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Vacua and walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$

Bum-Hoon Lee,[1,*] Chanyong Park,[2,3,†] and Sunyoung Shin[2,‡]

[1]*Department of Physics, Sogang University, Seoul 121-742, Korea*
[2]*Asia Pacific Center for Theoretical Physics, Pohang 790-784, Korea*
[3]*Department of Physics, Pohang University of Science and Technology, Pohang 790-784, Korea*

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We construct walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$, by using the moduli matrix formalism and the simple roots of $SO(2N)$. Penetrable walls are observed in the nonlinear sigma models on $SO(2N)/U(N)$ with $N > 3$.

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I. INTRODUCTION

The moduli matrix formalism, which was developed in Refs. [1,2], is one of the powerful tools for studying Bogomol'nyi-Prasad-Sommerfield (BPS) objects. The moduli matrices contain all the moduli parameters of BPS solutions. In Refs. [1,2], moduli matrices of walls are constructed in a mass-deformed hyper-Kähler nonlinear sigma model of which the target space is the cotangent bundle over the complex Grassmann manifold $T^*G_{N_F,N_C}$, which is the strong coupling limit of supersymmetric $U(N_C)$ gauge theory with $N_F > N_C$[1] [3]. It is also shown that the vacua and the walls are on the Kähler manifold.

In the moduli matrix formalism, walls are generated by operators, which are positive root generators of the system. Elementary wall operators are generators of simple roots [4]. Thus, elementary walls and compressed walls can be labeled by simple roots and by linear combinations of simple roots, respectively. We can also figure out structures of multiwalls from the linear combinations of simple roots.

Various BPS objects are constructed in the moduli matrix formalism. Domain walls and domain wall webs are discussed in Refs. [5–7]. Other BPS objects and composite objects are discussed in Refs. [8–14].

Moduli matrices, which are constrained by a single quadratic constraint $SO(2N)$ or $Sp(N)$, are constructed for walls in Ref. [6] and for kink monopoles [14]. In Ref. [6], walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$ with $N = 2, 3$ are discussed. It is known that there are isomorphisms,

$$SO(4)/U(2) \simeq CP^1, \qquad SO(6)/U(3) \simeq CP^3. \tag{1.1}$$

Therefore, the nonlinear sigma models discussed in Ref. [6] are Abelian theories. In Abelian theories, the ordering of walls is absolute. In non-Abelian theories, however, walls can pass through each other. Penetrable walls are observed in nonlinear sigma models on the Grassmann manifold [2].

The purpose of this paper is to construct walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$ with $N > 3$, which are non-Abelian theories. We use the convention and part of the formalism used in Ref. [14]. In Sec. II, we review the moduli matrix method discussed in Refs. [2,6,14]. In Sec. III, we review the results of Ref. [6] with the $O(2N)$ invariant tensor, which is used in Ref. [14]. We also identify elementary walls with the simple roots of $SO(2N)$. In Sec. IV, we study elementary walls of nonlinear sigma models on $SO(2N)/U(N)$ with $N = 4, 5, 6, 7$, which are labeled by the simple roots of $SO(2N)$. In Sec. V, we discuss the vacuum structure connected to the maximum number of elementary walls. In Sec. VI, we make some observations about walls of the nonlinear sigma model on $SO(12)/U(6)$, which is the simplest nontrivial case. In Sec. VII, we summarize our result.

II. MODULI MATRICES ON $SO(2N)/U(N)$

In this section, we review the formalism of Refs. [6,14]. The mass-deformed Kähler nonlinear sigma model on $SO(2N)/U(N)$ in $(2+1)$ dimensions [6] is obtained by dimensional reduction [15] of the $\mathcal{N}=1$ $(3+1)$-dimensional Kähler nonlinear sigma model on $SO(2N)/U(N)$ [16]. As we are interested in solitonic objects, we consider only the bosonic part of the model. The Lagrangian in $(2+1)$ dimensions is

$$\begin{aligned}\mathcal{L} = &-|D_\mu\phi_a{}^i|^2 - |i\phi_a{}^j M_j{}^i - i\Sigma_a{}^b\phi_b{}^i|^2 \\ &+ |F_a{}^i|^2 + (D_a{}^b\phi_b{}^i\bar{\phi}_i{}^a - D_a{}^a) \\ &+ ((F_0)^{ab}\phi_b{}^i J_{ij}\phi^{Tj}{}_a + \phi_0^{ab}F_b{}^i J_{ij}\phi_a^{Tj} \\ &+ (\phi_0)^{ab}\phi_b{}^i J_{ij}F^{Tj}{}_a + (\mathrm{H.c.})),\end{aligned} \tag{2.1}$$

where the greek letter μ denotes three-dimensional spacetime indices and the covariant derivative is defined by $(D_\mu\phi)_a{}^i = \partial_\mu\phi_a{}^i - iA_{\mu a}{}^b\phi_b{}^i$. The indices $i, j(= 1, \ldots, 2N)$

*bhl@sogang.ac.kr
†chanyong.park@apctp.org
‡sunyoung.shin@apctp.org

[1]This condition is required to define the Grassmann manifold $G_{N_F,N_C} = \frac{SU(N_F)}{SU(N_C)\times SU(N_F-N_C)\times U(1)}$.

are flavor indices, and the indices $a, b(=1, \ldots, N)$ are color indices. The last term (h.c.) stands for the Hermitian conjugate. The $O(2N)$ invariant tensor J is defined by

$$J = \sigma^1 \otimes I_N, \tag{2.2}$$

where I_N stands for the $N \times N$ identity matrix. In this basis, the $SO(2N)$ Cartan generators are

$$H_n = \left(\begin{array}{c|c} h_n & \\ \hline & -h_n \end{array} \right), \quad (n = 1, \ldots, N), \tag{2.3}$$

with $N \times N$ matrix h_n, which has a unit component only in the (n, n) element. The mass matrix is a linear combination of the Cartan generators. By defining vectors

$$\begin{aligned} \underline{m} &= (m_1, m_2 \cdots, m_N), \\ \underline{H} &= (H_1, H_2, \ldots, H_N), \end{aligned} \tag{2.4}$$

the mass matrix is formulated as

$$M = \underline{m} \cdot \underline{H}. \tag{2.5}$$

In the basis of $SO(2N)$ with the invariant tensor (2.2), the mass matrix (2.5) is

$$M = \sigma_3 \otimes \mathrm{diag}(m_1, m_2, \ldots, m_N). \tag{2.6}$$

The constraints of the Lagrangian (2.1) are

$$\phi_a{}^i \bar{\phi}_i{}^b - \delta_a{}^b = 0, \tag{2.7}$$

$$\phi_a{}^i J_{ij} \phi^{Tj}{}_b = 0, \quad (\text{Hermitian conjugate}) = 0. \tag{2.8}$$

Equation (2.7) is from the D-term constraint, which defines the Grassmann manifold $G_{2N,N}$, and Eqs. (2.8) are from the F-term constraints, which correspond to the (anti)holomorphic embedding of the $SO(2N)$ manifold. By eliminating auxiliary fields, the potential term of the Lagrangian becomes

$$V = |i\phi_a{}^j M_j{}^i - i\Sigma_a{}^b \phi_b{}^i|^2 + 4|(\phi_0)^{ab} \phi_b^i|^2. \tag{2.9}$$

Therefore, the vacuum conditions are

$$\phi_a{}^j M_j{}^i - \Sigma_a{}^b \phi_b{}^i = 0, \tag{2.10}$$

$$(\phi_0)^{ab} = 0. \tag{2.11}$$

Σ can be diagonalized as

$$\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_N), \tag{2.12}$$

by a $U(N)$ gauge transformation. Then, the vacua are labeled by

$$(\Sigma_1, \Sigma_2, \ldots, \Sigma_N) = (\pm m_1, \pm m_2, \ldots, \pm m_N). \tag{2.13}$$

Since the tensor (2.2) is invariant under $O(2N)$ transformation, only the half of the vacua in Eq. (2.13) belong to a nonlinear sigma model, and the other half belong to the other nonlinear sigma model, which is related by parity. Therefore, the number of vacua of the model is 2^{N-1}. This number is the Euler characteristic of the manifold on which the nonlinear sigma model lives [17–20].

The BPS equation for wall solutions is derived from the Bogomol'nyi completion of the Hamiltonian. It is assumed that fields are static and all the fields depend only on the $x_1 \equiv x$ coordinate. It is also assumed that there is Poincaré invariance on the two-dimensional world volume of walls to set $A_0 = A_2 = 0$. The energy density along the x direction is

$$\begin{aligned} \mathcal{E} &= (|(D\phi)_a{}^i|^2 + |\phi_a{}^j M_j{}^i - \Sigma_a{}^b \phi_b{}^i|^2 + 4|(\phi_0)^{ab} \phi_b{}^i|^2) \\ &= (|(D\phi)_a{}^i \mp (\phi_a{}^j M_j{}^i - \Sigma_a{}^b \phi_b{}^i)|^2 + 4|(\phi_0)^{ab} \phi_b{}^i|^2) \pm T \\ &\geq \pm T. \end{aligned} \tag{2.14}$$

This is also constrained by Eqs. (2.7) and (2.8). The tension of the wall is

$$T = \partial(\phi_a{}^i M_i{}^j \bar{\phi}_j{}^a). \tag{2.15}$$

We choose the upper sign for the BPS equation and the lower sign for the anti-BPS equation. The BPS equation is

$$(D\phi)_a{}^i - (\phi_a{}^j M_j{}^i - \Sigma_a{}^b \phi_b{}^i) = 0. \tag{2.16}$$

