Ground state for a massive scalar field in the BTZ spacetime with Robin boundary conditions

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We consider a real, massive scalar field in Bañados-Teitelboim-Zanelli spacetime, a $2 + 1$-dimensional black hole solution of Einstein's field equations with a negative cosmological constant. First, we analyze the space of classical solutions in a mode decomposition, and we characterize the collection of all admissible boundary conditions of Robin type which can be imposed at infinity. Second, we investigate whether, for a given boundary condition, there exists a ground state by constructing explicitly its two-point function. We demonstrate that for a subclass of the boundary conditions it is possible to construct a ground state that locally satisfies the Hadamard property. In all other cases, we show that bound state mode solutions exist and, therefore, such construction is not possible.

I. INTRODUCTION

Quantum field theory on curved backgrounds is a well-established branch of theoretical and mathematical physics which allows one to study matter systems in the presence of a nonvanishing gravitational field (for a recent review, see Ref. [1]). In this framework, it is always assumed both that no proper quantum gravitational effect has to be accounted for and that the backreaction in Einstein's equations is negligible.

Although this entails that the geometry of the spacetime is fixed, it is not at all necessary to consider metrics which are small perturbations over a flat background. Actually, quantum field theory in the presence of a strong gravitational field, e.g., a black hole, is of great interest since one can unveil some novel phenomena, the most famous example being Hawking radiation [2], which have no counterpart on Minkowski spacetime.

For this reason, a lot of attention has always been given to the investigation and to the formulation of quantized free field theories on black hole spacetimes. Especially under the additional assumption of spherical symmetry, many results have been obtained, leading to an almost complete understanding of these matter systems both at the structural and at the physical levels [3–9].

Much more complicated is the scenario when the underlying black hole solution of Einstein's equations is rotating, hence only axisymmetric, the most notable example being Kerr spacetime. In this case, even the analysis of free quantum theories is more elusive, and simple questions like the construction of a ground state in the region outside the event horizon are difficult to answer [10]. As a byproduct, the computation of renormalized physical observables has been a daunting task [11–14], and only very recently a promising renormalization scheme has been applied with success [15,16]. The main reason for this quandary can be ascribed to a peculiarity of such rotating geometries, namely the absence of a complete, everywhere timelike Killing field. If it existed, the latter would allow for the identification of a canonical and natural choice for the notion of positive frequency, which can be used in turn to select a distinguished two-point function for the underlying theory. This defines uniquely and unambiguously a full-fledged quantum state, dubbed the "ground state," with notable physical and structural properties [17]. Among them, we recall in particular that all quantum observables have finite fluctuations and that, starting from such a state, it is possible to construct the algebra of all Wick polynomials, including relevant objects such as the stress-energy tensor [18].

Therefore, on account of the lack of a global timelike Killing field, our understanding of quantum field theories in the presence of rotating black holes is not as advanced as one could hope. The main goal of this paper is to discuss a concrete scenario where most of the problems mentioned above can be circumvented. We refer to the so-called Bañados-Teitelboim-Zanelli (BTZ) black hole [19,20], a solution of the $(2 + 1)$-dimensional Einstein equations with a negative cosmological constant. This geometry possesses some rather peculiar features. On the one hand, it can be obtained directly from the anti–de Sitter (AdS) metric with an appropriate identification of boundaries—see Ref. [20]; hence, locally it is a region of constant curvature. On the

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other hand, the BTZ solution is both stationary and axisymmetric and possesses an inner and an outer horizon, as well as two canonical Killing fields, say ∂_t and ∂_ϕ, associated to these symmetries. In addition, contrary to what happens in the Kerr spacetime, although none of these vector fields is everywhere timelike, there exists a suitable linear combination which enjoys such a property everywhere in the region exterior of the black hole.

This feature prompts the possibility of analyzing free field theories in the BTZ background constructing an associated ground state. In this paper, we will address this issue thoroughly for the case of a real, massive Klein-Gordon field obeying Robin boundary conditions at conformal infinity. In this respect, we generalize and complement the results of Ref. [21], which considers a massless, conformally coupled scalar field with either Dirichlet or Neumann boundary conditions at infinity.

Our analysis starts from the construction, via a mode expansion, of the space of solutions for the underlying equation of motion. This must be approached delicately, since the underlying spacetime shares both locally and asymptotically the geometry of anti-de Sitter spacetime. In particular, this entails that a BTZ black hole spacetime is not globally hyperbolic, which is tantamount to saying that an acausal spacelike surface can be at most partially Cauchy (at least some complete timelike curves will never intersect it) and that the solutions of the equations of motion for a free field theory cannot be obtained only by imposing suitable initial data on such a partial Cauchy surface. As a matter of fact, the existence of a timelike conformal boundary at infinity requires additional boundary conditions thereon. Here, we follow the same path taken in the analysis of a pure AdS spacetime, recently investigated in Refs. [22,23]. This was based on a careful use of the Sturm-Liouville theory for ordinary differential equations (ODEs) [24,25] which complements the earlier analyses in Refs. [26–28]. By using and extending similar methods, we will show that also in the BTZ spacetime there exists a one parameter family of admissible boundary conditions, of Robin type, depending on the value of the effective squared mass in the Klein-Gordon equation. From a physical point of view, this guarantees vanishing energy flux through conformal infinity [29]. These conditions are different from the "transparent boundary conditions" used in Ref. [30] to compute the renormalized stress-energy tensor for a massless, conformally coupled scalar field in the BTZ black hole.

A thorough discussion of this feature is of paramount importance for the core goal of this work: the construction of the two-point function of a ground state. In fact, each different boundary condition identifies, for all practical purposes, a separate dynamical theory. To each of these, one can compute a distinguished two-point function associated to the ground state defined with respect to the timelike Killing field which exists in the region outside the

outer horizon. In the main body of the paper, we not only construct such two-point functions explicitly, but we also investigate their physical properties. Most notably, we show that, for a large class of the Robin boundary conditions, including the Dirichlet one, only positive frequencies contribute to the mode expansion of the two-point function. Hence, for each of these admissible boundary conditions, we identify a full-fledged ground state, which moreover is locally of Hadamard form on account of some structural results of quantum field theory on curved backgrounds proven in Ref. [17]. By saying "locally," we distinguish from the *global* feature observed in Refs. [22,23] for a free quantum field theory in the Poincaré patch of anti-de Sitter spacetime: on account of the presence of the boundary and independently of the chosen boundary condition, the two-point function is singular not only at those pairs of points connected by a null geodesic but also at those which can be reached after such a geodesic is reflected at the conformal boundary. The presence of these additional singularities cannot be inferred from the standard structural properties proven in Ref. [17], and it requires a more involved mathematical analysis, which is outside the scope of the present work. It is worth stressing once more that the two-point functions, that we shall construct, are all expressed in terms of a mode expansion, hence involving both a sum along a discrete Fourier parameter and an integral along a continuous one, representing the energy frequencies. One might wonder whether, at least for the simplest boundary condition, the Dirichlet one, it is possible to obtain a closed form expression for the two-point function. To the best of our knowledge, a direct evaluation, especially of the integral along the frequencies, does not appear to be possible. Yet, in this special case, one can resort to a different approach to the construction of the two-point function and of the propagators with Dirichlet boundary conditions. Such a procedure relies on BTZ being realizable from the three-dimensional AdS spacetime via a suitable identification. In Refs. [31–34], this property was used to construct the propagators with Dirichlet boundary conditions starting from the AdS counterpart and applying the method of images, ultimately obtaining an expression which involves only a sum along a remaining discrete Fourier parameter. Uniqueness of the propagators and of the ground state for the massive Klein-Gordon theory with Dirichlet boundary conditions entails that this and our method yield the same result. Yet they highlight different structural properties. On the one hand, the method of images allows for better and more direct control of auxiliary structures, such as the bulk-to-boundary propagators which are of relevance in the AdS/CFT correspondence; see, for example, Ref. [35]. On the other hand, the mode expansion allows one to individuate directly all admissible boundary conditions and to construct two-point functions which are associated to a ground state.

In addition, by using this method, we confirm the existence of a rather peculiar feature which was already observed in the analysis of a real, massive scalar field in the Poincaré patch of anti-de Sitter spacetime [22]. There exists a class of Robin boundary conditions for which the mode expansion of the two-point function necessarily includes the contribution of bound state mode solutions. For these boundary conditions, one cannot claim that the constructed two-point function is that of a ground state and, more importantly, that it is of Hadamard form. As a last remark, we observe that our results are consistent with those obtained in Ref. [36] by looking at the class of admissible self-adjoint extensions of the Hamiltonian operator, associated to a massive, real scalar field in the static region of a nonrotating BTZ spacetime.

The paper will be organized as follows. In Sec. II, we review the geometry of a BTZ black hole, emphasizing in particular the presence of an everywhere timelike Killing field in the exterior region of the black hole. In Sec. III, we analyze the massive Klein-Gordon equation with an arbitrary coupling to scalar curvature on this background. Via a Fourier expansion, the field equation is reduced to an ODE in the radial direction, which can be solved explicitly. The solutions are classified in terms of their square integrability near the horizon and the conformal infinity, which gives us the range of the effective squared mass of the scalar field for which Robin boundary conditions have to be imposed at conformal infinity. Finally, in Sec. IV, we obtain our main result, namely the explicit construction for the two-point function of the ground state for a large class of Robin boundary conditions. Those not in this set are shown not to possess a ground state, given the presence of bound state mode solutions. In Sec. V, we draw our conclusions. In Appendix A, we discuss how to handle a key technical problem in our construction of the two-point function: contrary to what happens when dealing with a scalar field in a static spacetime, the ODE obtained out of the Fourier analysis cannot be interpreted as a simple eigenvalue problem with ω^2 as the spectral parameter, where ω is the frequency. In fact, the ensuing equation, having also a linear dependence in ω, can be read as a so-called quadratic operator pencil. In Appendix B, we present all the steps of the calculation of the two-point function for the ground state, the results of which are presented in Sec. IV. We leave some of the most mathematical details for Appendices C, D, and E.

Throughout the paper, we employ natural units in which $c = G_N = \hbar = 1$ and use the signature $(- + +)$.

II. BTZ BLACK HOLE AND 2+1 GEOMETRY

The BTZ black hole is a stationary, axisymmetric, $(2 + 1)$-dimensional solution of the vacuum Einstein field equations with a negative cosmological constant $\Lambda = -1/\ell^2$ [19,20]. It is diffeomorphic as a manifold to
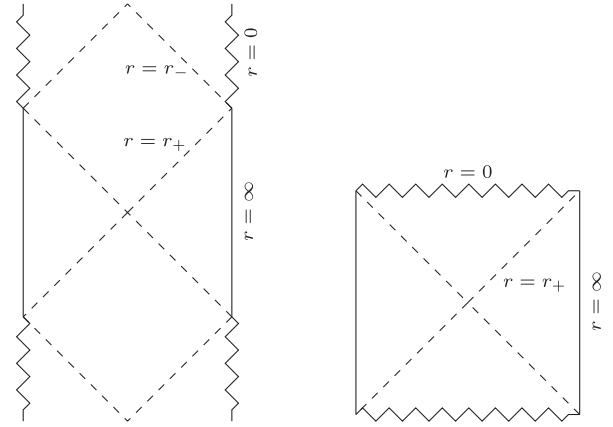

FIG. 1.   Penrose diagrams of the BTZ black hole for the rotating $0 < r_- < r_+$ (left) and the static $0 = r_- < r_+$ (right) cases.

$M \equiv \mathbb{R} \times I \times \mathbb{S}^1$, where I is an open interval of the real line. Its metric g can be realized in several, different, albeit equivalent ways, e.g., by a suitable identification of points in the Poincaré patch of the three-dimensional AdS spacetime [20]. The ensuing line element reads

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2, \quad (1)$$

where $t \in \mathbb{R}$, $\phi \in (0, 2\pi)$, $r \in (r_+, \infty)$, while

$$N^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \qquad N^\phi = -\frac{J}{2r^2}, \quad (2)$$

M being interpreted as the mass of the black hole and J being interpreted as its angular momentum. The value of r_+ can be inferred, observing that, in the range $M > 0$, $|J| \leq M\ell$, N vanishes at

$$r_\pm^2 = \frac{\ell^2}{2} \left(M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}} \right). \quad (3)$$

These *loci* are coordinate singularities, and, thus, as customary in rotating black hole spacetimes, the BTZ solution possesses an inner ($r = r_-$) and an outer horizon ($r = r_+$). The Penrose diagrams of this spacetime are shown in Fig. 1.

In addition, the event horizon turns out to be a Killing horizon of which the generator reads

$$\chi \doteq \partial_t + \Omega_{\mathcal{H}} \partial_\phi, \quad (4)$$

where $\Omega_{\mathcal{H}} \doteq N^\phi(r_+) = \frac{r_-}{\ell r_+}$ is the angular velocity of the horizon. It is of paramount relevance for this paper that χ is a well-defined, global, *timelike Killing vector field* across the whole exterior region ($r > r_+$) of BTZ spacetime. This is the sharpest difference in comparison to other models of rotating black hole spacetimes, e.g., the Kerr solution of Einstein's equation with vanishing cosmological constant.

In these cases, one is forced to cope with the existence of a speed of light surface at which the analog Killing field is null.

In view of the distinguished role of χ, it is natural to introduce the new coordinate system $(\tilde{t}, r, \tilde{\phi})$, which is related to (t, r, ϕ) in such a way that $\partial_{\tilde{t}} = \chi$. The simplest choice consists of defining $t = \tilde{t}$ and $\phi = \tilde{\phi} + \Omega_{\mathcal{H}}\tilde{t}$; the line element becomes

$$ds^2 = -N^2 d\tilde{t}^2 + N^{-2}dr^2 + (d\tilde{\phi} + (N^\phi + \Omega_{\mathcal{H}})d\tilde{t})^2. \quad (5)$$

Observe that, while the range of $\tilde{t}$ is still $\mathbb{R}$, that of $\tilde{\phi}$ is no longer simply the interval $(0, 2\pi)$, but rather $(-\Omega_{\mathcal{H}}\tilde{t}, 2\pi - \Omega_{\mathcal{H}}\tilde{t})$, with the end points still identified. Especially in the next section, we will be working mainly with (1), although when we address the construction of a ground state, Eq. (5) will turn out to be extremely useful.

