Direct test of the integral Yang-Mills equations through $SU(2)$ monopoles

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We use the $SU(2)$ 't Hooft-Polyakov monopole configuration, and its Bogomolny-Prasad-Sommerfield (BPS) version, to test the integral equations of the Yang-Mills theory. Those integral equations involve two (complex) parameters which do not appear in the differential Yang-Mills equations, and if they are considered to be arbitrary, it then implies that non-Abelian gauge theories (but not Abelian ones) possess an infinity of integral equations. For static monopole configurations, only one of those parameters is relevant. We expand the integral Yang-Mills equation in a power series of that parameter and show that the 't Hooft-Polyakov monopole and its BPS version satisfy the integral equations obtained in first and second order of that expansion. Our results point to the importance of exploring the physical consequences of such an infinity of integral equations on the global properties of the Yang-Mills theory.

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I. INTRODUCTION

The purpose of this paper is to perform a test of the integral equations of Yang-Mills theories, recently proposed in [1,2], using the $SU(2)$ 't Hooft-Polyakov monopole solution [3,4] as well as its exact analytical Bogomolny-Prasad-Sommerfield (BPS) version [5,6]. The main motivation for such a test is that these integral equations involve two complex parameters that are not present in the Yang-Mills partial differential equations. If those parameters are arbitrary, it means that contrary to Abelian electromagnetism, Yang-Mills theories possess, in fact, an infinity of integral equations. Indeed, by expanding the Yang-Mills integral equations in power series of those parameters, we check that the $SU(2)$ 't Hooft-Polyakov monopole, and its BPS version, do satisfy the integral equations appearing in that expansion, up to second order in one of the parameters. The cancellations involved in such a check are highly nontrivial and give strong evidence for the arbitrariness of those parameters.

As shown in [1,2], the integral Yang-Mills equations lead, in a quite natural way, to gauge-invariant conserved charges. Such charges involve those two parameters in a way that, if they are indeed arbitrary, it would imply that, in principle, the number of charges is infinite. However, due to some special properties of BPS multidyon solutions [7,8], shown in [9], the higher charges are not really independent for such solutions, being in fact powers of the first ones (the electric and magnetic charges). The same is true for the $SU(2)$ 't Hooft-Polyakov monopole. It remains to be investigated whether or not other non-BPS solutions also present such special properties and so possess an infinity of charges.

In order to discuss the role of such parameters in a more concrete way, let us start by the theory of electromagnetism described by the Maxwell equations,

$$\partial_\mu f^{\mu\nu} = j^\nu \qquad \partial_\mu \tilde{f}^{\mu\nu} = 0, \tag{1.1}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, $\tilde{f}^{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda} f^{\rho\lambda}$, j^μ being the external four current, and a_μ the electromagnetic four-vector potential. The integral version of those equations is obtained through the Abelian Stokes theorem for a rank-two antisymmetric tensor $b_{\mu\nu}$ on a spacetime 3-volume Ω, as $\int_{\partial\Omega} b = \int_\Omega d\wedge b$, where $\partial\Omega$ is the border of Ω. Taking $b_{\mu\nu}$ as a linear combination of $f^{\mu\nu}$ and its Hodge dual, and using (1.1), one gets

$$\int_{\partial\Omega} [\alpha f_{\mu\nu} + \beta \tilde{f}_{\mu\nu}] d\Sigma^{\mu\nu} = \int_\Omega \beta \tilde{j}_{\mu\nu\rho} dV^{\mu\nu\rho}, \tag{1.2}$$

where $\tilde{j}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\lambda} j^\lambda$ is the Hodge dual of the external current and α and β are arbitrary parameters used in the liner combination. By considering α and β to be arbitrary, the integral equations (1.2) correspond to the four usual integral equations of electromagnetic theory, which in fact preceded Maxwell differential equations. Indeed, taking $\alpha = 0$ and Ω to be a purely spatial 3-volume, one gets the Gauss law. On the hand, taking $\beta = 0$ and Ω to be a solid cylinder with its height in the time direction, and its base on a spatial plane, one gets the Faraday law, and so on. The role of the parameters α and β are not really important here because (1.2) is linear in them. The situation becomes more complex in a non-Abelian gauge theory.

The Yang-Mills theories were formulated *à la* Maxwell in terms of partial differential equations, the so-called Yang-Mills equations [10],

$$D_\mu F^{\mu\nu} = J^\nu \qquad D_\mu \tilde{F}^{\mu\nu} = 0, \tag{1.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$, with e being the gauge coupling constant, $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda} F^{\rho\lambda}$, J^μ being the external matter current, $D_\mu = \partial_\mu + ie[A_\mu,]$, and A_μ being the non-Abelian gauge field taking value on the Lie algebra of the gauge group G.

In order to construct the integral form of Yang-Mills equations (1.3), one needs the non-Abelian version of the Stokes theorem for a (non-Abelian) rank-two antisymmetric tensor $B_{\mu\nu}$ on a spacetime 3-volume Ω. Even though the non-Abelian Stokes theorem for a one-form connection on a 2-surface was known for some time, the same theorem for a two-form connection was constructed only more recently in [11,12] using concepts on generalized loop spaces. Conceptually, everything becomes more clear if one uses the two-form $B_{\mu\nu}$, defined on spacetime, to construct a one-form connection on the generalized loop space. Using such a generalized non-Abelian Stokes theorem, the integral form of Yang-Mills equations was constructed in [1,2]. The formulas involve path-, surface-, and volume-ordered integrals as follows.

Consider a spacetime 3-volume Ω, and choose a reference point x_R on its border $\partial\Omega$. Scan Ω with closed 2-surfaces based on x_R, labeled by a variable ζ, such that $\zeta = 0$ corresponds to the infinitesimal surface around x_R, and $\zeta = \zeta_0$ to the border $\partial\Omega$. Then scan each closed 2-surface with loops, starting and ending at x_R, labeled by a variable τ. Each loop is parametrized by a variable σ. The integral form of the Yang-Mills equations (1.3) is given by [1,2]

$$V(\partial\Omega) \equiv P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma W^{-1}(\alpha F_{\mu\nu} + \beta \tilde{F}_{\mu\nu})W \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}} \equiv U(\Omega), \tag{1.4}$$

where P_2 and P_3 mean surface- and volume-ordered integration, respectively, as explained above, and

$$\mathcal{J} = \int_{\sigma_i}^{\sigma_f} d\sigma \Big\{ ie\beta \tilde{J}^W_{\mu\nu\lambda} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2 \int_{\sigma_i}^{\sigma} d\sigma' [((\alpha-1)F^W_{\kappa\rho} + \beta\tilde{F}^W_{\kappa\rho})(\sigma'), (\alpha F^W_{\mu\nu} + \beta \tilde{F}^W_{\mu\nu})(\sigma)] \times \frac{dx^\kappa}{d\sigma'}\frac{dx^\mu}{d\sigma}\left(\frac{dx^\rho(\sigma')}{d\tau}\frac{dx^\nu(\sigma)}{d\zeta} - \frac{dx^\rho(\sigma')}{d\zeta}\frac{dx^\nu(\sigma)}{d\tau}\right)\Big\} \tag{1.5}$$

with $\tilde{J}_{\mu\nu\lambda} = \varepsilon_{\mu\nu\lambda\rho} J^\rho$, being the Hodge dual of the external matter current. In order to simplify the formulas, we have used the notation

$$X^W \equiv W^{-1} X W \tag{1.6}$$

with X standing for the field tensor, its Hodge dual, or the dual of the matter currents. The quantity W appearing above stands for the Wilson line, defined on a path parametrized by σ through the equation

$$\frac{dW}{d\sigma} + ieA_\mu \frac{dx^\mu}{d\sigma} W = 0 \tag{1.7}$$

and so

$$W = 1 - ie\int_{\sigma_i}^{\sigma} d\sigma' A_\mu(\sigma')\frac{dx^\mu}{d\sigma'} + (ie)^2 \int_{\sigma_i}^{\sigma} d\sigma' A_\mu(\sigma')\frac{dx^\mu}{d\sigma'}\int_{\sigma_i}^{\sigma'} d\sigma'' A_\nu(\sigma'')\frac{dx^\nu}{d\sigma''} - \dots. \tag{1.8}$$

The quantity V, called the Wilson surface, is defined on a surface parametrized by σ and τ, through the equation

$$\frac{dV}{d\tau} - VT(\tau) = 0, \tag{1.9}$$

with

$$T(\tau) = ie\int_{\sigma_i}^{\sigma_f} d\sigma W^{-1}(\alpha F_{\mu\nu} + \beta\tilde{F}_{\mu\nu})W \frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}, \tag{1.10}$$

and the integration being on the closed loops used in the scanning of Ω, as explained above. The initial and final values of σ, denoted σ_i and σ_f, respectively, correspond to the initial and final points of the loop, which in fact are the same point since the loop is always closed. Therefore, the solution of (1.9) is the surface-ordered series:

$$V(\tau) = 1 + \int_{\tau_i}^{\tau} d\tau' T(\tau') + \int_{\tau_i}^{\tau} d\tau' \int_{\tau_i}^{\tau'} d\tau'' T(\tau'')T(\tau') + \cdots. \tag{1.11}$$

The lhs of (1.4) is obtained by integrating (1.9) on the 2-surface $\partial\Omega$, i.e., the border of Ω. On the other hand the rhs of (1.4) is obtained by integrating the equation

$$\frac{dU}{d\zeta} - \mathcal{K}U = 0 \tag{1.12}$$

on the 3-volume Ω, and where

$$\mathcal{K} = \int_\Sigma d\tau V\mathcal{J}V^{-1}, \tag{1.13}$$

with Σ being the closed 2-surfaces scanning Ω, labeled by ζ, and $\mathcal{J}$ given by (1.5). The solution of (1.12) is given by the volume-ordered series

$$U(\zeta) = 1 + \int_0^\zeta d\zeta' \mathcal{K}(\zeta') + \int_0^\zeta d\zeta' \int_0^{\zeta'} d\zeta'' \mathcal{K}(\zeta')\mathcal{K}(\zeta'') + \cdots. \tag{1.14}$$