By introducing complex matrix functions $S_a{}^b(x)$ and $f_a{}^i(x)$, which are defined by

$$\Sigma_a{}^b - iA_a{}^b = (S^{-1}\partial S)_a{}^b, \qquad \phi_a{}^i = (S^{-1})_a{}^b f_a{}^i, \tag{2.17}$$

Eq. (2.16) is solved by

$$\phi_a{}^i = (S^{-1})_a{}^b H_{0b}{}^j (e^{Mx})_j{}^i. \tag{2.18}$$

The coefficient matrix H_0 is called the moduli matrix. Σ, A, and ϕ are invariant under transformations

$$S'_a{}^b = V_a{}^c S_c{}^b, \qquad H'^{\,i}_{0a} = V_a{}^c H^i_{0c}, \tag{2.19}$$

where $V \in GL(N, \mathbf{C})$. V defines an equivalent class of (S, H_0). This is called the world volume symmetry [1,2]. The constraints (2.7) and (2.8) correspond to

$$H_{0a}{}^i (e^{2Mx})_i{}^j H^{\dagger\, b}_{0j} = (S\bar{S})_a{}^b \equiv \Omega_a{}^b, \tag{2.20}$$

$$H_{0a}{}^i J_{ij} H^{Tj}{}_b = 0, \quad (\text{Hermitian conjugate}) = 0, \tag{2.21}$$

respectively. The world volume symmetry of H_0 in Eq. (2.19) defines the Grassmann manifold. The constraint

(2.21) is a holomorphic embedding of $SO(2N)$. Therefore, moduli matrices H_0's parametrize $SO(2N)/U(N)$.

There are quantities that we can consider:

$$\mathrm{tr}\Sigma = \frac{1}{2}\partial \ln \det \Omega,$$
$$T = \frac{1}{2}\partial^2 \ln \det \Omega. \tag{2.22}$$

In Ref. [2], walls are algebraically constructed from elementary walls. In the nonlinear sigma model on the Grassmann manifold, the two nearest vacua have the same color number and differ by one flavor number. The elementary wall interpolating the two nearest vacua, vacuum $\langle A\rangle$ and vacuum $\langle B\rangle$, which are in the flavor I and $I+1$ in the same color, is $H_{0\langle A\leftarrow B\rangle} = H_{0\langle A\rangle} e^{E_I(r)}$, where $E_I(r) \equiv e^r E_I$, $(r \in \mathbf{C})$. The elementary wall operator E_I is defined by

$$[cM, E_I] = c(m_I - m_{I+1})E_I = T_{\langle I\leftarrow I+1\rangle}E_I, \tag{2.23}$$

where c is a constant, M is the mass matrix, and E_I is an $N_f \times N_f$ square matrix generating an elementary wall. $T_{\langle I\leftarrow I+1\rangle}$ is the tension of the wall. E_I has a nonzero component only in the $(I, I+1)$th element. However, in the nonlinear sigma model on $SO(2N)/U(N)$, the definitions of the nearest vacua and the elementary wall are not valid due to the quadratic constraint (2.21). Moduli matrices, which are constrained by a single quadratic constraint $SO(2N)$ or $Sp(N)$, are studied in Refs. [6,14]. We review the methods for walls of nonlinear sigma models on $SO(2N)/U(N)$.

Since the mass matrix is a linear combination of the Cartan generators as it is defined in Eq. (2.6), we can generalize Eq. (2.23) as

$$c[M, E_i] = c(\underline{m}\cdot\underline{\alpha}_i)E_i = T_i E_i. \tag{2.24}$$

E_i are the elementary wall operators that are the positive root generators of the simple roots α_i of $SO(2N)$. The elementary wall $H_{0\langle A\leftarrow B\rangle} = H_{0\langle A\rangle} e^{E_i(r)}$ where $E_i(r) \equiv e^r E_i$, $(r \in \mathbf{C})$ can be labeled by the simple root α_i. The wall has tension T_i. We can restrict ourselves to the case in which $m_1 > m_2 > \cdots > m_N$. Then, the vector $\underline{m}$ of Eq. (2.4) is a vector in the interior of the positive Weyl chamber,

$$\underline{m}\cdot\underline{\alpha}_i > 0. \tag{2.25}$$

Elementary walls can be identified with simple roots [4]. The corresponding positive root generators are elementary wall operators. The formula (2.24) is valid in nonlinear sigma models on the Grassmann manifold as well.

We should only consider positive roots that correspond to the BPS solutions. Any linear combinations of positive roots and negative roots break supersymmetries. These linear combinations are called non-BPS sectors.

The simple roots of $SO(2N)$ are

$$\begin{aligned}
\underline{\alpha}_1 &= (1,-1,0,\ldots,0,0,0),\\
\underline{\alpha}_2 &= (0,1,-1,\ldots,0,0,0),\\
&\cdots\\
\underline{\alpha}_{N-1} &= (0,0,0,\ldots,0,1,-1),\\
\underline{\alpha}_N &= (0,0,0,\ldots,0,1,1).
\end{aligned} \tag{2.26}$$

These simple roots describe elementary walls of nonlinear sigma models on $SO(2N)/U(N)$. The corresponding elementary wall operators are

$$E_i = \begin{array}{c} \\ i \\ \\ i+N+1 \end{array}\left(\begin{array}{c|c} \overset{i+1}{1} & \overset{i+N}{} \\ \hline & -1 \end{array}\right), \quad (i = 1,\ldots,N-1),$$

$$E_N = \begin{array}{c} \\ N-1 \\ N \\ \\ \end{array}\left(\begin{array}{c|c} & \overset{2N-1 \;\; 2N}{\begin{matrix} & 1 \\ -1 & \end{matrix}} \\ \hline & \end{array}\right). \tag{2.27}$$

The elementary wall, which connects the two nearest vacua $\langle A\rangle$ and $\langle B\rangle$, is defined by

$$H_{0\langle A\leftarrow B\rangle} = H_{0\langle A\rangle} e^{E_a(r)},$$
$$E_a(r) \equiv e^r E_a, \quad (a = 1,\ldots,N), \tag{2.28}$$

where E_a is the elementary wall operator and r is a complex parameter ranging from $-\infty < \mathrm{Re}(r) < \infty$. Elementary walls can be compressed to a single compressed wall. A compressed wall of level n, which connects $\langle A\rangle$ and $\langle A'\rangle$, is defined by

$$H_{0\langle A\leftarrow A'\rangle} = H_{0\langle a\rangle} e^{[E_{a_1},[E_{a_2},[E_{a_3},[\cdots,[E_{a_n},E_{a_{n+1}}]]]\cdots]](r)},$$
$$(a_i = 1,\ldots,N). \tag{2.29}$$

There can exist multiwalls that connect two vacua $\langle A\rangle$ and $\langle B\rangle$,

$$H_{0\langle A\leftarrow B\rangle} = H_{0\langle A\rangle} e^{E_{a_1}(r_1)} e^{E_{a_2}(r_2)} \cdots e^{E_{a_n}(r_n)}, \tag{2.30}$$

where parameters $r_i (i = 1, 2, \ldots)$ are complex parameters ranging from $-\infty < \mathrm{Re}(r_i) < \infty$. Elementary walls can pass through each other if

$$[E_{a_i}, E_{a_j}] = 0. \tag{2.31}$$

These walls are called penetrable walls.

III. REVIEW ON $SO(2N)/U(N)$ WITH $N=2$, 3

In this section, we review the moduli matrices of vacua and walls of the nonlinear sigma models on $SO(2N)/U(N)$ with $N=2$, 3 [6] by using the formalism of Ref. [14] and the simple roots of $SO(2N)$ (2.26), which are summarized in Sec. II. The moduli matrices of the vacua are discussed in A. We label the moduli matrices of vacua in descending order as

$$\begin{aligned}
&(\Sigma_1,\Sigma_2,\ldots,\Sigma_{N-1},\Sigma_N)=(m_1,m_2,\ldots,m_{N-1},m_N),\\
&(\Sigma_1,\Sigma_2,\ldots,\Sigma_{N-1},\Sigma_N)=(m_1,m_2,\ldots,-m_{N-1},-m_N),\\
&\qquad\qquad\vdots\\
&(\Sigma_1,\Sigma_2,\ldots,\Sigma_{N-1},\Sigma_N)=(\pm m_1,-m_2,\ldots,-m_{N-1},-m_N),
\end{aligned}\tag{3.1}$$

where the sign $\pm$ is $+$ for odd N and $-$ for even N. $H_{0\langle A\rangle}$ and $\langle A\rangle$ denote a vacuum. $H_{0\langle A_1\leftarrow A_2\leftarrow\cdots\leftarrow A_p\rangle}$ and $\langle A_1\leftarrow A_2\leftarrow\cdots\leftarrow A_p\rangle$ denote a wall that connects vacua $\langle A_1\rangle\leftarrow\langle A_2\rangle\leftarrow\cdots\leftarrow\langle A_p\rangle$.

The vacua of the nonlinear sigma model on $SO(4)/U(2)$ (A4) are

$$\begin{aligned}
H_{0\langle 1\rangle}&=\left(\begin{array}{cc|cc}1&&0&\\&1&&0\end{array}\right), &&(\Sigma_1,\Sigma_2)=(m_1,m_2),\\
H_{0\langle 2\rangle}&=\left(\begin{array}{cc|cc}0&&1&\\&0&&1\end{array}\right), &&(\Sigma_1,\Sigma_2)=(-m_1,-m_2).
\end{aligned}\tag{3.2}$$

There is only one elementary wall operator:

$$E=\left(\begin{array}{cc|cc}0&&0&1\\&0&-1&0\\\hline 0&&0&\\&0&&0\end{array}\right).\tag{3.3}$$

The elementary wall is

$$H_{0\langle 1\leftarrow 2\rangle}=H_{0\langle 1\rangle}e^{E(r)}=\left(\begin{array}{cc|cc}1&&0&e^r\\&1&-e^r&0\end{array}\right).\tag{3.4}$$

This is the only wall of the nonlinear sigma model on $SO(4)/U(2)$.