III. MASSIVE SCALAR FIELD IN BTZ

A. Klein-Gordon equation

We consider a real, massive scalar field $\Phi: M \rightarrow \mathbb{R}$ satisfying the Klein-Gordon equation,

$$P\Phi = (\Box_g - m^2 - \xi R)\Phi = 0, \quad (6)$$

where $\Box_g$ and R are, respectively, the D'Alembert wave operator and the scalar curvature built out of (1), $\xi \in \mathbb{R}$ while m^2, is the mass parameter of the scalar field. Since $R = -6/\ell^2$, it is convenient to introduce the dimensionless parameter $\mu^2 \doteq m^2\ell^2 - 6\xi$. In addition, we assume that m^2 and ξ are such that the Breitenlohner-Freedman bound $\mu^2 \geq -1$ holds [37].

For our ultimate goal of quantizing (6) and constructing the associated ground state(s), the first step in this direction consists of a careful study of the solutions of the Klein-Gordon equation. Since the underlying spacetime is not globally hyperbolic, these cannot be constructed only by assigning initial data, for example, on a constant time-t hypersurface. One needs to supplement such information with the choice of an admissible boundary condition. *A priori*, it is not obvious how to proceed since one might wish to assign such a condition either at the horizon $r = r_+$, at infinity $r \rightarrow \infty$, or possibly at both ends. This quandary is easily solved by showing that (6) can be reduced to a second order ODE, the boundary conditions of which are much easier to analyze.

To this end, we work with the coordinates (t, r, ϕ), so that (6) reads

$$\left[-\frac{1}{N^2}\partial_t^2 + \frac{1}{r}\partial_r(rN^2)\partial_r + \left(\frac{1}{r^2} - \frac{N^\phi}{N}\right)^2\partial_\phi^2 \right.$$
$$\left. + 2\frac{N^\phi}{N^2}\partial_t\partial_\phi - \frac{\mu^2}{\ell^2} \right]\Phi = 0. \quad (7)$$

Since both ∂_t and ∂_ϕ are Killing fields of (1), we can take a Fourier expansion of Φ,

$$\Phi(t, r, \phi) = \frac{1}{2\pi}\sum_{k \in \mathbb{Z}}\int_{\mathbb{R}} d\omega e^{-i\omega t + ik\phi}\Psi_{\omega k}(r). \quad (8)$$

It is convenient to introduce a new coordinate, $z \in (0, 1)$,

$$z \doteq \frac{r^2 - r_+^2}{r^2 - r_-^2}, \quad (9)$$

so that, starting from (7), $\Psi_{\omega k}(z)$ obeys

$$L_\omega\Psi_{\omega k}(z) \doteq \frac{d}{dz}\left(z\frac{d\Psi_{\omega k}(z)}{dz}\right) + q(z)\Psi_{\omega k}(z) = 0, \quad (10)$$

with

$$q(z) = \frac{1}{4(1-z)}\left[\frac{\ell^2(\omega\ell r_+ - kr_-)^2}{(r_+^2 - r_-^2)^2 z}\right.$$
$$\left. - \frac{\ell^2(\omega\ell r_- - kr_+)^2}{(r_+^2 - r_-^2)^2} - \frac{\mu^2}{1-z}\right]. \quad (11)$$

This is indeed the sought second order ODE, written in Sturm-Liouville form, defined on the interval (0,1). We need to clarify which are the admissible boundary conditions that can be assigned at $z = 0$ (the horizon) or at $z = 1$ (infinity). For ordinary differential equations, this problem can be solved in full generality by using Sturm-Liouville theory; see, e.g., Refs. [24,25] or Ref. [22] for an application to the study of a real, massive scalar field in the Poincaré patch of anti-de Sitter spacetime of an arbitrary dimension. The nomenclature and the procedure that we will be using is the same employed in the last reference. For the sake of brevity, we will not recapitulate it fully here, and we refer the reader to the works cited above.

B. Solutions

The next step consists of identifying a basis of the vector space of solutions of (10). Using the Froebenius method, we infer that $\Psi_{\omega k}(z) = z^\alpha(1-z)^\beta F_{\omega k}(z)$, with

$$\alpha^2 = -\frac{\ell^4 r_+^2\tilde{\omega}^2}{4(r_+^2 - r_-^2)^2}, \qquad \beta^2 + \beta - \frac{\mu^2}{4} = 0, \quad (12)$$

where we define $\tilde{\omega} \doteq \omega - k\Omega_{\mathcal{H}}$ to be the square root of $\tilde{\omega}^2$ such that $\text{Im}[\omega] = \text{Im}[\tilde{\omega}] \geq 0$. By setting

$$\alpha = -i\frac{\ell^2 r_+\tilde{\omega}}{2(r_+^2 - r_-^2)}, \qquad \beta = \frac{1}{2}\left(1 + \sqrt{1 + \mu^2}\right) \quad (13)$$

and plugging $\Psi_{\omega k}(z)$ in (10), we obtain the Gaussian hypergeometric equation,

$$z(1-z)\partial_z^2 F_{\omega k} + [c - (a+b+1)z]\partial_z F_{\omega k} - ab F_{\omega k} = 0,$$
$$\tag{14}$$

where

$$\begin{cases} a = \frac{1}{2}\left(1 + \sqrt{1+\mu^2} - i\ell\frac{\tilde{\omega}\ell}{r_+ - r_-} + i\ell\frac{k}{r_+}\right), \\ b = \frac{1}{2}\left(1 + \sqrt{1+\mu^2} - i\ell\frac{\tilde{\omega}\ell}{r_+ + r_-} - i\ell\frac{k}{r_+}\right), \\ c = 1 - i\frac{\ell^2 r_+ \tilde{\omega}}{r_+^2 - r_-^2}. \end{cases} \tag{15}$$

For future convenience, we note that, under the substitution $\tilde{\omega} \mapsto \bar{\tilde{\omega}}$, these parameters behave as

$$\begin{aligned} a &\mapsto \overline{b-c+1}, & \alpha &\mapsto -\bar{\alpha}, \\ b &\mapsto \overline{a-c+1}, & \beta &\mapsto \beta, \\ c &\mapsto \overline{2-c}. \end{aligned} \tag{16}$$

Generic solutions of (14) can be written in closed form in terms of Gaussian hypergeometric functions that depend on the three parameters a, b, and c of the equation. When choosing two linearly independent solutions, the dependence on these parameters forces us to disentangle two cases, accordingly to the values of μ^2.

1. General case: $\mu^2 \neq (n-1)^2 - 1$, $n = 1,2,3,...$

In this case, we choose as a basis of solutions

$$\Psi_1(z) = z^\alpha (1-z)^\beta F(a, b, a+b-c+1; 1-z), \tag{17a}$$

$$\Psi_2(z) = z^\alpha (1-z)^{1-\beta}$$
$$\times F(c-a, c-b, c-a-b+1; 1-z), \tag{17b}$$

For future reference and inspired by the terminology used for Sturm-Liouville problems [24], we call Ψ_1 the "principal solution" at $z = 1$, that is, the unique solution (up to scalar multiples) such that $\lim_{z\to 1}\Psi_1(z)/\Psi(z) = 0$ for every solution Ψ that is not a scalar multiple of Ψ_1. Note that Ψ_2 is not defined when the third argument becomes a nonpositive integer [38]. From (15), we get $\sqrt{1+\mu^2} + 1 \notin \mathbb{N} \cup \{0\}$, which identifies exactly the special range of values for μ^2 which has been excluded. Observe in particular that this set includes the case $\mu^2 = -1$ which saturates the Breitenlohner-Freedman (BF) bound [37]. This is a very special case, which would require a lengthy analysis on its own. For this reason, we will not consider it further in this paper.

Note that the above definitions obey $\Psi_1 \mapsto \overline{\Psi_1}$ and $\Psi_2 \mapsto \overline{\Psi_2}$ under the substitution $\tilde{\omega} \mapsto \bar{\tilde{\omega}}$. This can be checked using the conjugation identities (16), the symmetry $F(a,b,c;z) = F(b,a,c;z)$ and the second equality from (15.10.13) of Ref. [38]:

$$F(a, b, a+b-c+1; 1-z)$$
$$= z^{1-c}F(a-c+1, b-c+1, a+b-c+1; 1-z).$$

2. Special cases: $\mu^2 = (n-1)^2 - 1$, $n = 2,3,...$

In this case, we choose the following basis of solutions for (10) (see Ref. [38], Sec. 15.10.8),

$$\Psi_1(z) = z^\alpha (1-z)^\beta F(a, b, n; 1-z), \tag{18a}$$

$$\Psi_2(z) = z^\alpha (1-z)^\beta$$
$$\times [F(a, b, n; 1-z)\log(1-z) + K_n(z)], \tag{18b}$$

where

$$K_n(z) = -\sum_{p=1}^{n-1}\frac{(n-1)!(p-1)}{(n-p-1)!(1-a)_p(1-b)_p}(z-1)^{-p}$$
$$+ \sum_{p=0}^{\infty}\frac{(a)_p(b)_p}{(n)_p p!}f_{p,n}(1-z)^p, \tag{19}$$

while $(a)_p = \Gamma(a+p)/\Gamma(a)$,

$$f_{p,n} = \psi(a+p) + \psi(b+p) - \psi(1+p) - \psi(n+p),$$

and ψ is the digamma function. Observe that, also in these special cases, Ψ_1 is the principal solution at $z = 1$.

Note that $\Psi_1 \mapsto \overline{\Psi_1}$ under the substitution $\tilde{\omega} \mapsto \bar{\tilde{\omega}}$, by the same argument as in the generic case. We do not need to check this property for Ψ_2 since, as will be clear in next section, Ψ_1 is the only solution which plays a role for the admissible boundary conditions.

C. End point classification

Having specified a basis of solutions of (10), we can continue in our quest to identify the admissible boundary conditions at the end points 0 and 1 for (6). These will depend on the square integrability of the solutions near the end points, in a way completely analogous to the case of the Poincaré patch of AdS analyzed in Ref. [22].

We start by identifying the falloff behavior of the solutions of (10) separately at the end points $z = 0$ and at $z = 1$. This allows us to classify the end points in the following way: we call the end point 0 (respectively, 1) the "limit circle" if, for some $\tilde{\omega} \in \mathbb{C}$, all solutions of (10) are in $L^2((0, z_0); \mathcal{J}(z)dz)$ for some $z_0 \in (0, 1)$ [respectively, $L^2((z_1, 1); \mathcal{J}(z)dz)$ for some $z_1 \in (0, 1)$]; otherwise, we call it the "limit point." The measure $\mathcal{J}(z)dz$, with

$$\mathcal{J}(z) = \frac{1}{1-z} + \frac{r_+^2}{z(r_+^2 - r_-^2)}, \tag{20}$$

satisfies the relation $d\nu(g) = \pi_I^*(\mathcal{J}(z)dz)d\varphi$, where $d\nu(g) = r/N^2 dr d\varphi$ and $\pi_I: M \to I$ is the projection along

TABLE I. Summary of the square integrability at $z = 0$ and at $z = 1$ of a basis of solutions for (10) depending on the parameters μ^2 and $\tilde{\omega}$ of the equation. The integration measure is $\mathcal{J}(z)\mathrm{d}z$ as per (20).

Range of μ^2	Range of $\tilde{\omega}$	L^2 at $z = 0$	L^2 at $z = 1$
$-1 < \mu^2 < 0$	$\mathrm{Im}[\tilde{\omega}] > 0$	Ψ_3	Ψ_1 and Ψ_2
	$\mathrm{Im}[\tilde{\omega}] = 0$	None	Ψ_1 and Ψ_2
	$\mathrm{Im}[\tilde{\omega}] < 0$	Ψ_4	Ψ_1 and Ψ_2
$\mu^2 \geq 0$	$\mathrm{Im}[\tilde{\omega}] > 0$	Ψ_3	Ψ_1
	$\mathrm{Im}[\tilde{\omega}] = 0$	None	Ψ_1
	$\mathrm{Im}[\tilde{\omega}] < 0$	Ψ_4	Ψ_1

the z direction. Notice that the operator $S_{\tilde{\omega}}\Psi(z) \doteq \frac{1}{\mathcal{J}(z)}L_{\tilde{\omega}}\Psi(z)$, with $L_{\tilde{\omega}}$ from (10), is Hermitian with respect to the measure $\mathcal{J}(z)\mathrm{d}z$.

A direct inspection of (17a) and (17b) as well as of (18a) and (18b), supplemented with the asymptotic behavior of the hypergeometric function at $z = 0$ and $z = 1$, yields the sought result for the basis elements of the space of solutions of (10). For convenience, we summarize the results described below in Table I.

1. End point z = 1

At $z = 1$, since the hypergeometric function is equal to 1 when evaluated at the origin, the behavior of (17a) and (17b) can be inferred from that of $(1 - z)^\beta$ and $(1 - z)^{1-\beta}$, respectively. By accounting also for the integration measure and using (13), it turns out that Ψ_1 lies in $L^2((z_1, 1); \mathcal{J}(z)\mathrm{d}z)$ for all values of $\mu^2 > -1$ and regardless of $z_1 \in (0, 1)$ and of $\tilde{\omega}$. On the contrary, Ψ_2 lies in $L^2((z_1, 1); \mathcal{J}(z)\mathrm{d}z)$ if $-1 < \mu^2 < 0$, again regardless of $z_1 \in (0, 1)$ and of $\tilde{\omega}$. Therefore, we say that $z = 1$ is the limit point if $\mu^2 \geq 0$, while it is the limit circle if $-1 < \mu^2 < 0$.

