator with gravity dual $\tilde{A}_y^2$, while normal phases describe CFTs at nonzero density and nonzero magnetic field.

There is a spontaneous breaking of $\mathrm{U}(1)_3$ symmetry by the corresponding v.e.v. in both the p-wave SC and its S-dual.

We can consider properties of the condensate with respect to discrete symmetries. In the parity transformation, either one or three coordinates change the sign. We adopt the former:

$$P\colon\ x \to -x, y \to y, z \to z, t \to t. \tag{76}$$

We summarize properties of the p-wave superconductor and its S-dual with respect to the parity transformation in Table IV.

From Eq. (76), the components of the vector potential $A_t^3, \tilde{A}_t^3, \tilde{A}_y^2, \tilde{A}_y^3$ are P even, and only A_x^1 is P odd. Also, the ϵ-symbol in the S-duality transformation changes the parity, that should be taken into account for the components $\tilde{A}_y^2, \tilde{A}_y^3$ calculated as the S-dual, as opposed to introducing $\tilde{A}_t^3$. Indeed, the magnetic field in the S-dual frame is $B \sim \epsilon q$ which is P odd and C odd.

On the E side, the p-wave superconducting condensate is a vector, while on the B side, the condensate is a pseudovector.

Next, we check that the gauge fields given in Table III, which are solutions of equations of motion, satisfy the duality relations (64)–(66). Equations of motion on the electric side are given in Eqs. (46) and (47). Equations of motion on the magnetic side are

$$\partial_z^2\tilde{A}_y^3 = \partial_x^2\tilde{A}_y^3 = 0, \tag{77}$$

$$\partial_z^2\tilde{A}_t^3 + (\tilde{A}_y^2)^2\tilde{A}_t^3 = 0, \tag{78}$$

$$\partial_z^2\tilde{A}_y^2 + (\tilde{A}_t^3)^2\tilde{A}_y^2 = 0. \tag{79}$$

To the leading order in small z, the gauge field components $A_t^3, \tilde{A}_t^3$ (chemical potentials) and $\tilde{A}_y^3$ (magnetic field) satisfy EOM and the first duality condition (64). For the gauge

TABLE IV. Properties of the p-wave superconductor and its S-dual under the parity transformation.

E side			B side		
μ	q	$a_x^{(1)}$	$\tilde{\mu}$	$\tilde{q}$	$\frac{\tilde{g}^2 a_x^{(1)}}{\tilde{\mu}}$
+	+	−	+	+	+

field components A_x^1, $\tilde{A}_y^2$ (condensates), we have the following EOM in the probe limit,

$$A'' + \mu^2 A = 0, \tag{80}$$

$$\tilde{A}'' + \tilde{\mu}^2 \tilde{A} = 0, \tag{81}$$

and the duality conditions (65) and (66),

$$\frac{1}{g^2} A' = \tilde{\mu} \tilde{A}, \tag{82}$$

$$\frac{1}{\tilde{g}^2} \tilde{A}' = -\mu A, \tag{83}$$

where we omitted the spacial and the SU(2) gauge group indices, $A_x^1 \equiv A$ and $\tilde{A}_y^2 \equiv \tilde{A}$, and $\partial_z A \equiv A'$. Solutions of the EOM are

$$A \sim \sin \mu z, \qquad \tilde{A} \sim \cos \tilde{\mu} z, \tag{84}$$

because A and $\tilde{A}$ satisfy the Dirichlet and Neumann boundary conditions, respectively, and the sources are switched off; $A(0) = 0$ and $\tilde{A}'(0) = 0$. Indeed, the solutions $\sin(\mu z)$ and $\cos(\tilde{\mu} z)$ with the appropriate choice of constants of integration satisfy the duality conditions (82) and (83). These constants of integration define the condensates on electric and magnetic sides.

C. IR asymptotics: AdS_2

Next, we consider the IR limit with $AdS_2 \times R^2$ metric (44) at large $z \to \infty$,

$$ds^2 = \frac{R_2^2}{z^2}\left[-\left(1 - \frac{z^2}{z_0^2}\right) dt^2 + \frac{dz^2}{1 - \frac{z^2}{z_0^2}}\right] + \frac{r_\star^2}{R^2}(dx^2 + dy^2), \tag{85}$$

where z is large. Again, we check that the gauge field solutions of EOM satisfy the duality constraints. Equations of motion on the electric side are

$$\partial_z^2 A_t^3 + \frac{2}{z}\partial_z A_t^3 - \frac{R^4}{6r_\star^2}\frac{(A_x^1)^2 A_t^3}{z^2(1 - \frac{z^2}{z_0^2})} = 0, \tag{86}$$

$$\partial_z^2 A_x^1 - \frac{2z}{z_0^2(1 - \frac{z^2}{z_0^2})}\partial_z A_x^1 + \frac{(A_t^3)^2 A_x^1}{(1 - \frac{z^2}{z_0^2})^2} = 0, \tag{87}$$

and EOM on the magnetic side are

$$\partial_z^2 \tilde{A}_y^3 - \frac{2z}{z_0^2(1 - \frac{z^2}{z_0^2})}\partial_z \tilde{A}_y^3 = 0, \tag{88}$$

$$\partial_x^2 \tilde{A}_y^3 = 0, \tag{89}$$

$$\partial_z^2 \tilde{A}_t^3 + \frac{2}{z}\partial_z \tilde{A}_t^3 - \frac{R^4}{6r_\star^2}\frac{(\tilde{A}_y^2)^2 \tilde{A}_t^3}{z^2(1 - \frac{z^2}{z_0^2})} = 0, \tag{90}$$

$$\partial_z^2 \tilde{A}_y^2 - \frac{2z}{z_0^2(1 - \frac{z^2}{z_0^2})}\partial_z \tilde{A}_y^2 + \frac{(\tilde{A}_t^3)^2 \tilde{A}_y^2}{(1 - \frac{z^2}{z_0^2})^2} = 0, \tag{91}$$

where $z/z_0 \ll 1$ is a small correction due to a small but nonzero temperature $T = \frac{1}{2\pi z_0}$. The EOM for the condensate components A_x^1 and $\tilde{A}_y^2$ are the same. We look for the two solutions of EOM which satisfy different boundary conditions. The duality relations that connect the electric and magnetic sides are

$$\partial_x \tilde{A}_y^3 = -\frac{1}{g^2}\frac{6r_\star^2}{R^4} z^2 \partial_z A_t^3, \tag{92}$$

$$\partial_z A_x^1 = -\frac{1}{\tilde{g}^2(1 - \frac{z^2}{z_0^2})}\tilde{A}_t^3 \tilde{A}_y^2, \tag{93}$$

$$\partial_z \tilde{A}_y^2 = -\frac{1}{g^2(1 - \frac{z^2}{z_0^2})} A_t^3 A_x^1. \tag{94}$$

To the leading order in $O(1/z)$, solutions for the temporal components in the probe limit are

$$A_t^3 = \frac{\mu}{6z}\left(1 - \frac{z}{z_0}\right), \qquad \tilde{A}_t^3 = \frac{\tilde{\mu}}{6z}\left(1 - \frac{z}{z_0}\right), \tag{95}$$

with $\partial_z A_t^3 = -\frac{\mu}{6z^2}$. It produces the duality relation (92)

$$\partial_x \tilde{A}_y^3 = \frac{r_\star^2}{g^2 R^4}\mu, \tag{96}$$

that gives the magnetic field perpendicular to the (x, y) plane,

$$\tilde{A}_y^3 = \frac{r_\star^2}{g^2 R^4}\mu x. \tag{97}$$

Using solutions for the temporal components (95), EOM for the condensate components are

$$A'' - \frac{2zA'}{z_0^2(1 - \frac{z^2}{z_0^2})} + \left(\frac{\mu}{6}\right)^2 \frac{A}{z^2(1 + \frac{z}{z_0})^2} = 0, \tag{98}$$

$$\tilde{A}'' - \frac{2z\tilde{A}'}{z_0^2(1 - \frac{z^2}{z_0^2})} + \left(\frac{\tilde{\mu}}{6}\right)^2 \frac{\tilde{A}}{z^2(1 + \frac{z}{z_0})^2} = 0, \tag{99}$$

and the duality relations (93) and (94) are

$$A' = \frac{1}{\tilde{g}^2}\frac{\tilde{\mu}}{6z(1 + \frac{z}{z_0})}\tilde{A}, \tag{100}$$

$$\tilde{A}' = \frac{1}{g^2}\frac{\mu}{6z(1+\frac{z}{z_0})}A, \tag{101}$$

where we omitted the space and group indices, $A_x^1 \equiv A$ and $\tilde{A}_y^2 \equiv \tilde{A}$, and $\partial_z A \equiv A'$. Solutions of EOM in the leading order of $O(1/z)$ and $O(z/z_0)$ are

$$A \sim \frac{1}{z}\left(1+\frac{z}{z_0}\right) + O\left(\frac{1}{z^2}\right), \tag{102}$$

$$\tilde{A} \sim \frac{1}{z}\left(1+\frac{z}{z_0}\right) + O\left(\frac{1}{z^2}\right). \tag{103}$$

Because $A' \sim -\frac{1}{z^2}$, these solutions satisfy the duality relations (100) and (101) in each order of perturbation theory in $1/z$.

Thus, we showed analytically that in the UV and IR limits, solutions of the EOM on electric and magnetic sides are related by the duality conditions. In the next section, we solve the EOM and check the duality constraint in the holographic bulk numerically.

III. NUMERICAL SOLUTIONS AND DUALITY MAPPING BETWEEN THEM

In Sec. II, we demonstrated the $\mathrm{SL}(2,Z)$ invariance on the level of the non-Abelian SU(2) action. Also, we demonstrated the S-duality for the equations of motion in the asymptotic UV and IR regimes. In general, the S-duality cannot be traced at the EOM level due to the covariant derivative that introduces the gauge field instead of the field strength for which the duality relation is written. Therefore, we solve the EOM directly and show that the physical solutions on different sides are connected by the S-duality relation.