The vacua of the nonlinear sigma model on $SO(6)/U(3)$ (A8) are

$$\begin{aligned}
H_{0\langle 1\rangle}&=\left(\begin{array}{ccc|ccc}1&&&0&&\\&1&&&0&\\&&1&&&0\end{array}\right), &&(\Sigma_1,\Sigma_2,\Sigma_3)=(m_1,m_2,m_3),\\
H_{0\langle 2\rangle}&=\left(\begin{array}{ccc|ccc}1&&&0&&\\&0&&&1&\\&&0&&&1\end{array}\right), &&(\Sigma_1,\Sigma_2,\Sigma_3)=(m_1,-m_2,-m_3),\\
H_{0\langle 3\rangle}&=\left(\begin{array}{ccc|ccc}0&&&1&&\\&1&&&0&\\&&0&&&1\end{array}\right), &&(\Sigma_1,\Sigma_2,\Sigma_3)=(-m_1,m_2,-m_3),\\
H_{0\langle 4\rangle}&=\left(\begin{array}{ccc|ccc}0&&&1&&\\&0&&&1&\\&&1&&&0\end{array}\right), &&(\Sigma_1,\Sigma_2,\Sigma_3)=(-m_1,-m_2,m_3).
\end{aligned}\tag{3.5}$$

The positive root generators of $SO(6)$, which are the elementary wall operators of $SO(6)/U(3)$, are

$$E_1=\left(\begin{array}{ccc|ccc}0&1&&&&\\&0&&&&\\&&0&&&\\\hline &&&0&&\\&&&-1&0&\\&&&&&0\end{array}\right),\quad E_2=\left(\begin{array}{ccc|ccc}0&&&&&\\&0&1&&&\\&&0&&&\\\hline &&&0&&\\&&&&0&\\&&&&-1&0\end{array}\right),\quad E_3=\left(\begin{array}{ccc|ccc}0&&&0&&\\&0&&&0&1\\&&0&&-1&0\\\hline &&&0&&\\&&&&0&\\&&&&&0\end{array}\right).\tag{3.6}$$

The elementary walls are

$$H_{0\langle 1\leftarrow 2\rangle} = H_{0\langle 1\rangle} e^{E_3(r_1)} = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 1 & & & 0 & e^{r_1} \\ & & 1 & & -e^{r_1} & 0 \end{array}\right),$$

$$H_{0\langle 2\leftarrow 3\rangle} = H_{0\langle 2\rangle} e^{E_1(r_1)} = \left(\begin{array}{ccc|ccc} 1 & e^{r_1} & & 0 & & \\ & 0 & & -e^{r_1} & 1 & \\ & & 0 & & & 1 \end{array}\right),$$

$$H_{0\langle 3\leftarrow 4\rangle} = H_{0\langle 3\rangle} e^{E_2(r_1)} = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & e^{r_1} & & 0 & \\ & & 0 & & -e^{r_1} & 1 \end{array}\right). \tag{3.7}$$

There are two compressed walls of level 1. These are generated by the following operators:

$$\tilde{E}_4 = [E_3, E_1], \qquad \tilde{E}_5 = [E_1, E_2]. \tag{3.8}$$

We have used $\sim$ to distinguish the operators from elementary wall operators. The walls are

$$H_{0\langle 1\leftarrow 3\rangle} = H_{0\langle 1\rangle} e^{\tilde{E}_4(r_1)},$$
$$H_{0\langle 2\leftarrow 4\rangle} = H_{0\langle 2\rangle} e^{\tilde{E}_5(r_1)}. \tag{3.9}$$

There is only one compressed wall of level 2, which is generated by

$$\tilde{E}_6 = [E_3, \tilde{E}_5] = [\tilde{E}_4, E_2]. \tag{3.10}$$

The moduli matrix for the wall is

$$H_{0\langle 1\leftarrow 4\rangle} = H_{0\langle 1\rangle} e^{\tilde{E}_6(r_1)}. \tag{3.11}$$

There are four double walls,

$$H_{0\langle 1\leftarrow 2\leftarrow 3\rangle} = H_{0\langle 1\leftarrow 2\rangle} e^{E_1(r_2)} = \left(\begin{array}{ccc|ccc} 1 & e^{r_2} & & 0 & & \\ & 1 & & & 0 & e^{r_1} \\ & & 1 & e^{r_1+r_2} & -e^{r_1} & 0 \end{array}\right),$$

$$H_{0\langle 2\leftarrow 3\leftarrow 4\rangle} = H_{0\langle 2\leftarrow 3\rangle} e^{E_2(r_2)} = \left(\begin{array}{ccc|ccc} 1 & e^{r_1} & e^{r_1+r_2} & 0 & & \\ & 0 & & -e^{r_1} & 1 & \\ & & 0 & & -e^{r_2} & 1 \end{array}\right),$$

$$H_{0\langle 1\leftarrow 2\leftarrow 4\rangle} = H_{0\langle 1\leftarrow 2\rangle} e^{\tilde{E}_5(r_2)} = \left(\begin{array}{ccc|ccc} 1 & & e^{r_2} & 0 & & \\ & 1 & & -e^{r_1+r_2} & 0 & e^{r_1} \\ & & 1 & & -e^{r_1} & 0 \end{array}\right),$$

$$H_{0\langle 1\leftarrow 3\leftarrow 4\rangle} = H_{0\langle 1\leftarrow 3\rangle} e^{E_2(r_2)} = \left(\begin{array}{ccc|ccc} 1 & & & 0 & e^{r_1+r_2} & -e^{r_1} \\ & 1 & e^{r_2} & & 0 & \\ & & 1 & e^{r_1} & & 0 \end{array}\right), \tag{3.12}$$

and one triple wall,

$$H_{0\langle 1\leftarrow 2\leftarrow 3\leftarrow 4\rangle} = H_{0\langle 1\leftarrow 2\leftarrow 3\rangle} e^{E_2(r_3)} = \left(\begin{array}{ccc|ccc} 1 & e^{r_2} & e^{r_2+r_3} & 0 & & \\ & 1 & e^{r_3} & & -e^{r_1+r_3} & e^{r_1} \\ & & 1 & e^{r_1+r_2} & -e^{r_1} & 0 \end{array}\right). \quad (3.13)$$

We investigate the walls by using positive roots. The $SO(6)$ Cartan generators are

$$\begin{aligned} H_1 &= \mathrm{diag}(1,0,0,-1,0,0), \\ H_2 &= \mathrm{diag}(0,1,0,0,-1,0), \\ H_3 &= \mathrm{diag}(0,0,1,0,0,-1), \end{aligned} \quad (3.14)$$

and the simple roots are

$$\begin{aligned} \underline{\alpha}_1 &\coloneqq (1,-1,0), \\ \underline{\alpha}_2 &\coloneqq (0,1,-1), \\ \underline{\alpha}_3 &\coloneqq (0,1,1). \end{aligned} \quad (3.15)$$

The subscripts correspond to the subscripts of the elementary wall operators in Eq. (3.6). We indicate the root that connects two vacua $\langle i\rangle \leftarrow \langle j\rangle$ as $g_{\langle i\leftarrow j\rangle}$. Then, the elementary walls are

$$\begin{aligned} g_{\langle 1\leftarrow 2\rangle} &= \underline{\alpha}_3, \\ g_{\langle 2\leftarrow 3\rangle} &= \underline{\alpha}_1, \\ g_{\langle 3\leftarrow 4\rangle} &= \underline{\alpha}_2, \end{aligned} \quad (3.16)$$

and the roots of compressed walls are

$$\begin{aligned} g_{\langle 1\leftarrow 3\rangle} &= \underline{\alpha}_3 + \underline{\alpha}_1, \\ g_{\langle 2\leftarrow 4\rangle} &= \underline{\alpha}_1 + \underline{\alpha}_2, \\ g_{\langle 1\leftarrow 4\rangle} &= \underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\alpha}_3. \end{aligned} \quad (3.17)$$

Diagrams of simple roots that connect the vacua of nonlinear sigma models on $SO(4)/U(2)$ and $SO(6)/U(3)$ are drawn in Fig. 1.

Once we identify the moduli matrices of vacua and the simple roots that connect the vacua, we can construct all the operators and the moduli matrices of walls. From Sec. IV, we concentrate only on the moduli matrices of vacua and the simple roots.

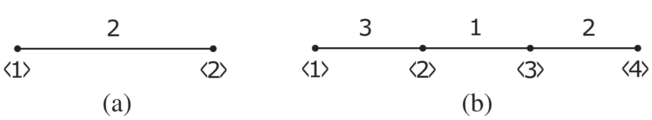


FIG. 1. (a) $SO(4)/U(2)$; (b) $SO(6)/U(3)$. The numbers indicate the subscript i's of roots $\underline{\alpha}_i$.

IV. $SO(2N)/U(N)$ WITH $N = 4, 5, 6, 7$

As discussed in the Introduction, nonlinear sigma models on $SO(2N)/U(N)$ with $N > 3$ are non-Abelian theories. Therefore, we expect that some of the walls are penetrable.