For the special cases $\mu^2 = (n - 1)^2 - 1, n = 2, 3, \ldots$, the first basis element Ψ_1 as in (18a) behaves exactly like (17a). At the same time, Ψ_2 as in (18b) never lies in $L^2((z_1, 1); \mathcal{J}(z)\mathrm{d}z)$ on account of the singularities of $K_n(z)$. Hence, $z = 1$ is always the limit point of $\mu^2 \geq 0$.

2. End point z = 0

In order to understand the behavior of the solutions of (10) at $z = 0$, we need to consider a different, more convenient basis,

$$\Psi_3(z) = z^\alpha (1 - z)^\beta F(a, b, c; z), \tag{21a}$$

$$\Psi_4(z) = z^{-\alpha}(1 - z)^\beta \\ \times F(a - c + 1, b - c + 1, 2 - c; z). \tag{21b}$$

where a, b, and c are defined in (15). Observe that Ψ_3 and Ψ_4 form a well-defined basis of solutions for all $\mu^2 > -1$,

except when $c = 1$ ($\alpha = 0$), the case of which is dealt with separately below.

Since the hypergeometric function is equal to 1 when evaluated at $z = 0$, the leading behavior of the two solutions at the origin is regulated by z^α in the first case and by $z^{-\alpha}$ in the second one. It is easy to verify that $\Psi_3 \in L^2((0, z_0), \mathcal{J}(z)\mathrm{d}z)$ for $\mathrm{Im}[\tilde{\omega}] > 0$, irrespectively of $z_0 \in (0, 1)$, while $\Psi_4 \in L^2((0, z_0), \mathcal{J}(z)\mathrm{d}z)$ if $\mathrm{Im}[\tilde{\omega}] < 0$. For $\mathrm{Im}[\tilde{\omega}] = 0$, none of the solutions is square integrable since a logarithmic singularity occurs. Therefore, we say that $z = 0$ is the limit point.

If $c = 1$, then $\omega = k\frac{r_-}{\ell r_+} = k\Omega_{\mathcal{H}}$ satisfies a synchronization condition with the black hole angular velocity, a case extensively studied in Ref. [29]. The solutions Ψ_3 and Ψ_4 no longer form a basis of solutions of (10), and hence we consider the following basis (Ref. [38], Sec. 15.10.8),

$$(1 - z)^\beta F(a, b, 1; z),$$

$$(1 - z)^\beta [F(a, b, 1; z)\log(z) + K_1(1 - z)], \tag{22}$$

where K_1 is as in (19). A close inspection of these two solutions unveils that the leading behavior at $z = 0$ is dominated by a constant in the first case and by $\log(z)$ in the second one. Hence, none of the solutions lies in $L^2((0, z_0), \mathcal{J}(z)\mathrm{d}z)$ regardless of $z_0 \in (0, 1)$. This is in agreement with the previous point.

Note that for $\tilde{\omega} \notin \mathbb{R}$, hence excluding the $c = 1$ case, the above definitions obey $\Psi_3 \mapsto \overline{\Psi_4}$ and $\Psi_4 \mapsto \overline{\Psi_3}$ under the substitution $\tilde{\omega} \mapsto \bar{\tilde{\omega}}$. This can be checked using the conjugation identities (16) and the symmetry $F(a, b, c; z) = F(b, a, c; z)$.

D. Robin boundary conditions

We can address finally the question of which are the admissible boundary conditions at the two end points $z = 0$ and $z = 1$. Tentatively, as in the simple example of a massive scalar field in the Poincaré patch of AdS studied in Ref. [22], we wish to impose Robin boundary conditions at $z = 1$ for a range of the mass parameter μ^2 of the scalar field. In fact, as pointed out in Ref. [29], imposing Robin boundary conditions is equivalent to requiring zero energy flux through the conformal boundary, a the natural physical condition.

To start with, we focus our attention on the ODE (10) at fixed values of $\tilde{\omega}$ and k. Since we deal with a singular Sturm-Liouville problem, it is not possible to assign Robin boundary conditions by specifying the value of a linear combination between a solution and its derivative. This statement is supported also by the observation that at $z \to 1$ both solutions $\Psi_2(z)$ as per (17b) and per (18b) are divergent.

This problem can be overcome by using Sturm-Liouville theory. While we do not wish to enter in a full explanation of the technical details, which are fully accounted for in

Refs. [22] and in [24], we outline the main idea of the procedure. The rationale consists of observing that, in a so-called regular Sturm-Liouville problem, a generic Robin boundary condition can be expressed equivalently either in terms of a linear combination between a solution and its derivative or in terms of a linear combination between the Wronskians of such a solution with respect to two linearly independent solutions, one of which is chosen to be the principal solution.

In the case at hand, this translates to the following: we say that a solution Ψ_ζ of (10) satisfies a *Robin boundary condition* at $z = 1$ parametrized by $\zeta \in [0, \pi)$ if

$$\lim_{z \to 1}\{\cos(\zeta)\mathcal{W}_z[\Psi_\zeta, \Psi_1] + \sin(\zeta)\mathcal{W}_z[\Psi_\zeta, \Psi_2]\} = 0, \qquad (23)$$

where Ψ_1 is the principal solution at $z = 1$ [(17a) or (18a)], Ψ_2 is a second linearly independent solution [for instance, Eq. (17b) or (18b)], and both are square integrable in a neighborhood of $z = 1$. Here, $\mathcal{W}_z[u, v] \doteq u(z)v'(z) - v(z)u'(z)$ is the Wronskian computed with respect to two differentiable functions u and v. As a consequence, the solution Ψ_ζ may be written as

$$\Psi_\zeta(z) = \cos(\zeta)\Psi_1(z) + \sin(\zeta)\Psi_2(z). \qquad (24)$$

We note that $\zeta = 0$ corresponds to the standard Dirichlet boundary condition since it guarantees that Ψ_ζ coincides with Ψ_1. At the same time, if $\zeta = \frac{\pi}{2}$, we say that Ψ_ζ satisfies a Neumann boundary condition, coinciding with Ψ_2. Yet, contrary to the Dirichlet boundary condition, this is not a universal assignment as it depends on the choice of Ψ_2.

The requirement of square integrability of both Ψ_1 and Ψ_2 near $z = 1$ implies that a Robin boundary condition can only be applied when $-1 < \mu^2 < 0$, as analyzed in the last section. For $\mu^2 \geq 0$, only the principal solution Ψ_1 is square integrable near $z = 1$, and, hence, no boundary condition is required. In practice, this is as if the Dirichlet boundary condition had been chosen.

A similar reasoning could be applied at $z = 0$, but, as we have shown in the preceding subsection, if we focus only on square integrable solutions, only one exists, provided that $\text{Im}[\tilde{\omega}] \neq 0$. Therefore, at $z = 0$, there is no need to impose any boundary condition.

IV. TWO-POINT FUNCTION

In this section, we address the main question of this paper, namely the construction of a class of two-point functions, investigating whether or not they define a ground state for a real, massive scalar field in the BTZ black hole spacetime. We will follow the same procedure employed in Ref. [22] in the Poincaré patch of an AdS spacetime of an arbitrary dimension. As we will point out in the subsequent discussion, the main structural difference lies in the

underlying metric being stationary, unless one considers the static case ($J = 0$) in (2).

Dropping for the moment the requirement of individuating a ground state, in general, by the *two-point function* (or Wightman function), we refer to a bidistribution $G^+ \in \mathcal{D}'(M \times M)$ such that

$$(P \otimes \mathbb{I})G^+ = (\mathbb{I} \otimes P)G^+ = 0, \qquad (25)$$

and

$$G^+(f, f) \geq 0, \quad \forall f \in C_0^\infty(M). \qquad (26)$$

In addition, the antisymmetric part of G^+ is constrained to coincide with the commutator distribution, in order to account for the canonical commutation relations (CCRs) of the underlying quantum field theory.

In order to make this last requirement explicit, let us consider the coordinate system (t, z, ϕ) introduced in (1) with r replaced by z as in (9). Working at the level of the integral kernel for G^+ and imposing the CCRs is tantamount to requiring that the antisymmetric part $iG(x, x')$, $x, x' \in M$, where

$$iG(x, x') = G^+(x, x') - G^+(x', x)$$

satisfies (25) together with the initial conditions

$$G(x, x')|_{t=t'} = 0, \qquad (27a)$$

$$-\partial_t G(x, x')|_{t=t'} = \partial_{t'} G(x, x')|_{t=t'} = \frac{\delta(z - z')\delta(\phi - \phi')}{\mathcal{J}(z)}, \qquad (27b)$$

with $\mathcal{J}(z)$ as in (20).

In order to construct explicitly the two-point function, we assume that G^+ admits a mode expansion,

$$G^+(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \frac{d\omega}{(2\pi)^2} e^{-i\omega(t-t'-i\epsilon)+ik(\phi-\phi')}$$
$$\times \hat{G}_{\omega k}(z, z'), \qquad (28)$$

where $x, x' \in M$, $i\epsilon$ has been added as a regularization, while the limit has to be taken in the weak sense. At this point, it is convenient to recall that, although both ∂_t and ∂_ϕ are global Killing vector fields, a more prominent physical role is played by the globally timelike Killing vector field χ defined in (4). More precisely, in the construction of a ground state, the notion of positive frequencies is played by $\tilde{\omega} = \omega - k\Omega_\mathcal{H}$ which is subordinated to χ. Hence, in order to make the role of $\tilde{\omega}$ manifest, following the discussion of Sec. II, we change from (ω, k) to $(\tilde{\omega}, k)$ and from the coordinates (t, r, ϕ) to $(\tilde{t}, r, \tilde{\phi})$, where $\tilde{\phi} = \phi - \Omega_\mathcal{H}t$ and $\tilde{t} = t$. Moreover, since only the positive $\tilde{\omega}$-frequencies

contribute to the two-point function of the ground state, we can write $\hat{G}_{\omega k}(z, z') \doteq \tilde{G}_{\tilde{\omega} k}(z, z')\Theta(\tilde{\omega})$, with $\tilde{G}_{\tilde{\omega} k}(z, z')$ defined for all $\tilde{\omega} \in \mathbb{R}$.

Taking into account these comments and recalling that the antisymmetric part ought to satisfy (27a), a natural requirement consists of looking for $\tilde{G}_{\tilde{\omega} k}(z, z')$ which is symmetric for exchange of z and z' and such that $\tilde{G}_{-\tilde{\omega}, -k}(z, z') = -\tilde{G}_{\tilde{\omega} k}(z, z')$. In this way, the commutator distribution reads

$$iG(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \frac{d\tilde{\omega}}{(2\pi)^2} e^{-i\tilde{\omega}(t-t'-i|\tilde{\omega}|\epsilon)+ik(\tilde{\phi}-\tilde{\phi}')}$$
$$\times \tilde{G}_{\tilde{\omega} k}(z, z'), \qquad (29)$$

where $\tilde{G}_{\omega k}(z, z')$ is a mode bidistribution chosen in such a way that, cf. Eq. (27b),

$$\int_{\mathbb{R}} \frac{d\tilde{\omega}}{2\pi} \tilde{\omega} \tilde{G}_{\tilde{\omega} k}(z, z') = \frac{\delta(z - z')}{\mathcal{J}(z)}. \qquad (30)$$

This identity, together with the Fourier series for the delta distribution along the angular coordinates, guarantees that finding $\tilde{G}_{\tilde{\omega} k}(z, z')$ is tantamount to constructing a full-fledged two-point function G^+, provided that positivity as in (26) is satisfied. In addition, Eq. (25) entails that the mode bidistribution is such that

$$(L_{\tilde{\omega}} \otimes \mathbb{1})\tilde{G}_{\tilde{\omega} k}(z, z') = (\mathbb{1} \otimes L_{\tilde{\omega}})\tilde{G}_{\tilde{\omega} k}(z, z') = 0,$$

where $L_{\tilde{\omega}}$ is defined in (10).

Our next goal will be to use this information to construct explicitly $\tilde{G}_{\tilde{\omega} k}(z, z')$ in terms of solutions of (10). Our strategy, as in Ref. [22], will be to obtain an integral representation for the delta distribution on the rhs of (30), from which we can read off $\tilde{G}_{\tilde{\omega} k}(z, z')$. However, and contrarily to the case of pure AdS analyzed in Ref. [22], we face a technical hurdle. When dealing with the static case $J = 0$, the ODE (10) can be treated as an eigenvalue problem with spectral parameter $\tilde{\omega}^2$, and it is possible to express the delta distribution as an expansion in terms of the eigenfunctions of $L_{\tilde{\omega}}$ (resolution of the identity). But this is not possible when dealing with the nonstatic case $J \neq 0$, in which case the ODE (10) has linear terms in $\tilde{\omega}$. Instead, we may treat $L_{\tilde{\omega}}$ as a *quadratic operator pencil*, i.e., a differential operator with quadratic dependence on the spectral parameter $\tilde{\omega}$. In Appendix A, it is described how to obtain the expansion of the delta distribution in terms of eigenfunctions of an operator of this type.

In the following, we present the results for the resolution of the identity and for the mode expansion of the two-point function for a fixed Robin boundary condition. We start from the simplest scenario, $\mu^2 \geq 0$, for which no boundary

condition needs to be imposed to the solutions of (10) at $z = 1$, and then consider the more interesting case $-1 < \mu^2 < 0$. The full details of the calculation can be consulted in Appendix B.

