Double-winding Wilson loops in the $SU(N)$ Yang-Mills theory

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(Received 1 July 2017; published 13 November 2017)

We consider double-winding, triple-winding, and multiple-winding Wilson loops in the $SU(N)$ Yang-Mills gauge theory. We examine how the area-law falloff of the vacuum expectation value of a multiple-winding Wilson loop depends on the number of color N. In sharp contrast to the difference-of-areas law recently found for a double-winding $SU(2)$ Wilson loop average, we show irrespective of the spacetime dimensionality that a double-winding $SU(3)$ Wilson loop follows a novel area law which is neither difference-of-areas nor sum-of-areas law for the area-law falloff and that the difference-of-areas law is excluded and the sum-of-areas law is allowed for $SU(N)$ ($N \geq 4$), provided that the string tension obeys the Casimir scaling for the higher representations. Moreover, we extend these results to arbitrary multiple-winding Wilson loops. Next, we argue that the area law follows a novel law, which is neither sum-of-areas nor difference-of-areas law when $N \geq 3$. In fact, such a behavior is exactly derived in the $SU(N)$ Yang-Mills theory in the two-dimensional spacetime. Finally, we introduce new Wilson loops whose averages are expected to follow the difference-of-areas law even in the $SU(N)$ Yang-Mills theory for $N \geq 3$.

DOI: 10.1103/PhysRevD.96.105011

I. INTRODUCTION

In this paper we discuss double-winding, triple-winding and more general multiple-winding Wilson loops [1] in the $SU(N)$ Yang-Mills gauge theory [2]. The "double-winding" Wilson loops consist of the contours which wind once around a loop C_1 and once around a loop C_2 where the two coplanar loops share one point in common and where C_2 lies entirely in the minimal area of C_1. Recently, the $SU(2)$ case for the double-winding Wilson loop [3] has been investigated to study the mechanism for quark confinement. See, e.g., [4,5] for reviews of quark confinement. It has been found that the area-law falloff of the vacuum expectation value (or average) of the double-winding Wilson loop follows a difference-of-areas law [3]. In this paper we examine how the area-law falloff of double-winding, triple-winding, and arbitrary multiple-winding Wilson loop averages depend on the number of color N in the $SU(N)$ Yang-Mills theory.

First, we discuss the case where the two loops C_1 and C_2 are identical for a double-winding Wilson loop and derive the exact operator relation which relates the double-winding Wilson loop operator in the fundamental representation to a single Wilson loop in the higher-dimensional representations depending on N. By taking the average of the relation, we find the relation among the Wilson loop averages. We find that the difference-of-areas law for the area-law falloff of a double-winding Wilson loop average recently claimed for $N = 2$ is excluded for $N \geq 3$, provided that the string tension obeys the Casimir scaling [6] for the higher representations. We show that a double-winding $SU(3)$ Wilson loop average follows a novel area law which is neither difference of areas nor sum of areas, while the difference-of-areas is excluded and the sum-of-areas law is allowed for $SU(N)$ ($N \geq 4$), although the double-winding $SU(2)$ Wilson loop average is consistent with the difference-of-areas law.

Next, we extend the analysis to a multiple-winding Wilson loop in the $SU(N)$ Yang-Mills gauge theory. We give a physical motivation to consider the multiple-winding Wilson loop and give the physical interpretation of the obtained results. This enables us to explain how the $SU(2)$ case is so different from the other cases.

These results are derived from the group theoretical consideration in the case where all loops are identical. In this case, a m-times-winding Wilson loop operator in the fundamental representation is rewritten as a linear combination of Wilson loop operators in the higher representations which are distinct from the fundamental representation. The results do not depend on the dimensionality of spacetime. This provides us with the useful information to analyze the area-law falloff of the multiple-winding Wilson loop average.

Next, we discuss the case where the two loops are distinct for a double-winding Wilson loop. In this case, we argue that the area law follows a novel law $(N-3)S_2/(N-1) + S_1$ with S_1 and S_2 ($S_2 < S_1$) being the minimal areas spanned respectively by the loops C_1 and C_2, which is neither sum-of-areas ($S_1 + S_2$) nor difference-of-areas ($S_1 - S_2$) law when $N \geq 3$. Indeed, we show that this behavior is exactly derived in the $SU(N)$ Yang-Mills theory in the two-dimensional spacetime. These results are consistent with the result obtained recently based on the

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leading order calculations of the strong-coupling expansion within the framework of the lattice gauge theory [7], which does not depend on the dimensionality of spacetime.

Finally, we introduce new Wilson loops whose averages are expected to follow the difference-of-areas law. In the $SU(N)$ Yang-Mills theory with $N \geq 3$, such Wilson loops can be used just as the double-winding Wilson loop was used in the $SU(2)$ Yang-Mills theory.

The results obtained in this paper will give useful information to investigate the true mechanism for quark confinement, which is to be tackled in the subsequent works.

This paper is organized as follows. In Secs. II and III, we give the main results of this paper with their physical interpretation. In Sec. II, we discuss the area-law falloff for a double-winding Wilson loop for the two identical loops. In Sec. III, we extend our analysis to multiple-winding Wilson loops for the m identical loops. In Sec. IV, we treat a double-winding Wilson loop with two distinct loops in the $SU(N)$ Yang-Mills theory in the two-dimensional spacetime and with Wilson loops whose expectation values are expected to follow the difference-of-areas law even in the $SU(N)$ Yang-Mills theory. Some of the details of the proofs of the main results are given in Appendices.

II. DOUBLE-WINDING WILSON LOOP WITH IDENTICAL LOOPS

For a single closed loop C, the Wilson loop operator in the fundamental representation is defined by

$$W(C) := \frac{1}{N}\mathrm{tr}[U_F(C)], \tag{1}$$

where $U_F(C)$ is the parallel transporter along the loop C, i.e., the path-ordered product of the group element along the loop C:

$$U_F(C) := P\exp\left\{ig\int_C dx^\mu \mathcal{A}_\mu\right\} \in G. \tag{2}$$

For two closed loops C_1 and C_2, a double-winding Wilson loop operator in the fundamental representation is defined by

$$W(C_1 \times C_2) := \frac{1}{N}\mathrm{tr}[U_F(C_1)U_F(C_2)]. \tag{3}$$

See Fig. 1.

In what follows, we consider what type of the area law follows for the double-winding Wilson loop average, irrespective of the lattice and continuum formulations. For this purpose, we consider the case of two identical loops, i.e., $C_1 = C_2 = C$. In the identical case, the double-winding Wilson loop operator is written as

$$W(C \times C) := \frac{1}{N}\mathrm{tr}[U_F(C)U_F(C)]. \tag{4}$$

The two loops C_1 and C_2 have the same direction. The two identical loops correspond to the world line of a pair of quarks in the fundamental representation. The direct product of two fundamental representations is decomposed into the irreducible representations of the color group $SU(N)$.

For $SU(2)$, the product of two fundamental representations $\mathbf{2}$ is decomposed into a singlet $\mathbf{1}$ and a triplet $\mathbf{3}$, i.e., adjoint representation:

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} = \mathbf{2} \otimes \mathbf{2}^*. \tag{5}$$

Since the color singlet state must not be confined and could be observed, the string tension must vanish and the area law would disappear. In fact, the double-winding Wilson loop operator in the fundamental representation is decomposed into a trivial term and the Wilson loop operator $W(C)_{\mathrm{Adj}}$ in the adjoint representation for a single Wilson loop C (see Appendix A for the derivation) as

$$W(C \times C) = -\frac{1}{2}\mathbf{1} + \frac{3}{2}W(C)_{\mathrm{Adj}}. \tag{6}$$

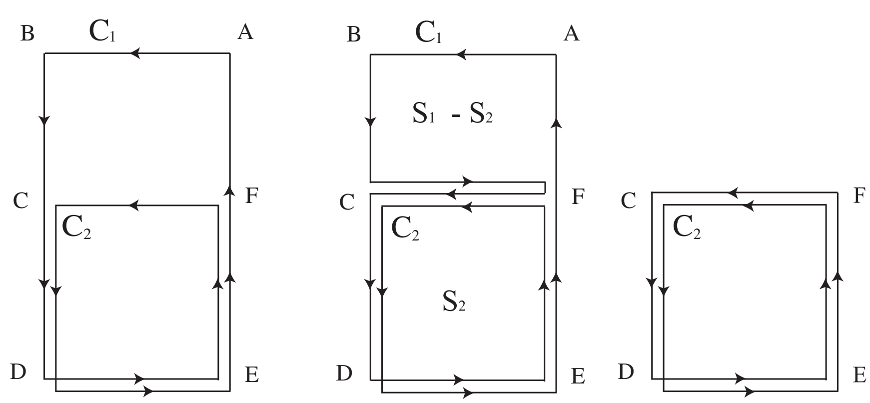


FIG. 1. The leftmost figure is a double-winding loop with two closed loops C_1 and C_2 winding in the same direction and the middle one is its deformation. The rightmost figure is the case of two identical loops.