For $N = 4$, there are $8(=2^3)$ vacua. The moduli matrices of vacua are defined by

$$\begin{aligned} &H_{0\langle 1\rangle}\colon\ (\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4) = (m_1,m_2,m_3,m_4), \\ &H_{0\langle 2\rangle}\colon\ (\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4) = (m_1,m_2,-m_3,-m_4), \\ &H_{0\langle 3\rangle}\colon\ (\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4) = (m_1,-m_2,m_3,-m_4), \\ &H_{0\langle 4\rangle}\colon\ (\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4) = (m_1,-m_2,-m_3,m_4), \\ &H_{0\langle 5\rangle}\colon\ (\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4) = (-m_1,m_2,m_3,-m_4), \\ &H_{0\langle 6\rangle}\colon\ (\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4) = (-m_1,m_2,-m_3,m_4), \\ &H_{0\langle 7\rangle}\colon\ (\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4) = (-m_1,-m_2,m_3,m_4), \\ &H_{0\langle 8\rangle}\colon\ (\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4) = (-m_1,-m_2,-m_3,-m_4). \end{aligned} \quad (4.1)$$

The simple roots of $SO(8)$ are

$$\begin{aligned} \underline{\alpha}_1 &\coloneqq (1,-1,0,0), \\ \underline{\alpha}_2 &\coloneqq (0,1,-1,0), \\ \underline{\alpha}_3 &\coloneqq (0,0,1,-1), \\ \underline{\alpha}_4 &\coloneqq (0,0,1,1). \end{aligned} \quad (4.2)$$

Then, the roots of elementary walls $g_{\langle i\leftarrow j\rangle}$ are

$$\begin{aligned} g_{\langle 3\leftarrow 4\rangle} &= g_{\langle 5\leftarrow 6\rangle} = \underline{\alpha}_1, \\ g_{\langle 2\leftarrow 3\rangle} &= g_{\langle 6\leftarrow 7\rangle} = \underline{\alpha}_2, \\ g_{\langle 3\leftarrow 5\rangle} &= g_{\langle 4\leftarrow 6\rangle} = \underline{\alpha}_3, \\ g_{\langle 1\leftarrow 2\rangle} &= g_{\langle 7\leftarrow 8\rangle} = \underline{\alpha}_4. \end{aligned} \quad (4.3)$$

The roots of penetrable walls are orthogonal

$$\underline{\alpha}_i \cdot \underline{\alpha}_j = 0. \quad (4.4)$$

The vacua and the roots are depicted in Fig. 2. A pair of orthogonal simple roots in this diagram makes a parallelogram. The roots $\underline{\alpha}_1$ and $\underline{\alpha}_3$ are orthogonal. Therefore, elementary walls $\langle 3\leftarrow 5\rangle$ and $\langle 5\leftarrow 6\rangle$ are penetrable. Elementary walls $\langle 3\leftarrow 4\rangle$ and $\langle 4\leftarrow 6\rangle$ are also penetrable. The observable quantities $\mathrm{tr}\Sigma$ and T (2.22) of double wall $\langle 3\leftarrow 5\leftarrow 6\rangle$ are plotted in Fig. 3.

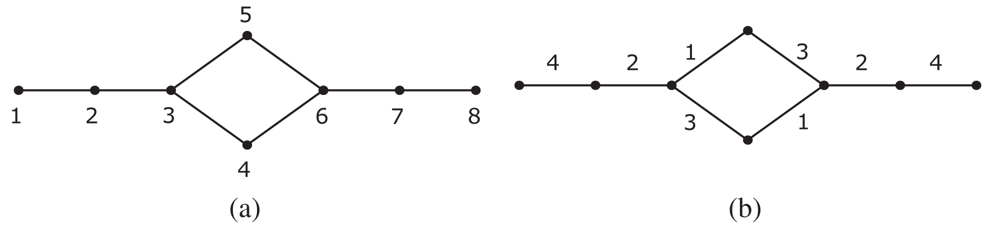


FIG. 2. (a) Vacua of $SO(8)/U(4)$. The numbers indicate the labels of vacua. (b) Elementary walls of $SO(8)/U(4)$. The numbers indicate the subscript i's of simple roots $\underline{\alpha}_i$ $(i = 1, \ldots, 4)$.

Double wall $\langle 2 \leftarrow 3 \leftarrow 5 \rangle$ can be compressed. The moduli matrix is

$$\begin{aligned} H_{0\langle 2\leftarrow 3\leftarrow 5\rangle} &= H_{0\langle 2\rangle} e^{E_2(r_1)} e^{E_1(r_2)} \\ &= \left(\begin{array}{cccc|cccc} 1 & e^{r_2} & & & 0 & & & \\ & 1 & e^{r_1} & & & 0 & & \\ & & 0 & & e^{r_1+r_2} & -e^{r_1} & 1 & \\ & & & 0 & & & & 1 \end{array}\right). \end{aligned} \tag{4.5}$$

Under the world volume symmetry transformation, $H_{0\langle 2\leftarrow 3\leftarrow 5\rangle}$ can be transformed to

$$\begin{aligned} H_{0\langle 2\leftarrow 3\leftarrow 5\rangle} &\to \begin{pmatrix} 1 & -e^{r_2} & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \left(\begin{array}{cccc|cccc} 1 & e^{r_2} & & & 0 & & & \\ & 1 & e^{r_1} & & & 0 & & \\ & & 0 & & e^{r_1+r_2} & -e^{r_1} & 1 & \\ & & & 0 & & & & 1 \end{array}\right) \\ &= \left(\begin{array}{cccc|cccc} 1 & & -e^{r_1+r_2} & & 0 & & & \\ & 1 & e^{r_1} & & & 0 & & \\ & & 0 & & e^{r_1+r_2} & -e^{r_1} & 1 & \\ & & & 0 & & & & 1 \end{array}\right). \end{aligned} \tag{4.6}$$

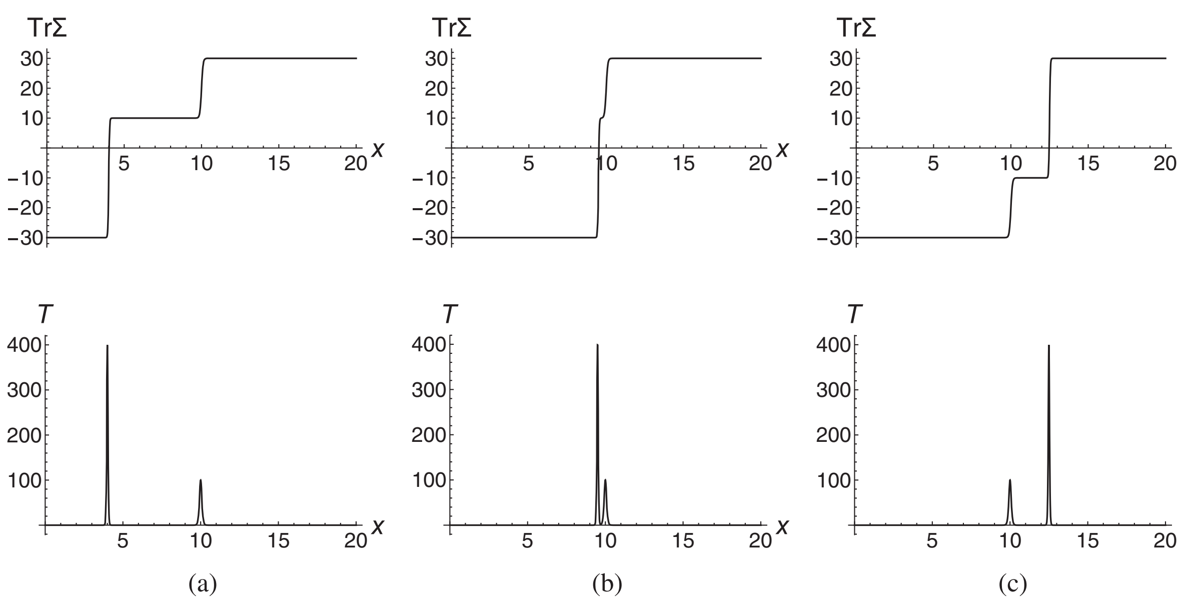
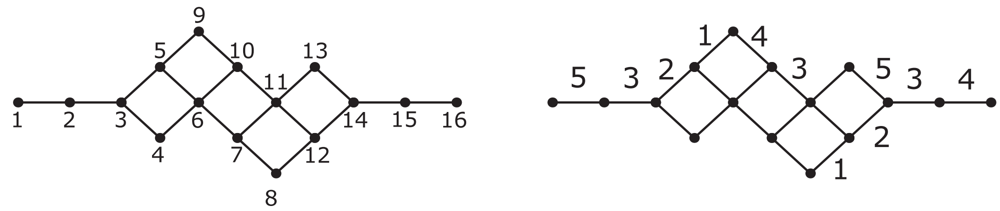


FIG. 3. Double wall $\langle 3 \leftarrow 5 \leftarrow 6 \rangle$ in $SO(8)/U(4)$, which consists of two penetrable walls. $m_1 = 70$, $m_2 = 50$, $m_3 = 30$, $m_4 = 20$. (a) $r_1 = 80$, $r_2 = 100$; (b) $r_1 = 200$, $r_2 = 100$; (c) $r_1 = 250$, $r_2 = 100$.



(a) (b)

FIG. 4. (a) Vacua of $SO(10)/U(5)$. The numbers indicate the labels of vacua. (b) Elementary walls of $SO(10)/U(5)$. The numbers indicate the subscript i of simple roots $\underline{\alpha}_i$ $(i = 1, \ldots, 5)$. Since facing sides are the same, only one side of each parallelogram is labeled.

In the limit where $r_1 + r_2 = r$ (finite) and $r_1 \to -\infty$, double wall $H_{0\langle 2\leftarrow 3\leftarrow 5\rangle}$ becomes a compressed wall of level 1, $H_{0\langle 2\leftarrow 5\rangle} = H_{0\langle 2\rangle} e^{[E_2,E_1](r)}$.