Note that (1.4) does reduce to (1.2) in the case that the gauge group G is $U(1)$. However, for non-Abelian gauge groups the dependence of both sides of (1.4) on the parameters α and β are highly nonlinear. Indeed, if such parameters are arbitrary one can expand both sides of (1.4) in a power series on them. The coefficient of each term of such series on the lhs of (1.4) will have to equal the corresponding coefficient of the series on the rhs, leading to an infinity of integral equations. Consequently any solution of the Yang-Mills equations (1.3) will have to satisfy such an infinity of integral equations. It is this test that we want to perform with the 't Hooft-Polyakov monopole, and its exact analytical BPS version [8]. We shall consider the 3-volume Ω to be purely spatial, and consequently only the spatial components of the field tensor and its dual, i.e., F_{ij} and $\tilde{F}_{ij}$, $i, j = 1, 2, 3$, will be present on both sides of (1.4). However, $\tilde{F}_{ij}$ is proportional to the electric field and so it vanishes for those static monopole solutions. In addition, only the component $\tilde{J}_{123} \sim J_0$ appears on the rhs of (1.4), and that vanishes because the solution is static and we shall work in the gauge where $A_0 = 0$. Remember that the only contribution for the matter current for such a solution comes from the triplet Higgs field ϕ, and that is of the form $J_\mu \sim [\phi, D_\mu \phi]$. Therefore, all terms involving the parameter β are not present on both sides of (1.4), for static monopoles when Ω is purely spatial, and so it reduces to

$$P_2 e^{ie\alpha \int_{\partial\Omega} d\tau d\sigma W^{-1} F_{ij} W \frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}} = P_3 e^{\alpha(\alpha-1)\int_\Omega d\zeta d\tau V \hat{\mathcal{J}} V^{-1}}, \qquad (1.15)$$

with

$$\hat{\mathcal{J}} = \frac{e^2}{2}\left[\int_{\sigma_i}^{\sigma_f} d\sigma' F^W_{k,l}(\sigma'), \int_{\sigma_i}^{\sigma_f} d\sigma F^W_{ij}(\sigma)\right]\frac{\partial x^k}{\partial\sigma'}\frac{\partial x^i}{\partial\sigma} \times\left(\frac{\partial x^l}{\partial\tau}(\sigma')\frac{\partial x^j}{\partial\zeta}(\sigma) - \frac{\partial x^l}{\partial\zeta}(\sigma')\frac{\partial x^j}{\partial\tau}(\sigma)\right) \qquad (1.16)$$

where i, j, k, $l = 1, 2, 3$, repeated indices are summed, and where we have denoted $F^W_{ij} \equiv W^{-1} F_{ij} W$. Note that we have explored the symmetry of $\hat{\mathcal{J}}$ in σ and σ' to replace $\int_{\sigma_i}^{\sigma_f} d\sigma \int_{\sigma_i}^{\sigma} d\sigma' \to \frac{1}{2}\int_{\sigma_i}^{\sigma_f} d\sigma \int_{\sigma_i}^{\sigma_f} d\sigma'$.

Equation (1.15) is what we call the *generalized integral Bianchi identity.* Note that one would expect the integral Bianchi identity to be (1.15) for $\alpha = 1$, i.e.,

$$P_2 e^{ie \int_{\partial\Omega} d\tau d\sigma W^{-1} F_{ij} W \frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}} = \mathbb{1}. \qquad (1.17)$$

Indeed, that is what leads to the quantization of the magnetic charge. From a physical point of view it is intriguing that by rescaling the field tensor (magnetic field) as $F_{ij} \to \alpha F_{ij}$, leads to the appearance of a term like the rhs of (1.15), making the magnetic flux through $\partial\Omega$ to change drastically. However, the validity of (1.4), and so of (1.15), is guaranteed by the generalized non-Abelian Stokes theorem for a two-form $B_{\mu\nu}$ and the partial differential Yang-Mills equations (1.3) as proved in [1,2,11,12]. The intriguing nonlinear phenomenon that we want to directly check in this paper, is if one can expand both sides of (1.15) in powers of α, and if the $SU(2)$ 't Hoot-Polyakov monopole and its exact analytical BPS version, satisfy each one of the integral equations obtained through such an expansion.

The paper is organized as follows. In Sec. II, we perform the expansion of the generalized integral Bianchi identity (1.15) in powers of the parameter α, and we show that each term of the expansion can be expressed solely in terms of the Wilson line operator. In Sec. III, we calculate explicitly the Wilson line operator for the $SU(2)$ 't Hooft-Polyakov and BPS monopoles using a suitable scanning of surfaces and volumes. The result is quite simple and it is given in (3.14). In Sec. IV, we check the validity of the integral equation in first order of the α expansion and, in Sec. V, we do the same for the integral equation in second order of that same expansion. We present our conclusions in Sec. VI, and in Appendix, we give the results of the numerical calculations of the integrals needed to perform the check of the integral equations.

II. THE EXPANSION OF THE YANG-MILLS INTEGRAL EQUATIONS

Assuming that α and β are indeed arbitrary we expand both sides of (1.4) in power series in those parameters. As we have said the lhs of (1.4) is obtained by integrating (1.9), and its rhs by integrating (1.12). By writing the expressions on the lhs and on the rhs of the integral equation (1.4) in terms of (1.11) and (1.4), and collecting the coefficients at first order in α and zeroth order in β, we get the integral equation at first order in α

$$\int_{\tau_i}^{\tau_f} d\tau \int_{\sigma_i}^{\sigma_f} d\sigma F^W_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\Big|_{\zeta=\zeta_0} = ie\int_0^{\zeta_0} d\zeta \int_{\tau_i}^{\tau_f} d\tau \int_{\sigma_i}^{\sigma_f} d\sigma \int_{\sigma_i}^{\sigma} d\sigma' [F^W_{\kappa\rho}(\sigma'), F^W_{\mu\nu}(\sigma)] \times \frac{dx^\kappa}{d\sigma'}\frac{dx^\mu}{d\sigma}\left(\frac{dx^\rho(\sigma')}{d\tau}\frac{dx^\nu(\sigma)}{d\zeta} - \frac{dx^\rho(\sigma')}{d\zeta}\frac{dx^\nu(\sigma)}{d\tau}\right), \qquad (2.1)$$

where ζ_0 is the value of ζ corresponding to the closed surface $\partial\Omega$, in the scanning of the 3-volume Ω, which is the border of Ω [see explanation of the scanning in the paragraph above (1.4)]. On the other hand, the integral equation appearing in order β and zeroth order in α, in the expansion of (1.4) is given by

$$\int_{\tau_i}^{\tau_f} d\tau \int_{\sigma_i}^{\sigma_f} d\sigma \tilde{F}^W_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}\Big|_{\zeta=\zeta_0} = \int_0^{\zeta_0} d\zeta \int_{\tau_i}^{\tau_f} d\tau \int_{\sigma_i}^{\sigma_f} d\sigma \Big\{ \tilde{J}^W_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + ie\int_{\sigma_i}^{\sigma} d\sigma' [F^W_{\kappa\rho}(\sigma'), \tilde{F}^W_{\mu\nu}(\sigma)] \times \frac{dx^\kappa}{d\sigma'}\frac{dx^\mu}{d\sigma}\left(\frac{dx^\rho(\sigma')}{d\tau}\frac{dx^\nu(\sigma)}{d\zeta} - \frac{dx^\rho(\sigma')}{d\zeta}\frac{dx^\nu(\sigma)}{d\tau}\right)\Big\}. \qquad (2.2)$$

Note that in the case where the gauge group G is the Abelian group $U(1)$, Eq. (2.1) corresponds to (1.2) for $\alpha = 1$ and $\beta = 0$. Equation (2.2) corresponds to (1.2) for $\alpha = 0$ and $\beta = 1$. Note in addition that in the case where the 3-volume Ω is purely spatial, the commutator term in (2.1) involving the field tensors can be interpreted as a density of non-Abelian magnetic charge associated to the gauge field configuration inside Ω. The commutator term in (2.2) involving the field tensor and its Hodge dual can be interpreted as a density of non-Abelian electric charge associated to the gauge field configuration inside Ω. In the case where Ω has time components, those commutators will be associated to flows of non-Abelian electric and magnetic charges. We have explored further those facts to obtain the integral form of the non-Abelian Gauss, Faraday, etc., laws, and the physical implications of these new terms (the commutator terms) should be further explored in some other opportunity.