This operator identity for the Wilson loops leads to the relation for their averages:

$$\langle W(C\times C)\rangle = -\frac{1}{2}+\frac{3}{2}\langle W(C)_{\mathrm{Adj}}\rangle. \tag{7}$$

The adjoint Wilson loop average exhibits the area law in the intermediate distance, since the adjoint quarks are screened by gluons in the long distance. In the intermediate region, we have

$$\langle W(C\times C)\rangle = -\frac{1}{2}+b_2 e^{-\sigma_{\mathrm{Adj}}S}+\cdots. \tag{8}$$

This is consistent with the difference-of-areas behavior and contradicts with the sum-of-areas one, as pointed out by [3].

The quadratic Casimir operator of a representation with an index $J=\frac{1}{2},1,\frac{3}{2},\ldots$ for $SU(2)$ is given by

$$C_2(J)=J(J+1), \tag{9}$$

which has the specific value for $J=\frac{1}{2}$ and $J=1$:

$$C_2\left(\frac{1}{2}\right)=\frac{3}{4},\qquad C_2(1)=2. \tag{10}$$

Suppose that the Casimir scaling for the string tension holds. Then we find that the adjoint string tension σ_{Adj} is obtained from the fundamental string tension σ_{F} using the ratio of the quadratic Casimir operators:

$$\sigma_{\mathrm{Adj}}=\frac{C_2(1)}{C_2(\frac{1}{2})}\sigma_{\mathrm{F}}=\frac{8}{3}\sigma_{\mathrm{F}}. \tag{11}$$

The fundamental representation and its conjugate representation are district in general. If they happen to coincide, the representation is called a *real representation*. Otherwise, they are called the *complex representation*. The group $SU(N)$ allows complex representations for $N\geq 3$. The $SU(2)$ group is very special, since **2** and $\mathbf{2}^*$ are equivalent:

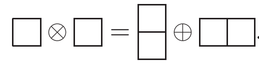
$$\mathbf{2}=\mathbf{2}^*. \tag{12}$$

In $SU(2)$, therefore, 2 quarks qq can make a singlet: $\mathbf{2}\otimes\mathbf{2}^*=\mathbf{2}\otimes\mathbf{2}=\mathbf{3}\oplus\mathbf{1}$. Thus, the composite particle $qq=q\bar{q}$ is regraded as a meson $q\bar{q}$ and a baryon qq simultaneously and there is no distinction between mesons and baryons for the $SU(2)$ group.

For $SU(3)$, the product of two fundamental representations **3** is decomposed into an antitriplet $\mathbf{3}^*$ and a sextet representation **6**:

$$\mathbf{3}\otimes\mathbf{3}=\mathbf{3}^*\oplus\mathbf{6}. \tag{13}$$

This is represented as the Young diagram:

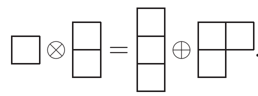


(14)

For $SU(3)$, there is no color singlet for a pair of two quarks, in sharp contrast with a pair of quark and antiquark where

$$\mathbf{3}\otimes\mathbf{3}^*=\mathbf{1}\oplus\mathbf{8}=\mathbf{3}^*\otimes\mathbf{3}, \tag{15}$$

which is represented as the Young diagram:



(16)

In fact, the double-winding Wilson loop operator in the fundamental representation **3** is decomposed into the Wilson loop $W^*(C)=W(C)_{[0,1]}$ in the (anti)fundamental representation $\mathbf{3}^*$ with the Dynkin indices [0, 1] and the Wilson loop operator $W(C)_{[2,0]}$ in the sextet representation **6** with the Dynkin indices [2, 0] (see Appendix A for the derivation)[1]:

$$W(C\times C)=-W(C)_{[0,1]}+2W(C)_{[2,0]}. \tag{20}$$

This identity leads to the relation for the average:

$$\langle W(C\times C)\rangle = -\langle W(C)_{[0,1]}\rangle+2\langle W(C)_{[2,0]}\rangle. \tag{21}$$

In the confinement phase, both Wilson loop averages $\langle W(C)_{[0,1]}\rangle$ and $\langle W(C)_{[2,0]}\rangle$ exhibit the area law for the loop C of any size larger than a critical size below which the Coulomb like behavior is dominant, since they are not screened by gluons which belong to the adjoint representation **8** with the Dynkin indices [1, 1]. Therefore, we have

[1]It is also possible to rewrite

$$2\mathrm{tr}(U_{[2,0]})=\mathrm{tr}(U_{[1,0]}^2)+(\mathrm{tr}[U_{[1,0]}])^2, \tag{17}$$

which is equal to

$$4W(C)_{[2,0]}=W(C\times C)+3W(C)_{[1,0]}^2. \tag{18}$$

This operator relation leads to the relation for the average:

$$4\langle W(C)_{[2,0]}\rangle=\langle W(C\times C)\rangle+3\langle W(C)_{[1,0]}^2\rangle. \tag{19}$$

This relation was used to examine the Casimir scaling for the representation [2, 0] on the lattice; see Eq. (5.15) of [6].

$$\langle W(C\times C)\rangle = a_3 e^{-\sigma_{\rm F}S} + b_3 e^{-\sigma_{[2,0]}S} + \cdots \quad (a_3<0, b_3>0). \tag{22}$$

In the intermediate region, we assume the Casimir scaling to estimate the string tension σ_R in the higher-dimensional representation R. The dimension of the representation with the Dynkin indices $[m,n]$ for $SU(3)$ is given by

$$D([m,n]) = \frac{1}{2}(m+1)(n+1)(m+n+2). \tag{23}$$

The quadratic Casimir operator of the representation with the Dynkin indices $[m,n]$ for $SU(3)$ is given by [8]

$$C_2([m,n]) = \frac{1}{3}(m^2+mn+n^2)+m+n, \tag{24}$$

with the specific values:

$$C_2([0,0])=0, \qquad C_2([1,0])=C_2([0,1])=\frac{4}{3},$$
$$C_2([2,0])=\frac{10}{3}, \qquad C_2([1,1])=3, \ldots. \tag{25}$$

Assuming the Casimir scaling for the string tension, therefore, the string tension $\sigma_{[2,0]}$ of the representation [2, 0] is obtained as the ratio to the fundamental string tension $\sigma_{\rm F}=\sigma_{[1,0]}$:

$$\sigma_{[2,0]} = \frac{C_2([2,0])}{C_2([1,0])}\sigma_{\rm F} = \frac{5}{2}\sigma_{\rm F}. \tag{26}$$

Therefore, the area-law falloff of the double-winding $SU(3)$ Wilson loop average is given in the intermediate region by

$$\langle W(C\times C)\rangle = a_3 e^{-\sigma_{\rm F}S} + b_3 e^{-\frac{5}{2}\sigma_{\rm F}S} + \cdots \quad (a_3<0, b_3>0). \tag{27}$$

In the asymptotic region, on the other hand, the string tension σ_R for quarks in the representation R is determined only through the N-ality k of the representation R (see, e.g., Sec. X.5 of [5]). Notice that the two representations $\mathbf{3}^*=[0,1]$ and $\mathbf{6}=[2,0]$ have the same N-ality, $k=2$. Therefore, the two string tensions $\sigma_{[0,1]}$ and $\sigma_{[2,0]}$ converge to the same asymptotic value which is expected to be the fundamental string tension:

$$\sigma_{[0,1]}, \sigma_{[2,0]} \to \sigma_{\rm F} = \sigma_{[1,0]}. \tag{28}$$

Thus, the area-law falloff of the double-winding $SU(3)$ Wilson loop average with two identical loops has the same dominant behavior as that of a single-winding Wilson loop average in the fundamental representation.

$$\langle W(C\times C)\rangle \simeq c_3 e^{-\sigma_{\rm F}S}. \tag{29}$$

This is not consistent with the difference-of-areas behavior and contradicts also with the sum-of-areas law.