In $SO(10)/U(5)$, there are $16(=2^4)$ vacua. The five simple roots of $SO(10)$ are

$$\begin{aligned}
\underline{\alpha}_1 &:= (1,-1,0,0,0),\\
\underline{\alpha}_2 &:= (0,1,-1,0,0),\\
\underline{\alpha}_3 &:= (0,0,1,-1,0),\\
\underline{\alpha}_4 &:= (0,0,0,1,-1),\\
\underline{\alpha}_5 &:= (0,0,0,1,1).
\end{aligned} \tag{4.7}$$

The roots of elementary walls are

$$\begin{aligned}
&g_{\langle 4\leftarrow 5\rangle} = g_{\langle 7\leftarrow 8\rangle} = g_{\langle 9\leftarrow 10\rangle} = g_{\langle 12\leftarrow 13\rangle} = \underline{\alpha}_1,\\
&g_{\langle 3\leftarrow 4\rangle} = g_{\langle 6\leftarrow 7\rangle} = g_{\langle 10\leftarrow 11\rangle} = g_{\langle 13\leftarrow 14\rangle} = \underline{\alpha}_2,\\
&g_{\langle 2\leftarrow 3\rangle} = g_{\langle 7\leftarrow 9\rangle} = g_{\langle 8\leftarrow 10\rangle} = g_{\langle 14\leftarrow 15\rangle} = \underline{\alpha}_3,\\
&g_{\langle 3\leftarrow 6\rangle} = g_{\langle 4\leftarrow 7\rangle} = g_{\langle 5\leftarrow 8\rangle} = g_{\langle 15\leftarrow 16\rangle} = \underline{\alpha}_4,\\
&g_{\langle 1\leftarrow 2\rangle} = g_{\langle 9\leftarrow 12\rangle} = g_{\langle 10\leftarrow 13\rangle} = g_{\langle 11\leftarrow 14\rangle} = \underline{\alpha}_5.
\end{aligned} \tag{4.8}$$

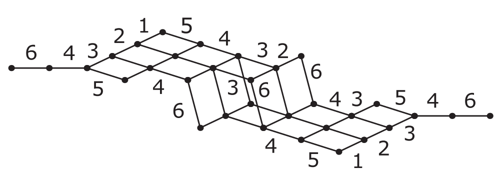


FIG. 5. Elementary walls of $SO(12)/U(6)$. The numbers indicate the subscript i of simple roots $\underline{\alpha}_i$ $(i = 1, \ldots, 6)$.

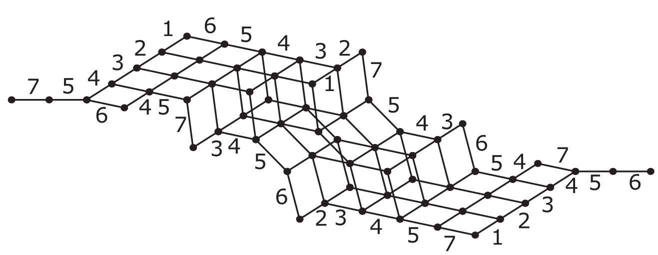


FIG. 6. Elementary walls of $SO(14)/U(7)$. The numbers indicate the subscript i of simple roots $\underline{\alpha}_i$ $(i = 1, \ldots, 7)$.

The roots of elementary walls are depicted in Fig. 4. As before, a pair of facing sides of each parallelogram is the same, and a pair of adjacent sides of each parallelogram is orthogonal.

The roots of elementary walls of $SO(12)/U(6)$ and $SO(14)/U(7)$ are depicted in Figs. 5 and 6. We present the same diagrams labeled by the vacua in B to avoid cluttering up pages.

V. GENERALIZATION

The vacua are parametrized by $(\Sigma_1, \ldots, \Sigma_N) = (\pm m_1, \ldots, \pm m_N)$ as mentioned previously. The half of the vacua that differ by even numbers of minus signs belong to one nonlinear sigma model, and the other half of the vacua belong to the other nonlinear sigma model, which is related by parity. Therefore, there are 2^{N-1} vacua in the nonlinear sigma models on $SO(2N)/U(N)$.

From the diagrams for $N = 2, \ldots, 7$, we make the following observations:

(i) $N = 2$:

$$\begin{aligned}
\langle 1\rangle &\leftarrow \underline{2},\\
\underline{2} &\leftarrow \langle 2\rangle.
\end{aligned} \tag{5.1}$$

(ii) $N = 3$:

$$\underline{3} \leftarrow \langle 2\rangle \leftarrow \underline{1} \leftarrow \langle 3\rangle \leftarrow \underline{2}. \tag{5.2}$$

(iii) $N = 4$:

$$\begin{aligned}
&\underline{4} \leftarrow \underline{2} \leftarrow \langle 3\rangle \leftarrow \{\underline{1},\underline{3}\} \leftarrow \cdots \leftarrow \{\underline{1},\underline{3}\} \leftarrow \langle 6\rangle\\
&\quad \leftarrow \underline{2} \leftarrow \underline{4}.
\end{aligned} \tag{5.3}$$

(iv) $N = 5$:

$$\begin{aligned}
&\underline{5} \leftarrow \cdots \leftarrow \{\underline{2},\underline{4}\} \leftarrow \langle 6\rangle \leftarrow \{\underline{1},\underline{3}\} \leftarrow \cdots\\
&\cdots \leftarrow \{\underline{1},\underline{3}\} \leftarrow \langle 11\rangle \leftarrow \{\underline{2},\underline{5}\} \leftarrow \cdots \leftarrow \underline{4}.
\end{aligned} \tag{5.4}$$

(v) $N = 6$:

$$\begin{aligned}
&\underline{6} \leftarrow \cdots \leftarrow \{\underline{2},\underline{4}\} \leftarrow \langle 11\rangle \leftarrow \{\underline{1},\underline{3},\underline{6}\} \leftarrow \cdots\\
&\cdots \leftarrow \{\underline{1},\underline{3},\underline{6}\} \leftarrow \langle 22\rangle \leftarrow \{\underline{2},\underline{4}\} \leftarrow \cdots \leftarrow \underline{6}.
\end{aligned} \tag{5.5}$$

(vi) $N=7$:

$$\begin{aligned}&\underline{7} \leftarrow \cdots \leftarrow \{\underline{2},\underline{4},\underline{7}\} \leftarrow \langle 22\rangle \leftarrow \{\underline{1},\underline{3},\underline{5}\} \leftarrow \cdots\\ &\cdots \leftarrow \{\underline{1},\underline{3},\underline{5}\} \leftarrow \langle 43\rangle \leftarrow \{\underline{2},\underline{4},\underline{6}\} \leftarrow \cdots \leftarrow \underline{6}.\end{aligned} \tag{5.6}$$

The vacuum structures that are connected to the maximum number of simple roots are as follows:

(i) $N=4m-2$ $(m\geq 2)$:

$$\begin{aligned}&\underline{N}(=4m-2) \leftarrow \cdots\\ &\quad\cdots \leftarrow \{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-4}}_{(2m-2)}\} \leftarrow \langle A\rangle \leftarrow\\ &\leftarrow \{\underbrace{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-5}}_{(2m-2)},\underline{4m-2}}_{(2m-1)}\} \leftarrow \cdots\\ &\quad\cdots \leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-5},\underline{4m-2}}_{(2m-1)}\} \leftarrow \langle B\rangle \leftarrow\\ &\leftarrow \{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-4}}_{(2m-2)}\} \leftarrow \cdots\\ &\quad\cdots \leftarrow \underline{N}(=4m-2).\end{aligned} \tag{5.7}$$

(ii) $N=4m-1$ $(m\geq 2)$:

$$\begin{aligned}&\underline{N}(=4m-1) \leftarrow \cdots\\ &\quad\cdots \leftarrow \{\underbrace{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-4}}_{(2m-2)},\underline{4m-1}}_{(2m-1)}\} \leftarrow \langle A\rangle \leftarrow\\ &\leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-5},\underline{4m-3}}_{(2m-1)}\} \cdots\\ &\quad\cdots \leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-5},\underline{4m-3}}_{(2m-1)}\} \leftarrow \langle B\rangle \leftarrow\\ &\leftarrow \{\underbrace{\underline{2},\underline{4},\ldots,4m-2}_{(2m-1)}\} \leftarrow \cdots\\ &\leftarrow \underline{N}-1(=4m-2).\end{aligned} \tag{5.8}$$

(iii) $N=4m$ $(m\geq 2)$:

$$\begin{aligned}&\underline{N}(=4m) \leftarrow \cdots\\ &\quad\cdots \leftarrow \{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-2}}_{(2m-1)}\} \leftarrow \langle A\rangle \leftarrow\\ &\leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-1}}_{(2m)}\} \leftarrow \cdots\\ &\quad\cdots \leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-1}}_{(2m)}\} \leftarrow \langle B\rangle \leftarrow\\ &\leftarrow \{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-2}}_{(2m-1)}\} \leftarrow \cdots\\ &\quad\cdots \leftarrow \underline{N}(=4m).\end{aligned} \tag{5.9}$$

(iv) $N=4m+1$ $(m\geq 2)$:

$$\begin{aligned}&\underline{N}(=4m+1) \leftarrow \cdots\\ &\quad\cdots \leftarrow \{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m}}_{(2m)}\} \leftarrow \langle A\rangle \leftarrow\\ &\leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-1}}_{(2m)}\} \leftarrow \cdots\\ &\quad\cdots \leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-1}}_{(2m)}\} \leftarrow \langle B\rangle \leftarrow\\ &\leftarrow \{\underbrace{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-2}}_{(2m-1)},\underline{4m+1}}_{(2m)}\} \leftarrow \cdots\\ &\quad\cdots \leftarrow \underline{N}-1(=4m).\end{aligned} \tag{5.10}$$

$\langle A\rangle$ and $\langle B\rangle$ are the vacua that are connected to the maximum number of elementary walls. $\langle A\rangle$ denotes the vacuum near $\langle 1\rangle$, and $\langle B\rangle$ denotes the vacuum near $\langle 2^{N-1}\rangle$. Equations (5.7), (5.8), (5.9), and (5.10) are proved in C.