We look for the nontrivial solutions of EOM describing the condensates: A_x^1 in Eq. (29) on the electric side and $\tilde{A}_y^2$ in Eq. (37) on the magnetic side. The gauge fields A_x^1 and $\tilde{A}_y^2$ satisfy the same equations. Therefore, we will be looking for two nontrivial condensate solutions of one EOM. As shown in Sec. II, one imposes for the solutions on the E and B sides two different UV boundary conditions, Dirichlet and Neumann b.c., respectively. This situation is known to arise in the holographic superconductor that is built using the bulk scalar field (the s-wave holographic superconductivity) [30–33] and in the Sakai-Sugimoto model [34]. In the former case, one obtains a "standard" hairy solution at a threshold charge density, i.e. for $\mu \geq \mu_c$, using the Dirichlet b.c. (standard quantization). Also a "new" instability occurs at small charge density—scalar hair, with Neumann b.c. (alternative quantization) [30,31]. In the literature, different explanations are given to what causes a new instability [30–33]. Here, we will find two types of instabilities in the holographic p-wave superconductor. However, contrary to AdS_5 where the analytic solution for the standard p-wave superconductor is known [35], there is no analytic solution in AdS_4, and we solve it numerically.

We use the metric

$$ds^2 = \frac{1}{z^2}\left(-f dt^2 + \frac{dz^2}{f} + dx_1^2 + dx_2^2\right), \tag{104}$$

where the redshift factor for the AdS_4-Reissner-Nordström black hole is

$$\begin{aligned} f &= 1 + q^2 z^4 - (1+q^2)z^3 \\ &= (1-z)(1+z+z^2-z^3q^2), \end{aligned} \tag{105}$$

where q is the charge of the black hole, $q = \sqrt{3}r_\star^2$. Equation (105) can be obtained from Eq. (39) by rescaling to make $r_0 = 1$ and $R = 1$ and changing the variable $r = 1/z$. For the extremal black hole $T = 0$, the redshift factor develops a double zero near the horizon,

$$f = 6(1-z)^2, \tag{106}$$

and the metric reduces to $\mathrm{AdS}_2 \times R^2$. As $q = 0$, we have the known metric of the AdS_4-Schwarzschild black hole with the redshift factor [16]

$$f = 1 - z^3 = (1-z)(1+z+z^2). \tag{107}$$

In both cases (105), (107), the black hole horizon is at $z = 1$, $f(z=1) = 0$, and the boundary is at $z = 0$. In this metric (104), the EOM for the magnetic field component $\tilde{A}_y^3$ are

$$\partial_z^2 \tilde{A}_y^3 + \frac{f'}{f}\partial_z \tilde{A}_y^3 = 0, \tag{108}$$

$$\partial_x^2 \tilde{A}_y^3 = 0, \tag{109}$$

and the duality relation takes the form

$$\partial_x \tilde{A}_y^3 = \frac{1}{g^2}\partial_z A_t^3. \tag{110}$$

EOM and the duality constraint are solved in the probe limit, $A_t^3 = \mu(1-z)$, and the constant magnetic field $\tilde{A}_y^3 = -\mu x$. A nontrivial task is to solve the EOM and check the duality for the condensate components. In the metric given by Eq. (104), the EOM for the temporal A_t^3 and condensate A_x^1 gauge field components on the electric side are

$$A_t'' - \frac{(A_x)^2 A_t}{f} = 0, \tag{111}$$

$$A_x'' + \frac{f'A_x'}{f} + \frac{(A_t)^2 A_x}{f^2} = 0. \qquad (112)$$

The same system of equations is obtained for the temporal $\tilde{A}_t^3$ and the condensate $\tilde{A}_y^2$ components on the magnetic side. For now, we omit the group indices by the gauge fields and denote $\partial_z A \equiv A'$. In the UV at $z = 0$, the asymptotic behavior of the solution is

$$A = A^{(0)} + zA^{(1)} + \cdots. \qquad (113)$$

To obtain a nontrivial condensate solution, we need to switch off the source. The Dirichlet boundary condition implies that the leading source term is $A^{(0)}$ and the subleading term $A^{(1)}$ is a condensate. For the Neumann boundary condition, the roles of the source and the v.e.v. are interchanged; i.e. $A^{(1)}$ is the source, and $A^{(0)}$ is the v.e.v.. Therefore, the UV behavior of the two solutions is

$$\text{Dirichlet (E side)}:\ A^{(0)} = 0, A'(z=0) = A^{(1)} = \text{v.e.v.}, \qquad (114)$$

$$\text{Neumann (B side)}:\ \tilde{A}^{(1)} = 0, \tilde{A}(z=0) = \tilde{A}^{(0)} = \text{v.e.v.}, \qquad (115)$$

where we read off the condensates as v.e.v.s on both sides of duality. The duality conditions for the condensate components read

$$A_x' = \frac{\tilde{A}_t \tilde{A}_y}{\tilde{g}^2 f}, \qquad (116)$$

$$\tilde{A}_y' = -\frac{A_t A_x}{g^2 f}, \qquad (117)$$

that relates the solutions of the EOM on the electric and magnetic sides A_x^1 and $\tilde{A}_y^2$ with each other. In what follows, we consider the probe limit, where solutions for the temporal gauge components read

$$A_t = \mu(1-z), \qquad \tilde{A}_t = \tilde{\mu}(1-z). \qquad (118)$$

Next, we check the asymptotic regimes in the UV and IR analytically. In the UV, at $z = 0$, the redshift factor is $f = 1$ for the AdS-RN and the Schwarzschild black holes, and thus it is asymptotically an AdS_4 metric. Therefore, the asymptotic EOM and the asymptotic duality relations are

$$z \sim 0:\ A_x'' + \mu^2 A_x = 0, \qquad \tilde{A}_y'' + \tilde{\mu}^2 \tilde{A}_y = 0, \qquad (119)$$

$$A_x' = \frac{\tilde{\mu}}{\tilde{g}^2}\tilde{A}_y, \qquad \tilde{A}_y' = -\frac{\mu}{g^2}A_x, \qquad (120)$$

solved by

$$A_x \sim \sin(\mu z), \qquad \tilde{A}_y \sim \cos(\tilde{\mu} z). \qquad (121)$$

One can also express, using the duality relation, the dual field $\tilde{A}$ on the B side through the original one A on the E side and substitute it in the EOM, to make sure that the EOM are satisfied. In the IR, at $z = 1$, and at small temperatures, the redshift factor for an AdS-RN black hole is $f = 6(1-z)^2$. In the $\mathrm{AdS}_2 \times R^2$ metric, the EOM and the duality relations are

$$z \sim 1:\ A_x'' - \frac{2}{1-z}A_x' + \frac{\mu^2}{36(1-z)^2}A_x = 0, \qquad (122)$$

$$\tilde{A}_y'' - \frac{2}{1-z}\tilde{A}_y' + \frac{\tilde{\mu}^2}{36(1-z)^2}\tilde{A}_y = 0, \qquad (123)$$

$$A_x' = \frac{\tilde{\mu}}{\tilde{g}^2}\frac{\tilde{A}_y}{6(1-z)}, \qquad (124)$$

$$\tilde{A}_y' = -\frac{\mu}{g^2}\frac{A_x}{6(1-z)}. \qquad (125)$$

Because of the second term $A'/(1-z)$ in the EOM, an expansion of the solution starts from $(1-z)^2$ to ensure the regularity,

$$A_x \sim (1-z)^2 + O((1-z)^3), \qquad (126)$$

$$\tilde{A}_y \sim (1-z)^2 + O((1-z)^3), \qquad (127)$$

that satisfies the duality relations in each order of the expansion. It happens due to the double zero in the redshift factor $f \sim (1-z)^2$ that leads to the duality relation having structure $A' \sim \tilde{A}/(1-z)$. Therefore, the S-duality holds in the IR for an AdS-RN black hole for small enough temperatures where the metric reduces to an AdS_2 throat.

To show that the solutions of the EOM satisfy the duality relations in the holographic bulk, we resort to a numerical study. It is convenient to rewrite the EOM for the condensate component in the form of the Riccati equation [36], that transforms the linear ordinary differential equation (ODE) of the second order into a nonlinear ODE of the first order.[2] In this way, one needs to specify only one boundary condition instead of two. Introducing $w = A_x'/A_x$ and $\tilde{w} = \tilde{A}_y'/\tilde{A}_y$ in Eq. (112), we obtain the following EOM and the duality relation,

[2] In the second order equation $\alpha(x)y'' + \beta(x)y' + \gamma(x)y = 0$, we make the substitution $w = -\frac{y'}{\alpha(x)y}$. Then, the Riccati equation is given by $w' = \alpha(x)w^2 + \frac{\alpha'(x)-\beta(x)}{\alpha(x)}w + \frac{\gamma(x)}{\alpha^2(x)}$.

$$w' + w^2 + \frac{f'}{f} w + \frac{A_t^2}{f^2} = 0, \tag{128}$$

$$\tilde{w}' + \tilde{w}^2 + \frac{f'}{f} \tilde{w} + \frac{\tilde{A}_t^2}{f^2} = 0, \tag{129}$$

$$w\tilde{w} = -\frac{A_t \tilde{A}_t}{f^2}, \tag{130}$$

where the metric is given by Eq. (105) with $f(z)$ specified for the AdS-RN/Schwarzschild BH and the probe limit solutions $A_t = \mu(1-z)$ and $\tilde{A}_t = \tilde{\mu}(1-z)$ are used. The EOM are supplemented by the IR boundary condition

$$z \sim 1: \; w(z \sim 1) = w_1(1-z), \tag{131}$$

$$\tilde{w}(z \sim 1) = -\frac{2}{1-z}, \tag{132}$$

where $w_1 = w'(z=1)$ is a constant. Note that at $z=1$, the redshift factor is $f = (1-z)(3-q^2) = 3(1-z)(1-r_\star^4)$ for the AdS-RN BH and $f = 3(1-z)$ for the Schwarzschild BH. This boundary condition ensures that the condensate gauge field solutions are regular in the IR. It corresponds to the following behavior of the condensate fields in the IR:

$$A_x = a^{(0)} + a^{(2)}(1-z)^2 + \cdots, \tag{133}$$

$$\tilde{A}_y = \tilde{a}^{(2)}(1-z)^2 + \cdots. \tag{134}$$

There is no boundary condition in the IR apart from the regularity condition. A regular solution is obtained when $A'(z=1) = 0$ in Eq. (112); that is, there is no $(1-z)$ term in the gauge field solution at $z=1$. To obtain the duality for w's, the original duality conditions (116) and (117) are Z_2 reflected (the no-tilde variables interchange with the tilde variables), and the relation $\tilde{g} = 1/g$ is used.