For $SU(N)$ $(N\geq 4)$, we have the decomposition:

$$\mathbf{N}\otimes\mathbf{N} = \left(\frac{\mathbf{N}^2-\mathbf{N}}{2}\right)_A \oplus \left(\frac{\mathbf{N}^2+\mathbf{N}}{2}\right)_S, \quad (N\geq 4). \tag{30}$$

The decomposition (30) shows that the $N=3$ case is a bit special: $\mathbf{3}\otimes\mathbf{3}=\mathbf{3}^*_A\oplus\mathbf{6}_S$, where the antisymmetric part belongs to $\mathbf{3}^*$ (not $\mathbf{3}$). In any case, the color singlet $\mathbf{1}$ does not occur for $N\geq 3$. This excludes the difference-of-areas law for the double-winding Wilson loop average for $N\geq 3$, because the difference-of-areas law contradicts with this fact in the identical case. The difference-of-areas law is possible only when the color singlet $\mathbf{1}$ occurs in the irreducible decomposition of two quarks.

In fact, the double-winding Wilson loop operator in the fundamental representation $\boldsymbol{N}$ is decomposed into the Wilson loop $W(C)_{[0,1,0,\ldots,0]}$ in the representation $\frac{1}{2}\mathbf{N}(\mathbf{N}-1)$ with the Dynkin indices $[0,1,0,\ldots,0]$ and the Wilson loop operator $W(C)_{[2,0,\ldots,0]}$ in the representation $\frac{1}{2}\mathbf{N}(\mathbf{N}+1)$ with the Dynkin indices $[2,0,\ldots,0]$ (see Appendix A for the derivation):

$$W(C\times C) = -\frac{N-1}{2}W(C)_{[0,1,0,\ldots,0]} + \frac{N+1}{2}W(C)_{[2,0,\ldots,0]}. \tag{31}$$

This operator relation leads to the relation for the average:

$$\langle W(C\times C)\rangle = -\frac{N-1}{2}\langle W(C)_{[0,1,0,\ldots,0]}\rangle + \frac{N+1}{2}\langle W(C)_{[2,0,\ldots,0]}\rangle. \tag{32}$$

The Wilson loop averages $\langle W(C)_{[0,1,0,\ldots,0]}\rangle$ and $\langle W(C)_{[2,0,\ldots,0]}\rangle$ exhibit the area-law for any size larger than a critical size below which the Coulomb-like behavior is dominant, since they are not screened by gluons which belong to the adjoint representation $\boldsymbol{N^2-1}$ with the Dynkin indices $[1,0,\ldots,0,1]$. Therefore, we have

$$\langle W(C\times C)\rangle = a_N e^{-\sigma_{[0,1,\ldots,0]}S} + b_N e^{-\sigma_{[2,0,\ldots,0]}S} + \cdots \quad (a_N<0, b_N>0). \tag{33}$$

In the intermediate region, we assume the Casimir scaling for the string tension σ_R in the higher-dimensional representation R. It is shown that the dimension of the representation with the Dynkin indices $[m_1,\ldots,m_{N-1}]$ for $SU(N)$ is given by [8]

$$
\begin{aligned}
D([m_1,\ldots,m_{N-1}]) \\
= &\frac{1}{2!\cdots(N-1)!}(m_1+1)(m_1+m_2+2)\cdots \\
&\times(m_1+\cdots+m_{N-1}+N-1) \\
&\times(m_2+1)(m_2+m_3+2)\cdots(m_2+\cdots+m_{N-1}+N-2) \\
&\times\cdots\times(m_{N-2}+m_{N-1}+2)(m_{N-1}+1),
\end{aligned} \tag{34}
$$

and the quadratic Casimir operator of the representation with the Dynkin indices $[m_1,\ldots,m_{N-1}]$ for $SU(N)$ is given by [9]

$$
C_2([m_1,\ldots,m_{N-1}]) = \frac{1}{2N}\sum_{k=1}^{N-1}\left[N(N-k)km_k + k(N-k)m_k^2 + \sum_{\ell=0}^{k-1}2\ell(N-k)m_\ell m_k\right], \tag{35}
$$

with the specific values:

$$
\begin{aligned}
C_2([0,\ldots,0]) &= 0, C_2([1,0,\ldots,0]) = \frac{N^2-1}{2N}, \\
C_2([0,1,0,\ldots,0]) &= \frac{(N-2)(N+1)}{N}, \\
C_2([2,0,\ldots,0]) &= \frac{(N+2)(N-1)}{N}, \ldots.
\end{aligned} \tag{36}
$$

Under the Casimir scaling, the area-law falloff of the double-winding $SU(N)$ Wilson loop average is described in the intermediate region by

$$
\begin{aligned}
\langle W(C\times C)\rangle = &a_N \exp\left(-2\frac{N-2}{N-1}\sigma_F S\right) \\
&+ b_N \exp\left(-2\frac{N+2}{N+1}\sigma_F S\right) + \cdots \\
&(a_N<0, b_N>0).
\end{aligned} \tag{37}
$$

Notice that the first term becomes dominant on the right-hand side for large S.

In the asymptotic region, on the other hand, the string tension σ_R for quarks in the representation R is determined only through the N-ality k of the representation R. Notice that the two representations $\frac{1}{2}\mathbf{N}(\mathbf{N}-1)=[0,1,0,\ldots,0]$ and $\frac{1}{2}\mathbf{N}(\mathbf{N}+1)=[2,0,\ldots,0]$ have the same N-ality $k=2$, since the Young diagram of (30) is the same as the $SU(3)$ case (14). [The N-ality of a representation of $SU(N)$ is equal to the number of boxes in the corresponding Young tableaux (mod N).] Therefore, the two string tensions $\sigma_{[0,1,0,\ldots,0]}$ and $\sigma_{[2,0,\ldots,0]}$ converge to the same asymptotic value, i.e., σ_k with $k=2$:

$$
\sigma_{[0,1,0,\ldots,0]}, \sigma_{[2,0,\ldots,0]} \to \sigma_k(k=2). \tag{38}
$$

If we assume the Casimir scaling also for the asymptotic string tension,

$$
\sigma_k = \frac{k(N-k)}{N-1}\sigma_{\mathrm{F}}, \tag{39}
$$

then the area-law falloff of the double-winding $SU(N)$ Wilson loop average with two identical loops has the dominant behavior in the intermediate and asymptotic regions given by

$$
\langle W(C\times C)\rangle \simeq c_N \exp\left(-2\frac{N-2}{N-1}\sigma_F S\right). \tag{40}
$$

If we adopt another scaling known as the *sine-law scaling* suggested by M theory fivebrane version of QCD and softly broken $\mathcal{N}=2$ [10],

$$
\sigma_k = \frac{\sin\frac{\pi k}{N}}{\sin\frac{\pi}{N}}\sigma_F, \tag{41}
$$

then the asymptotic behavior is given by

$$
\langle W(C\times C)\rangle \simeq c_N e^{-2\cos\frac{\pi}{N}\sigma_F S}. \tag{42}
$$

In any case, the result is not consistent with the difference-of-areas behavior and contradicts also with the sum-of-areas law. For $N\geq 3$, the area-law falloff obeys neither difference-of-areas nor sum-of-areas law.

In the large N limit, however, the result is consistent with the sum-of-areas law in the intermediate and asymptotic regions:

$$
\langle W(C\times C)\rangle \simeq e^{-k\sigma_F S}(k=2). \tag{43}
$$

However, this result is interpreted as just coming from the N-ality, rather than reflecting the dynamics of the Yang-Mills theory.

III. MULTIPLE-WINDING WILSON LOOP WITH IDENTICAL LOOPS

We can extend the above considerations for a double-winding Wilson loop to a triple-winding and more general multiple-winding Wilson loops.