As one goes higher in the expansion, the integral equations become more and more complex. However, for the case we are considering in this paper, namely the static 't Hooft-Polyakov monopole and its BPS version, and where the 3-volume Ω is purely spatial, there is an important simplification. As we have argued in the paragraph above (1.15), only the spatial components of the field tensor (magnetic field) appear in the formulas, since the spatial components of its Hodge dual (electric field) vanish. As explained in Sec. II of [11], or in the Appendix of [2], if one performs an infinitesimal variation, $x^\mu(\sigma) \to x^\mu(\sigma) + \delta x^\mu(\sigma)$, of the curve where the Wilson line W (1.7) is calculated, but keeping its end points fixed, the infinitesimal variation of the Wilson line operator is given by

$$W^{-1}(\sigma_f)\delta W(\sigma_f) = ie\int_{\sigma_i}^{\sigma_f} d\sigma W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma}\delta x^\nu. \tag{2.3}$$

The Wilson line operators W appearing in the Yang-Mills integral equations (1.4) are evaluated on the paths that scan the closed surfaces which in their turn scan the 3-volume Ω. Therefore, as we vary the parameter τ which labels the loops, we vary the loop along a given surface, and so $\delta x^\mu = \frac{dx^\mu}{d\tau}\delta\tau$. When we vary the parameter ζ which labels the surfaces, the loops vary perpendicular to that surface and so $\delta x^\mu = \frac{dx^\mu}{d\zeta}\delta\zeta$. Consequently, from (2.3) we get the following two useful formulas

$$\begin{aligned} ie\int_{\sigma_i}^{\sigma_f} d\sigma W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} &= W^{-1}\frac{dW}{d\tau} \\ ie\int_{\sigma_i}^{\sigma_f} d\sigma W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\zeta} &= W^{-1}\frac{dW}{d\zeta}. \end{aligned} \tag{2.4}$$

As we have shown, for the static 't Hooft-Polyakov monopole and its BPS version, and a purely spatial 3-volume Ω, the integral Yang-Mills equation (1.4) becomes the generalized integral Bianchi identity (1.15). Therefore, from (1.11), (1.10), and (2.4), one gets that the lhs of (1.15) is given by

$$\begin{aligned} V(\partial\Omega) &\equiv P_2 e^{ie\alpha\int_{\partial\Omega} d\tau d\sigma W^{-1} F_{ij} W \frac{\partial x^i}{\partial\sigma}\frac{\partial x^j}{\partial\tau}} \\ &= 1 + \alpha\int_{\tau_i}^{\tau_f} d\tau W^{-1}\frac{dW}{d\tau}(\tau)\Big|_{\zeta=\zeta_0} \\ &\quad + \alpha^2\int_{\tau_i}^{\tau_f} d\tau\int_{\tau_i}^{\tau} d\tau' W^{-1}\frac{dW}{d\tau'}(\tau') W^{-1}\frac{dW}{d\tau}(\tau)\Big|_{\zeta=\zeta_0} \\ &\quad + \cdots \\ &= 1 + \alpha V_{(1)} + \alpha^2 V_{(2)} + \cdots. \end{aligned} \tag{2.5}$$

From (2.4) one gets that (1.16) becomes

$$\hat{\mathcal{J}} = -\left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right]. \tag{2.6}$$

Therefore, from (1.14) and (2.6), the rhs of (1.15) becomes

$$\begin{aligned} U(\Omega) &\equiv P_3 e^{\alpha(\alpha-1)\int_\Omega d\zeta d\tau V\hat{\mathcal{J}}V^{-1}} \\ &= 1 - \alpha(\alpha-1)\int_0^{\zeta_0} d\zeta\int_{\tau_i}^{\tau_f} d\tau V\left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right]V^{-1} \\ &\quad + [\alpha(\alpha-1)]^2\int_0^{\zeta_0} d\zeta\int_0^{\zeta} d\zeta'\int_{\tau_i}^{\tau_f} d\tau\int_{\tau_i}^{\tau_f} d\tau'\left(V\left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right]V^{-1}\right)(\tau,\zeta) \\ &\quad \times\left(V\left[W^{-1}\frac{dW}{d\tau'}, W^{-1}\frac{dW}{d\zeta'}\right]V^{-1}\right)(\tau',\zeta') + \cdots \\ &= 1 + \alpha U_{(1)} + \alpha^2 U_{(2)} + \cdots, \end{aligned} \tag{2.7}$$

where V in (2.7) is evaluated with the same expansion as in (1.11) with $\beta = 0$, and so an expansion similar to (2.5).

Therefore, by equating (2.5) to (2.7), one gets that the term in first order in α leads to the integral equation

$$V_{(1)} = \int_{\tau_i}^{\tau_f} d\tau W^{-1} \frac{dW}{d\tau}\bigg|_{\zeta=\zeta_0} = \int_0^{\zeta_0} d\zeta \int_{\tau_i}^{\tau_f} d\tau \left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right] = U_{(1)}. \tag{2.8}$$

Similarly, the term in order α^2 gives the following integral equation:

$$\begin{aligned} V_{(2)} &= \int_{\tau_i}^{\tau_f} d\tau \int_{\tau_i}^{\tau} d\tau' W^{-1}\frac{dW}{d\tau'} W^{-1}\frac{dW}{d\tau}\bigg|_{\zeta=\zeta_0} \\ &= -\int_0^{\zeta_0} d\zeta \int_{\tau_i}^{\tau_f} d\tau \left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right] + \int_0^{\zeta_0} d\zeta \int_{\tau_i}^{\tau_f} d\tau \int_{\tau_i}^{\tau} d\tau' \left[W^{-1}\frac{dW}{d\tau'}, \left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right]\right] \\ &\quad + \int_0^{\zeta_0} d\zeta \int_0^{\zeta} d\zeta' \int_{\tau_i}^{\tau_f} d\tau \int_{\tau_i}^{\tau_f} d\tau' \left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right]\left[W^{-1}\frac{dW}{d\tau'}, W^{-1}\frac{dW}{d\zeta'}\right] = U_{(2)}. \end{aligned} \tag{2.9}$$

We are going to verify if the $SU(2)$ 't Hooft-Polyakov monopole and its BPS version [3–6,8], satisfy the integral equations (2.8) and (2.9). Note that the only quantity appearing in (2.8) and (2.9) is the Wilson loop W. In the next section, we evaluate it for those monopole solutions.

III. THE WILSON LOOP FOR 't HOOFT-POLYAKOV AND BPS MONOPOLES

The spherically symmetric 't Hooft-Polyakov ansatz [3,4] for a $SU(2)$ static magnetic monopole reads

$$\begin{aligned} \phi &= \frac{1}{er} H(\zeta)\hat{r}\cdot T \\ A_0 &= 0 \\ A_i &= -\frac{1}{e}\epsilon_{ijk}\frac{x_j}{r^2}(1-K(\zeta))T_k, \end{aligned} \tag{3.1}$$

with $r = \sqrt{x_1^2+x_2^2+x_3^2}$, $\hat{r}_i = x_i/r$, $\zeta = ear$, a being the vacuum expectation value of the Higgs field in the triplet representation, and T_i being the generators of the $SU(2)$ Lie algebra:

$$[T_i, T_j] = i\varepsilon_{ijk}T_k. \tag{3.2}$$

The exact analytical BPS monopole solution corresponds to the functions [5,6]

$$K(\zeta) = \frac{\zeta}{\sinh\zeta} \qquad H(\zeta) = \zeta\coth\zeta - 1. \tag{3.3}$$

For the 't Hooft-Polyakov monopole, the functions $K(\zeta)$ and $H(\zeta)$ are obtained numerically, but they have qualitatively the same behavior as (3.3); i.e., we have that $K(0) = 1$, and then it decays monotonically (exponentially) to zero as $r \to \infty$, and $H(0) = 0$, and then it grows monotonically with r and for $r \to \infty$, such grow is linear in r. The function $H(\zeta)$ will not be important in our calculations because the Higgs field does not appear in our integral equations for the case of static solutions and for Ω being purely spatial [see (2.8) and (2.9)]. The important simplification we obtain in our calculations is due to the fact that $K(\zeta)$ is a monotonic function of ζ, and so it admits an inverse function. We will then trade the parameter ζ by the function K, and our calculations will not depend upon the explicit form of the function $K(\zeta)$.

We have chosen to evaluate both sides of the integral equations (2.8) and (2.9) on a purely spatial 3-volume Ω which is a ball centered at the origin of the Cartesian coordinate system x_i, $i = 1, 2, 3$, used in the ansatz (3.1). We then scan that volume Ω with closed surfaces which are spheres also centered at the origin of the Cartesian coordinate system, with radii varying from zero to the radius of Ω. However, since the reference point x_R have to lie on the border $\partial\Omega$ of Ω, and since the surfaces scanning it have to be based at x_R, we shall attach to the ball Ω a infinitesimally thin cylinder lying on the negative x_1 axis, and locate the reference point x_R at $(x_1, x_2, x_3) = (-\infty, 0, 0)$. The cylinder has a radius ε, which will be taken to zero at the end of the calculations. The surfaces scanning Ω will have the form depicted in Fig. 2, i.e., an infinitesimally thin cylinder on the negative x_1 axis and a sphere centered at the origin of the Cartesian coordinate system. With the attachment of the thin cylinder we can keep the surfaces based at x_R, and centered at the origin. In addition, x_R being at infinity allows us to have the volume Ω with any radius. We shall label the surfaces scanning Ω with the parameter ζ, which is the same as the one appearing in the ansatz (3.1). Then $\zeta = 0$ corresponds to the surface made of the thin cylinder and a sphere of radius zero attached to it, and $\zeta = \zeta_0$ corresponds to the border $\partial\Omega$, made of the thin cylinder attached to a sphere of radius ζ_0, the same as the radius of Ω, centered at the origin. The loops will be labeled by a parameter τ, they start and end at the reference point x_R, and there will be three types of loops, as follows:

(1) *Loops of type (I), scanning the thin cylinder, as depicted in Fig. 1*. For such loops, the parameter τ

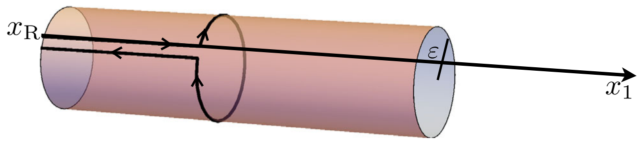

FIG. 1. Scanning of type (I). The gap between the straight lines is only a visual resource.

varies from $-\infty$ to $-\frac{\pi}{2}$, with $\tau = -\infty$ corresponding to the infinitesimal loop around x_R, and $\tau = -\frac{\pi}{2}$ corresponding to a straight line from x_R to the border of the sphere, then encircling the joint of the cylinder with the sphere, and coming back to x_R through the same straight line. The three parts of such loops will be denoted (I.1), the first straight line, (I.2), the circle and (I.3) the second straight line. We parametrize the loops with σ, such that the points on the loops have the following coordinates:

$$\text{(I.1)}\ x_1 = \tau + \sigma - \zeta + \frac{\pi}{2} \qquad x_2 = 0$$
$$x_3 = -\varepsilon \qquad (-\infty \leq \sigma \leq 0)$$
$$\text{(I.2)}\ x_1 = \tau - \zeta + \frac{\pi}{2} \qquad x_2 = \varepsilon \sin\sigma$$
$$x_3 = -\varepsilon\cos\sigma \qquad (0 \leq \sigma \leq 2\pi)$$
$$\text{(I.3)}\ x_1 = \tau + 2\pi - \sigma - \zeta + \frac{\pi}{2} \qquad x_2 = 0$$
$$x_3 = -\varepsilon \qquad (2\pi \leq \sigma \leq \infty)$$

with fixed ζ and $-\infty \leq \tau \leq -\frac{\pi}{2}$.