In the IR, the duality relation (130) gives

$$w(z=1)\tilde{w}(z=1) = \frac{\mu\tilde{\mu}}{(3-q^2)^2}, \tag{135}$$

for the AdS-RN and Schwarzschild black holes. Due to Eqs. (131) and (132), the IR duality condition (135) gives

$$w'(z=1) = -\frac{\mu\tilde{\mu}}{2(3-q^2)^2}. \tag{136}$$

This equation fixes a constant $w_1 = w'(z=1)$ in the IR boundary condition (131).

The UV behavior of the solutions of the Riccati equations is

$$z \sim 0: \; \text{Dirichlet b.c. (E side)} \; w(z \sim 0) = \frac{1}{z} \to \infty, \tag{137}$$

$$\text{Neumann b.c. (B side)} \; \tilde{w}(z \sim 0) = \tilde{w}_0 z \to 0, \tag{138}$$

where $\tilde{w}_0 = w'(z=0)$ is a constant. It translates into the following behavior of the gauge fields in the UV,

$$A_x = A^{(1)} z + \cdots, \tag{139}$$

$$\tilde{A}_y = \tilde{A}^{(0)} + \tilde{A}^{(2)} z^2 + \cdots, \tag{140}$$

with the sources being switched off on both sides of the duality. In the UV, the duality relation (130) amounts to

$$w(z=0)\tilde{w}(z=0) = -\mu\tilde{\mu}. \tag{141}$$

Due to Eqs. (137) and (138), the UV duality condition (141) gives

$$\tilde{w}'(z=0) = -\mu\tilde{\mu}. \tag{142}$$

We rewrite the S-duality equation (141) in the UV using the connection between the Riccati variable w calculated at the boundary and the Green function

$$w(z=0) = \frac{A_x'}{A_x}\bigg|_{\mathrm{UV}} \sim G_{xx}^{11}(\omega = k = 0)$$
$$\tilde{w}(z=0) = \frac{\tilde{A}_y'}{A_y}\bigg|_{\mathrm{UV}} \sim \tilde{G}_{yy}^{22}(\omega = k = 0), \tag{143}$$

where the retarded Green function is $G_{ij}(\omega) = -i\int d^2x dt e^{-i\omega t}\theta(t)\langle[J_i(t), J_j(0)]\rangle$ and J_i is the current. The duality relation (141) reads

$$G_{xx}^{11}\tilde{G}_{yy}^{22} = \mu\tilde{\mu}. \tag{144}$$

Therefore, in $2+1$ theory at nonzero density, the S-duality transformation $E \overset{S}{\to} B$ acts as follows,

$$\frac{G(\omega = k = 0)}{\mu} \overset{S}{\to} \left[\frac{G(\omega = k = 0)}{\mu}\right]^{-1}, \tag{145}$$

where the Green function is associated with the boundary directions, i.e. G_{xx}. Formally identifying the real part of the retarded Green function at zero frequency with the superfluid density $n_s = \mathrm{Re}[G^R(\omega = k = 0)]$ [16,37–41], we can rewrite Eq. (145),

$$n_s/\mu \overset{S}{\to} \frac{1}{n_s/\mu}, \tag{146}$$

when the duality transformation is performed. However, the meaning of Eq. (146) may be obscure because the superfluid density should be identified with the direction in the isospin space of the conserved charge which is $U(1)_3$. We leave Eq. (146) as a speculative suggestion that can be realized in other models at nonzero charge densities where the change of transport coefficients with duality transformation is considered.

To this end, we consider the duality relations for the electrical conductivity which arises from the non-Abelian current $J^3_{x,y}$ generated by the τ^3 component. The electrical conductivity is defined through Ohm's law,

$$J_i = \sigma_{ij} E^j, \tag{147}$$

where E^j is an external electric field and J_i is the current generated. The current $J^3_{x,y}$ is dual to fluctuations of the $\delta A^3_{x,y}$ fields. Because of the non-Abelian Yang-Mills action, the fluctuations in $\delta A^3_{x,y}$ will source other field components. We will keep all the modes which couple at a linearized level.

The gauge field includes the background and fluctuation components $A_i + \delta A_i$. We summarize the background gauge fields on the two sides of the duality:

$$(A^3_t, A^1_x; A^3_x) \xrightarrow{S} (\tilde{A}^3_t, \tilde{A}^2_y; \tilde{A}^3_y). \tag{148}$$

We consider fluctuations that have the same charge as $\delta A^3_{x,y}$ under $U(1)_3$ action. There will be decoupled equations involving the following fluctuation fields on the two sides of the duality:

$$(\delta A^3_x, \delta A^2_t; \delta A^1_x) \xrightarrow{S} (\delta \tilde{A}^3_y, \delta \tilde{A}^1_t; \delta \tilde{A}^2_y). \tag{149}$$

In Eqs. (148) and (149), a semicolon separates the gauge fields responsible for the magnetic fields. Additionally, the fluctuations $\delta A^{1,2}_z$ arise in coupled equations of motion. We use a background field gauge transformation to set $\delta A^{1,2}_z = 0$ [42]. All fluctuation fields are taken to have an overall time dependence of $e^{-i\omega t}$.

Integrating the fields to the UV, we can read off the dual currents and external electric fields. The current and charge densities are obtained from

$$F^a_{z\mu} = \langle J^a_\mu \rangle + \cdots, \tag{150}$$

where μ runs over the boundary directions t, x, y. The background equilibrium values are $\langle J^1_x \rangle = J$, $\langle \tilde{J}^2_y \rangle = \tilde{J}$ and $\langle J^3_t \rangle = \rho$, $\langle \tilde{J}^3_t \rangle = \tilde{\rho}$. The external electric fields are obtained from

$$F^a_{ti} = -E^a_i + \cdots. \tag{151}$$

We summarize the duality relations for the background fields in the UV (i.e. omitting the metric factors and the Yang-Mills coupling constant),

$$\tilde{F}^2_{zy} \sim F^2_{tx}, \tag{152}$$

$$\tilde{F}^1_{ty} \sim F^1_{zx}, \tag{153}$$

$$\tilde{F}^3_{xy} \sim F^3_{zt}, \tag{154}$$

$$\tilde{F}^3_{zt} \sim F^3_{xy}, \tag{155}$$

where the field strengths are

$$\tilde{F}^2_{zy} = \partial_z \tilde{A}^2_y, \qquad F^2_{tx} = A^3_t A^1_x, \tag{156}$$

$$\tilde{F}^1_{ty} = -\tilde{A}^3_t \tilde{A}^2_y, \qquad F^1_{zx} = \partial_z A^1_x, \tag{157}$$

$$\tilde{F}^3_{xy} = \partial_x \tilde{A}^3_y, \qquad F^3_{zt} = \partial_z A^3_t, \tag{158}$$

$$\tilde{F}^3_{zt} = \partial_z \tilde{A}^3_t, \qquad F^3_{xy} = \partial_y A^3_x. \tag{159}$$

The first two equations (152) and (153) provide the relation in the symmetric form for the Green functions when the duality transformation is done, while in Eqs. (154) and (155), the charge density generates the magnetic field on the other side of the duality.

The duality relations for the gauge field fluctuations are written (again omitting the metric factors and the coupling constant) as follows,

$$\tilde{F}^3_{ty} \sim F^3_{zx}, \tag{160}$$

$$\tilde{F}^3_{zy} \sim F^3_{tx}, \tag{161}$$

$$\tilde{F}^2_{xy} \sim F^2_{zt}, \tag{162}$$

$$\tilde{F}^1_{zt} \sim F^1_{xy}, \tag{163}$$

where the field strengths on the linearized level are

$$\begin{aligned} \tilde{F}^3_{ty} &= -\tilde{E}^3_y = \partial_t \delta \tilde{A}^3_y + \tilde{A}^2_y \delta \tilde{A}^1_t, \\ F^3_{zx} &= J^3_x = \partial_z \delta A^3_x, \end{aligned} \tag{164}$$

$$\begin{aligned} \tilde{F}^3_{zy} &= \tilde{J}^3_y = \partial_z \delta \tilde{A}^3_y, \\ F^3_{tx} &= -E^3_x = \partial_t \delta A^3_x - A^1_x \delta A^2_t, \end{aligned} \tag{165}$$

$$\tilde{F}^2_{xy} = \partial_x \delta \tilde{A}^2_y, \qquad F^2_{zt} = \partial_z \delta A^2_t, \tag{166}$$

$$\tilde{F}^1_{zt} = \partial_z \delta \tilde{A}^1_t, \qquad F^1_{xy} = \partial_y \delta A^1_x, \tag{167}$$

where we simplified $\langle J^3_x \rangle = J^3_x$ and $\langle \tilde{J}^3_y \rangle = \tilde{J}^3_y$.