For $SU(3)$, we introduce a triple-winding Wilson loop. In the identical case, the triple-winding Wilson loop average for $SU(3)$ is related to the baryon potential. Baryons are color singlet composite particles to be observed in experiments. Therefore, the baryon potential should be nonconfining and the string tension must be zero. Indeed, we have

$$\begin{aligned}\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}&=(\mathbf{3}\otimes\mathbf{3})\otimes\mathbf{3}\\&=(\mathbf{3}^*_A\oplus\mathbf{6}_S)\otimes\mathbf{3}\\&=\mathbf{3}^*_A\otimes\mathbf{3}\oplus\mathbf{6}_S\otimes\mathbf{3}\\&=\mathbf{1}_A\oplus\mathbf{8}_{\mathrm{MA}}\oplus\mathbf{8}_{\mathrm{MS}}\oplus\mathbf{10}_S.\end{aligned}\tag{44}$$

Thus, we can identify the baryon with the color singlet $\mathbf{1}_A$:

$$B=\varepsilon_{abc}q^aq^bq^c.\tag{45}$$

Thus, for the gauge group $G=SU(3)$, a baryon is constructed from three quarks as the color singlet object. Therefore, both baryons and mesons are colorless combinations to be observed, whereas the respective color and the colorful particle as a constituent cannot be observed according to the hypothesis of color confinement. Thus, the Wilson loop average with a trivial representation is most dominant and does not exhibit the area law, that is to say, string tension is zero.

For $SU(N)$ $(N\geq 4)$, a baryon cannot be constructed from three quarks, since the three product does not contain the singlet for $N\geq 4$:

$$\begin{aligned}\mathbf{N}\otimes\mathbf{N}\otimes\mathbf{N}=&\frac{1}{3}\mathbf{N}(\mathbf{N}+\mathbf{1})(\mathbf{N}-\mathbf{1})\oplus\frac{1}{3}\mathbf{N}(\mathbf{N}+\mathbf{1})(\mathbf{N}-\mathbf{1})\\&\oplus\frac{1}{6}\mathbf{N}(\mathbf{N}+\mathbf{1})(\mathbf{N}+\mathbf{2})\oplus\frac{1}{6}\mathbf{N}(\mathbf{N}-\mathbf{1})(\mathbf{N}-\mathbf{2}).\end{aligned}\tag{46}$$

For $SU(4)$, incidentally, we can check the following results:

$$\begin{aligned}\mathbf{4}\otimes\mathbf{4}^*&=\mathbf{15}\oplus\mathbf{1},\\\mathbf{4}\otimes\mathbf{4}&=\mathbf{10}_S\oplus\mathbf{6}_A,\\\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4}&=(\mathbf{10}_S\oplus\mathbf{6}_A)\otimes\mathbf{4}\\&=\mathbf{20}_{\mathrm{MS}}\oplus\mathbf{20}_S\oplus\mathbf{20}_{\mathrm{MA}}\oplus\mathbf{4}_A.\end{aligned}\tag{47}$$

For $SU(4)$, a quark-antiquark pair $q\bar{q}$ can form a color singlet, while the three quarks qqq is unable to form a color singlet. This is because there are $\frac{4\cdot3\cdot2}{3\cdot2\cdot1}=4$ ways of forming a completely antisymmetric wave function using 3 colors from 4 colors. For $SU(N)$, therefore, we need N quarks to make a color singlet:

$$B=\varepsilon_{a_1\cdots a_N}q_1^a\cdots q_N^a,\quad (N\geq 3).\tag{48}$$

This is examined by considering the N-times-winding Wilson loop operator.

In view of these, we consider the general multiple-winding Wilson loop operator of m-times-winding loops, $W(C_1\times C_2\times\ldots\times C_m)$. We show that the m-times-winding Wilson loop operator $W(C\times C\times\ldots\times C)=W(C^m)$ in the fundamental representation is written as the linear combination of a single Wilson loop operator $W_{R_\ell}(C)$ in higher representations R_ℓ when all loops are identical:

$$W(C^m)=\sum_{\ell=1}^{\min(m,N)}(-1)^{\ell+1}\frac{D(R_\ell)}{N}W_{R_\ell}(C),\tag{49}$$

where the representation R_ℓ is specified by the Dynkin indices:

$$R_\ell\coloneqq\begin{cases}[m,0,\ldots,0] & \text{for } \ell=1,\\ [\underbrace{m-\ell,0,\ldots,0,1}_{\ell},0,\ldots,0] & \text{for } \ell=2,\ldots,\min(m,N-1),\\ [m-N,0,\ldots,0] & \text{for } \ell=N, m\geq N,\end{cases}\tag{50}$$

and $D(R_\ell)$ is the dimension of R_ℓ, i.e.,

$$D(R_\ell)=\frac{(N+m-\ell)!}{m(\ell-1)!(m-\ell)!(N-\ell)!}.\tag{51}$$

The proof is given in Appendix B. For a given $SU(N)$, especially, the case $m=N$ is an important physical case corresponding to the baryon potential.

Then we have the relation for the average

$$\langle W(C^m)\rangle=\sum_{\ell=1}^{\min(m,N)}(-1)^{\ell+1}\frac{(N+m-\ell)!}{mN(\ell-1)!(m-\ell)!(N-\ell)!}\times\langle W_{R_\ell}(C)\rangle.\tag{52}$$

Assuming the area-law falloff with the string tension obeying the Casimir scaling, therefore, the most dominant term is given by

$$\langle W(C^m)\rangle\simeq\begin{cases}(-1)^{m-1}c_{Nm}\exp\left(-\frac{m(N-m)}{N-1}\sigma_FS\right) & \text{for } m<N,\\ (-1)^{N-1}c_{Nm} & \text{for } m=N,\\ (-1)^{N-1}c_{Nm}\exp\left(-\frac{m(m-N)}{N+1}\sigma_FS\right) & \text{for } m>N,\end{cases}\tag{53}$$

where S is the minimal area of the loop and c_{Nm} are positive constants.

In particular, a triple-winding Wilson loop for the $SU(3)$ Yang-Mills theory is written as

$$\begin{aligned}\langle W(C^3)\rangle &= \frac{10}{3}\langle W(C)_{[3,0]}\rangle - \frac{8}{3}\langle W(C)_{[1,1]}\rangle + \frac{1}{3}\langle W(C)_{[0,0]}\rangle \\ &= \frac{10}{3}\langle W(C)_{[3,0]}\rangle - \frac{8}{3}\langle W(C)_{[1,1]}\rangle + \frac{1}{3}, \end{aligned} \tag{54}$$

where we have used $W(C)_{[0,0]} = \mathbf{1}$. This is consistent with (44). The triple-winding Wilson loop operator is related to the baryonic Wilson loop operator; see, e.g., [11].

For the loop of the asymptotic size, the expectation value is expected to be

$$\langle W(C^m)\rangle \simeq (-1)^{k-1} c_{Nm} \exp\left(-\frac{k(N-k)}{N-1}\sigma_F S\right) \quad \text{for } m = k \mod N. \tag{55}$$

Therefore, the difference between the loop of intermediate size and that of asymptotic size appears if the winding number is greater than N.

IV. MULTIPLE-WINDING WILSON LOOPS WITH NONIDENTICAL LOOPS

In this section, first, we consider the general double-winding Wilson loop where the two loops are distinct and see that the double-winding Wilson loop follows the novel law when the gauge group is $SU(N)$ ($N \geq 3$). Next, we introduce new Wilson loops whose expectation values are expected to follow the difference-of-areas law even in the $N \geq 3$ case.

In the two-dimensional spacetime we can exactly calculate the double-winding Wilson loop average. This fact is first demonstrated by Bralic in [12] for the $U(N)$ gauge theory. The exact result for the double-winding Wilson loop average for $U(N)$ is

$$\begin{aligned}\langle W(C_1 \times C_2)\rangle = &\frac{N+1}{2}\exp\left[-\frac{\tilde{g}^2 N}{2}\left(S_1 + \frac{N+2}{N}S_2\right)\right] \\ &- \frac{N-1}{2}\exp\left[-\frac{\tilde{g}^2 N}{2}\left(S_1 + \frac{N-2}{N}S_2\right)\right],\end{aligned} \tag{56}$$

where $\tilde{g}$ is the coupling constant in the $SU(N)$ gauge theory. Incidentally, the $U(1)$ case reads

$$\langle W(C_1 \times C_2)\rangle = \exp\left[-\frac{g^2}{2}(S_1 + 3S_2)\right], \tag{57}$$

which reduces for the identical loops $S_1 = S_2$ to

$$\langle W(C \times C)\rangle = \exp\left[-\frac{g^2}{2}(4S)\right]. \tag{58}$$

Notice that the area-law falloff for the double-winding $U(1)$ Wilson loop average in two-dimensional spacetime does not follow the sum-of-areas law.