(2) *Loops of type (II), scanning the thin sphere, as depicted in Fig. 2*. For such loops the parameter τ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. A loop of this type is made of a straight line from x_R to the border of the sphere, then making a circle on the surface of the sphere, starting and ending at the junction of the cylinder with the sphere, and lying on a plane perpendicular to the plane x_1x_3, that makes an angle τ with the plane x_1x_2. Finally, it returns to x_R through the same straight line. Again, the three parts of such loops will be denoted as (II.1) for the first straight line, (II.2) for the circle and (II.3) labels the second straight line. We parametrize the loops with σ, such that the points on the loops have the following coordinates:

$$\text{(II.1)}\ x_1 = \sigma - \zeta \qquad x_2 = 0$$
$$x_3 = -\varepsilon \qquad (-\infty \leq \sigma \leq 0)$$
$$\text{(II.2)}\ x_1 = \zeta(\cos^2\tau(1-\cos\sigma) - 1)$$
$$x_2 = \zeta\cos\tau\sin\sigma \qquad (0 \leq \sigma \leq 2\pi)$$
$$x_3 = \zeta\cos\tau\sin\tau(1-\cos\sigma)$$
$$\text{(II.3)}\ x_1 = -\sigma + 2\pi - \zeta \qquad x_2 = 0$$
$$x_3 = -\varepsilon \qquad (2\pi \leq \sigma \leq \infty).$$

with fixed ζ and where in (II.2) the parameter τ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

(3) *Loops of type (III), scanning the thin cylinder backwards, as depicted in Fig. 3*. For such loops the parameter τ varies from $\frac{\pi}{2}$ to ∞, and they are made of two straight lines. The first one starting at x_R and ending on some point on the side of the cylinder with coordinates $(x_1, x_2, x_3) = (x_1, 0, -\varepsilon)$. The second part of the loop is the same straight line (reversed) going back to x_R. We shall denote the first straight line (III.1) and the second (III.2). We parametrize the loops with σ, such that the points on the loops have the following coordinates:

$$\text{(III.1)}\ x_1 = \frac{\pi}{2} - \tau - \zeta + \sigma \qquad x_2 = 0$$
$$x_3 = -\varepsilon \qquad (-\infty \leq \sigma \leq 0)$$
$$\text{(III.2)}\ x_1 = \frac{\pi}{2} - \tau - \zeta - \sigma \qquad x_2 = 0$$
$$x_3 = -\varepsilon \qquad (0 \leq \sigma \leq \infty)$$

with fixed ζ, and where $\frac{\pi}{2} \leq \tau \leq \infty$.

An important simplification is made by observing that the Wilson line is constant along loops scanning the thin cylinder. Indeed, we observe that on the segments (I.1), (I.3), (II.1), (II.3), (III.1) and (III.2), the coordinate x_1 is linear in σ, and x_2 and x_3 are independent of it. Therefore, using (3.1), we have that

$$A_i \frac{dx^i}{d\sigma}\bigg|_{\text{straight lines}} = \pm A_1 = \mp \frac{1}{e}\frac{\varepsilon}{r^2}(1-K)T_2 \to 0 \ \text{ as } \varepsilon \to 0 \tag{3.4}$$

with the upper signs valid for the segments (I.1), (II.1) and (III.1), and the lower signs valid for (I.3), (II.3) and (III.2). On the segment (I.2), on the other hand, we get that

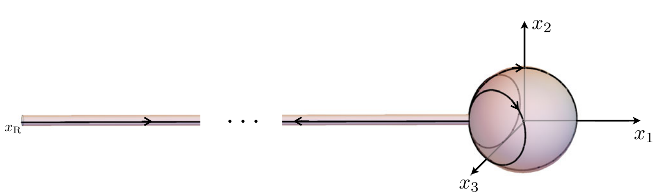

FIG. 2. Scanning of type (II).

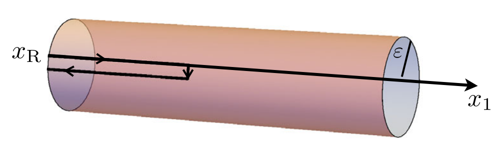

FIG. 3. Scanning of type (III).

$$A_i \frac{dx^i}{d\sigma}\bigg|_{(\mathrm{I}.2)} = \varepsilon[\cos\sigma A_2 + \sin\sigma A_3] = -\frac{\varepsilon}{e}\frac{(1-K)}{r^2}\left[-\varepsilon T_1 + \left(\tau - \zeta + \frac{\pi}{2}\right) \times (-\cos\sigma T_3 + \sin\sigma T_2)\right] \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.5}$$

The only nonvanishing contribution comes from the segment (II.2), which gives

$$A_i \frac{dx^i}{d\sigma}\bigg|_{(\mathrm{II}.2)} = \frac{1}{e}(1-K)\cos\tau[\cos\tau\sin\tau(1-\cos\sigma)T_1 + \sin\tau\sin\sigma T_2 + (\sin^2\tau(1-\cos\sigma)-1)T_3] = -\frac{1}{e}(1-K)\cos\tau e^{i\tau T_2}e^{i\sigma T_3}e^{-i\tau T_2} \times T_3 e^{i\tau T_2}e^{-i\sigma T_3}e^{-i\tau T_2}. \tag{3.6}$$

Therefore, integrating (1.7) one gets that the Wilson lines on the loops of type I and III are trivial, i.e., $W(\mathrm{I}) = W(\mathrm{III}) = \mathbb{1}$. On the loops of type II one gets that $W(\mathrm{II}) = W_3 W_2 W_1$, where W_a are the Wilson lines obtained by integrating (1.7) on the segments (II.a), $a = 1, 2, 3$. Due to (3.4) we have that $W_1 = W_3 = \mathbb{1}$. Under a gauge transformation $A_i \to \bar{A}_i = g A_i g^{-1} + \frac{i}{e}\partial_i g g^{-1}$, with $g = e^{i\tau T_2}e^{-i\sigma T_3}e^{-i\tau T_2}$, one gets that

$$W_2 \to \bar{W}_2 = g_f W_2 g_i^{-1} = e^{i\tau T_2}e^{-i2\pi T_3}e^{-i\tau T_2}W_2, \tag{3.7}$$

where g_i and g_f are the values of g at the initial and final points of the loop (II.2), and so $g_i = \mathbb{1}$, and $g_f = e^{i\tau T_2}e^{-i2\pi T_3}e^{-i\tau T_2}$. Therefore, one gets that

$$A_i \frac{dx^i}{d\sigma}\bigg|_{(\mathrm{II}.2)} \to \bar{A}_i \frac{dx^i}{d\sigma}\bigg|_{(\mathrm{II}.2)} = \frac{1}{e}[K\cos\tau T_3 - \sin\tau T_1], \tag{3.8}$$

and the Eq. (1.7) for $\bar{W}_2$ becomes

$$\frac{d\bar{W}_2}{d\sigma} + i[K\cos\tau T_3 - \sin\tau T_1]\bar{W}_2 = 0. \tag{3.9}$$

Since the connection term $[K(\zeta)\cos\tau T_3 - \sin\tau T_1]$ is independent of σ it follows that the path ordering is unimportant and the integration on the loops (II.2) gives

$$\bar{W}_2 = e^{-i2\pi[K\cos\tau T_3 - \sin\tau T_1]}. \tag{3.10}$$

Using the fact that $e^{i2\pi T_3} = \pm\mathbb{1}$, depending if the representation used is of integer (+) or half-integer (−) spin, we get that $g_f = \pm\mathbb{1}$, and so

$$W = W(\mathrm{II}) = W_2 = \pm e^{-i2\pi[K(\zeta)\cos\tau T_3 - \sin\tau T_1]}, \tag{3.11}$$

where we have equated W to $W(\mathrm{II})$, because, as shown above $W(\mathrm{I}) = W(\mathrm{III}) = \mathbb{1}$. Therefore, in (3.11) we have τ varying from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The calculations concerning the Wilson line can be simplified defining γ as

$$\cos\gamma = \frac{K\cos\tau}{F}; \qquad \sin\gamma = \frac{\sin\tau}{F} \tag{3.12}$$

with

$$F(\zeta,\tau) = \sqrt{K^2(\zeta)\cos^2\tau + \sin^2\tau}. \tag{3.13}$$

Then (3.11) can be written as

$$W = \pm e^{i\gamma T_2}e^{-i2\pi F T_3}e^{-i\gamma T_2}, \tag{3.14}$$

from which we get

$$W^{-1}\partial W = ie^{i\gamma T_2}\{-2\pi\partial F T_3 + \partial\gamma[(\cos(2\pi F)-1)T_2 + \sin(2\pi F)T_1]\}e^{-i\gamma T_2} = i\{[2\pi\partial F\sin\gamma + \partial\gamma\sin(2\pi F)\cos\gamma]T_1 + \partial\gamma[\cos(2\pi F)-1]T_2 + [-2\pi\partial F\cos\gamma + \partial\gamma\sin(2\pi F)\sin\gamma]T_3\}. \tag{3.15}$$