We are interested in the electrical conductivity of the $U(1)$ subgroup of SU(2) generated by τ^3. Therefore, we consider currents $J^3_{x,y}$ that result from external sources in the τ^3 direction only. Therefore, we read off the linearized electric response to a time varying external electric field E_i^3 only. However, if we integrate equations of motion to the boundary, we would not obtain electric fields $E_i^{1,2}$; we would obtain a source $\delta A_t^{1,2}$. Therefore, a gauge transformation in the bulk should be done that sets the boundary value of δA_t^2, $\delta\tilde{A}_t^1$ to zero [42], which results in the new scalar potentials and the new field strengths. In what follows, the specific form of the field strengths is not important for us. The first two duality equations (160), (161) give relation between the electric conductivities

$$\sigma_{xx}^3 = \frac{J_x^3}{E_x^3} = \lim_{z\to 0}\frac{F_{zx}^3}{F_{tx}^3}, \tag{168}$$

$$\tilde{\sigma}_{yy}^3 = \frac{\tilde{J}_y^3}{\tilde{E}_y^3} = \lim_{z\to 0}\frac{\tilde{F}_{zy}^3}{\tilde{F}_{ty}^3}, \tag{169}$$

when written in the symmetric form is

$$\sigma_{xx}^3\tilde{\sigma}_{yy}^3 = 1. \tag{170}$$

This relation holds when all the metric factors and coupling constant are restored. Alternatively, this equation shows how the electric conductivity transforms when the S-duality transformation $E\overset{S}{\rightarrow}B$ is done,

$$\sigma\overset{S}{\rightarrow}\frac{1}{\sigma}. \tag{171}$$

The relation for conductivities was first established for the self-dual CFTs in Ref. [43] and then in Ref. [44] for the CFTs where the EM self-duality was lost. Later, it was shown by W. Witczak-Krempa and S. Sachdev in Ref. [44] that the particle vortex or S-duality interchanges the locations of the conductivity zeros and poles. Specifically, the poles of the dual conductivity $\tilde{\sigma}\sim 1/\sigma$ correspond to the zeros of the conductivity σ in the ω complex plane and vice versa. It was also shown that the S-duality transformation corresponds to the metal-insulator transition. We consider Eq. (171) as a generalization of the duality relation for conductivities obtained by W. Witczak-Krempa and S. Sachdev to theories at nonzero densities.

The last two duality equations (162) and (163) relate the charge density and the magnetic field on the two sides of the duality.

We solve the Riccati equation (128) numerically using the adaptive algorithm with the chemical potential as a parameter. First, we adjust $\tilde{\mu}$ to obtain the required UV behavior of the dual solution (138): $\tilde{w}$ is a straight line going through the origin (138). It corresponds to switching off the source $\tilde{a}^{(1)} = 0$ on the magnetic side. Then, solving the Riccati equation numerically, we adjust μ to obtain the required UV behavior of the solution on the electric side (137): A is a straight line going through the origin (139), that corresponds to switching off the source $a^{(0)} = 0$ on the electric side. This step is done provided that a constant w_1 in the IR boundary condition satisfies the duality relation (136). The gauge fields A are obtained from the solutions of the Riccati equation w imposing the IR boundary conditions that provides proper normalization: $A(z=1) = 1$ on electric side, and $\tilde{A}(z=1)$ satisfies the duality condition. We find that in the Schwarzschild metric, we satisfy the required UV boundary conditions for the following chemical potentials:

$$\text{E side:}\ \mu \geq 3.656, \tag{172}$$

$$\text{B side:}\ \tilde{\mu} \approx 1.05. \tag{173}$$

The two solutions of the Riccati equations have the UV asymptotic behavior $w\sim 1/z$ ("threshold" solution) and $w\sim z$ (new solution) as depicted in Fig. 1. They are the gauge field solutions of the equations of motion with UV behavior $A\sim z$ (the threshold solution)—Dirichlet b.c. and $A\to \text{const}$ (the new solution)—Neumann b.c. shown in Fig. 2.

Using the adaptive algorithm for a range of AdS-RN BH charges, we find the mapping between the two chemical potentials $\tilde{\mu}(\mu)$, Fig. 6. In other words, we find that for each

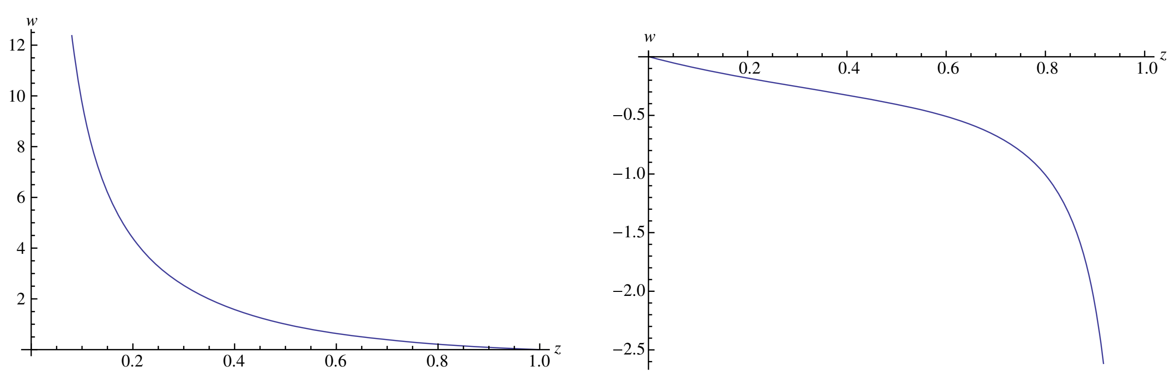

FIG. 1. Two solutions w of the Riccati equation in the AdS$_4$-Schwarzschild metric: a standard solution for $\mu = 3.66$, $w_1 = 0.21$ (left) and a new solution for $\mu = 1.05$, $\tilde{w}_1 = -2$ (right).

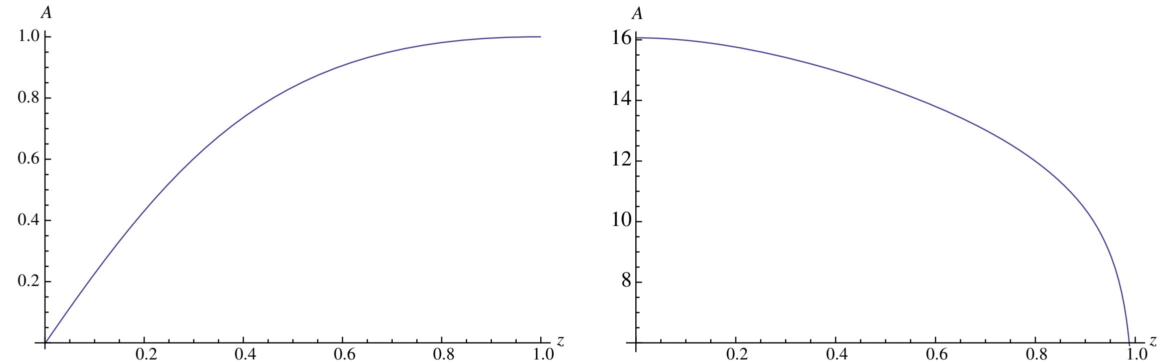

FIG. 2. Two solutions for the gauge field A of the equation of motion in the AdS_4-Schwarzschild metric: a standard solution for $\mu = 3.656$, $w_1 = 0.21$ (left) and a new solution for $\tilde{\mu} = 1.05$, $\tilde{w}_1 = -2$ (right).

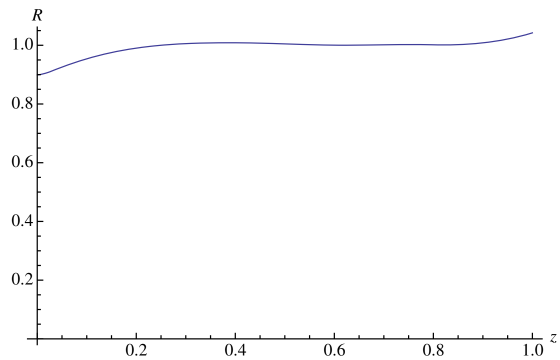

FIG. 3. Combination $R(z) = -w(z)\tilde{w}(z)(1 + z + z^2 - z^3 q^2)^2$ for the two solutions of the Riccati equation in the Schwarzschild metric $q = 0$. Combination R being constant means that the duality relation (130) is satisfied in the AdS-RN space.

solution of Riccati equation $w = A_x^{1\prime}/A_x^1$ at μ with Dirichlet b.c., there exists a solution of the Riccati equation $w = \tilde{A}_y^{2\prime}/\tilde{A}_y^2$ at $\tilde{\mu}$ with Neumann b.c. These two solutions are related by the duality equations (116) and (117), and there is a duality mapping $\mu \to \tilde{\mu}$ described by a function $\mu(\tilde{\mu})$. The S-duality transformation $E \overset{S}{\to} B$ acts on the gauge field solution as

$$\text{electric side} \overset{S}{\to} \text{“magnetic” side} \tag{174}$$

$$\text{large}\,\mu \overset{S}{\to} \text{small}\,\mu \tag{175}$$

$$\text{Dirichlet b.c.} \overset{S}{\to} \text{Neumann b.c.} \tag{176}$$

$$\text{source} \overset{S}{\to} \text{v.e.v.} \tag{177}$$

and on the superfluid density and the conductivity as

$$n_s/\mu \overset{S}{\to} \frac{1}{n_s/\mu} \tag{178}$$

$$\sigma \overset{S}{\to} \frac{1}{\sigma} \tag{179}$$

$$\text{poles}\,\sigma \overset{S}{\to} \text{zeros}\,\frac{1}{\sigma} \tag{180}$$