A. Case $\mu^2 \geq 0$

For $\mu^2 \geq 0$, both $z = 0$ and $z = 1$ in the Sturm-Liouville problem associated to (10) are of the limit point type. Using the results of Appendix B in the case $\zeta = 0$, it is possible to obtain an integral representation of $\delta(z - z')$ in terms of eigenfunctions of $L_{\tilde{\omega}}$,

$$\frac{\delta(z - z')}{\mathcal{J}(z)} = \int_{\mathbb{R}} \frac{d\tilde{\omega}}{2\pi i} \tilde{\omega} \left(\frac{A}{B} - \frac{\bar{A}}{\bar{B}}\right) C \Psi_1(z) \Psi_1(z'),$$

where the constants A, B, and C are defined as

$$A = \frac{\Gamma(c - 1)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \qquad (31a)$$

$$B = \frac{\Gamma(c - 1)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}, \qquad (31b)$$

$$C = \frac{\ell^4}{4(r_+^2 - r_-^2)\sqrt{1 + \mu^2}}. \qquad (31c)$$

Comparing with (30), we can read off $\tilde{G}_{\tilde{\omega} k}(z, z')$ and write the two-point function as

$$G^+(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} e^{ik(\tilde{\phi}-\tilde{\phi}')} \int_0^\infty \frac{d\tilde{\omega}}{(2\pi)^2} e^{-i\tilde{\omega}(\tilde{t}-\tilde{t}'-i\epsilon)}$$
$$\times \left(\frac{A}{B} - \frac{\bar{A}}{\bar{B}}\right) C \Psi_1(z) \Psi_1(z'). \qquad (32)$$

The mode decomposition of G^+ in (32) contains only positive $\tilde{\omega}$-frequencies, and, per construction, its antisymmetric part satisfies (27). Hence, it is legitimate to call the state associated with G^+ the ground state for a real, massive scalar field in the BTZ spacetime with $\mu^2 \geq 0$.

An alternative procedure to construct such a state consists of starting from its counterpart on the full AdS spacetime, applying subsequently the method of images; see, e.g., Refs. [31–34]. The net advantage of this procedure is that the final expression of the two-point function contains only a sum over a discrete Fourier mode. While getting rid of the integral along the frequencies hides the properties of being a ground state, the ensuing result is more versatile and easily applicable to compute related quantities, such as the bulk-to-boundary propagator, which are of interest in the framework of the AdS/CFT correspondence [35].

An important related question consists of whether G^+ is locally of the Hadamard form. Such a property is desirable not only at a structural level but also for constructing Wick polynomials, the building blocks for dealing with interactions at a perturbative level. In Ref. [17], it is proven under rather general hypotheses that a ground state, such as the one defined by (32) in particular, is always of *local* Hadamard form; namely, G^+ identifies a Hadamard state in every globally hyperbolic subregion of BTZ (for the definition of the Hadamard state, refer to Ref. [18]). A more difficult task is to verify if this ground state satisfies a global Hadamard condition such as the one proposed in Refs. [22,23] for a quantum state in anti-de Sitter space-time. Although we conjecture that to be the case, we leave a rigorous verification for future work.

B. Case $-1 < \mu^2 < 0$

For $-1 < \mu^2 < 0$, a Robin boundary condition needs to be imposed on solutions at $z = 1$, and therefore the analysis of the previous section is changed as we obtain a different two-point function for each possible Robin boundary condition. We have to consider separately two regimes, $\zeta \in [0, \zeta_*)$ and $\zeta \in [\zeta_*, \pi)$, with

$$\zeta_* \doteq \arctan\left(\frac{\Gamma(2\beta-1)|\Gamma(1-\beta+i\ell\frac{k}{r_+})|^2}{\Gamma(1-2\beta)|\Gamma(\beta+i\ell\frac{k}{r_+})|^2}\right), \quad (33)$$

where $\beta = \frac{1}{2} + \frac{1}{2}\sqrt{1+\mu^2}$ was defined in (13). Since $\mu^2 \in (-1, 0)$ and thus $\beta \in (\frac{1}{2}, 1)$, it follows that $\zeta_* \in (\frac{\pi}{2}, \pi)$.

1. Case $\zeta \in [0, \zeta_*)$

For Robin boundary conditions such that $\zeta \in [0, \zeta_*)$, it turns out that the spectrum of the operator $L_{\tilde{\omega}}$ in (10) is only $\tilde{\omega} \in \mathbb{R}$ and does not include any isolated eigenvalue in $\mathbb{C}\backslash\mathbb{R}$, which would correspond to poles in the Green's distribution associated with $L_{\tilde{\omega}}$ (see Appendixes B and D for more details). Observe that, since $\zeta_* \in (\frac{\pi}{2}, \pi)$, this scenario includes both the Dirichlet and the Neumann boundary conditions. This situation is structurally identical to the one investigated in the previous section for $\mu^2 \geq 0$. Using the results of Appendix B, we obtain the following resolution of the identity,

$$\frac{\delta(z-z')}{\mathcal{J}(z)} = \int_{\mathbb{R}} \frac{d\tilde{\omega}}{2\pi i}\,\tilde{\omega}\,\frac{(A\bar{B}-\bar{A}B)C}{|\cos(\zeta)B-\sin(\zeta)A|^2}\Psi_\zeta(z)\Psi_\zeta(z'), \quad (34)$$

where the constants A, B, and C are the same as in (31). We can use this result in combination with (28) and (30) to obtain, for each $\zeta \in [0, \zeta_*)$,

$$G_\zeta^+(x, x') = \lim_{\epsilon\to 0^+} \sum_{k\in\mathbb{Z}} e^{ik(\tilde{\phi}-\tilde{\phi}')} \int_0^\infty \frac{d\tilde{\omega}}{(2\pi)^2} e^{-i\tilde{\omega}(\tilde{t}-\tilde{t}'-i\epsilon)}$$
$$\times \frac{(A\bar{B}-\bar{A}B)C}{|\cos(\zeta)B-\sin(\zeta)A|^2}\Psi_\zeta(z)\Psi_\zeta(z'). \quad (35)$$

Note that this two-point function, valid for scalar fields with $-1 < \mu^2 < 0$, coincides with the one for scalar fields with $\mu^2 \geq 0$ obtained in (32) if $\zeta = 0$, that is, for Dirichlet boundary conditions.

2. Case $\zeta \in [\zeta_*, \pi)$

For Robin boundary conditions such that $\zeta \in [\zeta_*, \pi)$, it turns out that the spectrum of the operator $L_{\tilde{\omega}}$ in (10) not only contains all $\tilde{\omega} \in \mathbb{R}$ but includes also two isolated eigenvalues in $\mathbb{C}\backslash\mathbb{R}$, complex conjugate to each other, which correspond to poles in the Green's distribution associated with $L_{\tilde{\omega}}$ (see Appendixes B and D for more details). Denote those eigenvalues by $\tilde{\omega}_\zeta$ and $\overline{\tilde{\omega}_\zeta}$ such that $\text{Im}[\tilde{\omega}_\zeta] > 0$. They are dubbed "bound state frequencies," and their corresponding eigensolutions are called "bound state mode solutions." The existence of bound state mode solutions was also verified in Ref. [22] for the case of a massive scalar field in the Poincaré patch of AdS when Robin boundary conditions parametrized with $\zeta \in (\frac{\pi}{2}, \pi)$ are imposed at conformal infinity.

Unfortunately, an analytic expression for $\tilde{\omega}_\zeta$ cannot be found since, for $\text{Im}[\tilde{\omega}_\zeta] > 0$ and fixed ζ, one needs to invert the equality

$$\tan(\zeta) = \frac{B}{A}\bigg|_{\tilde{\omega}=\tilde{\omega}_\zeta},$$

where the constants A and B are the same as in (31). This operation can only be completed numerically (except in very particular cases such as $\zeta = 0$ and $\zeta = \pi/2$), and a representative example is shown in Fig. 2. A more qualitative discussion of the behavior of the solutions $\tilde{\omega}_\zeta$ as a function of ζ can be found in Appendix D.

As a consequence of these bound state frequencies, the resolution of the identity acquires an extra term in comparison to (34), which, following Appendix B, can be computed via Cauchy's residue theorem, yielding

$$\frac{\delta(z-z')}{\mathcal{J}(z)} = \int_{\mathbb{R}} \frac{d\tilde{\omega}}{2\pi i}\,\tilde{\omega}\,\frac{(A\bar{B}-\bar{A}B)C}{|\cos(\zeta)B-\sin(\zeta)A|^2}\Psi_\zeta(z)\Psi_\zeta(z')$$
$$+ \Re[\tilde{\omega}CD(\tilde{\omega})\Psi_\zeta(z)\Psi_\zeta(z')]|_{\tilde{\omega}=\tilde{\omega}_\zeta}, \quad (36)$$

where we used the identity $\Psi_\zeta(z)|_{\tilde{\omega}=\overline{\tilde{\omega}_\zeta}} = \overline{\Psi_\zeta(z)|_{\tilde{\omega}=\tilde{\omega}_\zeta}}$. The remaining term $D(\tilde{\omega}_\zeta)$ cannot be expressed analytically, but can be defined implicitly (see Appendix B).
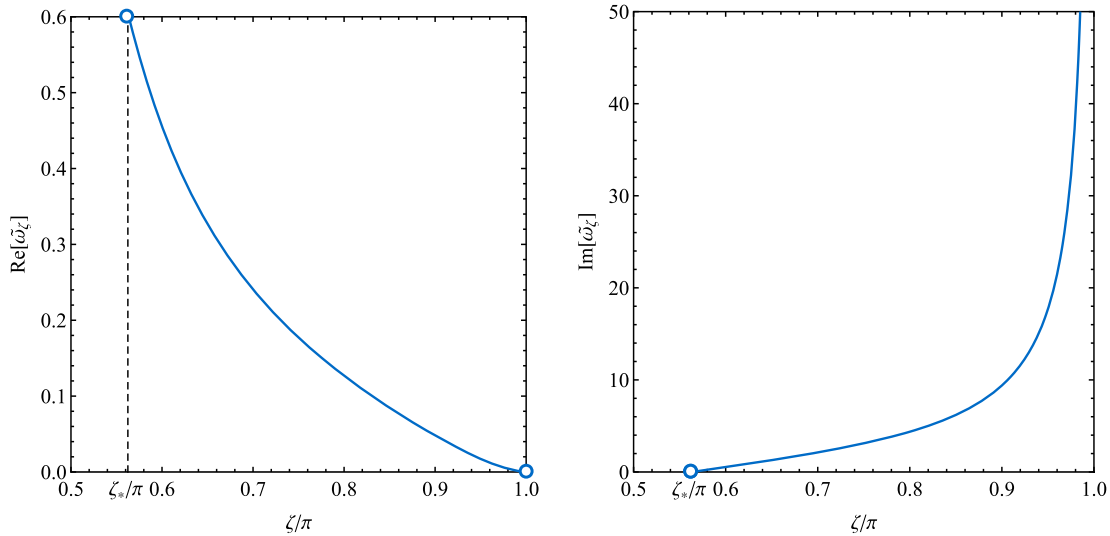
FIG. 2. Real and imaginary part of the bound state frequency $\tilde{\omega}_\zeta$ as a function of the parameter ζ defining the Robin boundary condition for a BTZ black hole with $\ell = 1$, $r_+ = 5$, and $r_- = 3$ and a scalar field with $\mu^2 = -0.65$ and $k = 1$. The bound state mode solutions exist for values of ζ between $\zeta_* \approx 0.5625\pi$ and π.

Finally, the bound state mode solutions will also contribute to the two-point function so that its antisymmetric part still obeys (27), and, consequently, the CCRs of the quantum field theory are satisfied. Using all the above information in combination with (28) and (30), the two-point function for the putative ground state may be written, for each $\zeta \in [\zeta_*, \pi)$,

$$G_\zeta^+(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} e^{ik(\tilde{\phi} - \tilde{\phi}')} \int_0^\infty \frac{d\tilde{\omega}}{(2\pi)^2} e^{-i\tilde{\omega}(\tilde{t} - \tilde{t}' - i\epsilon)} \frac{(A\bar{B} - \bar{A}B)C}{|\cos(\zeta)B - \sin(\zeta)A|^2} \Psi_\zeta(z)\Psi_\zeta(z')$$
$$+ i\sum_{k \in \mathbb{Z}} e^{ik(\tilde{\phi} - \tilde{\phi}')} (e^{-i\tilde{\omega}_\zeta(\tilde{t} - \tilde{t}')} + e^{-i\bar{\tilde{\omega}}_\zeta(\tilde{t} - \tilde{t}')}) \Re[CD(\tilde{\omega})\Psi_\zeta(z)\Psi_\zeta(z')]|_{\tilde{\omega} = \tilde{\omega}_\zeta}. \tag{37}$$

Notice that for $\zeta = \zeta_*$ the two bound state frequencies both coincide with the real value $\tilde{\omega} = 0$. In this case, the integral over positive $\tilde{\omega}$-frequencies has to be interpreted as a Cauchy principal value for $\tilde{\omega} = 0$, while the contribution of the bound state mode solutions is calculated using the Sokhotsky-Plemelj formula for distributions.

To conclude this section, we comment on the physical significance of the two-point functions obtained in (35) and (37). In the first case, we are dealing with a generalization of (32) to Robin boundary conditions. Hence, Eq. (35) is a genuine ground state built only out of positive $\tilde{\omega}$-frequencies, and, using once more the results of Ref. [17], it satisfies the local Hadamard condition. On the contrary, in (37), there is an additional contribution due to bound state frequencies $\tilde{\omega}_\zeta$, the existence of which spoils the property of G_ζ^+ of being a ground state. For this reason, it is not possible to conclude directly whether, in the presence of bound sate frequencies, we have constructed a Hadamard, and hence physically satisfactory, state. We plan to investigate this issue in future work.