We then have

$$W^{-1}\frac{dW}{d\tau} = i\cos\tau N_j(K,\tau)T_j \tag{3.16}$$

with

$$N_1(K,\tau) = \frac{1}{F^2}\left[2\pi(1-K^2)\sin^2\tau + \frac{K^2\sin(2\pi F)}{F}\right]$$
$$N_2(K,\tau) = -\frac{K}{F^2\cos\tau}[1-\cos(2\pi F)]$$
$$N_3(K,\tau) = \frac{K\sin\tau}{F^2\cos\tau}\left[-2\pi(1-K^2)\cos^2\tau + \frac{\sin(2\pi F)}{F}\right]. \tag{3.17}$$

In addition,

$$\left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right] = 2\pi i\left(\frac{dF}{d\tau}\frac{d\gamma}{d\zeta} - \frac{d\gamma}{d\tau}\frac{dF}{d\zeta}\right) \times e^{i\gamma T_2}[(1-\cos(2\pi F))T_1 + \sin(2\pi F)T_2]e^{-i\gamma T_2} = -i2\pi K'\cos^2\tau M_j(K,\tau)T_j \quad (3.18)$$

with

$$M_1(K,\tau) = \frac{K\cos\tau}{F^2}[1-\cos(2\pi F)]$$
$$M_2(K,\tau) = \frac{\sin(2\pi F)}{F}$$
$$M_3(K,\tau) = \frac{\sin\tau}{F^2}[1-\cos(2\pi F)], \quad (3.19)$$

where K' stands for $\frac{dK}{d\zeta}$, and where we have used the formulas

$$\frac{dF}{d\tau} = \frac{\sin\tau\cos\tau}{F}(1-K^2), \qquad \frac{dF}{d\zeta} = \frac{K\cos^2\tau}{F}K',$$
$$\frac{d\gamma}{d\tau} = \frac{K}{F^2}, \qquad \frac{d\gamma}{d\zeta} = -\frac{\sin\tau\cos\tau}{F^2}K'. \quad (3.20)$$

With these expressions we are ready to perform the calculations of Sec. II for the $SU(2)$ monopoles.

IV. CHECK OF FIRST-ORDER INTEGRAL EQUATIONS FOR $SU(2)$ MONOPOLES

The integral equation for a purely spatial volume Ω, in first order in α, for the $SU(2)$ monopoles ('t Hooft-Polyakov or BPS) is given by expression (2.8). However, since the Wilson line is unit for the loops of type I and III (see Sec. III), we get that (2.8) is only nontrivial for loops of type II, where τ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and so (2.8) becomes

$$V_{(1)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau W^{-1}\frac{dW}{d\tau}\Big|_{\zeta=\zeta_0} = \int_0^{\zeta_0} d\zeta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right] = U_{(1)}, \quad (4.1)$$

where the lhs is an integration on a closed surface of radius ζ_0 and the rhs is an integration in the volume contained inside that surface. Our goal is to evaluate both sides of this equation using the results obtained in the expressions (3.16) and (3.18). In order to perform the integration of the lhs term, a better choice of variables is the following:

$$y = \sin\tau; \qquad -1 \le y \le 1;$$
$$z = K(\zeta)\cos\tau; \qquad 0 \le z \le 1, \quad (4.2)$$

with $0 \le \zeta \le \infty$, $0 \le K(\zeta) \le 1$, and $-\frac{\pi}{2} \le \tau \le \frac{\pi}{2}$. Note that we are using the fact that $K(\zeta)$ is monotonically decreasing function of ζ for both, the 't Hooft-Polyakov and BPS monopole solutions. The explicit form of the function $K(\zeta)$ is not important here. In these variables we get

$$F^2 = y^2 + z^2 = K^2 + (1-K^2)y^2, \quad (4.3)$$

and so using (3.16) and (3.17) we get that the lhs of (4.1) becomes

$$V_{(1)} = i\int_{-1}^{1} dy N_j(K_0, y)T_j, \quad (4.4)$$

with $K_0 \equiv K(\zeta_0)$, and

$$N_1(K,y) = \frac{2\pi}{F^2}\left\{y^2(1-K^2) + \frac{K^2\sin(2\pi F)}{2\pi F}\right\}$$
$$N_2(K,y) = -\frac{K}{\sqrt{1-y^2}F^2}\{1-\cos(2\pi F\}$$
$$N_3(K,y) = 2\pi\frac{Ky}{\sqrt{1-y^2}F^2}\left\{F^2 - 1 + \frac{\sin(2\pi F)}{2\pi F}\right\}. \quad (4.5)$$

Note that N_3 is an odd function of y and, thus, integrating we get

$$V_{(1)}(K_0) = iJ_1(K_0)T_1 + iJ_2(K_0)T_2, \quad (4.6)$$

with

$$J_1(K_0) = 2\pi\int_{-1}^{1} dy \frac{1}{K_0^2 + (1-K_0^2)y^2} \times \left\{(1-K_0^2)y^2 + \frac{K_0^2\sin\left(2\pi\sqrt{K_0^2+(1-K_0^2)y^2}\right)}{2\pi\sqrt{K_0^2+(1-K_0^2)y^2}}\right\}$$
$$J_2(K_0) = -\int_{-1}^{1} dy \frac{K_0}{\sqrt{1-y^2}(K_0^2+(1-K_0^2)y^2)} \times \left\{1-\cos\left(2\pi\sqrt{K_0^2+(1-K_0^2)y^2}\right)\right\}.$$

Note that as ζ varies from 0 to ζ_0, one has that K varies from 1 to $K_0 \equiv K(\zeta_0) < 1$. Therefore, the integration domain on the rhs of (4.1) is a truncated semidisc shown in Fig. 4. The absolute value of the Jacobian of the variable transformation $(\zeta,\tau) \to (y,z)$, given in (4.2), is $|K'|\cos^2\tau = -K'\cos^2\tau$, since K' is strictly negative. In addition, it is more appropriate to perform a further change of variables to evaluate the integration on the rhs of (4.1). We define the polar type coordinates (s,θ) as

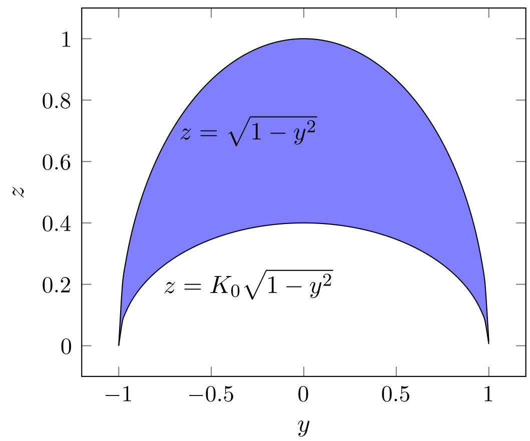

FIG. 4. The integration domain in the new "polar" coordinates. Each value of K_0 fixes a new domain by shortening the area of the disk from below.

$$y = s\cos\theta; \qquad z = s\sin\theta;$$
$$S(K_0,\theta) \le s \le 1; \qquad 0 \le \theta \le \pi \tag{4.7}$$

with

$$S(K,\theta) \equiv \frac{K}{\sqrt{1-\cos^2\theta(1-K^2)}}. \tag{4.8}$$

Therefore, one has that

$$\int_0^{\zeta_0} d\zeta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau K' \cos^2\tau = -\int_{\text{truncated semidisc}} dz dy$$
$$= -\int_0^{\pi} d\theta \int_{S(K_0,\theta)}^{1} ds s. \tag{4.9}$$

We then have that the M_i's, defined in (3.19), become

$$M_1 = \frac{z}{F^2}[1-\cos(2\pi F)] = \frac{\sin\theta}{s}[1-\cos(2\pi s)]$$
$$M_2 = \frac{\sin(2\pi F)}{F} = \frac{\sin(2\pi s)}{s}$$
$$M_3 = \frac{y}{F^2}[1-\cos(2\pi F)] = \frac{\cos\theta}{s}[1-\cos(2\pi s)]. \tag{4.10}$$

Using (3.18) we get that in these coordinates the rhs of (4.1), denoted by $U_{(1)}$, reads

$$U_{(1)} = i2\pi \int_0^{\pi} d\theta \int_{S(K_0,\theta)}^{1} ds\{\sin\theta(1-\cos(2\pi s))T_1 + \sin(2\pi s)T_2 + \cos\theta(1-\cos(2\pi s))T_3\}, \tag{4.11}$$

from which we can easily perform the integration in s, obtaining

$$U_{(1)}(K_0) = iI_1(K_0)T_1 + iI_2(K_0)T_2, \tag{4.12}$$

where

$$I_1(K_0) = \int_0^{\pi} d\theta \sin\theta\left\{2\pi - \frac{2\pi K_0}{\sqrt{1-\cos^2\theta(1-K_0^2)}} + \sin\left(\frac{2\pi K_0}{\sqrt{1-\cos^2\theta(1-K_0^2)}}\right)\right\}$$
$$I_2(K_0) = -\int_0^{\pi} d\theta\left\{1-\cos\left(\frac{2\pi K_0}{\sqrt{1-\cos^2\theta(1-K_0^2)}}\right)\right\} \tag{4.13}$$

The integral along the T_3 direction in (4.11) vanishes since the integrand is odd, under reflection around $\theta = \frac{\pi}{2}$, in the interval $0 \le \theta \le \pi$ (note that $S_0(\theta)$ is even in that interval).

Therefore, in order to check the validity of the integral equation at first order in α, given in (4.1), we have to verify the equalities of the coefficients of T_i in (4.6) and in (4.12). We have performed the numerical integration of the quantities $I_i(K_0)$ and $J_i(K_0)$ for several values of K_0, covering the range $1 \ge K_0 \ge 0$, corresponding to $0 \le \zeta_0 \le \infty$. Note that the actual value of K_0 for a given value of ζ_0 is different for the 't Hooft-Polyakov and BPS monopoles. However, the fact that $K(\zeta)$ is a monotonically decreasing function of ζ, for both solutions, allowed us to trade the coordinate ζ by K, and perform one check that is valid for the two monopole solutions. In Sec. A 1 we give the results of the numerical integrations of the quantities $I_i(K_0)$ and $J_i(K_0)$, $i = 1, 2$. As one observes in those tables, the values of $I_i(K_0)$ and $J_i(K_0)$ are remarkably identical, differing in the worst case around the eighth decimal place, due to the numerical approximation. This indicates that the 't Hooft-Polyakov and BPS $SU(2)$ monopoles are indeed solutions of the first-order integral equation (2.1), or equivalently (4.1), appearing in the expansion in α of the integral non-Abelian Gauss law in (2.5) and (2.7).