Fortunately, we can apply this method to the $SU(N)$ gauge theory. Indeed, by replacing the relations among the generators of $U(N)$ by the ones valid for generators T_A of $SU(N)$ ($A = 1, \ldots, N^2 - 1$):

$$\begin{aligned}\delta^{AB} T_A T_B &= \frac{N^2-1}{2N}\mathbf{1}, \\ \delta^{AB}(T_A)^{\alpha_1}_{\beta_1}(T_B)^{\alpha_2}_{\beta_2} &= \frac{1}{2}\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1} - \frac{1}{2N}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2},\end{aligned} \tag{59}$$

we can obtain the exact result for the double-winding Wilson loop average for $SU(N)$:

$$\begin{aligned}\langle W(C_1 \times C_2)\rangle = &\frac{N+1}{2}\exp\left[-\frac{g^2}{2}\frac{N^2-1}{2N}\left(S_1 + \frac{N+3}{N+1}S_2\right)\right] \\ &- \frac{N-1}{2}\exp\left[-\frac{g^2}{2}\frac{N^2-1}{2N}\left(S_1 + \frac{N-3}{N-1}S_2\right)\right].\end{aligned} \tag{60}$$

In the large N limit, both $U(N)$ and $SU(N)$ cases agree[2]

$$\langle W(C_1 \times C_2)\rangle = (1 - \tilde{g}^2 N S_2)\exp\left[-\frac{\tilde{g}^2 N}{2}(S_1 + S_2)\right]. \tag{61}$$

See, e.g., [13] for the large N result of $SU(N)$ based on the Makeenko-Migdal loop equation.

In view of these facts, we give a conjecture for the area-law falloff of the double-winding $SU(N)$ Wilson loop average with two loops C_1, C_2:

$$\langle W(C_1 \times C_2)\rangle \simeq -c_N \exp\left[-\sigma_F\left(S_1 + \frac{N-3}{N-1}S_2\right)\right] \quad (c_N > 0). \tag{62}$$

This follows assuming the factorization of the expectation value $\langle W(C_1 \times C_2)\rangle = \langle W(C_1 \times C_2^{-1} \times C_2^2)\rangle \simeq \langle W(C_1 \times C_2^{-1})\rangle\langle W(C_2^2)\rangle$ from the product of the two area-law falloffs for an ordinary single-winding loop with the area $S_1 - S_2$ and a double-winding loop with the identical area S_2 obeying (40):

$$\begin{aligned}\langle W(C_1 \times C_2)\rangle \simeq &\exp[-\sigma_F(S_1 - S_2)] \\ &\times (-c_N)\exp\left[-2\frac{N-2}{N-1}\sigma_F S_2\right] \quad (c_N > 0).\end{aligned} \tag{63}$$

This is suggested from the middle diagram of Fig. 1. This conjecture is consistent with the above considerations for

[2]The agreement occurs if $\tilde{g}^2 = g^2/2$.

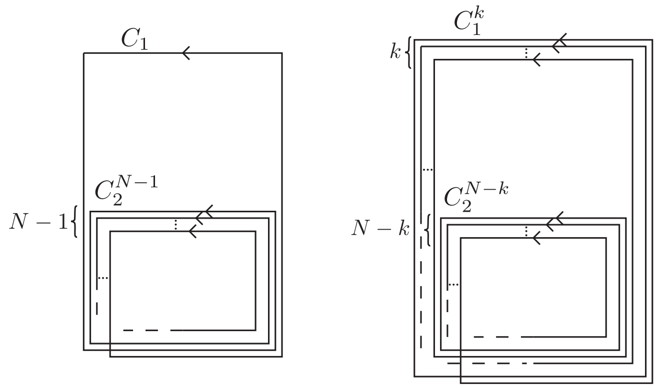


FIG. 2. The left figure is the loop winding once around C_1 and $N-1$ times around C_2. The right figure is the loop winding k times around C_1 and $N-k$ times around C_2. The Wilson loop averages for these loops are expected to decrease exponentially with the difference of areas $S_1 - S_2$.

the identical loops $S_1 = S_2$ and reduces to the ordinary area law for $S_2 = 0$. We expect that this result holds also in four dimensions. Indeed, this leading behavior could hold irrespective of the spacetime dimension, which is also suggested from the strong-coupling expansion of the lattice gauge theory [7].

If the gauge group is $SU(N)$ $(N \geq 3)$, the double-winding Wilson loop does not show the difference-of-areas law, and cannot be used just as in the $SU(2)$ case. Fortunately, however, we can construct the other types of Wilson loops whose averages show the difference-of-areas law. One of these loops is shown in the left panel of Fig. 2, which shows the loop winding once around C_1 and $N-1$ times around C_2. For the loop of intermediate size, by assuming the Casimir scaling and the factorization and using Eq. (53), the Wilson loop average for this loop behaves as

$$\langle W(C_1 \times C_2^{N-1}) \rangle \propto (-1)^{N-1} \exp\left[-\sigma_F (S_1 - S_2)\right]. \tag{64}$$

For the loop of asymptotic size, the same behavior is expected to hold.

Next we consider the loop shown in the right panel of Fig. 2, which winds k times around C_1 and $N-k$ times around C_2. In this case, we also expect the difference-of-areas behavior. For the loop of intermediate size, the Wilson loop average is expected to behave as

$$\langle W(C_1^k \times C_2^{N-k}) \rangle \propto (-1)^{N+k-2} \exp\left[-\frac{k(N-k)}{N-1} \sigma_F (S_1 - S_2)\right]. \tag{65}$$

For the loop of asymptotic size, the Wilson loop average is expected to behave as

$$\langle W(C_1^k \times C_2^{N-k}) \rangle \propto (-1)^{N+k-2} \exp\left[-\sigma_k (S_1 - S_2)\right], \tag{66}$$

where σ_k is the asymptotic string tension for a representation whose N-ality is k.

ACKNOWLEDGMENTS

The authors would like to thank one of the referees for giving valuable comments on the new Wilson loops whose averages are expected to follow the difference-of-areas law. R. M. was supported by Grant-in-Aid for JSPS Research Fellows, Grant No. 17J04780. K.-I. K. was supported by Grant-in-Aid for Scientific Research, JSPS KAKENHI Grant (C) No. 15K05042.

APPENDIX A: DOUBLE-WINDING CASE: THE DERIVATION OF EQS. (6) AND (31)

First, we consider the case $N = 2$. Let U be an element of $SU(2)$. There exists a group element V such that VUV^{-1} is diagonal. Let this diagonal matrix be $\mathrm{diag}(\exp(i\theta/2), \exp(-i\theta/2))$. Thus, we can write

$$\mathrm{tr}U^2 = \mathrm{tr}(VUV^{-1})^2 = e^{i\theta} + e^{-i\theta} = \mathrm{tr}U_A - 1 \tag{A1}$$

where U_A denotes the adjoint representation of U. Here we have used the adjoint representation of VUV^{-1}, which is $\mathrm{diag}(\exp(i\theta), 1, \exp(-i\theta))$. Therefore, in the case of the gauge group $SU(2)$, the double-winding Wilson loop operator $W(C \times C)$ can be written using the single-winding Wilson loop operator W_A in the adjoint representation as

$$W(C \times C) = \frac{3}{2} W_A - \frac{1}{2} \mathbf{1}. \tag{A2}$$

When the gauge group is $SU(N)$ $(N \geq 3)$, we show that the double-winding Wilson loop operator $W(C \times C)$ can be written using the higher-dimensional representation as

$$W(C \times C) = \frac{N+1}{2} W_{[2,0,\ldots,0]} - \frac{N-1}{2} W_{[0,1,0,\ldots,0]} \tag{A3}$$

by showing

$$\mathrm{tr}U^2 = \mathrm{tr}U_{[2,0,\ldots,0]} - \mathrm{tr}U_{[0,1,0,\ldots,0]}, \tag{A4}$$

where U is an arbitrary element of $SU(N)$.