V. CONCLUSIONS

In this paper, we have addressed two different, albeit related, questions. The first concerns the structural properties of a real, massive scalar field in BTZ spacetime, with an arbitrary coupling to scalar curvature. More precisely, since the underlying background is not globally hyperbolic, the equation of motion ruling the dynamics cannot be solved only assigning initial data on a partial Cauchy surface (a codimension-1, acausal, spacelike surface that is intersected by any complete timelike curve at most once), but also a boundary condition at infinity has to be imposed. In this work, we focused our attention on those of the Robin type, proving under which constraints on the parameters of the theory they can be imposed and subsequently constructing explicitly the associated solutions of the equation of motion.

In the second part of the paper, we used this result to address whether it is possible to associate to a real, massive scalar field in BTZ spacetime a two-point function, which can be in turn read as the building block of a ground state. We have given a positive and explicit answer to this query for a large class of Robin boundary conditions. Nonetheless, we have highlighted that there exists of a range of boundary conditions that must be excluded, those for which bound state mode solutions occur. When this is not the case, the two-point function possesses some nice physical properties, the most notable one being of local Hadamard form. Hence, the states that we have constructed

are suitable for defining an algebra of Wick polynomials which are the key ingredient to discussing interactions at a perturbative level.

Besides offering one of the first examples of a ground state for a quantum field theory in the exterior region of a rotating black hole, this work prompts several future directions of investigation. On the one hand, one could prove the existence of a thermal counterpart of our ground states, hence obtaining in this framework the analog of the Hartle-Hawking state in Schwarzschild spacetime. On the other hand, one could investigate Hawking radiation in this context and its interplay with the rotation of the black hole, by using the method of Parikh and Wilczek [39], recently extended to the framework of algebraic quantum field theory in Ref. [40]. A more long-term and ambitious goal is the explicit construction of a regularized stress-energy tensor, to be used in the analysis of the semiclassical Einstein equations, extending the work of Ref. [41]. We hope to come back to these problems in the near future.

APPENDIX A: DELTA FUNCTION AS AN EXPANSION IN EIGENFUNCTIONS OF A QUADRATIC EIGENVALUE PROBLEM

Our goal in this Appendix is to give a formula for the expansion of the delta distribution in terms of eigenfunctions of a differential operator with quadratic dependence on the spectral parameter like in (10), as it is necessary in the calculations of Sec. IV. More precisely, we want to obtain the spectral resolution of the identity for quadratic operator pencils, specifically concentrating on the case of unbounded operators coming from Sturm-Liouville ODEs as the one above. While the spectral theory of polynomial operator pencils has been widely studied [42,43], it is not a topic often covered in standard references on spectral theory [44,45].

Consider a family of operators defined on a Hilbert space $\mathcal{H}$, referred to as a quadratic operator pencil,

$$S_{\tilde{\omega}} = P + \tilde{\omega}\mathcal{R}_1 + \tilde{\omega}^2\mathcal{R}_2, \tag{A1}$$

with (S1) $\mathcal{R}_1, \mathcal{R}_2$, and $\mathcal{R}_2^{-1}$ all bounded and self-adjoint and P unbounded, closed, and Hermitian on a dense domain $D(S_{\tilde{\omega}}) \subset \mathcal{H}$, as is the case with our main example $S_{\tilde{\omega}} = \mathcal{J}^{-1}L_{\tilde{\omega}}$ on $\mathcal{H} = L^2((0,1); \mathcal{J}(z)\mathrm{d}z)$, where $L_{\tilde{\omega}}$ is defined in (10) and $\mathcal{J}(z)$ is defined in (20).

Define the *resolvent* of $S_{\tilde{\omega}}$ as $T_{\tilde{\omega}} = S_{\tilde{\omega}}^{-1}$, when it exists. The *resolvent set* $\rho(S_{\tilde{\omega}}) \subset \mathbb{C}$ consists of all values of $\tilde{\omega} \in \mathbb{C}$ such that $T_{\tilde{\omega}}$ exists and is a bounded operator. As usual, we define the *spectrum* $\sigma(S_{\tilde{\omega}}) = \mathbb{C}\backslash\rho(S_{\tilde{\omega}})$. We will show that, when (S2) $\sigma(S_{\tilde{\omega}})$ consists only of a subset of $\mathbb{R}$ together with a finite number of isolated points in $\mathbb{C}\backslash\mathbb{R}$ symmetric with respect to complex conjugation, the identity operator can be represented by the integral

$$I = \lim_{\varsigma\to\infty}\int_{-\varsigma}^{\varsigma}\frac{\mathrm{d}\tilde{\omega}}{2\pi i}\lim_{\epsilon\to 0^+}\tilde{\omega}(T_{\tilde{\omega}-i\epsilon} - T_{\tilde{\omega}+i\epsilon})\mathcal{R}_2 + \oint_{C}\frac{\mathrm{d}\tilde{\omega}}{2\pi i}\tilde{\omega}T_{\tilde{\omega}}, \tag{A2}$$

where the contour $\overset{\circ}{C}$ illustrated in Fig. 3 positively and simply encircles the nonreal part of the spectrum, the inner $\epsilon \to 0^+$ limit is taken in the sense of distributions in $\tilde{\omega}$ (boundary values of holomorphic functions define a special kind of distribution, Ref. [46], Chap. IX), and the outer $\varsigma \to \infty$ limit is taken in the sense of the strong operator topology.

The key idea is to *linearize* the quadratic operator pencil to a linear operator pencil $\mathbf{S}_{\tilde{\omega}}$, while doubling the size of the Hilbert space, in a way that keeps the spectral problems of $S_{\tilde{\omega}}$ and $\mathbf{S}_{\tilde{\omega}}$ equivalent. Since the spectral theory of linear operator pencils (essentially, generalized eigenvalue problems) is well known, we can leverage this equivalence to obtain the desired formulas for $S_{\tilde{\omega}}$. More precisely, consider the following linear operator pencil defined on $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$:
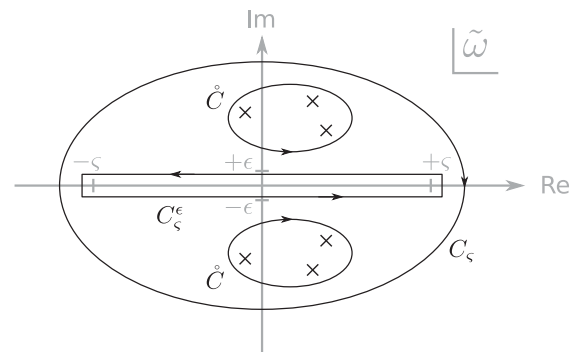

FIG. 3. Contour for the integral representation of the identity operator in (A2).

$$\mathbf{S}_{\tilde{\omega}} = \mathbf{P} + \tilde{\omega}\mathbf{R} = \begin{bmatrix} P & 0 \\ 0 & -\mathcal{R}_2 \end{bmatrix} + \tilde{\omega} \begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ \mathcal{R}_2 & 0 \end{bmatrix}. \quad \text{(A3)}$$

The linear pencil $\mathbf{S}_{\tilde{\omega}}$ is related to the quadratic one $S_{\tilde{\omega}}$ by the basic identities

$$\mathbf{S}_{\tilde{\omega}} \begin{bmatrix} I \\ \tilde{\omega} \end{bmatrix} \Psi = \begin{bmatrix} I \\ 0 \end{bmatrix} S_{\tilde{\omega}} \Psi, \quad \text{(A4)}$$

$$S_{\tilde{\omega}} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ \Phi \end{bmatrix} = \begin{bmatrix} I & \tilde{\omega} \end{bmatrix} \mathbf{S}_{\tilde{\omega}} \begin{bmatrix} \Psi \\ \Phi \end{bmatrix}. \quad \text{(A5)}$$

It is easy to see that, when $\mathcal{R}_1$ and $\mathcal{R}_2$ are bounded and self-adjoint, so is $\mathbf{R}$, and when in addition P is closed and Hermitian on $D(S_{\tilde{\omega}})$, so is $\mathbf{P}$ on $D(\mathbf{S}_{\tilde{\omega}}) = D(S_{\tilde{\omega}}) \oplus \mathcal{H}$. While there are many possible linearizations of a quadratic operator pencil, we have chosen this one to preserve these self-adjointness properties.

Define the resolvent $\mathbf{T}_{\tilde{\omega}} = \mathbf{S}_{\tilde{\omega}}^{-1}$, when it exists. The spectrum and resolvent set $\sigma(\mathbf{S}_{\tilde{\omega}})$, $\rho(\mathbf{S}_{\tilde{\omega}}) \subset \mathbb{C}$ are defined in the usual way, essentially exactly as above. Direct calculation shows that, when both exist, the resolvents of $S_{\tilde{\omega}}$ and $\mathbf{S}_{\tilde{\omega}}$ are related to each other by

$$\begin{aligned} \mathbf{T}_{\tilde{\omega}} &= \begin{bmatrix} I \\ \tilde{\omega} \end{bmatrix} T_{\tilde{\omega}} \begin{bmatrix} I & \tilde{\omega} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\mathcal{R}_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} T_{\tilde{\omega}} & \tilde{\omega} T_{\tilde{\omega}} \\ \tilde{\omega} T_{\tilde{\omega}} & \tilde{\omega}^2 T_{\tilde{\omega}} - \mathcal{R}_2^{-1} \end{bmatrix}, \end{aligned} \quad \text{(A6)}$$

$$T_{\tilde{\omega}} = \begin{bmatrix} I & 0 \end{bmatrix} \mathbf{T}_{\tilde{\omega}} \begin{bmatrix} I \\ 0 \end{bmatrix} = \frac{1}{\tilde{\omega}} \begin{bmatrix} I & 0 \end{bmatrix} \mathbf{T}_{\tilde{\omega}} \begin{bmatrix} 0 \\ I \end{bmatrix}. \quad \text{(A7)}$$

From the above formulas, it is clear that when $T_{\tilde{\omega}}$ exists and is bounded, so is $\mathbf{T}_{\tilde{\omega}}$, and vice versa. Thus, $\rho(S_{\tilde{\omega}}) = \rho(\mathbf{S}_{\tilde{\omega}})$, and, necessarily, $\sigma(S_{\tilde{\omega}}) = \sigma(\mathbf{S}_{\tilde{\omega}})$, which makes precise the sense in which the spectral problems of the two operator pencils equivalent. Once we know what $\rho(S_{\tilde{\omega}})$ is, using the boundedness of $\mathbf{R}$, a variant of Theorem VI.5 of Ref. [44] shows that $\mathbf{T}_{\tilde{\omega}}$ is analytic on $\rho(S_{\tilde{\omega}}) = \rho(\mathbf{S}_{\tilde{\omega}})$, which implies by the explicit relationship between them that $T_{\tilde{\omega}}$ is also analytic on $\rho(S_{\tilde{\omega}})$.

Let $\nu \in \rho(\mathbf{S}_{\tilde{\omega}}) = \rho(S_{\tilde{\omega}})$, and let $C_\nu \subset \rho(\mathbf{S}_{\tilde{\omega}}) = \rho(S_{\tilde{\omega}})$ be a contour that simply encircles ν, though in the negative direction, meaning that, upon deformation, C_ν has a chance of simply and positively encircling $\sigma(\mathbf{S}_{\tilde{\omega}})$. However, since our $\sigma(\mathbf{S}_{\tilde{\omega}})$ is unbounded, the deformation of the contour will have to go through a limiting procedure. There is no need for C_ν to be connected. In fact, it is advantageous to have a connected component of C_ν contained in each connected component of $\rho(\mathbf{S}_{\tilde{\omega}})$. Provided that the resolvent $\mathbf{G}_{\tilde{\omega}}$ is analytic on $\rho(\mathbf{S}_{\tilde{\omega}})$, the Cauchy residue formula gives

$$\mathbf{T}_\nu \mathbf{R} = -\oint_{C_\nu} \frac{d\tilde{\omega}}{2\pi i} \frac{1}{\tilde{\omega} - \nu} \mathbf{T}_{\tilde{\omega}} \mathbf{R}. \quad \text{(A8)}$$

Multiplying both sides by $\mathbf{R}^{-1}\mathbf{S}_\nu$, we get

$$\mathbf{I} = \oint_{C_\nu} \frac{d\tilde{\omega}}{2\pi i} \left(\mathbf{T}_{\tilde{\omega}} \mathbf{R} - \frac{\mathbf{I}}{\tilde{\omega} - \nu} \right), \quad \text{(A9)}$$

where the contour C_ν can be deformed at will, as long as it remains within $\rho(\mathbf{S}_{\tilde{\omega}}) \setminus \{\nu\}$.

We can deform the contour C_ν to the desired limiting form in (A2) if we can take advantage of an abstract spectral representation for the operator $\mathbf{R}^{-1}\mathbf{P}$, that is (S3) there exists a projection operator valued measure $\mathbf{E}(\nu)$ on $\sigma(\mathbf{S}_{\tilde{\omega}})$, satisfying the usual commutation and monotonicity conditions, giving the spectral representation $\mathbf{R}^{-1}\mathbf{P} = \int_{\sigma(\mathbf{S}_{\tilde{\omega}})} \nu d\mathbf{E}(\nu)$. As a consequence, we also get the spectral representation $\mathbf{T}_{\tilde{\omega}}\mathbf{R} = \int_{\sigma(\mathbf{S}_{\tilde{\omega}})} \frac{1}{\nu + \tilde{\omega}} d\mathbf{E}(\nu)$. If we let $\mathbf{E}_\varsigma = \mathbf{E}(\{\nu \in \mathbb{C} | |\nu| < \varsigma\})$, then $\mathbf{E}_\varsigma \to \mathbf{I}$ strongly as $\varsigma \to \infty$ and $\bigcup_{\varsigma > 0} \text{ran } \mathbf{E}_\varsigma$ is dense in $\mathcal{H}^2$.