V. CHECK OF SECOND-ORDER INTEGRAL EQUATIONS FOR $SU(2)$ MONOPOLES

The integral equation for a purely spatial volume Ω, in second order in α, for the $SU(2)$ monopoles ('t Hooft-Polyakov or BPS) is given by expression (2.9). However, since the Wilson line is unit for the loops of type I and III (see Sec. III) we get that (2.9) is only nontrivial for loops of type II, where τ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and so (2.9) becomes

$$
\begin{aligned}
V_{(2)} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \int_{-\frac{\pi}{2}}^{\tau} d\tau' W^{-1}\frac{dW}{d\tau'}W^{-1}\frac{dW}{d\tau}\Big|_{\zeta=\zeta_0} \\
&= -\int_0^{\zeta_0} d\zeta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right] + \int_0^{\zeta_0} d\zeta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \int_{-\frac{\pi}{2}}^{\tau} d\tau' \left[W^{-1}\frac{dW}{d\tau'}, \left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right]\right] \\
&\quad + \int_0^{\zeta_0} d\zeta \int_0^{\zeta} d\zeta' \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau' \left[W^{-1}\frac{dW}{d\tau}, W^{-1}\frac{dW}{d\zeta}\right]\left[W^{-1}\frac{dW}{d\tau'}, W^{-1}\frac{dW}{d\zeta'}\right] \\
&\equiv -U_{(1)} + G_2 + G_3 = U_{(2)}, \qquad (5.1)
\end{aligned}
$$

where we have denoted G_2 and G_3 the terms appearing on the second and third lines, respectively, of (5.1). In addition, we have used the fact that the first term on rhs of the first line of (5.1) is the same (up to a minus sign) as $U_{(1)}$ given on the rhs of (4.1).

We start by evaluating the lhs of (5.1), using (3.16), and (4.5) to get

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \int_{-\frac{\pi}{2}}^{\tau} d\tau' W^{-1}\frac{dW}{d\tau'}W^{-1}\frac{dW}{d\tau}\Big|_{\zeta=\zeta_0} &= -\int_{-1}^{1} dy \int_{-1}^{y} dy' \sum_{i,j=1}^{3} N_i(K_0,y')N_j(K_0,y)T_iT_j \\
&= -\frac{1}{2}\sum_{i=1}^{3}\left[\int_{-1}^{1} dy N_i(K_0,y)\right]^2 T_i^2 - \int_{-1}^{1} dy \int_{-1}^{y} dy' \sum_{i\neq j=1}^{3} N_i(K_0,y')N_j(K_0,y)T_iT_j, \qquad (5.2)
\end{aligned}
$$

where in the first term on the rhs of (5.2) we have used the symmetry of the integrand in y and y' to transform the integral on the triangle $-1\le y\le 1$ and $y'\le y$, to the integral on the square $-1\le y, y'\le 1$. We now use the fact that $N_i(K_0,-y)=\varepsilon_i N_i(K_0,y)$, with $\varepsilon_i=1$ for $i=1$, 2 and $\varepsilon_3=-1$ [see (4.5)]. Then we can write

$$
\int_{-1}^{1} dy \int_{-1}^{y} dy' N_i(K_0,y')N_j(K_0,y) = \frac{1}{2}\int_{-1}^{1} dy \int_{-1}^{y} dy' N_i(K_0,y')N_j(K_0,y) + \frac{\varepsilon_i\varepsilon_j}{2}\int_{-1}^{1} dy \int_{y}^{1} dy' N_i(K_0,y')N_j(K_0,y).
$$

Therefore, for the case where $\varepsilon_i\varepsilon_j=1$, one can write further that

$$
\int_{-1}^{1} dy \int_{-1}^{y} dy' N_i(K_0,y')N_j(K_0,y) = \frac{1}{2}\int_{-1}^{1} dy \int_{-1}^{1} dy' N_i(K_0,y')N_j(K_0,y); \qquad \varepsilon_i\varepsilon_j = 1. \qquad (5.3)
$$

For the case $\varepsilon_i\varepsilon_j=-1$, we do not use (5.3), but instead write

$$
T_iT_j = \frac{1}{2}\{T_i,T_j\} + \frac{1}{2}[T_i,T_j] = \frac{1}{2}\{T_i,T_j\} + i\varepsilon_{ijk}T_k. \qquad (5.4)
$$

Note that we are dealing here with products, and not only commutators, of the $SU(2)$ Lie algebra generators. We have, therefore, to work with a basis in the enveloping algebra of $SU(2)$, which in the case of quadratic terms we shall take to be the nine quantities T_i, and the anticommutators $\{T_i,T_j\}$, i, $j=1$, 2, 3. If one works with the spinor representation given by the Pauli matrices σ_i, $i=1$, 2, 3, then one has $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1}$, and nondiagonal terms vanish, i.e., $\{\sigma_i,\sigma_j\}=0$, for $i\neq j$. However, if one works with the triplet or higher representations one has $\{T_i,T_j\}\neq 0$ even for $i\neq j$. So, we have to consider the coefficients of all the nine elements of the basis of the enveloping algebra to be independent. Therefore, using (5.4), one gets that

$$
\begin{aligned}
V_{(2)} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \int_{-\frac{\pi}{2}}^{\tau} d\tau' W^{-1}\frac{dW}{d\tau'}W^{-1}\frac{dW}{d\tau}\Big|_{\zeta=\zeta_0} \\
&= -[\mathcal{N}_1(K_0)T_1^2 + \mathcal{N}_2(K_0)T_2^2 + \mathcal{N}_{12}(K_0)\{T_1,T_2\} + \mathcal{N}_{13}^{+}(K_0)\{T_1,T_3\} + \mathcal{N}_{23}^{+}(K_0)\{T_2,T_3\} \\
&\quad - i\mathcal{N}_{13}^{-}(K_0)T_2 + i\mathcal{N}_{23}^{-}(K_0)T_1], \qquad (5.5)
\end{aligned}
$$

where

$$\begin{aligned}
\mathcal{N}_i(K_0) &= \frac{1}{2}\left(\int_{-1}^{1} dy N_i(K_0,y)\right)^2 \quad i=1,2\\
\mathcal{N}_{12}(K_0) &= \frac{1}{2}\left(\int_{-1}^{1} dy N_1(K_0,y)\right)\left(\int_{-1}^{1} dy' N_2(K_0,y')\right)\mathcal{N}_{13}^{\pm}(K_0)\\
&= \frac{1}{2}\left(\int_{-1}^{1} dy \int_{-1}^{y} N_1(K_0,y')N_3(K_0,y) \pm \int_{-1}^{1} dy \int_{-1}^{y} dy' N_3(K_0,y')N_1(K_0,y)\right)\\
\mathcal{N}_{23}^{\pm}(K_0) &= \frac{1}{2}\left(\int_{-1}^{1} dy \int_{-1}^{y} dy' N_2(K_0,y')N_3(K_0,y) \pm \int_{-1}^{1} dy \int_{-1}^{y} dy' N_3(K_0,y')N_2(K_0,y)\right),
\end{aligned} \tag{5.6}$$

with the N_i's defined in (4.5), and where we have dropped the term proportional to T_3^2 because N_3 is an odd function of y, and so its integral on the interval $-1 \le y \le 1$, vanishes.

Using (3.16), (3.18) and (4.2), the term on the second line of (5.1), denoted G_2, becomes

$$\begin{aligned}
G_2 &= -i2\pi\varepsilon_{ijk}T_k \int_1^{K_0} dK \int_{-1}^{1} dy \sqrt{1-y^2} M_j(K,y)\\
&\quad\times \int_{-1}^{y} dy' N_i(K,y')\\
&\equiv -i4\pi^2 R_k(K_0)T_k.
\end{aligned} \tag{5.7}$$

Using (3.18) and (4.10) the term on the third line of (5.1), denoted G_3, becomes

$$\begin{aligned}
G_3 &= -4\pi^2 \int_0^{\pi} d\theta \int_{S(K_0,\theta)}^{1} dss \int_0^{\pi} d\theta'\\
&\quad\times \int_{S(K,\theta')}^{1} ds's' \sum_{i,j=1}^{3} M_i(s,\theta)M_j(s',\theta')T_iT_j,
\end{aligned} \tag{5.8}$$

with $K \ge K_0$, and so $\zeta \le \zeta_0$. Note that in the (θ', s') integration, K has to be taken as a function of θ and s. From (4.2) and (4.7) one gets that $K = \frac{s\sin\theta}{\sqrt{1-s^2\cos^2\theta}}$. Note that the (θ', s') integration is the same as the one performed in (4.11), with K_0 replaced by K. Therefore, similar to what happened, there will be no terms in the direction of T_j for $j=3$, since $M_3(s',\theta')$ is odd under reflection of θ' around $\theta' = \frac{\pi}{2}$ [see (4.10)]. Since K and $S(K_0,\theta)$ are even under the reflection of θ around $\theta = \frac{\pi}{2}$, there will be no terms in the direction of T_i for $i=3$, since $M_3(s,\theta)$ is odd under that reflection. Using (5.4) one gets that

$$\begin{aligned}
G_3 = -4\pi[&S_1(K_0)T_1^2 + S_2(K_0)T_2^2\\
&+ S_{12}\{T_1,T_2\} + iS_3(K_0)T_3]
\end{aligned}$$

with

$$\begin{aligned}
S_a(K_0) &= \pi \int_0^{\pi} d\theta \int_{S(K_0,\theta)}^{1} dss \int_0^{\pi} d\theta'\\
&\quad\times \int_{S(K,\theta')}^{1} ds's' M_a(s,\theta)M_a(s',\theta'); \qquad a=1,2\\
S_{12}(K_0) &= \frac{\pi}{2}\int_0^{\pi} d\theta \int_{S(K_0,\theta)}^{1} dss \int_0^{\pi} d\theta' \int_{S(K,\theta')}^{1} ds's'\\
&\quad\times [M_1(s,\theta)M_2(s',\theta') + M_2(s,\theta)M_1(s',\theta')]\\
S_3(K_0) &= \frac{\pi}{2}\int_0^{\pi} d\theta \int_{S(K_0,\theta)}^{1} dss \int_0^{\pi} d\theta' \int_{S(K,\theta')}^{1} ds's'\\
&\quad\times [M_1(s,\theta)M_2(s',\theta') - M_2(s,\theta)M_1(s',\theta')].
\end{aligned} \tag{5.9}$$