$$\text{metal} \overset{S}{\to} \text{insulator}, \tag{181}$$

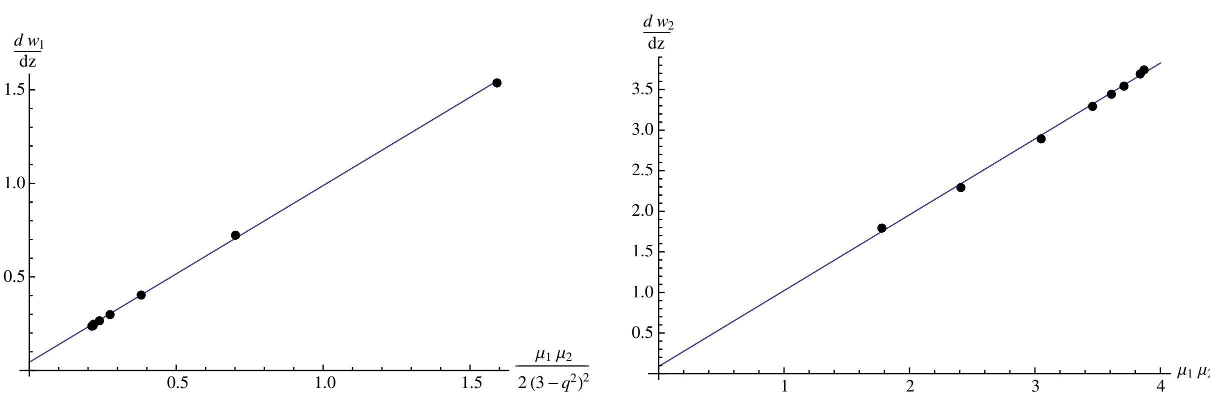

FIG. 4. The duality relation for the RN-AdS BH metric in the IR and the UV asymptotics with the charge increased from $q = 0$ to $q = 1.5$. The duality in the IR is $-w'(z = 1)$ vs $\mu\tilde{\mu}/2(3 - q^2)^2$ (left). The duality in the UV is $-\tilde{w}'(z = 0)$ vs $\mu\tilde{\mu}$ (right). Both are straight lines, which verifies the duality conditions (136) and (142).

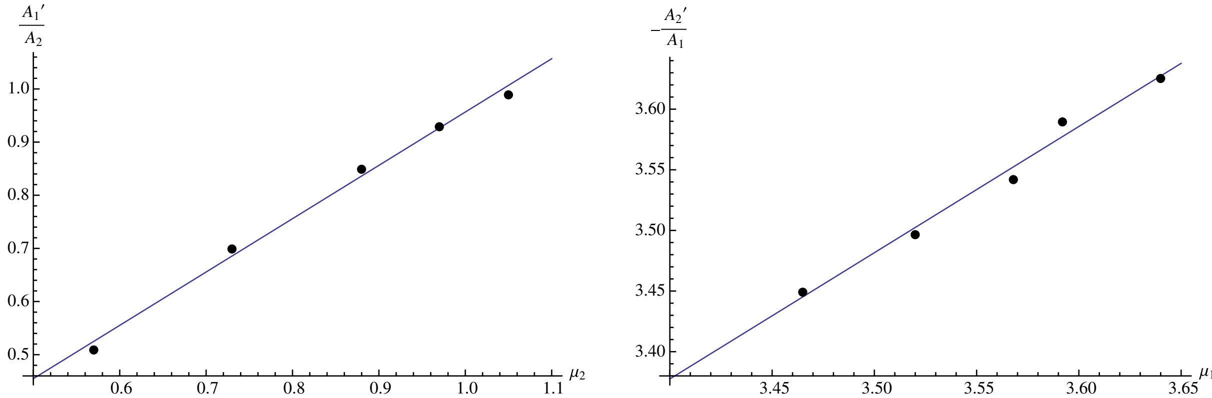

FIG. 5. The duality relation for the RN-AdS BH metric in the UV asymptotics with the charge increased from $q = 0$ to $q = 1.5$: $A'(z = 0)/\tilde{A}(z = 0)$ vs $\tilde{\mu}$ (left) and $-\tilde{A}'(z = 0)/A(z = 0)$ vs μ (right). Linear behavior confirms the duality conditions for the gauge fields (120).

and the reverse transformation "$\overset{S}{\leftarrow}$" is also true. It was shown for Abelian $U(1)$ gauge fields in Ref. [1] and for non-Abelian gauge fields in Ref. [28] that the generator S of $\mathrm{SL}(2, Z)$ exchanges electric and magnetic fields. Further, the "electric" side with $B = 0$ b.c. for the gauge fields corresponds to the Dirichlet b.c. (standard quantization), vector potential A vanishing at the boundary. While the magnetic side with $E = 0$ b.c. corresponds to the Neumann b.c. (alternative quantization), boundary values of A remaining unrestricted. This means that swapping between the Dirichlet and the Neumann b.c. leads to swapping the identification of the source and the v.e.v. [29].

The transformation $\overset{S}{\rightarrow}$ corresponds to a superconductor-insulator quantum critical point of bosons in $(2 + 1)$ dimensions [44]. Thus, we have demonstrated that the duality relations (116) and (117) are satisfied for the AdS-RN and Schwarzschild black holes in the bulk for all z, see Fig. 3, and in the IR/UV asymptotics, Fig. 4. We checked the duality condition for the original gauge fields $A, \tilde{A}$ in the UV, Fig. 5, while the duality relation in the IR serves as boundary conditions to normalize the solutions. Therefore, we can state that S-duality acts on the Riccati

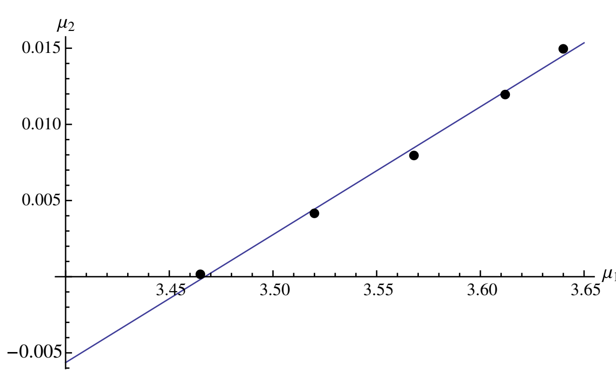

FIG. 6. The mapping between the chemical potentials, $\tilde{\mu}$ vs μ, as the charge is increased from $q = 0$ to $q = 1.5$.

equation and its physically relevant solutions in a known way.

IV. FLAVOR S-DUALITY IN HOLOGRAPHIC QCD WITH ISOTOPIC AND BARYONIC CHEMICAL POTENTIALS

Let us briefly discuss the possible counterpart of the S-duality for the p-wave superconductor in holographic QCD. First, let us remind the reader what kind of phenomenon is relevant for this issue. For low-energy QCD, the flavor group $U(N_f)_L \times U(N_f)_R$ plays the role of the gauge group in the holographic dual which is broken to the diagonal one by the chiral condensate. To fit with the previous discussion, we restrict ourselves by $N_f = 2$. The Abelian $U(1)_B, U(1)_A$ factors correspond to the baryon charge and axial singlet symmetry broken by the anomaly. The most popular holographic models for QCD are the Sakai-Sugimoto models and D3/D7 models which can be thought of as the chiral Lagrangian supplemented by the infinite tower of the massive vector mesons.

In the chirally broken phase, the symmetry involves global $\mathrm{U}(1)_B \times \mathrm{SU}(2)_I$. We would like to have dopped system. Therefore we introduce the isotopic and baryonic chemical potentials μ_I, μ_B. If $\mu_B \neq 0$ at small μ_I, the pion condensate gets emerged, and there is a kind of rotation between chiral and pion condensates discussed in details in Ref. [45]. The pion condensate yields the nontrivial supercurrent and the superfluid component. If μ_I increases, the new phase with the vector condensate appears which is the analog of p-wave superconductor. This ρ-meson condensate has been identified both in the Sakai-Sugimoto model [23] and D3/D7 model [24]. The physics of this phase is quite clear—the mass of the vector meson decreases with μ_I, and at some critical value of the chemical potential, it vanishes, allowing the condensation. The appearance of the vector condensate can be seen geometrically in the D3/D7 model as follows [24]. At small isotopic chemical potential, the strings from D7-branes are attracted to the horizon making it isotopically charged. However, increasing the μ_I leads that the strings rearrange and form 7-7 strings which are vector mesons. The vector condensate is formed from the flow of 7-7 strings in the bulk.

One more relevant phenomenon concerns the effects of the external magnetic field. The electromagnetic charge is included in the isotopic group as $Q = \mathrm{diag}(2/3, -1/3)$, and therefore the magnetic field B_{em} has the baryonic and isotopic components $QB = B_{\mathrm{bar}} I + B_{\mathrm{iso}} \tau_3$. The magnetic field changes the critical values of the chemical potential for the pion condensate formation. Moreover, there are arguments in favor of formation of the vector meson condensate above some critical B_{crit} [46,47]. This is the analog of the p-wave superconductivity once again; however, this phenomenon is still questionable.

Turn now to our conjecture concerning the flavor S-duality in low-energy QCD. Since the S-duality is expected to be a generalization of particle-vortex duality in $2+1$ dimensions, we have to identify what "particle" and "vortex" mean in the $3+1$ case and in its $4+1$ dual. In $4+1$ dual non-Abelian gauge theory, there are two pointlike objects with different charges—gauge bosons and instantons. The corresponding global symmetry group generated by conserved currents is $U(1)_{\mathrm{top}} \times U(1)_{el}$ where the topologically conserved current is defined as

$$j_\beta = \epsilon_{\nu\mu\gamma\delta\beta} Tr F_{\mu\nu} F_{\gamma\delta}. \tag{182}$$

These Abelian groups are the $4+1$-dimensional counterpart of the pair of Abelian groups involved in S-duality transformation in the $2+1$ case. In boundary low-energy QCD, particle and vortex are identified as charged vector mesons and baryons which are the instantons in the flavor group in $4+1$ dimensions and realize Skyrmions in $3+1$ dimensions [48]. There is, however, some subtlety concerning the identification of the baryon as the instanton in $4+1$ dimensions, and there are serious arguments for its treatment not as the instanton but as the dyonic instanton [49]. Such interpretation fits better with the effects of the chiral condensate on the baryon state.