Another consequence of the abstract spectral representation is that $\mathbf{T}_{\tilde{\omega}}\mathbf{R}\mathbf{E}_\varsigma$ is now analytic for $|\tilde{\omega}| > \varsigma$ and has the strong asymptotic expansion $\mathbf{T}_{\tilde{\omega}}\mathbf{R}\mathbf{E}_\varsigma = \frac{1}{\tilde{\omega}}\mathbf{E}_\varsigma + \mathcal{O}(\frac{1}{\tilde{\omega}^2})$. Multiplying both sides of (A9) by $\mathbf{E}_\varsigma$, we get

$$\mathbf{E}_\varsigma = \oint_{C_\nu} \frac{d\tilde{\omega}}{2\pi i} \left(\mathbf{E}_\varsigma \mathbf{T}_{\tilde{\omega}} \mathbf{R} - \frac{\mathbf{E}_\varsigma}{\tilde{\omega} - \nu} \right) - \oint_{C_\nu} \frac{d\tilde{\omega}}{2\pi i} \frac{\mathbf{I} - \mathbf{E}_\varsigma}{\tilde{\omega} - \nu}.$$

The second integral can be evaluated immediately and combined with the left-hand side. In the first integral, we can deform the contour C_ν to the contour $C_\varsigma \cup C_\varsigma^\epsilon \cup \overset{\circ}{C}$, as illustrated in Fig. 3. Because of the asymptotics mentioned above, the integral over the large circle C_ς contributes at the order $\mathcal{O}(\frac{1}{\varsigma})$. On the other hand, the term $\frac{\mathbf{E}_\varsigma}{\tilde{\omega} - \nu}$ is analytic over the contours $\overset{\circ}{C}$, C_ς^ϵ and their interiors, so its contribution vanishes, which leaves us with

$$\mathbf{I} = \oint_{C_\varsigma^\epsilon} \frac{d\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R}\mathbf{E}_\varsigma + \oint_{\overset{\circ}{C}} \frac{d\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R}\mathbf{E}_\varsigma + \mathcal{O}(\varsigma^{-1}). \quad \text{(A10)}$$

Next, before taking the limits $\epsilon \to 0^+$ and $\varsigma \to \infty$, we multiply both sides of (A10) by an arbitrary $\mathbf{v}_{\varsigma'} \in \mathcal{H}^2$ such that $\mathbf{v}_{\varsigma'} = \mathbf{E}_\varsigma \mathbf{v}_{\varsigma'}$ for any $\varsigma > \varsigma'$, so that

$$\begin{aligned} \mathbf{v}_{\varsigma'} &= \oint_{C_\varsigma^\epsilon} \frac{d\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R}\mathbf{E}_\varsigma \mathbf{v}_{\varsigma'} + \oint_{\overset{\circ}{C}} \frac{d\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R}\mathbf{E}_\varsigma \mathbf{v}_{\varsigma'} + \mathcal{O}(\varsigma^{-1}) \\ &= \int_{-\varsigma}^{\varsigma} \frac{d\tilde{\omega}}{2\pi i} \lim_{\epsilon \to 0^+} (\mathbf{T}_{\tilde{\omega} - i\epsilon} - \mathbf{T}_{\tilde{\omega} + i\epsilon}) \mathbf{R}\mathbf{E}_\varsigma \mathbf{v}_{\varsigma'} \\ &\quad + \oint_{\overset{\circ}{C}} \frac{d\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R}\mathbf{E}_\varsigma \mathbf{v}_{\varsigma'} + \mathcal{O}(\varsigma^{-1}) \\ &= \lim_{\varsigma \to \infty} \int_{-\varsigma}^{\varsigma} \frac{d\tilde{\omega}}{2\pi i} \lim_{\epsilon \to 0^+} (\mathbf{T}_{\tilde{\omega} - i\epsilon} - \mathbf{T}_{\tilde{\omega} + i\epsilon}) \mathbf{R}\mathbf{v}_{\varsigma'} \\ &\quad + \oint_{\overset{\circ}{C}} \frac{d\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}} \mathbf{R}\mathbf{v}_{\varsigma'}. \end{aligned}$$

Note that the $\epsilon \to 0^+$ limit is taken in the distributional sense with respect to $\tilde{\omega}$. Finally, using a variant of the Banach-Steinhaus theorem (Theorem 2.11.4 of Ref. [47]), we obtain the following strong limit,

$$\mathbf{R}^{-1} = \lim_{\varsigma \to \infty} \int_{-\varsigma}^{\varsigma} \frac{d\tilde{\omega}}{2\pi i} \lim_{\epsilon \to 0^+} (\mathbf{T}_{\tilde{\omega}-i\epsilon} - \mathbf{T}_{\tilde{\omega}+i\epsilon}) + \oint_C \frac{d\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}},$$

where to apply the theorem we need to recall that finite linear combinations of vectors like $\mathbf{v}_{\zeta'}$ are dense in $\mathcal{H}^2$ and note that, due to (A10), the norms of the integrals

$$\int_{-\varsigma}^{\varsigma} \frac{d\tilde{\omega}}{2\pi i} \lim_{\epsilon \to 0^+} (\mathbf{T}_{\tilde{\omega}-i\epsilon} - \mathbf{T}_{\tilde{\omega}+i\epsilon}) + \oint_C \frac{d\tilde{\omega}}{2\pi i} \mathbf{T}_{\tilde{\omega}}$$

are uniformly bounded for large σ.

Using the second equality in (A6) and the formula

$$\mathbf{R}^{-1} = \begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ \mathcal{R}_2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \mathcal{R}_2^{-1} \\ \mathcal{R}_2^{-1} & -\mathcal{R}_2^{-1}\mathcal{R}_1\mathcal{R}_2^{-1} \end{bmatrix} \quad (A11)$$

finally gives us the desired identity (A2).

The argument we have just presented, for the linear operator pencil, mimics that of Ref. [25], Chap. 9. There, the existence of the spectral measure $\mathbf{E}(\nu)$ followed from the standard spectral theorem for self-adjoint operators on a Hilbert space, with the operator $\mathbf{R}^{-1}\mathbf{P}$ being self-adjoint with respect to the weighted inner product $[\mathbf{v}, \mathbf{u}] = (\mathbf{v}, \mathbf{R}\mathbf{u})$, which was assumed to be positive definite. In our case, $[\mathbf{v}, \mathbf{u}]$ is clearly indefinite and thus defines a Krein space $\mathcal{K} = (\mathcal{H}^2, [-, -])$ rather than a Hilbert space. Fortunately, in the Krein space setting, we can still appeal to a spectral theorem, provided that the operator $\mathbf{R}^{-1}\mathbf{P}$ is definitizable. This will indeed be the case for the specific operators defined in Appendix B. However, since verifying the necessary hypothesis is rather technical, we relegate them to Appendix E. A more hands-on alternative to hypothesis (S3) would be a direct estimate of the form $\mathbf{T}_{\tilde{\omega}} = \frac{1}{\tilde{\omega}} \mathbf{R}^{-1} + \mathcal{O}(\frac{1}{\tilde{\omega}^2})$ that is uniform over a neigborhood of $\tilde{\omega} = \infty$ minus a sector of a positive angle containing the real axis. Such an estimate could be obtained by a WKB analysis of the differential operators discussed in Appendix B, which may be considered in future work.

APPENDIX B: EXPLICIT CALCULATION OF THE DELTA INTEGRAL REPRESENTATION

In this Appendix, we show in detail the procedure to compute the delta integral representation (30) and, hence, the mode expansion of the two-point function (28) for the

case in which the mass parameter is such that $-1 < \mu^2 < 0$ and Robin boundary conditions parametrized by $\zeta \in [0, \pi)$ are imposed at $z = 1$. The results for $\mu^2 \geq 0$ may be simply obtained by setting $\zeta = 0$.

Now, let us apply the general discussion from Appendix A to the differential operator $L_{\tilde{\omega}}$ introduced in (10), which we write for convenience as

$$L_{\tilde{\omega}}\Psi(z) = \frac{d}{dz}\left(z\frac{d\Psi(z)}{dz}\right) - \left[\frac{\ell^2 k^2(1-z) - r_+^2\mu^2}{4r_+^2(1-z)}\right.$$
$$\left. - \frac{\tilde{\omega}\ell^3 k r_-}{2r_+(r_+^2 - r_-^2)(1-z)} - \frac{\tilde{\omega}^2\ell^4 \mathcal{J}(z)}{4(r_+^2 - r_-^2)}\right]\Psi(z),$$
$$(B1)$$

with $\mathcal{J}(z)$ the same as in (20). We let the Hilbert space be $\mathcal{H} = L^2((0, 1); \mathcal{J}(z)dz)$, and we let the quadratic operator pencil be

$$S_{\tilde{\omega}}\Psi(z) = \frac{1}{\mathcal{J}(z)}L_{\tilde{\omega}}\Psi(z). \quad (B2)$$

This operator satisfies the hypotheses (S1), (S2), and (S3) from Appendix A. The verification of the hypotheses is of a much more technical nature and is relegated to Appendixes C, D, and E, respectively.

We want to construct a Green's distribution $\mathcal{G}_{\tilde{\omega},\zeta}$ associated to $L_{\tilde{\omega}}$ consisting of the product of square integrable solutions of $L_{\tilde{\omega}}\Psi = 0$ at both $z = 0$ and $z = 1$. For that, we introduce

$$u_{\tilde{\omega}}(z) = \begin{cases} \Psi_3(z), & \text{Im}[\tilde{\omega}] > 0, \\ \Psi_4(z), & \text{Im}[\tilde{\omega}] < 0, \end{cases} \quad (B3)$$

with Ψ_3 and Ψ_4 defined in (21), which is uniquely chosen by the property of being L^2 at $z = 0$, as seen in Sec. III C. We also introduce

$$\Psi_{\tilde{\omega},\zeta}(z) = \cos(\zeta)\Psi_1(z) + \sin(\zeta)\Psi_2(z), \quad (B4)$$

with Ψ_1 and Ψ_2 defined either by (17) or (18), which is uniquely chosen by the property of being L^2 at $z = 1$ when $-1 < \mu^2 < 0$ and satisfying Robin boundary conditions parametrized by $\zeta \in [0, \pi)$. Note that, given the identity $\overline{L_{\tilde{\omega}}} = L_{\bar{\tilde{\omega}}}$, one has $u_{\bar{\tilde{\omega}}} = \overline{u_{\tilde{\omega}}}$ and $\Psi_{\bar{\tilde{\omega}},\zeta} = \overline{\Psi_{\tilde{\omega},\zeta}}$.

The Green's distribution $\mathcal{G}_{\tilde{\omega},\zeta}$ may then be written as

$$\mathcal{G}_{\tilde{\omega},\zeta}(z, z') = \begin{cases} \mathcal{N}_{\tilde{\omega},\zeta}^{-1} u_{\tilde{\omega}}(z)\Psi_{\tilde{\omega},\zeta}(z'), & z \leq z', \\ \mathcal{N}_{\tilde{\omega},\zeta}^{-1} u_{\tilde{\omega}}(z')\Psi_{\tilde{\omega},\zeta}(z), & z \geq z', \end{cases} \quad (B5)$$

with

$$\mathcal{N}_{\tilde{\omega},\zeta} = -z\mathcal{W}_z[u_{\tilde{\omega}}, \Psi_\zeta] = \begin{cases} \cos(\zeta)\frac{\Gamma(c)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} + \sin(\zeta)\frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}, & \text{Im}[\tilde{\omega}] > 0, \\ \cos(\zeta)\frac{\Gamma(2-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)} + \sin(\zeta)\frac{\Gamma(2-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)}, & \text{Im}[\tilde{\omega}] < 0, \end{cases} \quad (B6)$$

where the parameters a, b, and c are as in (15). The normalization constant $\mathcal{N}_{\tilde{\omega},\zeta}$ was evaluated using the intermediate result

$$\mathcal{W}_z[\Psi_1, \Psi_2] = \frac{a+b-c}{z} = \frac{\sqrt{1+\mu^2}}{z},$$

and the following connection formulas of hypergeometric functions (see Eqs. (15.10.17–18) of Ref. [38]):

$$\Psi_3(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Psi_1(z)$$
$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\Psi_2(z), \qquad \text{(B7a)}$$

$$\Psi_4(z) = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\Psi_1(z)$$
$$+ \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}\Psi_2(z). \qquad \text{(B7b)}$$

By inspection of (B5) and (B6), one has that $\overline{\mathcal{N}_{\tilde{\omega},\zeta}} = \mathcal{N}_{\bar{\tilde{\omega}},\zeta}$ and $\overline{\mathcal{G}_{\tilde{\omega},\zeta}}(z,z') = \mathcal{G}_{\bar{\tilde{\omega}},\zeta}(z',z)$. Moreover, as noted in Appendix D, $\mathcal{N}_{\tilde{\omega}}$ is analytic on $\text{Im}[\tilde{\omega}] \neq 0$ and has at most two isolated zeros, the *bound state frequencies*, that are reflection symmetric about the real axis, forming a set $\text{BS}_\zeta \subset \mathbb{C}$ such that $\text{BS}_\zeta = \text{BS}_\zeta^+ \cup \overline{\text{BS}_\zeta^+}$ with $\text{Im}[\text{BS}_\zeta^+] > 0$.