The s' integration can be performed analytically and so, using (4.10) and the fact that $K = \frac{s\sin\theta}{\sqrt{1-s^2\cos^2\theta}}$, we get

$$\begin{aligned}
&\int_{S(K,\theta')}^{1} ds's' M_1(s',\theta')\\
&\quad= \sin\theta'\left[1 - \frac{s\sin\theta}{\sqrt{s^2\sin^2\theta + (1-s^2)\sin^2\theta'}}\right.\\
&\qquad\left. - \frac{1}{2\pi}\sin\left(\frac{2\pi s\sin\theta}{\sqrt{s^2\sin^2\theta + (1-s^2)\sin^2\theta'}}\right)\right],
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
&\int_{S(K,\theta')}^{1} ds's' M_2(s',\theta')\\
&\quad= \frac{1}{2\pi}\left[-1 + \cos\left(\frac{2\pi s\sin\theta}{\sqrt{s^2\sin^2\theta + (1-s^2)\sin^2\theta'}}\right)\right].
\end{aligned} \tag{5.11}$$

Note that the above integrals are symmetric under the reflection of θ and θ' around $\frac{\pi}{2}$. The quantities $M_1(s,\theta)$, $M_2(s,\theta)$, $S(K,\theta)$ and $K(s,\theta)$ are also symmetric under the reflection of θ around $\frac{\pi}{2}$. Therefore, the integration in θ and θ' can be performed in the interval from zero to $\frac{\pi}{2}$, by multiplying the result by two. So, we then get that

$$S_1(K_0) = 2\int_0^{\frac{\pi}{2}} d\theta \int_{S(K_0,\theta)}^1 ds \int_0^{\frac{\pi}{2}} d\theta' (1-\cos(2\pi s))\sin\theta\sin\theta' \times \left[2\pi - \frac{2\pi s\sin\theta}{\sqrt{s^2\sin^2\theta + (1-s^2)\sin^2\theta'}} + \sin\left(\frac{2\pi s\sin\theta}{\sqrt{s^2\sin^2\theta+(1-s^2)\sin^2\theta'}}\right)\right] \tag{5.12}$$

$$P_{12}(K_0) = 2\int_0^{\frac{\pi}{2}} d\theta \int_{S(K_0,\theta)}^1 ds \int_0^{\frac{\pi}{2}} d\theta' (1-\cos(2\pi s))\sin\theta\sin\theta' \left[-1 + \cos\left(\frac{2\pi s\sin\theta}{\sqrt{s^2\sin^2\theta+(1-s^2)\sin^2\theta'}}\right)\right] \tag{5.13}$$

$$P_{21}(K_0) = 2\int_0^{\frac{\pi}{2}} d\theta \int_{S_0(\theta)}^1 ds \int_0^{\frac{\pi}{2}} d\theta' \sin(2\pi s)\left[2\pi - \frac{2\pi s\sin\theta}{\sqrt{s^2\sin^2\theta + (1-s^2)\sin^2\theta'}} + \sin\left(\frac{2\pi s\sin\theta}{\sqrt{s^2\sin^2\theta+(1-s^2)\sin^2\theta'}}\right)\right] \tag{5.14}$$

$$S_2(K_0) = 2\int_0^{\frac{\pi}{2}} d\theta \int_{S(K_0,\theta)}^1 ds \int_0^{\frac{\pi}{2}} d\theta' \sin(2\pi s)\left[-1 + \cos\left(\frac{2\pi s\sin\theta}{\sqrt{s^2\sin^2\theta+(1-s^2)\sin^2\theta'}}\right)\right], \tag{5.15}$$

where we have introduced

$$S_{12} = \frac{1}{2}(P_{12}(K_0) + P_{21}(K_0)) \qquad S_3 = \frac{1}{2}(P_{12}(K_0) - P_{21}(K_0)).$$

Summarizing, we have obtained both sides of the integral equation in second order in α for a given K_0, given in (5.1). From (5.5) we have that

$$V_{(2)} = -i\mathcal{N}_{23}^{-}T_1 + i\mathcal{N}_{13}^{-}T_2 - \mathcal{N}_1 T_1^2 - \mathcal{N}_2 T_2^2 - \mathcal{N}_{12}\{T_1,T_2\} - \mathcal{N}_{13}^{+}\{T_1,T_3\} - \mathcal{N}_{23}^{+}\{T_2,T_3\}, \tag{5.16}$$

and, from (5.1), (4.12), (5.7), and (5.9), we have that

$$U_{(2)} = -i(I_1 + 4\pi^2 R_1)T_1 + -i(I_2 + 4\pi^2 R_2)T_2 - i(4\pi^2 R_3 + 4\pi S_3)T_3 - 4\pi S_1 T_1^2 - 4\pi S_2 T_2^2 - 4\pi S_{12}\{T_1,T_2\}. \tag{5.17}$$

We have to check the equality between the coefficients of each element of the basis of the $SU(2)$ enveloping algebra on the expansion of $V_{(2)}$ and $U_{(2)}$. Those coefficients involve integrals which are calculated numerically for a set of values of K_0. The results are presented in the tables of Sec. A 2 in the Appendix. The consistency is remarkable and with that check we can state clearly that the 't Hooft-Polyakov monopole and its BPS version satisfy the integral Yang-Mills equations up to second order in α.

VI. CONCLUSIONS

The integral Yang-Mills equations appeared from an attempt to understand integrability in higher dimensional spacetimes [1,2]. Through a loop space formulation [11,12] one can construct a suitable generalization of the non-Abelian Stokes theorem for two-form fields that can be used naturally to define conservation laws, thus mimicking the so-called zero curvature representation of integrable field theories in $(1+1)$ dimensions. That has led us to consider the applications of such non-Abelian Stokes theorem to construct the integral equations for non-Abelian gauge theories, generalizing the well-known Abelian version of such integral equations used to describe the laws of electrodynamics. That was indeed possible, as we have shown in [1,2], and the usual differential Yang-Mills equations are obtained from these integral equations when the appropriate limit is taken.

The present paper shows that there is more to be explored. The integral Yang-Mills equations allow the introduction of two c-numbers as parameters which arise naturally in the construction of the equations, and as being nonlinear, produce a quite nontrivial dependence on those parameters of the surface- and volume-ordered integrals appearing on both sides of the equation.

We have tested the assumption that the integral Yang-Mills equations are in fact a collection of an infinite number of equations, each one corresponding to the coefficients of the above-mentioned expansion in powers of those parameters. This was done by considering the fact that, by construction, a solution of the differential Yang-Mills equation is also a solution of the integral Yang-Mills equation. Thus, using the 't Hooft-Polyakov and BPS monopoles as such configurations, we tested the validity of the equations arising at first and second order in the parameter expansion of the integral Yang-Mills equation.

Despite the quite different structures of the terms resulting from the surface- and volume-ordered integrals, we have checked their equalities with a high numerical precision of at least one part in 10^7. In addition, much of the check has been done analytically, and we have obtained an exact expression for the Wilson line operator, on each loop scanning the surfaces and volumes, for the $SU(2)$ 't Hooft-Polyakov monopole solution and its BPS version [see (3.14)]. That result can certainly be useful in many other applications.

The fact that those configurations are solutions of both of the highly nontrivial equations at each order of the expansion, indicates that the parameters could indeed be arbitrary. The arbitrariness of the parameters leads to a variety of important consequences which can now be considered, such as their role in the conserved charges that arise dynamically from the integral equations and the significance of having an infinite number of integral equations.

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APPENDIX: NUMERICAL RESULTS

In this section, we show the results of the numerical integrations related to the terms on the lhs and rhs of the expansion of the integral equation performed at first and second order in α. The coefficients of the generators of the algebra (eventually, up to a common factor of $i \equiv \sqrt{-1}$) are compared for different values of K_0 and the results are presented in Tables I, II, III, IV, and V below. For each integral estimative, there is an associated upper bound on the error, which we represent by using the following notation: $1.372 \pm 0.008 \equiv 1.37(2 \pm 8)$.

1. Equation $V_{(1)} = U_{(1)}$

TABLE I. Numerical verification of the validity of Eq. (4.1): the coefficients of T_1 and T_2 in (4.6) and in (4.12) agree up to the eighth order.