Before proceeding to the general N case, we consider the $N = 3$ case. As in the $SU(2)$ case, a group element U can be diagonalized. Let this diagonal matrix be $\exp(i\boldsymbol{v} \cdot \boldsymbol{H})$, $\boldsymbol{v} \cdot \boldsymbol{H} \coloneqq v_1 H_1 + v_2 H_2$ where H_1 and H_2 are the Cartan generators and v_1, $v_2 \in \mathbb{R}$. Therefore, the trace of U^2 is

$$\mathrm{tr}U^2 = \sum_i \langle \nu^i | e^{2i\boldsymbol{v}\cdot\boldsymbol{H}} | \nu^i \rangle = e^{2i\boldsymbol{v}\cdot\nu^1} + e^{2i\boldsymbol{v}\cdot\nu^2} + e^{2i\boldsymbol{v}\cdot\nu^3}, \tag{A5}$$

where ν^1, ν^2, and ν^3 are the weights of the fundamental representation and $|\nu^i\rangle$ is the normalized state corresponding to ν^i. To write this as the sum of the traces in higher-dimensional representations, we must find the representation which has the weights $2\nu^1$, $2\nu^2$, and $2\nu^3$. To do this, let us consider the representation corresponding to the Young diagram

$$\square\square. \tag{A6}$$

A state in this representation can be obtained by symmetrizing the tensor product of two states in the fundamental representation, that is to say,

$$|\nu^i\rangle \otimes |\nu^j\rangle + |\nu^j\rangle \otimes |\nu^i\rangle \tag{A7}$$

belongs to this representation. Therefore, the weights of this representation are $2\nu^1$, $2\nu^2$, $2\nu^3$, $\nu^1+\nu^2$, $\nu^1+\nu^3$, $\nu^2+\nu^3$, and the degeneracy of each state is one. Since the highest weight of this representation is $2\nu^1 = 2\mu^1$, this representation is [2, 0], where μ^i denotes a fundamental weight.[3] Generally the trace in the representation R can be written as

$$\mathrm{tr}U_R = \sum_\mu d_\mu e^{i\boldsymbol{v}\cdot\mu}, \tag{A8}$$

where the sum is over the weights μ of the representation R and d_μ is degeneracy of the weight μ. Then the trace of U in this representation is

$$\begin{aligned}\mathrm{tr}U_{[2,0]} &= e^{i\boldsymbol{v}\cdot 2\nu^1} + e^{i\boldsymbol{v}\cdot 2\nu^2} + e^{i\boldsymbol{v}\cdot 2\nu^3} + e^{i\boldsymbol{v}\cdot(\nu^1+\nu^2)} \\ &\quad + e^{i\boldsymbol{v}\cdot(\nu^1+\nu^3)} + e^{i\boldsymbol{v}\cdot(\nu^2+\nu^3)}.\end{aligned} \tag{A9}$$

Because $\nu^1+\nu^2+\nu^3=0$, the sum of the last three terms is the trace in the complex conjugate of the fundamental representation. Therefore we obtain

$$\mathrm{tr}U^2 = \mathrm{tr}U_{[2,0]} - \mathrm{tr}U_{[0,1]}. \tag{A10}$$

Now we consider the general N case. In this case, we can write $VUV^{-1} = \exp(i\boldsymbol{v}\cdot\boldsymbol{H})$, where $\boldsymbol{v}\cdot\boldsymbol{H} \coloneqq \boldsymbol{v}_a H_a, H_1, \ldots, H_{N-1}$ are the Cartan generators and $v_a \in \mathbb{R}$. Therefore,

$$\mathrm{tr}U^2 = \sum_i \langle \nu^i | e^{2i\boldsymbol{v}\cdot\boldsymbol{H}} | \nu^i\rangle = \sum_i e^{2i\boldsymbol{v}\cdot\nu^i}, \tag{A11}$$

where $\nu^1, \ldots, \nu^{N-1}$ are the weights of the fundamental representation and $|\nu^i\rangle$ is the normalized state corresponding to ν^i. From this expression, it turns out that we must find the representation with the doubled weights $2\nu^1, \ldots, 2\nu^{N-1}$. As in the $N=3$ case we consider the representation corresponding to the Young diagram

$$\square\square. \tag{A12}$$

A state in this representation can be obtained by symmetrizing the tensor product of two states in the fundamental representation, that is to say,

$$|\nu^i\rangle \otimes |\nu^j\rangle + |\nu^j\rangle \otimes |\nu^i\rangle \tag{A13}$$

belongs to this representation. Therefore, the weights of this representation are

$$\nu^i + \nu^j \quad (i,j = 1, \ldots, N, \quad i < j) \tag{A14}$$

and the degeneracy of each state is one. Because the highest weight is $2\nu^1 = 2\mu^1$, this representation is $[2, 0, \ldots, 0]$. Then the trace in this representation is

$$\mathrm{tr}U_{[2,0,\ldots,0]} = \sum_{i\le j} e^{i\boldsymbol{v}\cdot(\nu^i+\nu^j)} = \mathrm{tr}U^2 + \sum_{i<j} e^{i\boldsymbol{v}\cdot(\nu^i+\nu^j)}. \tag{A15}$$

Next let us consider the representation corresponding the Young diagram

$$\begin{array}{c}\square\\ \square\end{array}. \tag{A16}$$

A state in this representation can be obtained by antisymmetrizing the tensor product of two states in the fundamental representation, that is to say,

$$|\nu^i\rangle \otimes |\nu^j\rangle - |\nu^j\rangle \otimes |\nu^i\rangle \tag{A17}$$

belongs to this representation. Therefore, the weights of this representation are

$$\nu^i + \nu^j \quad (i,j = 1, \ldots, N, \quad i \neq j), \tag{A18}$$

and the degeneracy of each state is one. Because the highest weight is $\nu^1 + \nu^2 = \mu^2$, this representation is $[0, 1, 0, \ldots, 0]$. Then the trace in this representation is

$$\mathrm{tr}U_{[0,1,0,\ldots,0]} = \sum_{i<j} e^{i\boldsymbol{v}\cdot(\nu^i+\nu^j)}. \tag{A19}$$

[3] The fundamental weights μ^i are defined as $N-1$ dimensional vectors that satisfy

$$\frac{2\mu^i\cdot\alpha^j}{\alpha^k\cdot\alpha^k} = \delta_{ij},$$

where α^j are roots of $SU(N)$. The highest weight of the representation $[m_1, m_2, \ldots, m_{N-1}]$ is

$$\sum_{i=1}^{N-1} m_i\mu^i.$$

Therefore, by subtracting Eq. (A19) from Eq. (A15) we obtain Eq. (A4).

Since the dimensions of $[2,0,\ldots,0]$ and $[0,1,0,\ldots,0]$ are $N(N+1)/2$ and $N(N-1)/2$, respectively, the double-winding Wilson loop can be written as

$$W(C\times C)=\frac{N+1}{2}W_{[2,0,\ldots,0]}-\frac{N-1}{2}W_{[0,1,0,\ldots,0]}. \quad \text{(A20)}$$

APPENDIX B: MULTIPLE-WINDING CASE: DERIVATION OF EQ. (49)

The trace of the mth power of U can be written as

$$\mathrm{tr}U^m=\sum_{\ell=1}^{\min(m,N)}(-1)^{\ell-1}\mathrm{tr}U_{R_\ell} \quad \text{(B1)}$$

where

$$R_\ell := \begin{cases} [m,0,\ldots,0] & \text{for } \ell=1,\\ [\underbrace{m-\ell,0,\ldots,0,1}_{\ell},0,\ldots,0] & \text{for } \ell=2,\ldots,\min(m,N-1),\\ [m-N,0,\ldots,0] & \text{for } \ell=N, m\geq N. \end{cases} \quad \text{(B2)}$$

By denoting the representations using Young diagram, we can also write it as for $m>N$

$$\mathrm{tr}\,U^m=U_{\begin{ytableau} \, & \, & \cdots & \, \end{ytableau}}-U_{\begin{ytableau} \, & \, & \cdots & \, \\ \, \end{ytableau}}+\cdots+(-1)^{l-1}U_{\begin{ytableau} \, & \, & \cdots & \, \\ \, \\ \vdots \\ \, \end{ytableau}}+\cdots+(-1)^{m-1}U_{\begin{ytableau} \, \\ \, \\ \vdots \\ \, \end{ytableau}}, \quad \text{(B3)}$$

where there are m boxes in all diagrams and there are ℓ raws in the diagram in the ℓth term, and for $m\leq N$

$$\mathrm{tr}\,U^m=U_{\begin{ytableau} \, & \, & \cdots & \, \end{ytableau}}-U_{\begin{ytableau} \, & \, & \cdots & \, \\ \, \end{ytableau}}+\cdots+(-1)^{l-1}U_{\begin{ytableau} \, & \, & \cdots & \, \\ \, \\ \vdots \\ \, \end{ytableau}}+\cdots+(-1)^{N-1}U_{\begin{ytableau} \, & \, & \cdots & \, \\ \, \\ \vdots \\ \, \end{ytableau}}, \quad \text{(B4)}$$

where there are only N terms.