Let us emphasize that we consider the analog of S-duality in QCD for the flavor and not color group. Therefore instead of "W-bosons" and monopoles which represent S-dual pair in color gauge group in $3+1$ dimensions we consider the vector mesons and baryons. One could wonder if such S-duality is natural from the stringy viewpoint. To this aim, it is useful to remind the reader that non-Abelian five-dimensional gauge theory in the IIB picture is represented by the (p,q) 5-brane web. In the brane language, vector gauge bosons and instantons in five-dimensional gauge theory indeed form the S-dual pair being represented by the F1 nd D1 strings attached to the 5-brane web. In the IIA picture, the vector mesons and baryons are presented more asymmetrically.

Remark that in the $4+1$ dual theory, there is natural "flux attachment" procedure via the five-dimensional Witten effect. Indeed, due to the five-dimensional Chern-Simons term ($N_f > 2$) at level N_c, the state with the topological charge Q_{top} gets the electric charge

$$Q_e = N_c Q_{\mathrm{top}}, \tag{183}$$

where N_c is the rank of the color group. This is a familiar picture for the strings attachment to the baryonic vertex in the bulk [50].

Hence, our candidates for the S-dual pair are the (ρ, baryon), and therefore we can assume that (μ_I, μ_B) play the same role as the $(\mu, \tilde{\mu})$ in the rest of the paper. What kind of condensates are available? As was mentioned, at large enough μ_I, the electric p-wave condensate gets formed. The same p-wave condensate presumably gets formed at large magnetic field. In this respect, the situation resembles the quantum Hall effect (QHE) case if we treat (μ_I, B) as the dual pair. More interestingly, if we consider (μ_I, μ_B) as the dual pair, we have nontrivial phase structure in this plane when the pion condensate gets substituted by the condensate of the vector mesons. At large enough μ_B, the pion condensate gets formed; hence, we could attempt to treat $\langle \partial_i \pi \rangle$ as the pseudovector dual required by the S-duality.

Could we present more arguments supporting duality between QCD with isotopic and baryonic chemical potentials? A bit surprisingly, the orbifold equivalence provides some supporting evidence [51–53]. The orbifold equivalence relates the theories with the different groups supplemented to the projection to the particular sector of the theory. It is well established in the perturbation theory, while its status at the nonperturbative level is more subtle. Nevertheless, assuming its validity at the nonperturbative level, it was argued in Refs. [51–53] that there is the large N-duality between QCD with isotopic chemical potential and QCD with baryonic chemical potential at some regions in the (μ_I, μ_B) plane. Our interpretation of these orbifold equivalences as the version of S-duality is new. We could also speculate that the experimentally observed equal intercepts of meson and baryon Regge trajectories could be related to our duality conjecture. Usually, this observation is treated differently as the consequence of the large diquark component inside the baryon.

Our consideration suggests that there could be the relation between the superfluid/superconducting densities in the dual descriptions that is the superconducting components of densities for isotopic and the baryonic charge carriers in the chirally broken phase. Their product presumably can be proportional to $\mu_I \mu_B$,

$$n_I^s n_B^s \sim \mu_I \mu_B. \tag{184}$$

The special attention is to the large μ_B limit when the chiral symmetry is restored and QCD is in the color-flavor locking phase supporting the color superconductivity. This issue deserves further investigation, and we hope to discuss it elsewhere.

In conclusion we note that the S-duality followed from five-dimensional dual which we conjecture as flavor S-duality in low-energy QCD could have clear-cut counterpart in $1+1$ dimensionals case. Indeed, we could start with the $2+1$ gauge dual involving the Yang-Mills and Chern-Simons terms. In the bulk, we can define the electric U(1) and topologically conserved U(1) current

$$j_\mu = \epsilon_{\mu,\nu\alpha} F_{\nu\alpha}. \tag{185}$$

We have gauge bosons and instanton particles in the bulk theory which carry the topological charge. They form the S-dual pair in the bulk. In the boundary theory, the instantons become the Skyrmions. The boundary S-duality presumably exchanges the $1+1$ theory with chemical potentials for vector mesons and for Skyrmions.

V. CONCLUSIONS

Recently, there has been a considerable effort made to establish duality in $(2+1)$-dimensional theories [13, 54–57]. In Ref. [54], a particle-vortex duality used to study bosonic systems has been proposed to find duality for Dirac fermions in $(2+1)$ dimensions. There is also an alternative description of this duality which is accessed via an electromagnetic duality in the topological superconductor. In Refs. [13,57], a whole class of $(2+1)$-dimensional dualities is derived from an elemental duality between a bosonic field theory and a fermionic field theory, where the duality between the two fermion theories appears as one particular case. There is also a connection with duality between $(2+1)$-dimensional Chern-Simons theories [58]. Duality for the Dirac fermion in $(2+1)$ dimensions relates fermionic theories at nonzero density and in the magnetic field, that gives a powerful tool to study the fractional quantum Hall effect, in particular particle-hole symmetric quantum Hall states [59].

In this paper, we suggest a duality between fermion theories in $(2+1)$ dimensions where the charge density and the magnetic field are interchanged on the two sides of the duality. We establish this duality via electromagnetic duality in a holographic dual theory in $(3+1)$ dimensions. Specifically, we suggested some version of the generalization of S-duality in the holographic theories involving non-Abelian SU(2) gauge field in the bulk. This permits considering a class of theories which includes the condensates and exhibit phase transitions. Using the p-wave holographic superconductor, we construct the electric and magnetic sides of the electric-magnetic duality (EM) duality, where the holographic gauge fields are solutions of the EOM and they are related by the S-duality transformation. As in the Abelian case, for the non-Abelian gauge fields, the generator S exchanges the electric and magnetic fields. The electric side has $B=0$ b.c. for the gauge fields which are equivalent to Dirichlet b.c., where the vector potential vanishes on the boundary, while the magnetic side has $E=0$ b.c. which are equivalent to Neumann b.c., that leave the boundary value of the vector potential unrestricted. The action of the S transformation leads to the exchange between the source and v.e.v. in the solution.

We found analytically the gauge field components and asymptotics of the solutions, while we solved numerically for the solutions in the AdS bulk on both sides of the duality. The electric side is characterized by a standard p-wave superconductor solution which persists from a critical chemical potentials $\mu_c \approx 3.656$—a threshold solution. The magnetic side exhibits a new type of instability for very small chemical potentials—a new superconductor solution. A "neutral" superconductor was found in different AdS gravitational theories in Refs. [30–33], but always at small μ and with Neumann b.c. Assuming the EM-duality condition between the bulk solutions, we obtain the mapping between the chemical potentials of the threshold and new solutions which is a functional dependence μ vs $\tilde{\mu}$. It would be interesting to extend the S-duality transformation by the T-duality to get the $SL(2,Z)$ group.

We find the indication for the S-duality relation between superfluid densities which is similar to the relation for conductivities. Specifically, we observe that the dimensionless ratio n_s/μ and its S-dual CFT pair $\tilde{n}_s/\tilde{\mu}$ are the inverses of each other $n_s/\mu \cdot \tilde{n}_s/\tilde{\mu} \sim 1$. The relation for conductivities is $\sigma\tilde{\sigma} \sim 1$. It is consistent with the fact that S-duality corresponds to the metal-insulator transition [44].

We made a conjecture on the possible S-duality in the SU(2) flavor sector in large N_c QCD based on the holographic picture. On the holographic side, S-duality maps electrically and topologically charged states in $4+1$ bulk non-Abelian gauge theory which substitute the electric and magnetic frames in $3+1$ dimensions. Physically, at the boundary, the topological sector corresponds to baryons, while the electric sector corresponds to the vector mesons. The corresponding chemical potentials are isotopic and baryonic ones, and the large isotopic chemical potential yields the p-wave superconductor for the chirally broken phase. On the other hand, the baryonic chemical potential yields a pion condensate in the chirally broken phase and the Cooper condensates in the color-flavor locking phase. Therefore, it would be interesting to explore further the interplay between the isotopic and baryonic sectors at the chirally broken phase treated as the version of S-duality and on the other hand the interplay between the chirally broken phase supplemented by the magnetic field and the phase with the color superconductivity.

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APPENDIX A: SL(2,Z) INVARIANCE OF THE AXIODILATON SU(2) GAUGE ACTION

The Einstein-Maxwell action coupled to an axion and a dilaton field in $3+1$ dimensions has been considered in Refs. [3,60]. We generalize their action to the case of SU(2) non-Abelian gauge fields. The gauge-gravity action coupled to an axiodilaton is given by

$$S_{\phi,\chi} = -\int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}\left[R-2\Lambda+\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi+\mathrm{e}^{2\phi}\partial_\mu\chi\partial^\mu\chi)\right]+\frac{1}{4}\mathrm{e}^{-\phi}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}\chi F_{\mu\nu}*F^{\mu\nu},\right) \tag{A1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$, with non-Abelian gauge field $A_\mu=A_\mu^a\tau_a$, and τ_a are generators of the SU(2) group, $[\tau^a,\tau^b]=i\epsilon^{abc}\tau^c$, related to the Pauli matrices by $\tau_a=\sigma_a/2i$. We introduce the covariant derivative $D_\mu=I\partial_\mu-ig\tau^aA_\mu^a$; therefore, the field strength is $[D_\mu,D_\nu]=-ig\tau^aF_{\mu\nu}^a$ where the coupling g is defined further. The dual field strength is obtained by applying the Hodge star operation $*F_{\mu\nu}\coloneqq\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$, where the completely antisymmetric Levi-Cività tensor $\epsilon_{\mu\nu\rho\lambda}$ has a factor of $\sqrt{-g}$ with $g=\det g_{\mu\nu}$ extracted and transforms as a tensor and not as a tensor density, and indices are freely raised and lowered using the metric $g_{\mu\nu}$ of which the signature is Lorentzian $(-+++)$. Since we are working in four dimensions transforming 2-form $F_{\mu\nu}$, the Hodge dual applied twice results in $**=s=-1$ where s is the signature of the inner product on the manifold. In Eq. (A1), two scalar fields are the dilaton ϕ and axion χ. The constant $\Lambda=3/L^2$ is the AdS cosmological constant, and $\kappa^2=8\pi G$ is Newton's constant. The weak curvature means $\kappa^2/L^2\ll 1$. The relation to the gauge coupling g and the θ-angle is

$$\mathrm{e}^{-\phi}=\frac{1}{g_E^2}=\frac{1}{g^2},\qquad \chi=\frac{1}{g_B^2}=\theta, \tag{A2}$$

where subscripts E and B stand for the electric and magnetic part. Therefore, weak coupling corresponds to $\mathrm{e}^{\phi}\ll 1$.