VI. WALLS OF NONLINEAR SIGMA MODEL ON $SO(12)/U(6)$

We have studied the vacuum structures that are connected to the maximum number of elementary walls for general N. Walls can be penetrable or compressed to a single wall. We discuss some physical consequences on $SO(12)/U(6)$, which is the simplest nontrivial case. As it was shown previously, the vacua that are connected to the maximum number of elementary walls are $\langle 11\rangle$ and $\langle 22\rangle$. The structure near $\langle 11\rangle$ is

$$\begin{matrix} \langle 7\rangle \leftarrow \underline{2} \\ \langle 10\rangle \leftarrow \underline{4} \end{matrix} \leftarrow \langle 11\rangle \leftarrow \begin{matrix} \underline{1} \leftarrow \langle 19\rangle \\ \underline{3} \leftarrow \langle 13\rangle \\ \underline{6} \leftarrow \langle 12\rangle. \end{matrix} \tag{6.1}$$

In Eq. (6.1), $\underline{\alpha}_4 \cdot \underline{\alpha}_1 = 0$. Therefore, the elementary wall that interpolates $\langle 10\rangle$ and $\langle 11\rangle$ and the elementary wall that interpolates $\langle 11\rangle$ and $\langle 19\rangle$ are penetrable. The moduli matrix of the double wall that connects $\langle 10\rangle$, $\langle 11\rangle$, and $\langle 19\rangle$ is

$$H_{0\langle 10\leftarrow 11\leftarrow 19\rangle} = H_{0\langle 10\rangle} e^{E_4(r_1)} e^{E_1(r_2)}$$
$$= \left(\begin{array}{cccccc|cccccc} 1 & e^{r_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -e^{r_2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & e^{r_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{r_1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{6.2}$$

The tension of the walls is plotted in Fig. 7. Elementary wall $\langle 10 \leftarrow 11\rangle$ and elementary wall $\langle 11 \leftarrow 19\rangle$ pass through each other.

In Eq. (6.1), $\underline{\alpha}_4 \cdot \underline{\alpha}_3 \neq 0$. Therefore, elementary wall $\langle 10 \leftarrow 11\rangle$ and elementary wall $\langle 11 \leftarrow 13\rangle$ can be compressed to a single wall. The moduli matrix of double wall $\langle 10 \leftarrow 11 \leftarrow 13\rangle$ is

$$H_{0\langle 10\leftarrow 11\leftarrow 13\rangle} = H_{0\langle 10\rangle} e^{E_4(r_1)} e^{E_3(r_2)}$$
$$= \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & e^{r_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & e^{r_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{r_1+r_2} & -e^{r_1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{6.3}$$

By using the world volume symmetry, the moduli matrix can be transformed to

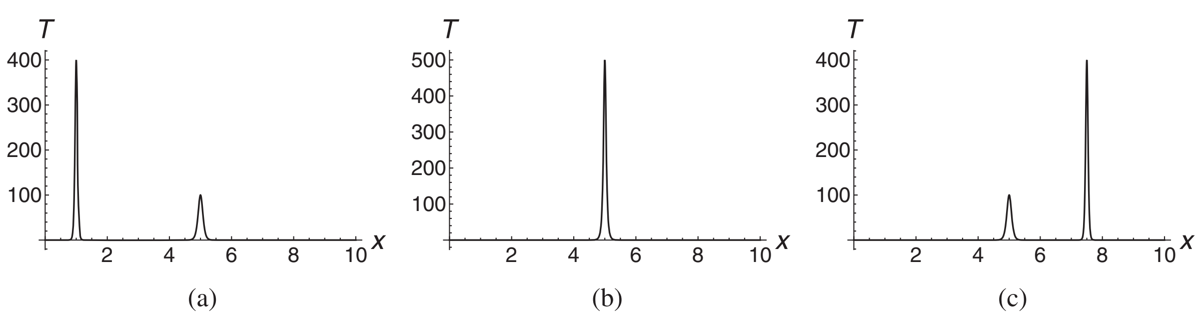
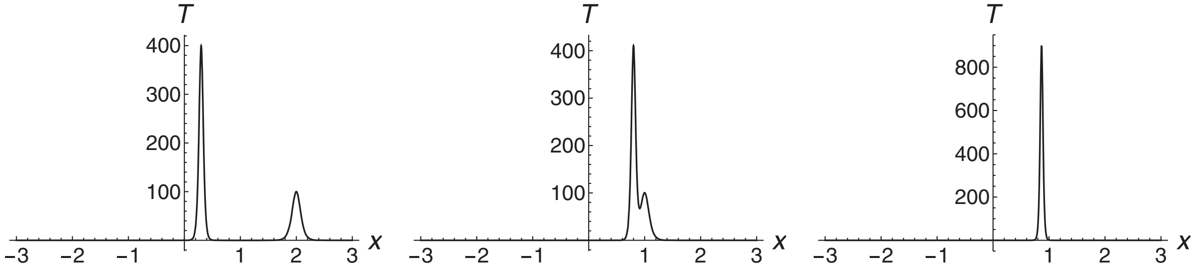


FIG. 7. Double wall $\langle 10 \leftarrow 11 \leftarrow 19\rangle$ in $SO(12)/U(6)$, which consists of two penetrable walls. $m_1 = 80$, $m_2 = 60$, $m_3 = 40$, $m_4 = 20$, $m_5 = 10$, $m_6 = 5$. (a) $r_1 = 50$, $r_2 = 20$; (b) $r_1 = 50$, $r_2 = 100$; (c) $r_1 = 50$, $r_2 = 150$.



(a) (b) (c)

FIG. 8. Double wall $\langle 10 \leftarrow 11 \leftarrow 13 \rangle$ in $SO(12)/U(6)$, which consists of two penetrable walls. $m_1 = 80$, $m_2 = 60$, $m_3 = 40$, $m_4 = 20$, $m_5 = 10$, $m_6 = 5$. (a) $r_1 = 20$, $r_2 = 6$; (b) $r_1 = 10$, $r_2 = 16$; (c) $r_1 = -10$, $r_2 = 36$.

$$H_{0\langle 10\leftarrow 11\leftarrow 13\rangle} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -e^{r_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & e^{r_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & e^{r_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{r_1+r_2} & -e^{r_1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$= \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -e^{r_1+r_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & e^{r_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{r_1+r_2} & -e^{r_1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{6.4}$$

In the limit where $r_1 + r_2 = r$ (finite) and $r_1 \rightarrow -\infty$, double wall $\langle 10 \leftarrow 11 \leftarrow 13 \rangle$ becomes a compressed wall of level 1. The tension of double wall $\langle 10 \leftarrow 11 \leftarrow 13 \rangle$ is plotted in Fig. 8. Two elementary walls are compressed to a single wall.

VII. DISCUSSION

We have discussed the vacua and the walls of mass-deformed Kähler nonlinear sigma models on $SO(2N)/U(N)$ with $N \geq 2$ by using the moduli matrix formalism and the simple roots of $SO(2N)$. We have observed that there are penetrable walls in the cases of $N > 3$, which are non-Abelian theories. We have discussed the vacuum structures that are connected to the maximum number of elementary walls and proved them by induction.

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APPENDIX A: VACUUM STRUCTURE OF NONLINEAR SIGMA MODELS ON $SO(2N)/U(N)$ WITH $N = 2, 3$

The vacua of nonlinear sigma models are obtained from the vacuum condition (2.10) in Sec. II.

The vacua of nonlinear sigma model on $SO(4)/U(2)$ are

$$\Phi_1 = \left(\begin{array}{cc|cc} 1 & & 0 & \\ & 1 & & 0 \end{array}\right), \qquad (\Sigma_1, \Sigma_2) = (m_1, m_2),$$
$$\Phi_2 = \left(\begin{array}{cc|cc} 0 & & 1 & \\ & 0 & & 1 \end{array}\right), \qquad (\Sigma_1, \Sigma_2) = (-m_1, -m_2),$$
$$\Phi_3 = \left(\begin{array}{cc|cc} 1 & & 0 & \\ & 0 & & 1 \end{array}\right), \qquad (\Sigma_1, \Sigma_2) = (m_1, -m_2),$$
$$\Phi_4 = \left(\begin{array}{cc|cc} 0 & & 1 & \\ & 1 & & 0 \end{array}\right), \qquad (\Sigma_1, \Sigma_2) = (-m_1, m_2). \tag{A1}$$

There are four solutions to the vacuum condition (2.10), but only the half of them are the vacua of a nonlinear sigma model on $SO(4)/U(2)$, and the other half are the vacua of the other nonlinear sigma model, which is related by a parity transformation. Let us define a rotation transformation R and a parity transformation P as follows:

$$R = \left(\begin{array}{cc|cc} 0 & & 1 & \\ & 0 & & 1 \\ \hline 1 & & 0 & \\ & 1 & & 0 \end{array}\right), \quad P = \left(\begin{array}{cc|cc} 1 & & 0 & \\ & 0 & & 1 \\ \hline 0 & & 1 & \\ & 1 & & 0 \end{array}\right). \tag{A2}$$

The vacua are related by

$$\Phi_2 = \Phi_1 \cdot R, \qquad \Phi_4 = \Phi_3 \cdot R,$$
$$\Phi_3 = \Phi_1 \cdot P, \qquad \Phi_4 = \Phi_2 \cdot P. \tag{A3}$$