This is proven as follows. The trace of the mth power of an element of $SU(N)$ can be written as

$$\mathrm{tr}U^m=\sum_i e^{im\nu\cdot\nu^i}. \quad \text{(B5)}$$

Here $m\nu^1,\ldots,m\nu^{N-1}$ belong to the set of weights of the representation $[m,0,\ldots,0]$ because the highest weight is $m\mu^1=m\nu^1$ and $m\nu^1,\ldots,m\nu^{N-1}$ are related by Weyl reflections. Therefore, as in the second power case the trace of mth power of U can be obtained by subtracting the part which contains the weights other than $m\nu^1,\ldots,m\nu^{N-1}$ from the trace of $U_{[m,0,\ldots,0]}$. The next step is finding the representation which contains the states corresponding to the weights of $[m,0,\ldots,0]$ other than $m\nu^1,\ldots,m\nu^{N-1}$.

To do this, we consider tensor representations. Let $|i\rangle$ be a vector in the fundamental representation space whose weight is ν^i. A vector belonging to mth tensor power of the fundamental representation space can be written as

$$|i_1\rangle\otimes|i_2\rangle\otimes\cdots|i_m\rangle, \quad \text{(B6)}$$

and we denote this by

$$|i_1 i_2\ldots i_m\rangle. \quad \text{(B7)}$$

It is known that an irreducible representation subspace of the tensor product space corresponds to a Young diagram. We can obtain a state belonging to an irreducible representation subspace as follows. First, put factors of a tensor product in each of the boxes of the Young diagram. Second, symmetrize in the factors in the same raws of the Young diagram. Lastly antisymmetrize in the factors in the same columns. The obtaining state belongs to an irreducible representation subspace. For example, let us consider the Young diagram

$$\begin{ytableau} \, & \, \\ \, \end{ytableau} \quad \text{(B8)}$$

and a state $|j_1 j_2 j_3\rangle$. First put j_1, j_2, and j_3 into the boxes of the diagram as

$$\begin{ytableau} j_1 & j_2 \\ j_3 \end{ytableau}. \quad \text{(B9)}$$

By symmetrizing in j_1 and j_2, we obtain

$$|j_1 j_2 j_3\rangle + |j_2 j_1 j_3\rangle. \tag{B10}$$

By antisymmetrizing in j_1 and j_3, we obtain

$$|j_1 j_2 j_3\rangle + |j_2 j_1 j_3\rangle - |j_3 j_2 j_1\rangle - |j_2 j_3 j_1\rangle. \tag{B11}$$

This belongs to an irreducible representation subspace. It is also known that a basis of an irreducible representation subspace corresponds to a set of *semistandard Young tableaux* (see, e.g., [14]). A semistandard Young tableau is obtained by filling in the boxes of a Young diagram with numbers which weakly increase along each row and strictly increase down each column. In fact, if $N = 3$, the basis of the representation in the example,

$$\begin{aligned}
&\{2|112\rangle - |211\rangle - |121\rangle, \quad 2|113\rangle - |311\rangle - |131\rangle, \\
&|122\rangle + |212\rangle - 2|221\rangle, \quad 2|223\rangle - |322\rangle - |232\rangle, \\
&|133\rangle + |313\rangle - 2|331\rangle, \quad |233\rangle + |323\rangle - 2|332\rangle, \\
&|123\rangle + |213\rangle - |321\rangle - |231\rangle, \\
&|132\rangle + |312\rangle - |231\rangle - |321\rangle\},
\end{aligned} \tag{B12}$$

corresponds to the set of the semistandard Young tableaux,

$$\left\{\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right\}. \tag{B13}$$

The weights of this representation are

$$\begin{aligned}
&2\nu^1 + \nu^2, \quad 2\nu^1 + \nu^3, \quad 2\nu^2 + \nu^1, \quad 2\nu^2 + \nu^3, \\
&2\nu^3 + \nu^1, \quad 2\nu^3 + \nu^2, \quad \nu^1 + \nu^2 + \nu^3.
\end{aligned} \tag{B14}$$

The weight space with $\nu^1 + \nu^2 + \nu^3$ is the two-dimensional space whose basis is the set of the last two elements of Eq. (B12).

Before proceeding with the general m and N case, we consider the case $m = 3$ and $N = 3$. Let us consider the representation [3, 0], which corresponds to the Young diagram

$$\yng(3). \tag{B15}$$

Since states in this representation are symmetric in the factors of tensor products, the weights are

$$\begin{aligned}
&3\nu^1, \quad 3\nu^2, \quad 3\nu^3, \quad 2\nu^1 + \nu^2, \quad 2\nu^1 + \nu^3, \\
&2\nu^2 + \nu^1, \quad 2\nu^2 + \nu^3, \quad 2\nu^3 + \nu^1, \\
&2\nu^3 + \nu^2, \quad \nu^1 + \nu^2 + \nu^3,
\end{aligned} \tag{B16}$$

and the degeneracy of each state is one. Therefore, by using Eq. (A8) we obtain the trace in this representation as

$$\begin{aligned}
\operatorname{tr} U_{\yng(3)} = {}& e^{i\boldsymbol{v}\cdot 3\nu^1} + e^{i\boldsymbol{v}\cdot 3\nu^2} + e^{i\boldsymbol{v}\cdot 3\nu^3} + + e^{i\boldsymbol{v}\cdot(2\nu^1+\nu^2)} + e^{i\boldsymbol{v}\cdot(2\nu^1+\nu^3)} + e^{i\boldsymbol{v}\cdot(2\nu^2+\nu^1)} + e^{i\boldsymbol{v}\cdot(2\nu^2+\nu^3)} + e^{i\boldsymbol{v}\cdot(2\nu^3+\nu^1)} \\
&+ e^{i\boldsymbol{v}\cdot(2\nu^3+\nu^2)} + e^{i\boldsymbol{v}\cdot(\nu^1+\nu^2+\nu^3)}.
\end{aligned} \tag{B17}$$

Next we consider the representation corresponding to the Young diagram, Eq. (B8). By using Eq. (A8) and the fact that the weights of this representation are Eq. (B14), the degeneracy of $\nu^1 + \nu^2 + \nu^3$ is two, and the degeneracies of other weights are one, we obtain the trace in this representation as

$$\operatorname{tr} U_{\yng(2,1)} = e^{i\boldsymbol{v}\cdot(2\nu^1+\nu^2)} + e^{i\boldsymbol{v}\cdot(2\nu^1+\nu^3)} + e^{i\boldsymbol{v}\cdot(2\nu^2+\nu^1)} + e^{i\boldsymbol{v}\cdot(2\nu^2+\nu^3)} + e^{i\boldsymbol{v}\cdot(2\nu^3+\nu^1)} + e^{i\boldsymbol{v}\cdot(2\nu^3+\nu^2)} + 2e^{i\boldsymbol{v}\cdot(\nu^1+\nu^2+\nu^3)}. \tag{B18}$$

Therefore,

$$\begin{aligned}
\operatorname{tr} U_{\yng(3)} - \operatorname{tr} U_{\yng(2,1)} &= e^{i\boldsymbol{v}\cdot 3\nu^1} + e^{i\boldsymbol{v}\cdot 3\nu^2} + e^{i\boldsymbol{v}\cdot 3\nu^3} - e^{i\boldsymbol{v}\cdot(\nu^1+\nu^2+\nu^3)} \\
&= \operatorname{tr} U^3 - 1,
\end{aligned} \tag{B19}$$

where we have used Eq. (B5) and $\nu^1 + \nu^2 + \nu^3 = 0$. By adding the trace of the trivial representation, i.e., one, we obtain

$$\operatorname{tr} U^3 = \operatorname{tr} U_{\yng(3)} - \operatorname{tr} U_{\yng(2,1)} + \operatorname{tr} U_{\yng(1,1,1)}, \tag{B20}$$

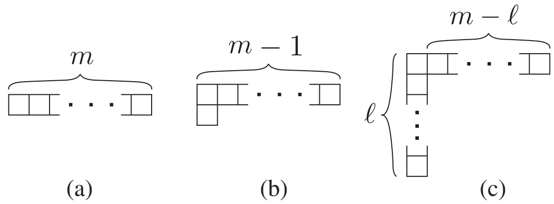
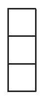


FIG. 3. The Young diagrams associated with m-times-winding Wilson loop operator for $SU(N)$ group. (a) $[m, 0, \ldots, 0]$. (b) $[m-2, 1, 0, \ldots, 0]$. (c) R_ℓ.

where we have used the fact that the Young diagram

 (B21)

corresponds to the trivial representation. Because of the degeneracy, when $m \geq 3$ we need more than two representations.