We can now apply formula (A2) to write the following integral representation of the delta distribution,

$$\frac{4(r_+^2 - r_-^2)}{\ell^4 \mathcal{J}(z)}\delta(z - z') = -\int_{\mathbb{R}}\frac{d\tilde{\omega}}{2\pi i}\tilde{\omega}\Delta\mathcal{G}_{\tilde{\omega},\zeta}(z,z')$$
$$+ \oint_C \frac{d\tilde{\omega}}{2\pi i}\tilde{\omega}\mathcal{G}_{\tilde{\omega},\zeta}(z,z'), \qquad \text{(B8)}$$

where the contour $\overset{\circ}{C}$ illustrated in Fig. 3 positively and simply encircles the bound state frequencies in BS_ζ, and

$$\Delta\mathcal{G}_{\tilde{\omega},\zeta}(z,z') \doteq \lim_{\epsilon \to 0^+}[\mathcal{G}_{\tilde{\omega}+i\epsilon,\zeta}(z,z') - \mathcal{G}_{\tilde{\omega}-i\epsilon,\zeta}(z,z')] \qquad \text{(B9)}$$

should be interpreted as a distribution in $\tilde{\omega}$. An application of Cauchy's residue theorem gives the integral representation

$$\frac{\delta(z-z')}{\mathcal{J}(z)} = -\frac{\ell^4}{4(r_+^2-r_-^2)}\left[\int_{\mathbb{R}}\frac{d\tilde{\omega}}{2\pi i}\tilde{\omega}\Delta\mathcal{G}_{\tilde{\omega},\zeta}(z,z')\right.$$
$$\left.+ \sum_{\tilde{\omega}' \in \text{BS}_\zeta}\text{Res}_{\tilde{\omega}=\tilde{\omega}'}[\tilde{\omega}\mathcal{G}_{\tilde{\omega},\zeta}(z,z')]\right]. \qquad \text{(B10)}$$

Both integrands in (B10) can be computed rather explicitly, except for analytic expressions for the bound state frequencies (see Appendix D). Introducing

$$A = \frac{\Gamma(c-1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad B = \frac{\Gamma(c-1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},$$

and using the connection formulas (B7a) and (B7b), one may write

$$u_{\tilde{\omega}}(z) = \begin{cases} (c-1)[A\Psi_1(z) + B\Psi_2(z)], & \text{Im}[\tilde{\omega}] > 0, \\ (1-c)[\bar{A}\Psi_1(z) + \bar{B}\Psi_2(z)], & \text{Im}[\tilde{\omega}] < 0, \end{cases}$$

and

$$\mathcal{N}_{\tilde{\omega},\zeta} = \begin{cases} (1-c)\sqrt{1+\mu^2}[\cos(\zeta)B - \sin(\zeta)A], & \text{Im}[\tilde{\omega}] > 0, \\ (c-1)\sqrt{1+\mu^2}[\cos(\zeta)\bar{B} - \sin(\zeta)\bar{A}], & \text{Im}[\tilde{\omega}] < 0. \end{cases}$$

Hence, for $z < z'$,

$$\Delta\mathcal{G}_{\tilde{\omega}}(z,z') = -\frac{1}{\sqrt{1+\mu^2}}\left[\frac{A\Psi_1(z) + B\Psi_2(z)}{\cos(\zeta)B - \sin(\zeta)A}\right.$$
$$\left. - \frac{\bar{A}\Psi_1(z) + \bar{B}\Psi_2(z)}{\cos(\zeta)\bar{B} - \sin(\zeta)\bar{A}}\right]\Psi_\zeta(z')$$
$$= \frac{\bar{A}B - A\bar{B}}{|\cos(\zeta)B - \sin(\zeta)A|^2}\frac{\Psi_\zeta(z)\Psi_\zeta(z')}{\sqrt{1+\mu^2}}, \qquad \text{(B11)}$$

and the result is also valid for $z > z'$.

Now, let us consider the residues at a bound state frequency $\tilde{\omega}_\zeta \in \text{BS}_\zeta^+$. When it exists, it is an isolated root of $\mathcal{N}_{\tilde{\omega},\zeta} = 0$ and

$$\text{Res}_{\tilde{\omega}=\tilde{\omega}_\zeta}[\tilde{\omega}\mathcal{G}_{\tilde{\omega},\zeta}(z,z')] = \frac{\tilde{\omega}_\zeta}{2}D(\tilde{\omega}_\zeta)\Psi_{\tilde{\omega}_\zeta,\zeta}(z)\Psi_{\tilde{\omega}_\zeta,\zeta}(z'), \qquad \text{(B12)}$$

where $D(\tilde{\omega}_\zeta) = D_2(\tilde{\omega}_\zeta)/D_1(\tilde{\omega}_\zeta)$. From the Laurent series of $\mathcal{N}_{\tilde{\omega},\zeta}$, we get

$$D_1(\tilde{\omega}_\zeta) \doteq \frac{\ell^2\sqrt{1+\mu^2}}{i(r_+^2-r_-^2)}\{\sin(\zeta)A[(r_+ + r_-)\psi(c-a)$$
$$+ (r_+ - r_-)\psi(c-b) - 2r_+\psi(c)](1-c)$$
$$- \cos(\zeta)B[(r_+ + r_-)\psi(b) + (r_+ - r_-)\psi(a)$$
$$- 2r_+\psi(c)](1-c)\}|_{\tilde{\omega}=\tilde{\omega}_\zeta},$$

where ψ is the digamma function. Since $\mathcal{N}_{\tilde{\omega}_\zeta,\zeta} = 0$, the solutions $u_{\tilde{\omega}_\zeta}$ and $\Psi_{\tilde{\omega}_\zeta,\zeta}$ are no longer linearly independent, and their ratio (recall that $\text{Im}[\tilde{\omega}_\zeta] > 0$) is

$$D_2(\tilde{\omega}_\zeta) \doteq \frac{u_{\tilde{\omega}_\zeta}(z)}{\Psi_{\tilde{\omega}_\zeta,\zeta}(z)}$$
$$= \begin{cases} \sec(\zeta)(c-1)A|_{\tilde{\omega}=\tilde{\omega}_\zeta}, & \cos(\zeta) \neq 0, \\ \csc(\zeta)(c-1)B|_{\tilde{\omega}=\tilde{\omega}_\zeta}, & \sin(\zeta) \neq 0. \end{cases}$$

Finally, the spectral resolution of the delta distribution takes the form

$$
\frac{\delta(z - z')}{\mathcal{J}(z)} = \frac{\ell^4}{4(r_+^2 - r_-^2)}
$$
$$
\times \left[\int_{\mathbb{R}} \frac{\mathrm{d}\tilde{\omega}}{2\pi i} \tilde{\omega} \frac{A\bar{B} - \bar{A}B}{|\cos(\zeta)B - \sin(\zeta)A|^2} \frac{\Psi_\zeta(z)\Psi_\zeta(z')}{\sqrt{1 + \mu^2}} \right.
$$
$$
\left. + \sum_{\tilde{\omega}_\zeta \in \mathrm{BS}_\zeta^+} \Re[\tilde{\omega}_\zeta D(\tilde{\omega}_\zeta) \Psi_{\tilde{\omega}_\zeta, \zeta}(z) \Psi_{\tilde{\omega}_\zeta, \zeta}(z')] \right].
$$

$$(B13)$$

We have taken advantage of the fact that bound state frequencies come in complex conjugate pairs, $\mathrm{BS}_\zeta = \mathrm{BS}_\zeta^+ \cup \overline{\mathrm{BS}_\zeta^+}$, and of the identities $\overline{D(\tilde{\omega}_\zeta)} = D(\overline{\tilde{\omega}}_\zeta)$, $\overline{\Psi_{\tilde{\omega}_\zeta, \zeta}(z)} = \Psi_{\overline{\tilde{\omega}}_\zeta, \zeta}(z)$.

APPENDIX C: CHECK OF HYPOTHESIS (S1)

In this Appendix, we show that hypothesis (S1) of Appendix A is verified for the quadratic operator pencil $S_{\tilde{\omega}}$.

First, we discuss the relation of the domain of $S_{\tilde{\omega}}$, $D(S_{\tilde{\omega}}) \subset \mathcal{H} = L^2((0,1); \mathcal{J}(z)\mathrm{d}z)$, to the choice of boundary conditions for $L_{\tilde{\omega}}$ in (B2). By standard arguments (Ref. [25], Chap. 3), each choice of boundary conditions will give a closed operator realization of $S_{\tilde{\omega}}$ on a dense domain $D(S_{\tilde{\omega}})$. Then, if there exists at least one $\tilde{\omega} \in \mathbb{C}$ such that $\tilde{\omega}, \bar{\tilde{\omega}} \in \rho(S_{\tilde{\omega}})$ and the corresponding bounded resolvents satisfy $T_{\tilde{\omega}}^* = T_{\bar{\tilde{\omega}}}$, the closed operator $S_{\tilde{\omega}}$ will be self-adjoint, in the sense that $S_{\tilde{\omega}}^* = S_{\bar{\tilde{\omega}}}$ and $D(S_{\tilde{\omega}}^*) = D(S_{\tilde{\omega}})$. Hence, we need to check that (a) the Green's distribution associated to $L_{\tilde{\omega}}$, $\mathcal{G}_{\tilde{\omega}}$ exists for at least one $\tilde{\omega} \in \mathbb{C}$, that

(b) $T_{\tilde{\omega}} = \mathcal{G}_{\tilde{\omega}}\mathcal{J}$ is bounded for at least one $\tilde{\omega} \in \mathbb{C}$, and that (c) we can satisfy $\overline{\mathcal{G}_{\tilde{\omega}}(z, z')} = \mathcal{G}_{\bar{\tilde{\omega}}}(z', z)$ and hence $T_{\tilde{\omega}}^* = T_{\bar{\tilde{\omega}}}$. The properties (a) and (c) are explicitly checked in Appendix B for each choice of Robin boundary conditions parametrized by ζ.

In order to check property (b), we need to prove the boundedness of the resolvent $T_{\tilde{\omega}} = \mathcal{G}_{\tilde{\omega}}\mathcal{J}$. Using the same notation of Appendix B, for a given $\tilde{\omega}$ with $\mathrm{Im}[\tilde{\omega}] \neq 0$, provided that $u_{\tilde{\omega}}$ and $\Psi_{\tilde{\omega}, \zeta}$ introduced in (B3) and (B4) are linearly independent, that is, $\mathcal{N}_{\tilde{\omega}, \zeta}$ in (B6) does not vanish, we can get boundedness starting from the more precise asymptotic estimates:

$$
|u_{\tilde{\omega}}(z)| \lesssim z^\lambda (1 - z)^{1-\beta-\epsilon}, \qquad (C1a)
$$

$$
|\Psi_{\tilde{\omega}, \zeta}(z)| \lesssim \begin{cases} z^{-\lambda}(1 - z)^\beta, & \zeta = 0, \\ z^{-\lambda}(1 - z)^{1-\beta-\epsilon}, & \zeta \neq 0. \end{cases} \qquad (C1b)
$$

Here, $\lambda \doteq \ell^2 r_+ |\mathrm{Im}\tilde{\omega}| / 2(r_+^2 - r_-^2)$, and the symbol $\lesssim$ denotes an inequality up to a multiplicative constant, uniform over $z \in (0, 1)$ where applicable. The constant $\epsilon > 0$ helps to cover the cases with logarithmic singularities, and it could be chosen to depend on other parameters. Using the same notation, we also have

$$
|\mathcal{J}(z)| \lesssim z^{-1}(1 - z)^{-1}. \qquad (C2)
$$

The strategy to show boundedness of $T_{\tilde{\omega}} = \mathcal{G}_{\tilde{\omega}}\mathcal{J}$ is to apply the so-called *weighted Schur test* (Theorem 5.2 of Ref. [48]). The inequalities, where, after a factorization, we apply the Cauchy-Schwarz inequality,

$$
\|T_{\tilde{\omega}}\Psi\|^2 = \int_0^1 \mathrm{d}z \mathcal{J}(z) \left| \int_0^1 \mathrm{d}z' \mathcal{G}_{\tilde{\omega}}(z, z')\mathcal{J}(z')\Psi(z') \right|^2
$$
$$
\leq \int_0^1 \mathrm{d}z \mathcal{J}(z) \left(\int_0^1 \mathrm{d}z' |\mathcal{G}_{\tilde{\omega}}(z, z')| \mathcal{J}(z')\mathcal{J}_1(z') \right) \left(\int_0^1 \mathrm{d}z' |\mathcal{G}_{\tilde{\omega}}(z, z')| \frac{\mathcal{J}(z')}{\mathcal{J}_1(z')} |\Psi(z)|^2 \right)
$$
$$
\leq \int_0^1 \mathrm{d}z' \left(\int_0^1 \mathrm{d}z \mathcal{J}(z)\mathcal{J}_2(z) |\mathcal{G}_{\tilde{\omega}}(z, z')| \right) \frac{\mathcal{J}(z')}{\mathcal{J}_1(z')} |\Psi(z')|^2
$$
$$
\leq \int_0^1 \mathrm{d}z' \frac{\mathcal{J}_3(z')}{\mathcal{J}_1(z')} \mathcal{J}(z') |\Psi(z')|^2,
$$

show that $\|T_{\tilde{\omega}}\Psi\|^2 \lesssim \|\Psi\|^2$, provided we can find functions $\mathcal{J}_1(z)$, $\mathcal{J}_2(z)$, $\mathcal{J}_3(z)$ satisfying the estimates

$$
\int_0^1 \mathrm{d}z' |\mathcal{G}_{\tilde{\omega}}(z, z')| \mathcal{J}(z')\mathcal{J}_1(z') \lesssim \mathcal{J}_2(z),
$$

$$
\int_0^1 \mathrm{d}z \mathcal{J}(z)\mathcal{J}_2(z) |\mathcal{G}_{\tilde{\omega}}(z, z')| \lesssim \mathcal{J}_3(z'),
$$

$$
\frac{\mathcal{J}_3(z')}{\mathcal{J}_1(z')} \lesssim 1.
$$

The only free choice is actually in $\mathcal{J}_1$, since $\mathcal{J}_2$ and $\mathcal{J}_3$ (or rather their lower bounds) are then determined by the properties of $\mathcal{G}_{\tilde{\omega}}(z, z')$. Given the estimates (C1) and formula (B5), it is straightforward to show that the following choices work as desired,

$$
\zeta = 0: \begin{cases} \mathcal{J}_1(z) = 1, \\ \mathcal{J}_2(z) = (1 - z)^{\min(\beta, 1-2\epsilon)}, \\ \mathcal{J}_3(z) = (1 - z)^{\min(\beta, 2-4\epsilon)}, \end{cases} \qquad (C3)
$$

$$\zeta \neq 0: \begin{cases} \mathcal{J}_1(z) = 1, \\ \mathcal{J}_2(z) = (1-z)^{1-\beta-\epsilon}, \\ \mathcal{J}_3(z) = (1-z)^{1-\beta-\epsilon}, \end{cases} \quad \text{(C4)}$$

where, for $\zeta \neq 0$, we restrict to $\beta \in (\frac{1}{2}, 1)$ and we choose $\epsilon < 1 - \beta$.