Coefficients of T_1		
K_0	$I_1(K_0)$	$J_1(K_0)$
0.01	$12.5614010(8 \pm 2)$	12.561401086
0.1	$12.077187419(9 \pm 6)$	12.0771874199
0.2	$10.70071291(6 \pm 2)$	10.7007129168
0.3	$8.6878758(4 \pm 8)$	8.68787584542
0.4	$6.38863592(8 \pm 4)$	6.38863592858
0.5	4.169079306	4.169079306
0.6	$2.3285155680(5 \pm 6)$	2.32851556805
0.7	$1.0380042(9 \pm 1)$	1.0380042978
0.8	$0.3151104413(2 \pm 9)$	0.315110441326
0.9	0.039159443823	0.039159443823
0.99	$0.000037982(0 \pm 3)$	$0.00003798206260(6 \pm 1)$

(Table continued)

TABLE I. *(Continued)*

Coefficients of T_2		
K_0	$I_2(K_0)$	$J_2(K_0)$
0.01	$-0.19171684(4 \pm 3)$	$-0.1917168(4 \pm 1)$
0.1	$-1.85828511(5 \pm 9)$	$-1.85828511(2 \pm 2)$
0.2	$-3.3769476(8 \pm 1)$	$-3.3769476(8 \pm 4)$
0.3	-4.29785670058	$-4.2978567(0 \pm 6)$
0.4	$-4.50418166(9 \pm 8)$	$-4.5041816(6 \pm 2)$
0.5	-4.04299388345	$-4.0429938(8 \pm 2)$
0.6	$-3.1056196(0 \pm 2)$	$-3.1056196(0 \pm 2)$
0.7	$-1.97241848(8 \pm 6)$	$-1.9724184(8 \pm 1)$
0.8	$-0.9381858850(1 \pm 6)$	$-0.93818588(5 \pm 8)$
0.9	$-0.23901332(5 \pm 3)$	$-0.23901332(5 \pm 9)$
0.99	$-0.00233660398(4 \pm 4)$	$-0.00233660(3 \pm 7)$

2. Equation $V_{(2)} = U_{(2)}$

The tables below show the values of the coefficients of the algebra elements of (5.16) and (5.17). The fact that they agree implies on the validity of the equation obtained after expanding the Yang-Mills integral equation to second order in α and, therefore, on the validity of the integral equation itself for any value of α, at least up to that order.

TABLE II. Comparison between the coefficients of T_1 and T_2 of Eqs. (5.16) and (5.17).

Coefficients of T_1		
K_0	$I_1(K_0) + 4\pi^2 R_1(K_0)$	$\mathcal{N}_{23}^{-}(K_0)$
0.01	$0.0106581(0 \pm 2)$	$0.010658107(5 \pm 3)$
0.1	$1.013915115(8 \pm 6)$	$1.01391511(5 \pm 4)$
0.2	$3.47985977(0 \pm 2)$	$3.4798597(7 \pm 1)$
0.3	$6.0261611(3 \pm 8)$	$6.0261611(4 \pm 6)$
0.4	$7.31167856(3 \pm 4)$	$7.3116785(6 \pm 1)$
0.5	$6.7762177776(6 \pm 5)$	$6.77621777(7 \pm 9)$
0.6	$4.8526187817(7 \pm 6)$	$4.85261878(1 \pm 4)$
0.7	$2.5695934(4 \pm 1)$	2.56959344117
0.8	$0.8721002092(6 \pm 9)$	$0.872100209(2 \pm 9)$

(Table continued)

TABLE II. *(Continued)*

Coefficients of T_1		
K_0	$I_1(K_0) + 4\pi^2 R_1(K_0)$	$\mathcal{N}^-_{23}(K_0)$
0.9	0.115252473857	$0.115252473(8 \pm 3)$
0.99	$0.000113925(3 \pm 3)$	$0.00011392533(9 \pm 1)$

Coefficients of T_2		
K_0	$-(I_2(K_0) + 4\pi^2 R_2(K_0))$	$\mathcal{N}^-_{13}(K_0)$
0.01	$-0.46994245(7 \pm 3)$	$-0.46994245(7 \pm 1)$
0.1	$-4.38319034(2 \pm 9)$	$-4.3831903(4 \pm 1)$
0.2	$-7.0567939(8 \pm 1)$	$-7.0567939(8 \pm 1)$
0.3	−7.20852026286	$-7.2085202(6 \pm 1)$
0.4	$-5.26717871(7 \pm 8)$	$-5.26717871(7 \pm 6)$
0.5	$-2.5185733549(6 \pm 9)$	$-2.51857335(4 \pm 2)$
0.6	$-0.2961954(3 \pm 2)$	$-0.29619543(4 \pm 7)$
0.7	$0.71488621(2 \pm 6)$	$0.71488621(2 \pm 7)$
0.8	$0.6675901030(5 \pm 6)$	$0.66759010(3 \pm 4)$
0.9	$0.22170728(6 \pm 3)$	$0.22170728(6 \pm 1)$
0.99	$0.00233492080(0 \pm 4)$	$0.00233492(0 \pm 8)$

TABLE III. Comparison between the coefficients of T_1^2 and T_2^2 of Eqs. (5.16) and (5.17).

Coefficients of T_1^2		
K_0	$4\pi S_1(K_0)$	$\mathcal{N}_1(K_0)$
0.01	$78.894398(6 \pm 2)$	78.8943986254
0.1	$72.92922798(7 \pm 8)$	72.9292279882
0.2	$57.2526284(6 \pm 2)$	57.2526284639
0.3	37.7395933527	37.7395933527
0.4	$20.4073345(1 \pm 3)$	20.4073345139
0.5	$8.690611129(8 \pm 2)$	8.69061112984
0.6	$2.7109923(7 \pm 3)$	2.71099237533
0.7	$0.5387264(6 \pm 1)$	0.538726461126
0.8	$0.0496472951(1 \pm 2)$	0.0496472951165
0.9	$0.0007667310202(6 \pm 1)$	0.000766731020264
0.99	$7.21318539905 \times 10^{-10}$	$7.213185398(9 \pm 2) \times 10^{-10}$

Coefficients of T_2^2		
K_0	$4\pi S_2(K_0)$	$\mathcal{N}_2(K_0)$
0.01	$0.018377674(1 \pm 2)$	$0.0183776(7 \pm 1)$
0.1	$1.7266117(8 \pm 1)$	$1.7266117(8 \pm 2)$
0.2	$5.7018878(2 \pm 4)$	$5.7018878(2 \pm 7)$
0.3	$9.2357861093(6 \pm 1)$	$9.235786(1 \pm 1)$
0.4	$10.1438262(5 \pm 4)$	$10.1438262(5 \pm 5)$
0.5	$8.172899770(8 \pm 4)$	$8.1728997(7 \pm 4)$

(Table continued)

TABLE III. *(Continued)*

Coefficients of T_2^2		
K_0	$4\pi S_2(K_0)$	$\mathcal{N}_2(K_0)$
0.6	4.82243658089	$4.8224365(8 \pm 3)$
0.7	$1.9452173(4 \pm 2)$	$1.9452173(4 \pm 1)$
0.8	$0.4400963774(2 \pm 5)$	$0.4400963(7 \pm 1)$
0.9	0.0285636848378	$0.0285636(8 \pm 1)$
0.99	2.72985908954e-06	$2.729(8 \pm 8)$e-06

TABLE IV. Comparison between the coefficients of $\{T_1, T_2\}$ of Eqs. (5.16) and (5.17).

Coefficients of $\{T_1, T_2\}$		
K_0	$4\pi S_{12}(K_0)$	$\mathcal{N}_{12}(K_0)$
0.01	$-1.2041160882(4 \pm 2)$	−1.20411608813
0.1	$-11.2214288(0 \pm 5)$	−11.2214288084
0.2	$-18.0678738(4 \pm 7)$	−18.0678738427
0.3	−18.669622708	−18.6696227077
0.4	$-14.3877884(2 \pm 1)$	−14.3877884224
0.5	$-8.4277810668(8 \pm 6)$	−8.42778106684
0.6	$-3.6157418(0 \pm 4)$	−3.6157418051
0.7	$-1.02368943(3 \pm 4)$	−1.02368943394
0.8	$-0.1478160841(3 \pm 1)$	−0.147816084139
0.9	$-0.0046798144427(1 \pm 3)$	−0.00467981444135
0.99	$-4.43745194072 \times 10^{-8}$	$-4.437470(7 \pm 7) \times 10^{-8}$

TABLE V. The coefficients above are the ones that should vanish in the equation obtained at second order in α; indeed, within a numerical precision, they are zero.

Coefficients of T_3, $\{T_2, T_3\}$ and $\{T_1, T_3\}$			
K_0	$4\pi^2 R_3(K_0) + 4\pi S_3(K_0)$	$\mathcal{N}^+_{23}(K_0)$	$\mathcal{N}^+_{13}(K_0)$
0.01	$\pm 2 \times 10^{-8}$	$\pm 7 \times 10^{-11}$	$\pm 9 \times 10^{-10}$
0.1	$\pm 7 \times 10^{-8}$	$\pm 2 \times 10^{-9}$	$\pm 6 \times 10^{-9}$
0.2	$\pm 1 \times 10^{-7}$	$\pm 7 \times 10^{-9}$	$\pm 1 \times 10^{-8}$
0.3	$\pm 2 \times 10^{-11}$	$\pm 1 \times 10^{-8}$	$\pm 4 \times 10^{-9}$
0.4	$\pm 7 \times 10^{-8}$	$\pm 1 \times 10^{-8}$	$\pm 6 \times 10^{-9}$
0.5	$\pm 7 \times 10^{-10}$	$\pm 9 \times 10^{-9}$	$\pm 7 \times 10^{-9}$
0.6	$\pm 5 \times 10^{-9}$	$\pm 4 \times 10^{-9}$	$\pm 7 \times 10^{-9}$
0.7	$-4.44(5 \pm 2) \times 10^{-11}$	$\pm 6 \times 10^{-11}$	$\pm 5 \times 10^{-9}$
0.8	$-1 \pm 7 \times 10^{-11}$	$\pm 9 \times 10^{-10}$	$\pm 3 \times 10^{-9}$
0.9	$-3.(7 \pm 7) \times 10^{-13}$	$\pm 3 \times 10^{-10}$	$\pm 5 \times 10^{-9}$
0.99	$9.599(0 \pm 4) \times 10^{-16}$	$-1.(6 \pm 2) \times 10^{-11}$	$\pm 8 \times 10^{-9}$

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