Now we consider the general m and N case. The representation $[m, 0, \ldots, 0]$ corresponds to the Young diagram shown in Fig. 3(a) because the highest weight of the representation corresponding to the Young diagram is $m\nu^1$, which is the highest weight of $[m, 0, \ldots, 0]$. Therefore, a weight of $[m, 0, \ldots, 0]$ can be written as

$$\sum_{\ell=1}^{m} \nu^{i_\ell} \quad (1 \leq i_1 \leq \cdots \leq i_m \leq N), \tag{B22}$$

and the degeneracy of each weight is one. This is because the states in this representation can be represented as the symmetric tensor products of m states in the fundamental representation, and there is only one symmetric tensor product which contains $|i_1\rangle, \ldots, |i_m\rangle$ as the factors.

Next let us consider the representation $[m-2, 1, 0, \ldots, 0]$, which corresponds to the Young diagram shown in Fig. 3(b). This representation contains the states which have the weights, Eq. (B22), other than $m\nu^1, \ldots, m\nu^N$ because at least two different states of the fundamental representation must appear as the factors of the tensor products in each state in this representation. The degeneracy of the weights which have k different weights of the fundamental representation in the sum, i.e.,

$$\sum_{i=1}^{k} \ell_i \nu^{j_i} \left(\ell_i \in \mathbb{N}, \quad \sum_{i=1}^{k} \ell_i = m, \quad 1 \leq j_1 < \cdots < j_k \leq N \right) \tag{B23}$$

is $k-1$ [notice that $2 \leq k \leq \min(m, N)$]. This fact is proven as follows. The degeneracy of the weights, Eq. (B23), is the number of the semistandard Young tableaux where the integer j_i appears ℓ_i times for $i = 1, \ldots, k$. In the semistandard Young tableaux, j_1, which is the smallest integer in $j_1, \ldots, j_k$, must appear in the first box of the first raw, and since the same number must not appear in the same column, the second box in the first column must be filled by any one of $j_2, \ldots, j_k$. The entries in the remaining boxes are automatically determined. This means that the semistandard Young tableau is determined by what is the entry of the second box in the first column. Thus, the number of the corresponding semistandard Young tableau is $k-1$. Therefore, if we subtract $\mathrm{tr}U_{[m-1,1,0,\ldots,0]}$ from $\mathrm{tr}U_{[m,0,\ldots,0]}$, we subtract too much. We need to consider another representation.

Consider the representation corresponding to the Young diagram shown in Fig. 3(c). Notice that $\ell \leq m$ and $\ell \leq N$ because there are m boxes in the diagrams and there are no representations corresponding to the Young diagrams which has more than N rows when the group is $SU(N)$. Since at least ℓ different states of the fundamental representation must appear as the factors of the tensor product in each state in this representation, the weights of this representation are Eq. (B23) for $k = \ell, \ldots, \min(m, N)$. The degeneracy of the weights Eq. (B23) is ${}_{k-1}C_{\ell-1}$. This is because, by putting j_1 into the first box and $l-1$ of $j_2, \ldots, j_k$ into the boxes in the first column other than first box in ascending order, the numbers which should be put in the remaining boxes are determined and then the corresponding semistandard Young tableau is obtained. This means that the semistandard Young tableau is determined by what is the entry of all boxes except the first one in the first column. Thus, the number of the corresponding semistandard Young tableau is ${}_{k-1}C_{\ell-1}$. This representation is R_ℓ since the highest weight of this representation is $(m-\ell+1)\nu^1 + \nu^2 + \cdots + \nu^\ell = (m-\ell)\mu^1 + \mu^\ell$ for $l < N$, and $(m-N+1)\nu^1 + \nu^2 + \cdots + \nu^N = (m-N)\nu^1$ for $\ell = N$, where we have used $\nu^1 + \cdots + \nu^\ell = \mu^\ell$ for $\ell < N$ and $\nu^1 + \cdots + \nu^N = 0$.

Because

$$\sum_{\ell=1}^{k} {}_{k-1}C_{\ell-1} (-1)^{\ell-1} = (1-1)^{k-1} = 0, \tag{B24}$$

the contribution from the weights, Eq. (B23), for $k = 2, \ldots, \min(m, N)$ cancels in Eq. (B1). Since Eq. (B23) for $k = 2, \ldots, \min(m, N)$ is all weights of $[m, 0, \ldots, 0]$ except $m\nu^1, \ldots, m\nu^N$, Eq. (B1) is proven.

By using Eq. (B1) we can write the m-times-winding Wilson loop operator by using the single-winding Wilson loop operator for the higher-dimensional representations: the m-times-winding Wilson loop operator can be written as

$$W(C^m) = \sum_{\ell=1}^{\min(m,N)} (-1)^{\ell-1} \frac{D(R_\ell)}{N} W_{R_\ell}, \tag{B25}$$

where $D(R_\ell)$ is the dimension of R_ℓ, i.e.,

$$D(R_\ell) = \frac{(N+m-\ell)!}{m(\ell-1)!(m-\ell)!(N-\ell)!}. \tag{B26}$$

[1] K. Wilson, Confinement of quarks, Phys. Rev. D **10**, 2445 (1974).

[2] C. N. Yang and R. L. Mills, Conservation of isotopic spin and isotopic gauge invariance, Phys. Rev. **96**, 191 (1954).

[3] J. Greensite and R. Höllwieser, Double-winding Wilson loops and monopole confinement mechanisms, Phys. Rev. D **91**, 054509 (2015).

[4] J. Greensite, The confinement problem in lattice gauge theory, Prog. Part. Nucl. Phys. **51**, 1 (2003).

[5] K.-I. Kondo, S. Kato, A. Shibata, and T. Shinohara, Quark confinement: Dual superconductor picture based on a non-Abelian Stokes theorem and reformulations of Yang-Mills theory, Phys. Rep. **579**, 1 (2015).

[6] G. Bali, QCD forces and heavy quark bound states, Phys. Rep. **343**, 1 (2001).

[7] S. Kato, K.-I. Kondo, and A. Shibata (to be published).

[8] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Dover Publications, New York, 2012).

[9] R. Slansky, Group theory for unified model building, Phys. Rep. **79**, 1 (1981); P. L. Whited, Discrete symmetries from broken SU(N) and the MSSM, Nucl. Phys. **B403**, 141 (1993); D. D. Dietrich and F. Sannino, Conformal window of SU(N) gauge theories with fermions in higher dimensional representations, Phys. Rev. D **75**, 085018 (2007).

[10] M. R. Douglas and S. H. Shenker, Dynamics of SU(N) supersymmetric gauge theory, Nucl. Phys. **B447**, 271 (1995); A. Hanany, M. J. Strassler, and A. Zaffaroni, Confinement and strings in MQCD, Nucl. Phys. **B513**, 87 (1998).

[11] M. Creutz, Quarks, *Gluons and Lattices* (Cambridge University Press, Cambridge, 1985).

[12] N. E. Bralic, Exact computation of loop averages in two-dimensional Yang-Mills theory, Phys. Rev. D **22**, 3090 (1980).

[13] Y. Makeenko, *Methods of Contemporary Gauge Theory* (Cambridge University Press, Cambridge, 2002).

[14] W. Fulton, *Young Tableaux: With Applications to Representation Theory and Geometry* (Cambridge University Press, Cambridge, 1996).