The matter fields, dilaton and axion, provide couplings of the gauge field action. They are added to ensure the existence of a duality group. The SL(2,Z) duality of Maxwell U(1) gauge fields in gravity-dual four dimensions has been shown by Witten in Ref. [1]. He also suggested that generalization to non-Abelian gauge fields is possible when "special collections of matter fields are included."

As in Ref. [3], we define the axiodilaton by a complex variable

$$\tau\coloneqq\chi+i\mathrm{e}^{-\phi}=\theta+\frac{i}{g^2}, \tag{A3}$$

[not to be confused with the SU(2) generator], and the axiodilaton action is rewritten

$$L_{\phi,\chi}\sim\partial_\mu\phi\partial^\mu\phi+\mathrm{e}^{2\phi}\partial_\mu\chi\partial^\mu\chi=\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{(\mathrm{Im}\tau)^2}. \tag{A4}$$

We define the matrix of $\mathrm{SL}(2,R)$ transformation

$$M=\begin{pmatrix}p & q\\ r & s\end{pmatrix}\in\mathrm{SL}(2,\mathrm{R}), \tag{A5}$$

where p, q, r, s are real numbers and $\det M=1$, that is $sp-qr=1$. The SL(2,R) transformation acts on a axiodilaton and on the metric as [3]

$$\tau\to\tilde{\tau}=\frac{p\tau+q}{r\tau+s},\qquad g_{\mu\nu}\to\tilde{g}_{\mu\nu}=g_{\mu\nu}, \tag{A6}$$

where the tilde distinguishes the variables which underwent the SL(2, R) transformation. We rewrite Eq. (A6),

$$\tau\to\frac{pr\|\tau\|^2+qs+ps2\mathrm{Re}\tau-\bar{\tau}}{\|r\tau+s\|^2}, \tag{A7}$$

where we used $sp-qr=1$. Therefore,

$$\partial_\mu\tau\partial^\mu\bar{\tau}\to\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{\|r\tau+s\|^4},\qquad \mathrm{Im}\tau\to\frac{\mathrm{Im}\tau}{\|r\tau+s\|^2}, \tag{A8}$$

and the axiodilaton action $S_{\phi,\chi}$ (A4) is $\mathrm{SL}(2,R)$ invariant. To see the $\mathrm{SL}(2,R)$ action on the gauge field, we define, as in Ref. [3,60],

$$G^{\mu\nu}\coloneqq-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta F_{\mu\nu}} \tag{A9}$$

and obtain

$$G^{\mu\nu} = \mathrm{e}^{-\phi} F^{\mu\nu} - \chi * F^{\mu\nu}. \tag{A10}$$

Written in terms of complex quantities [3]

$$\mathcal{F}_{\mu\nu} \coloneqq F_{\mu\nu} + i * F_{\mu\nu}, \tag{A11}$$

$$\mathcal{G}_{\mu\nu} \coloneqq *G_{\mu\nu} - iG_{\mu\nu}, \tag{A12}$$

Eq. (A10) takes a compact form,

$$\mathcal{G}^{\mu\nu} = \bar{\tau}\mathcal{F}^{\mu\nu}. \tag{A13}$$

The $\mathrm{SL}(2,Z)$ acts on the gauge fields as [3]

$$\begin{pmatrix} \mathcal{G}_{\mu\nu} \\ \mathcal{F}_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} \tilde{\mathcal{G}}_{\mu\nu} \\ \tilde{\mathcal{F}}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \mathcal{G}_{\mu\nu} \\ \mathcal{F}_{\mu\nu} \end{pmatrix}, \tag{A14}$$

where the tilde denotes the transformed variables. Relation (A13) is invariant under SL(2,R). Indeed, after transformation, we have

$$p\mathcal{G} + q\mathcal{F} = \frac{p\bar{\tau} + q}{r\bar{\tau} + s}(r\mathcal{G} + s\mathcal{F}), \tag{A15}$$

which is reduced to Eq. (A13) provided $sp - qr = 1$. From Eq. (A14) follows SL(2,R) transformation for the field strengths

$$F^a_{\mu\nu} \to \tilde{F}^a_{\mu\nu} = sF^a_{\mu\nu} + r * G^a_{\mu\nu}, \tag{A16}$$

$$G^a_{\mu\nu} \to \tilde{G}^a_{\mu\nu} = pG^a_{\mu\nu} - q * F^a_{\mu\nu}, \tag{A17}$$

where we used for the double Hodge duality $** = -1$. Using the definition for G (A10) and Eqs. (A16) and (A17), the SL(2,R)-dual field strength $F_{\mu\nu}$ is

$$F^a_{\mu\nu} \to \tilde{F}^a_{\mu\nu} = r\mathrm{e}^{-\phi} * F^a_{\mu\nu} + (s + r\chi)F^a_{\mu\nu}. \tag{A18}$$

The gauge action

$$\begin{aligned} L_{F,*F} \sim F_{\mu\nu}G^{\mu\nu} &\to (sF_{\mu\nu} + r * G_{\mu\nu})(pG^{\mu\nu} - q * F^{\mu\nu}) \\ &= (sp - qr)F_{\mu\nu}G^{\mu\nu} = F_{\mu\nu}G^{\mu\nu} \end{aligned} \tag{A19}$$

is invariant under SL(2,R) transformation. We used $*G_{\mu\nu} * F^{\mu\nu} = F_{\mu\nu}G^{\mu\nu}$ and $sp - qr = 1$, and the Lagrangian satisfies a differential constraint $*G_{\mu\nu}G^{\mu\nu} = F_{\mu\nu} * F^{\mu\nu}$ with $G_{\mu\nu}$ given by Eq. (A10) and $\|\tau\|^2 = 1$ [60].

The SL(2,R) invariance can be shown using the electric and magnetic fields. We define the electric intensity $E^a_i = F^a_{i0}$ and magnetic induction $B^a_i = \frac{1}{2}\epsilon_{ijk}F^{jka}$, with $*\mathbf{E} = -\mathbf{B}$ and $*\mathbf{B} = \mathbf{E}$, and the electric induction $D^a_i = G^a_{i0}$ and magnetic intensity $H^a_i = \frac{1}{2}\epsilon_{ijk}G^{jka}$ where $G_{\mu\nu}$ is given by Eq. (A10),

$$\mathbf{D} = \frac{1}{\sqrt{-g}}\frac{\partial S}{\partial \mathbf{E}} = \mathrm{e}^{-\phi}\mathbf{E} + \chi\mathbf{B}, \tag{A20}$$

$$\mathbf{H} = -\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial \mathbf{B}} = \mathrm{e}^{-\phi}\mathbf{B} - \chi\mathbf{E}, \tag{A21}$$

where $\mathbf{E} = \mathbf{E}^a\tau^a$ with $\tau^a \in \mathrm{SU}(2)$ and the same is true for other vector fields. From Eq. (A14), SL(2,R) transformation for the fields is

$$\mathbf{E}^a \to s\mathbf{E}^a - r\mathbf{H}^a, \qquad \mathbf{B}^a \to s\mathbf{B}^a + r\mathbf{D}^a, \tag{A22}$$

and

$$\mathbf{D}^a \to p\mathbf{D}^a + q\mathbf{B}^a, \qquad \mathbf{H}^a \to p\mathbf{H}^a - q\mathbf{E}^a. \tag{A23}$$

Therefore, the action

$$\begin{aligned} L_{\mathbf{E},\mathbf{B}} \sim \mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H} &\to (s\mathbf{E} - r\mathbf{H})(p\mathbf{D} + q\mathbf{B}) \\ &\quad + (s\mathbf{B} + r\mathbf{D})(p\mathbf{H} - q\mathbf{E}) \\ &= (sp - qr)(\mathbf{ED} + \mathbf{BH}) \\ &= \mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H} \end{aligned} \tag{A24}$$

is invariant under SL(2,R).

In general, only Abelian U(1) gauge action and the corresponding Maxwell equations are invariant under the $\mathrm{SL}(2,R)$-duality [60,61]. Indeed, performing the $\mathrm{SL}(2,R)$ transform (A16), (A17) in the Yang-Mills equation $D * G = 0$,

$$(D_\mu * G^{\mu\nu})^a = 0, \tag{A25}$$

and the Bianchi identity $DF = 0$,

$$(D_\mu F_{\nu\kappa})^a + (D_\kappa F_{\mu\nu})^a + (D_\nu F_{\kappa\mu})^a = 0, \tag{A26}$$

the problem is caused by a covariant derivative, where the $\mathrm{SL}(2,R)$ transformation should be written for the gauge field and not for the field strength; therefore, this transformation has an integral, nonlocal character. However, there exist solutions which are S invariant in all the space or in the asymptotic regions. One example is an AdS instanton solution which is self-dual in the whole space. Further, we show that the p-wave superconductor is S invariant in the UV and IR. Equations of motion for axion and dilaton are SL(2,R) invariant.

Also, one should have in mind the following remark. In string theory, the $\mathrm{SL}(2,R)$ group generally holds in the classical approximation and is broken down to a discrete subgroup by the quantum effects [3]. Specifically, the symmetry is broken by the presence of objects of which the charges are quantized; for example, the (m,n)-string breaks $\mathrm{SL}(2,R)$ down to $\mathrm{SL}(2,Z)$ [3]. In what follows, we consider $\mathrm{SL}(2,Z)$ for our applications.