The vacua Φ_1 and Φ_2 are on the same manifold. Φ_3 and Φ_4 are on the other manifold and are related by the parity transformation P. Therefore, we can focus only on Φ_1 and Φ_2 without loss of generality. The corresponding moduli matrices of the vacua, which are related to the vacuum solution by (2.18), are

$$H_{0\langle 1\rangle} = \left(\begin{array}{cc|cc} 1 & & 0 & \\ & 1 & & 0 \end{array}\right), \quad (\Sigma_1, \Sigma_2) = (m_1, m_2),$$
$$H_{0\langle 2\rangle} = \left(\begin{array}{cc|cc} 0 & & 1 & \\ & 0 & & 1 \end{array}\right), \quad (\Sigma_1, \Sigma_2) = (-m_1, -m_2). \tag{A4}$$

The vacua of the nonlinear sigma model on $SO(6)/U(3)$ are

$$\Phi_1 = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, m_2, m_3),$$
$$\Phi_2 = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, -m_2, -m_3),$$
$$\Phi_3 = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 0 & & & 1 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, m_2, -m_3),$$
$$\Phi_4 = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 1 & & & 0 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, -m_2, m_3),$$
$$\Phi_5 = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 0 & & & 1 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, m_2, -m_3),$$
$$\Phi_6 = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 1 & & & 0 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, -m_2, m_3),$$

$$\Phi_7 = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, m_2, m_3),$$

$$\Phi_8 = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, -m_2, -m_3). \tag{A5}$$

There are eight vacua, but only the half of them are the vacua of a nonlinear sigma model on $SO(6)/U(3)$. We can identify them by rotational transformations. We have three rotational transformations:

$$R_1 = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \\ \hline 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{array}\right), \quad R_2 = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 0 & & & \\ \hline 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 1 & & & 1 \end{array}\right), \quad R_3 = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 1 & & & 0 \\ \hline 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 0 & & & 1 \end{array}\right). \tag{A6}$$

The vacua are related by

$$\Phi_2 = \Phi_1 \cdot R_1, \qquad \Phi_3 = \Phi_1 \cdot R_2, \qquad \Phi_4 = \Phi_1 \cdot R_3. \tag{A7}$$

The solutions Φ_1, Φ_2, Φ_3, and Φ_4 are the vacua of a nonlinear sigma model on $SO(6)/U(3)$, and the others are the vacua of a nonlinear sigma model on the other $SO(6)/U(3)$ and are related by parity. The moduli matrices of the vacua are

$$H_{0\langle 1\rangle} = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, m_2, m_3),$$

$$H_{0\langle 2\rangle} = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (m_1, -m_2, -m_3)$$

$$H_{0\langle 3\rangle} = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 0 & & & 1 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, m_2, -m_3)$$

$$H_{0\langle 4\rangle} = \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & 1 & & & 0 \end{array}\right), \qquad (\Sigma_1, \Sigma_2, \Sigma_3) = (-m_1, -m_2, m_3). \tag{A8}$$

We use the same method to compute moduli matrices of vacua for any N. All the vacua are labeled by sets of $(\Sigma_1, \ldots, \Sigma_N) = (\pm m_1, \ldots, \pm m_N)$ of which the numbers of minus signs differ by even numbers.

APPENDIX B: VACUA

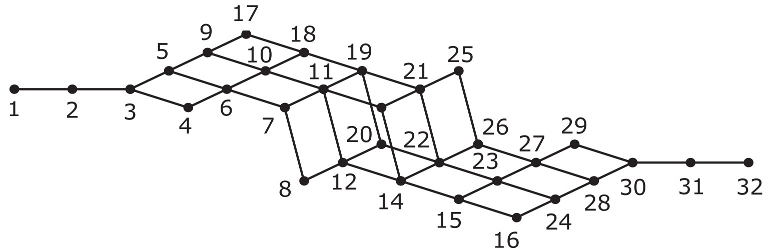


FIG. 9. Vacua of $SO(12)/U(6)$. The numbers indicate the labels of vacua.

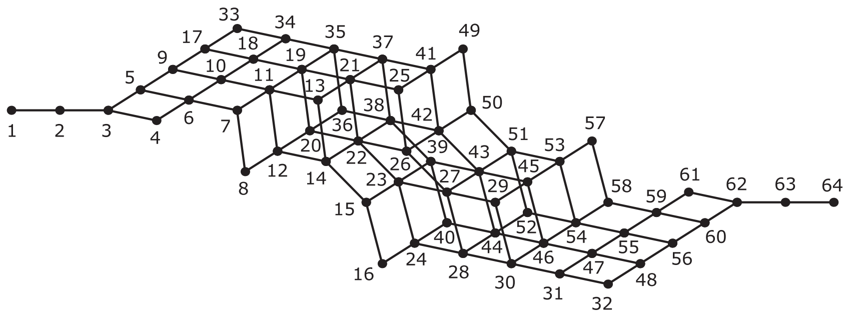


FIG. 10. Vacua of $SO(14)/U(7)$. The numbers indicate the labels of vacua.

APPENDIX C: VACCUUM STRUCTURES

We prove Eqs. (5.7), (5.8), (5.9), and (5.10). The vacua that are connected to the maximum number of elementary walls can be found by decomposing the diagrams into two-dimensional diagrams. The only rule is that simple roots that already have appeared in the previous diagrams should not be repeated.

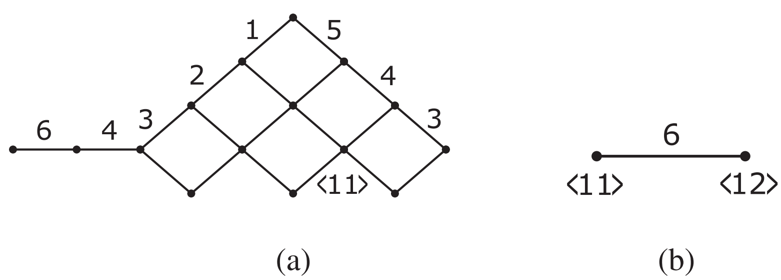


FIG. 11. $N = 6$ case. The vacuum structure near $\langle 1 \rangle$ decomposes into two diagrams. Diagram (b) is the same as Fig. 1(a).

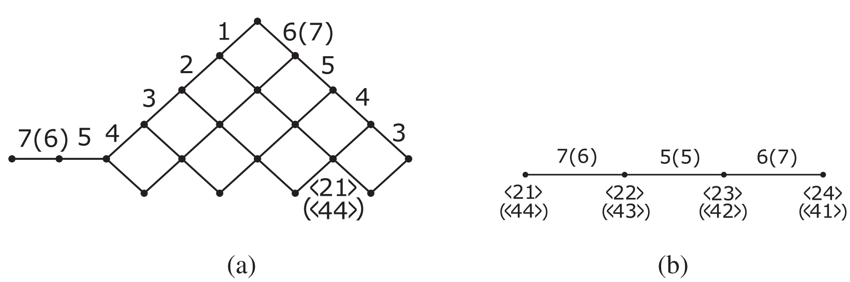
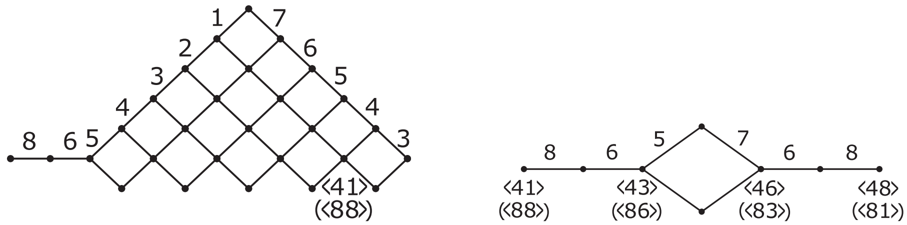
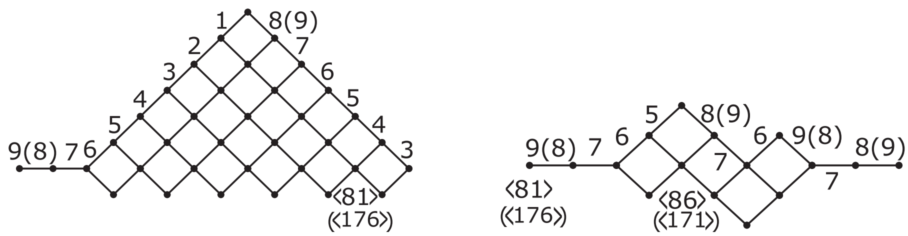


FIG. 12. $N = 7$ case. Diagrams near $\langle 1 \rangle(\langle 64 \rangle)$. Diagram (b) is the same as Fig. 1(b).



(a) (b)

FIG. 13. $N = 8$ case. Diagrams near $\langle 1\rangle(\langle\langle 128\rangle\rangle)$. Diagram (b) is the same as Fig. 2.

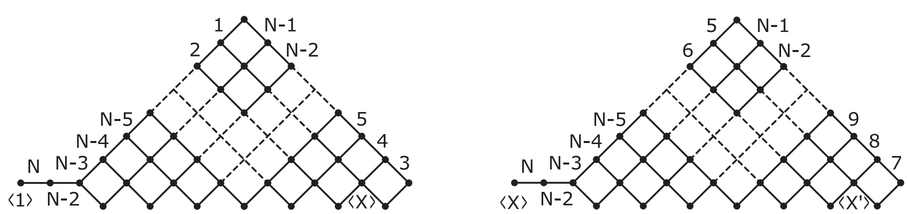


(a) (b)

FIG. 14. $N = 9$ case. Diagrams near $\langle 1\rangle(\langle\langle 256\rangle\rangle)$. Diagram (b) is the same as Fig. 4.