APPENDIX D: CHECK OF HYPOTHESIS (S2)

In this Appendix, we show that the hypothesis (S2) of Appendix A is verified, namely that the spectrum of $S_{\tilde{\omega}}$ consists only of $\mathbb{R}$ together with at most two isolated points in $\mathbb{C} \backslash \mathbb{R}$, symmetric with respect to complex conjugation.

The Green's distribution $\mathcal{G}_{\tilde{\omega},\zeta}$ computed in Appendix B has a branch cut at $\text{Im}[\tilde{\omega}] = 0$, and for certain values of ζ, it can have poles with $\text{Im}[\tilde{\omega}] \neq 0$, which from the explicit calculations of Appendix B coincide with the zeros of the normalization coefficient $\mathcal{N}_{\tilde{\omega},\zeta}$ in (B6).

By direct inspection, we know that $\mathcal{N}_{\tilde{\omega},\zeta}$ has at most isolated zeros, that are reflection symmetric about the real axis. These bound state frequencies form a set $\text{BS}_\zeta \subset \mathbb{C}$, with $\text{BS}_\zeta = \text{BS}_\zeta^+ \cup \overline{\text{BS}_\zeta^+}$ with $\text{Im}[\text{BS}_\zeta^+] > 0$. We conclude that $\sigma(S_{\tilde{\omega}}) = \mathbb{R} \cup \text{BS}_\zeta$. By general arguments from Appendix A, the resolvent $T_{\tilde{\omega}} = S_{\tilde{\omega}}^{-1}$ is analytic on its resolvent set $\rho(S_{\tilde{\omega}}) = \mathbb{C} \backslash \sigma(S_{\tilde{\omega}})$.

We will now argue that either $\text{BS}_\zeta^+ = \varnothing$ or $\text{BS}_\zeta^+ = \{\tilde{\omega}_\zeta\}$ consists of a single point. Using the notation from Appendix B, the zeros of $\mathcal{N}_{\tilde{\omega},\zeta}$ are precisely the solutions of the transcendental equation

$$\tan(\zeta) = \frac{B}{A} \doteq \Theta(\tilde{\omega}) \quad \text{(D1)}$$

in the upper half complex plane, $\text{Im}[\tilde{\omega}] > 0$ and $\zeta \in [0, \pi)$, together with their complex conjugates. A and B are as in IV A. When $\zeta = \pi/2$, we interpret any $\tilde{\omega}$ at which $\Theta(\tilde{\omega})$ has a pole as a solution of (D1). When written out explicitly, the rhs of (D1) is a ratio of products of gamma functions with $\tilde{\omega}$-dependent parameters. Its main characteristics are that, for generic values of the parameters, it has only the simple zeros at $\tilde{\omega}_\pm(n)$ and the simple poles at $\tilde{\omega}^\pm(n)$ for $n = 0, 1, 2, \dots$, where

$$\tilde{\omega}_\pm(n) = \pm\frac{k}{\ell} - k\Omega_{\mathcal{H}} - 2i(n+\beta)\frac{(r_+ \mp r_-)}{\ell^2}, \quad \text{(D2)}$$

$$\tilde{\omega}^\pm(n) = \pm\frac{k}{\ell} - k\Omega_{\mathcal{H}} - 2i(n+1-\beta)\frac{(r_+ \mp r_-)}{\ell^2}, \quad \text{(D3)}$$

as well as the asymptotic behavior

$$\Theta(\tilde{\omega}) = \frac{\Gamma(\sqrt{\mu^2+1})}{\Gamma(-\sqrt{\mu^2+1})} \left(\frac{\ell^4(-i\tilde{\omega})^2}{4(r_+^2 - r_-^2)}\right)^{-\sqrt{\mu^2+1}} \times [1 + \mathcal{O}(|\tilde{\omega}|^{-1})] \quad \text{(D4)}$$

for $|\tilde{\omega}| \to \infty$, which follows from the Stirling asymptotic formula. The branch of the power function must agree with the principal branch when $-i\tilde{\omega} > 0$. Some of the poles or zeros may merge for special values of the parameters.

The zeros and poles of $\Theta(\tilde{\omega})$ give us the explicit solutions of (D1), respectively, for $\zeta = 0$ (Dirichlet) and $\zeta = \pi/2$ (Neumann) boundary conditions. For a general value of ζ, the transcendental nature of Eq. (D1) prevents us from giving explicit solutions. Although this equation could certainly be solved numerically for any value of the parameters μ^2, ℓ, r_+, r_-, and k describing the BTZ black hole and the scalar field, we can make the following qualitative conclusions.

Since ζ is always real, $\tilde{\omega} \in \mathbb{C}$ for which $\Theta(\tilde{\omega}) \notin \mathbb{R}$ is never a solution of (D1). On the other hand, when $\Theta(\tilde{\omega})$ is real, Eq. (D1) is certainly satisfied for $\zeta = \arctan(\Theta(\tilde{\omega}))$. Thus, for fixed ζ, the solutions of (D1) exist and lie on the lines of real phase $\arg[\Theta(\tilde{\omega})] = 0$ or π. Roughly speaking, lines of the real phase stretch between the poles and zeros of $\Theta(\tilde{\omega})$, also with one such line stretching to ∞ through the upper half plane from the pole with the largest $\text{Im}[\tilde{\omega}]$, as can be deduced by (D4).

In the case $\mu^2 \geq 0$, only the $\zeta = 0$ (Dirichlet) boundary condition is allowed (see Sec. III C), which corresponds to zeros of $\Theta(\tilde{\omega})$. As can be seen from (D2), all of the zeros are confined to the lower half complex plane, and so there are no solutions of (D1) with $\text{Im}[\tilde{\omega}] > 0$. Therefore, in this case, there are no bound state frequencies, $\text{BS}_\zeta^+ = \varnothing$.

When $-1 < \mu^2 < 0$, all the poles and zeros lie in the lower half complex plane, and closest to the real axis is the pole at

$$\tilde{\omega}^+(0) = \frac{k}{\ell} - k\Omega_{\mathcal{H}} - i(1 - \sqrt{\mu^2+1})\frac{(r_+ - r_-)}{\ell^2}.$$

The solutions with $\text{Im}[\tilde{\omega}] > 0$ must lie on the single line of the real phase stretching from this pole and are parametrized by $\zeta \in [\zeta_*, \pi)$. This phase line crosses the $\text{Im}[\tilde{\omega}] = 0$ line at $\tilde{\omega} = 0$, where

$$\Theta(0) = \frac{\Gamma(2\beta-1)|\Gamma(1-\beta+i\ell\frac{k}{r_+})|^2}{\Gamma(1-2\beta)|\Gamma(\beta+i\ell\frac{k}{r_+})|^2} = \tan(\zeta_*).$$

Since $\beta \in (\frac{1}{2}, 1)$, then $\zeta_* \in (\frac{\pi}{2}, \pi)$. Qualitatively, we also see that the solution $\tilde{\omega} = \tilde{\omega}_\zeta$ is simple[1] and of course isolated. Hence, in this case, $\text{BS}_\zeta = \{\tilde{\omega}_\zeta, \bar{\tilde{\omega}}_\zeta\}$. The real and

---

[1]This could be rigorously established by a careful application of the *argument principle*, which we omit for brevity, to the function $f(\tilde{\omega}) = \tan(\zeta) - \Theta(\tilde{\omega})$, which confirms the existence of a single simple zero $\tilde{\omega}_\zeta \in \text{Im}[\tilde{\omega}]$ provided the integrals $\oint \frac{f'(\tilde{\omega})}{f(\tilde{\omega})} \frac{d\tilde{\omega}}{2\pi i}$ stabilize to the value 1 over a sequence of simple closed and positive contours of which the interior exhausts the upper half complex plane.

imaginary parts of $\tilde{\omega}_\zeta$ are plotted as a function of ζ in Fig. 2 for a particular value of other parameters.

APPENDIX E: CHECK OF HYPOTHESIS (S3)

In this Appendix, we show that the hypothesis (S3) of Appendix A is verified, namely that there exists a spectral measure for the linearized pencil $\mathbf{S}_{\tilde{\omega}}$ in (A3).

Following the notation of Appendix A, the inner product space $\mathcal{K} = (\mathcal{H}^2, [-, -])$, with bounded bilinear form $[\mathbf{v}, \mathbf{u}] = (\mathbf{v}, \mathbf{R}\mathbf{u})$, defines a *Krein space* [49,50], that is, a Banach (in this case Hilbert) space with a bounded Hermitian scalar product that need not be positive definite. The spectral problem of the linear operator pencil $\mathbf{S}_{\tilde{\omega}} = \mathbf{P} + \tilde{\omega}\mathbf{R}$ is equivalent to the standard spectral problem $-\mathbf{R}^{-1}\mathbf{P} = \tilde{\omega}\mathbf{I}$, where the operator $\mathbf{A} \doteq -\mathbf{R}^{-1}\mathbf{P}$ is now self-adjoint with respect to the Krein space scalar product $[-, -]$.

Unfortunately, unlike the Hilbert space case, there is no spectral theorem available for an arbitrary self-adjoint operator on a Krein space. However, there are some special cases where the spectral theorem, and hence the existence of a spectral measure $\mathbf{E}(\nu)$ as requested by hypothesis (S3) in Appendix A, is available. One such case is when $\mathbf{A}$ is *definitizable*, that is, when there exists a degree k polynomial $p(\tilde{\omega})$ with real coefficients such that $[\mathbf{u}, p(\mathbf{A})\mathbf{u}] \geq 0$ for each $\mathbf{u} \in D(\mathbf{A}^k)$. The corresponding spectral theorem can be found in Refs. [49,51]. Below, we give a brief argument verifying that the operator $\mathbf{A}$ discussed in Appendixes A and B is definitizable, hence fulfilling hypothesis (S3).

The argument is as follows. First, suppose that there exists a definitizable closed restriction $\mathbf{A}_0$ of $\mathbf{A}$ to a smaller domain $D(\mathbf{A}_0) \subset D(\mathbf{A})$, since $[\mathbf{u}, (-\mathbf{A}_0)\mathbf{u}] \geq 0$ for all $\mathbf{u} \in D(\mathbf{A}_0)$. While $\mathbf{A}_0$ itself may no longer be self-adjoint, the Krein space analog of the Friedrichs extension [52] then gives us a self-adjoint extension $\mathbf{A}_1$, that still satisfies $[\mathbf{u}, (-\mathbf{A}_1)\mathbf{u}] \geq 0$ on its domain. Second, since $\mathbf{A}$ is essentially defined by an ordinary differential operator, the difference of the resolvents

$$(\mathbf{A}_1 - \tilde{\omega}\mathbf{I})^{-1} - (\mathbf{A} - \tilde{\omega}\mathbf{I}) \qquad \text{(E1)}$$

is an operator of finite rank, which is described by the so-called *Krein resolvent formula* ([53], Sec. 106). The finiteness of the rank comes from the fact that an ordinary differential operator has a finite-dimensional space of solutions. Finally, it is also known that when at least one of the Krein self-adjoint operators $\mathbf{A}_1$ or $\mathbf{A}$ is definitizable and the difference of their resolvents (E1) has finite rank for at least one $\tilde{\omega}$ common to both resolvent sets, then both operators are definitizable [54].

Recall that we are working with $\mathcal{H} = L^2((0, 1); \mathcal{J}(z)\mathrm{d}z)$, and consider $\mathbf{u} = [\Psi \quad \Phi]^T \in D(\mathbf{A}_0)$ consisting of smooth functions with compact support. Unwinding all the definitions from Appendixes A and B, and writing out $[\mathbf{u}, (-\mathbf{A})\mathbf{u}]$ explicitly and using integration by parts, we get

$$
\begin{aligned}
&(\Psi, (-\mathcal{J}^{-1}L_{\tilde{\omega}=0})\Psi) + (\Phi, \mathcal{R}_2\Phi) \\
&= \int_0^1 \mathrm{d}z \Bigg[z \left| \frac{\mathrm{d}\Psi(z)}{\mathrm{d}z} \right|^2 \\
&\quad + \left(\frac{\ell^2 k^2}{4r_+^2} + \frac{\mu^2}{4(1-z)} \right) |\Psi(z)|^2 + \frac{\ell^4 \mathcal{J}(z)|\Phi(z)|^2}{4(r_+^2 - r_-^2)} \Bigg].
\end{aligned}
$$

When $\mu^2 \geq 0$, all the terms appearing under the integral are manifestly non-negative, meaning that so is the whole integral. When $-1 < \mu^2 < 0$, the integrand is still non-negative. This can be proven by observing that the term proportional to k^2 in the integrand is strictly greater than 0 and by showing positivity for $-\mathcal{J}^{-1}L_{\tilde{\omega}=0}$ when $k = 0$. Yet, in this case, in view of (B1) and of the results of Appendix A, $-\mathcal{J}^{-1}L_{\tilde{\omega}=0}$ is a self-adjoint operator with a strictly positive spectrum, which is tantamount to saying that it is a positive operator. Thus, the restriction of $\mathbf{A}$ to $D(\mathbf{A}_0)$ does satisfy $[\mathbf{u}, (-\mathbf{A}_0)\mathbf{u}] \geq 0$ for all $\mathbf{u} \in D(\mathbf{A}_0)$. By the preceding reasoning, this finally implies that $\mathbf{A}$ is definitizable.

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