$\mathrm{SL}(2,Z)$ is generated by the two matrices [1], S is the analog of electric-magnetic duality

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A27}$$

and T acts on a topological term $\theta \int d^4 x F * F$ by shifting $\theta \to \theta + 2\pi$,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{A28}$$

The $S^2 = -1$ is the central element, and $(ST)^3 = 1$, while S and T do not commute. The action of S and T transformation on the boundary conformal field theories and the relation to the AdS gravitational theories has been discussed in Ref. [1]. On the S-operation, gravitational theory in AdS space has two different CFT duals on the boundary, depending on which boundary condition one chooses to impose. It has been shown for the Abelian $U(1)$ gauge fields [1] that the generator S of $\mathrm{SL}(2,Z)$ exchanges electric and magnetic fields, and it corresponds from the AdS point of view to replacing the boundary condition $\mathbf{B} = 0$ (generally where B is specified) which is the electric side with $\mathbf{E} = 0$ (generally where E is specified) which is the magnetic side of electric-magnetic duality. Further [1], $\mathbf{B} = 0$ boundary conditions for the gauge fields are the analogs to Dirichlet boundary conditions. They say that vector potential $\mathbf{A}$ vanishes on the boundary, up to a gauge transformation. At the same time $\mathbf{E} = 0$ boundary conditions are analogous to free or Neumann boundary conditions. They leave the boundary values of $\mathbf{A}$ unrestricted. We will use this fact later to identify the source and the v.e.v. in the asymptotic behavior of gauge fields at the AdS boundary.

The S-operator acts on the axiodilaton (A7) as

$$\tau \to \tilde{\tau} = -\frac{1}{\tau}, \tag{A29}$$

where we used $p = s = 0$ and $q = -r = 1$ in the SL(2,R) matrix (A5). While the S-duality transformation is interesting to perform on the full gauge and axiodilaton action (A1), we restrict ourselves to the vanishing axion field

$$\chi = 0. \tag{A30}$$

According to Eq. (A29) and the definition of $\tau = \chi + i\mathrm{e}^{-\phi}$, the axion field is not generated by the S-duality transformation. In this case, the S-operator acts on the gauge field strength (A16) as

$$F^a_{\mu\nu} \to \tilde{F}^a_{\mu\nu} = - * G^a_{\mu\nu}, \tag{A31}$$

or written explicitly,

$$F^a_{\mu\nu} \to \tilde{F}^a_{\mu\nu} = -\mathrm{e}^{-\phi} * F^a_{\mu\nu}. \tag{A32}$$

Equations (A29) and (A32) express a familiar electric-magnetic duality where the field strength transforms into a Hodge-dual one and the coupling transformation is $g^2 \to \frac{1}{g^2}$; therefore, the weak-strong coupling regimes are interchanged. In $(3+1)$ dimensions, the Hodge-dual is defined as $\star F = \frac{\sqrt{-g}}{4} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} dx^\mu \wedge dx^\nu$.

APPENDIX B: *p*-WAVE SUPERCONDUCTOR IN AdS_5

We write the Riccati equation for the AdS_5 p-wave superconductor and obtain the two solutions imposing Dirichlet and Neumann boundary conditions [17,19,62]. However, these solutions are not related to each other by the EM duality. We discuss the AdS_5 case here, because there is a known p-wave superconducting solution for it, and we can test our numerical solution against the analytical one. As we discuss in the main text, the equations of motion for the "condensate" components on the E and B sides of duality transformation, A^1_x and A^2_y, respectively, are the same. Also, we substitute for $A^3_t = \mu(1 - z^2)$ and $\tilde{A}^3_t = \tilde{\mu}(1 - z^2)$. In what follows, we use one letter for both gauge components $A^1_x, A^2_y \to A$, and one μ. In the Schwarzschild metric with the AdS_5 asymptotic behavior, the EOM reads

$$A'' + \left(\frac{f'}{f} + \frac{1}{z}\right) A' + \frac{\mu^2 (1 - z^2)^2}{z^4 f^2} A = 0, \tag{B1}$$

$$f(z) = \frac{1}{z^2} - z^2, \tag{B2}$$

and explicitly it is

$$A'' - \frac{1 + 3z^4}{z(1 - z^4)} A' + \frac{\mu^2}{(1 + z^2)^2} A = 0 \tag{B3}$$

$$A(z = 1) = \mathrm{const}, \tag{B4}$$

$$A'(z = 1) = 0, \tag{B5}$$

where we specified the IR boundary conditions, which mean regularity.

In the UV $z = 0$, the asymptotic behavior of the gauge field is

$$A = A^{(0)} + A^{(1)} z^2 + \cdots. \tag{B6}$$

In the UV, the Dirichlet and Neumann boundary conditions for the gauge field and its derivative are given in the main text. There is one boundary condition (again Dirichlet and Neumann) in the UV for the Riccati equation. The Riccati

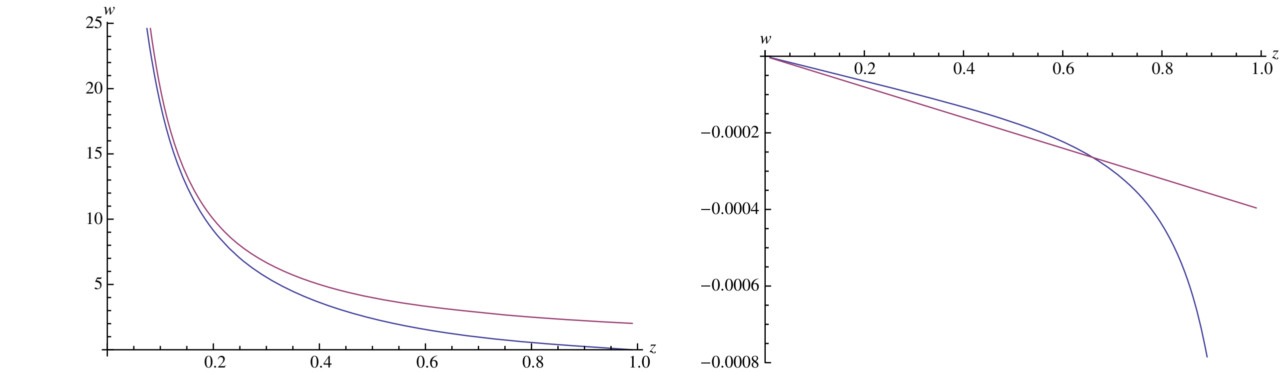

FIG. 7. Two solutions w of the Riccati equation in the AdS_5: a standard threshold solution for $\mu = 4$, $w_1 = 0$ (left) and a new solution for $\mu = 0.001$, $w_1 = -0.009$ (right). The upper curve in the right plot is $w = 2/z$. The straight line in the left plot is $w = -0.0004z$.

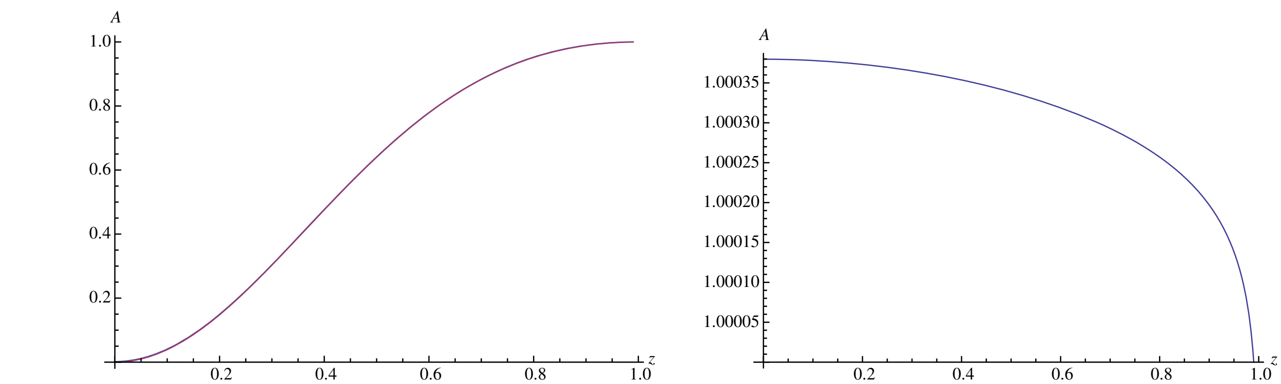

FIG. 8. Two solutions for the gauge field A of the equation of motion in the AdS_5: a standard threshold solution for $\mu = 4$ $w_1 = 0$ (left) and a new solution for $\mu = 0.001$, $w_1 = -0.009$ (right). Two coinciding curves for numerical and analytical solutions $4z^2/(1+z^2)^2$ are shown in the left panel.

equation corresponding to Eq. (B1) for the variable $w = A'/A$ reads

$$w' - \frac{1+3z^4}{z(1-z^4)}A' + \frac{\mu^2}{(1+z^2)^2}A = 0 \tag{B7}$$

$$w(z=1) = w_1, \tag{B8}$$

where we impose regularity at the IR by taking w_1 as a constant.

We vary two parameters μ and w_1 to obtain the needed UV behavior for the Dirichlet and Neumann boundary conditions:

$$\text{Dirichlet:}\; w(z=0) = \frac{2}{z} \to \infty, \tag{B9}$$

$$\text{Neumann:}\; w(z=0) = 0. \tag{B10}$$

As noted before, the analytic solution for the p-wave superconductor satisfying the Dirichlet boundary condition is known to be [35]

$$A = \frac{4z^2}{(1+z^2)^2}, \tag{B11}$$

$$\mu_c = 4, \tag{B12}$$

where μ_c is the critical chemical potential.

We solve the Riccati equation (B7) in AdS_5 numerically and obtain the two solutions, a threshold solution for $\mu = 4$ and a new solution for $\mu = 0.001$, Fig. 7, which satisfy the Dirichlet and Neumann b.c. (B9), (B10), respectively. The corresponding solutions for the gauge fields are depicted in Fig. 8.

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