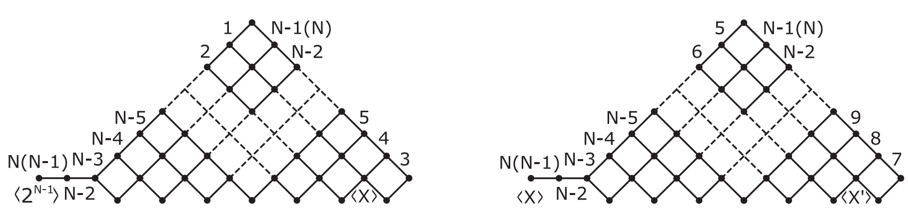
In Figs. 5 and 9, the vacuum that is connected to the maximum number of simple roots near $\langle 1\rangle$ is $\langle 11\rangle$. The vacuum structure near $\langle 1\rangle$ decomposes into two diagrams, as is shown in Fig. 11. From this, we can conclude that $\langle 11\rangle$ is connected to the maximum number of elementary walls. Figure 11(b) is the same as Fig. 1(a). In Figs. 5 and 9, the vacuum, which is connected to the maximum number of simple roots near $\langle 32\rangle$, is $\langle 22\rangle$. This can also be seen by decomposing the vacuum structure near $\langle 32\rangle$. We get the same two-dimensional diagrams as in Fig. 11 by replacing $\langle 11\rangle$ and $\langle 12\rangle$ with $\langle 22\rangle$ and $\langle 21\rangle$. In this case, the vacua on the left-hand side are in the $x \to -\infty$ limit, and the vacua on the right-hand side are in the $x \to \infty$ limit. In the same manner, Figs. 6 and 10 decompose as is shown in Fig. 12. The vacua that are connected to the maximum number of simple roots are $\langle 22\rangle$ and $\langle 43\rangle$. Figure 12(b) is the same as Fig. 1(b). The $N = 8$ and $N = 9$ cases are depicted in Figs. 13 and 14. The vacuum structures repeat the four diagrams in Figs. 1, 2, and 4.

The vacuum structure near $\langle 1\rangle$ decomposes into the diagrams in Fig. 15, while the vacuum structure near $\langle 2^{N-1}\rangle$ decomposes into the diagrams in Fig. 16. In both figures,



(a) (b)

FIG. 15. First two diagrams of the vacuum structure near $\langle 1\rangle$ for N.



(a) (b)

FIG. 16. First two diagrams of the vacuum structure near $\langle 2^{N-1}\rangle$ for N.

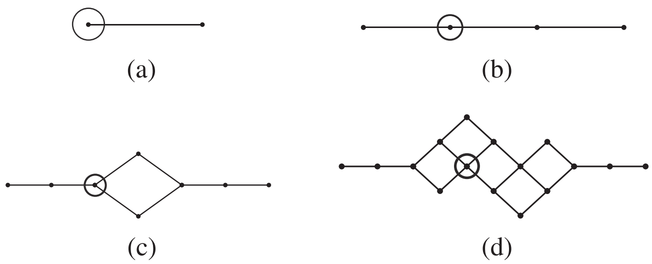


FIG. 17. Four types of vacuum structures. The circle indicates the vacuum that is connected to the maximum number of simple roots.

FIG. 18. Common part of the vacuum structure near $\langle A\rangle$.

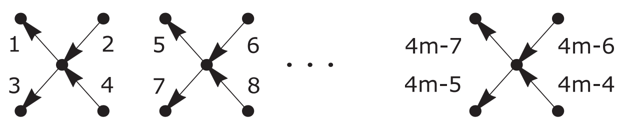


FIG. 19. Common part of the vacuum structure near $\langle B\rangle$.

only the first two diagrams are shown. By repeating them all, the cases fall into four categories. The vacuum that is connected to the maximum number of simple roots is circled in each diagram in Fig. 17.

There are two vacua that are connected to the maximum number of elementary walls. $\langle A\rangle$ denotes the vacuum near $\langle 1\rangle$, and $\langle B\rangle$ denotes the vacuum near $\langle 2^{N-1}\rangle$. The common parts of each vacuum structure are shown in Figs. 18 and 19. The rest of the vacuum structure can be derived from Fig. 17. The vacuum structures near $\langle A\rangle$ and $\langle B\rangle$ are illustrated in Figs. 20 and 21, respectively.

Figures 18 and 20 lead to the following:

(i) $N = 4m-2$, $(m\geq 2)$:

$$\{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-4}}_{2m-2}\}\leftarrow\langle A\rangle \leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-5},\underline{4m-2}}_{2m-2}\}. \tag{C1}$$

(ii) $N = 4m-1$, $(m\geq 2)$:

$$\{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-4}}_{2m-2},\underline{4m-1}\}\leftarrow\langle A\rangle \leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-5},\underline{4m-3}}_{2m-1}\}. \tag{C2}$$

(iii) $N = 4m$, $(m\geq 2)$:

$$\{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-2}}_{2m-1}\}\leftarrow\langle A\rangle \leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-1}}_{2m}\}. \tag{C3}$$

(iv) $N = 4m+1$, $(m\geq 2)$:

$$\{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m}}_{2m}\}\leftarrow\langle A\rangle \leftarrow \{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-1}}_{2m}\}. \tag{C4}$$

Each case with $m = 2$ is shown in Figs. 11, 12, 13, and 14. Let us assume that Eqs. (C1), (C2), (C3), and (C4) are true. Then, these are true for $m' = m+1$ since it corresponds to adding one more diagram in Fig. 18. Therefore, Eqs. (C1), (C2), (C3), and (C4) are true.

Figures 19 and 21 lead to the following:

(i) $N = 4m-2$, $(m\geq 2)$:

$$\{\underbrace{\underline{1},\underline{3},\ldots,\underline{4m-5}}_{2m-2},\underline{4m-2}\}\leftarrow\langle B\rangle \leftarrow \{\underbrace{\underline{2},\underline{4},\ldots,\underline{4m-4}}_{2m-2}\}. \tag{C5}$$

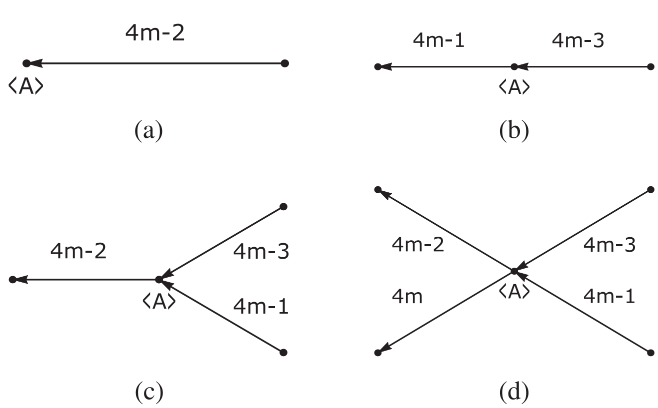


FIG. 20. Remaining part of the vacuum structure near $\langle A\rangle$.

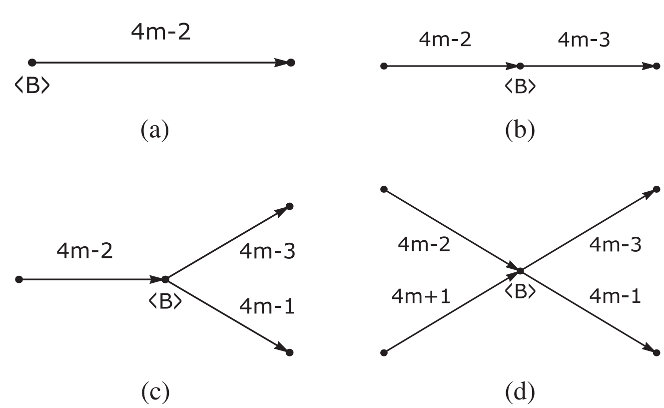


FIG. 21. Remaining part of the vacuum structure near $\langle B\rangle$.

(ii) $N = 4m - 1$, $(m \geq 2)$:

$$\{\underbrace{\underline{1}, \underline{3}, \ldots, \underline{4m-5}, \underline{4m-3}}_{2m-1}\}. \leftarrow \langle B \rangle \leftarrow \{\underbrace{\underline{2}, \underline{4}, \ldots, \underline{4m-2}}_{2m-1}\}. \tag{C6}$$

(iii) $N = 4m$, $(m \geq 2)$:

$$\{\underbrace{\underline{1}, \underline{3}, \ldots, \underline{4m-1}}_{2m}\} \leftarrow \langle B \rangle \leftarrow \{\underbrace{\underline{2}, \underline{4}, \ldots, \underline{4m-2}}_{2m-1}\}: \tag{C7}$$

(iv) $N = 4m + 1$, $(m \geq 2)$:

$$\{\underbrace{\underline{1}, \underline{3}, \ldots, \underline{4m-1}}_{2m}\} \leftarrow \langle B \rangle \leftarrow \{\underbrace{\underline{2}, \underline{4}, \ldots, \underline{4m-2}}_{2m-1}, \underline{4m+1}\}. \tag{C8}$$

Each case with $m = 2$ is shown in Figs. 11, 12, 13, and 14. Let us assume that Eqs. (C5), (C6), (C7), and (C8) are true. Then, these are true for $m' = m + 1$ since it corresponds to adding one more diagram in Fig. 19. Therefore, Eqs. (C5), (C6), (C7), and (C8) are true.

The vacuum structure is

$$\underline{N} \leftarrow \cdots \leftarrow \langle A \rangle \leftarrow \langle B \rangle \leftarrow \underline{N} \tag{C9}$$

for even N and

$$\underline{N} \leftarrow \cdots \leftarrow \langle A \rangle \leftarrow \langle B \rangle \leftarrow \underline{N} - 1 \tag{C10}$$

for odd N.

Therefore, Eqs. (5.7), (5.8), (5.9), and (5.10) are proved.

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