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Ghosts in extended quasidilaton theories

Alexey Golovnev* and Aleksandr Trukhin†

Faculty of Physics, Saint Petersburg State University, Ulyanovskaya ulitsa, dom 1, Saint Petersburg 198504, Russia

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We report on our independent investigations of the puzzle of cosmological perturbations in extended quasidilaton. We confirm the claims of presence of the Boulware-Deser ghost. We use both the language of cosmological perturbations with broken diffeomorphisms and the Stückelberg approach.

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I. INTRODUCTION

The history of massive gravity dates back to the classical Fierz-Pauli model [1] in which it was shown that, out of two possible quadratic mass term for a metric fluctuation $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ around Minkowski spacetime, only one combination $(h_{\mu\nu}h^{\mu\nu} - (h^{\mu}_{\mu})^2)$ gives healthy number of five degrees of freedom for a massive spin-two particle, whereas other options feature an extra mode with negative kinetic energy. Later it was realized that the problematic mode generically reappears after nonlinear corrections have been taken into account, and it acquired the name of Boulware-Deser ghost after the work of Ref. [2].

In recent years we have witnessed great progress in the theory of massive gravity. In particular, the healthy nonlinear extension of the classical Fierz-Pauli model [1] has been found [3,4] and proven [5–7] to be free of the Boulware-Deser ghost [2]. The theory is very peculiar in its mathematical features: healthy potentials are elementary symmetric polynomials of eigenvalues of a square root of the matrix $g^{\alpha\mu}f_{\alpha\nu}$ where $f_{\mu\nu}$ is a fiducial metric which is often taken to be $\eta_{\mu\nu}$. Of course, beside the formal interest, there were natural hopes to explain the accelerated expansion of the Universe via Yukawa attenuation of the gravitational force at large distances. Unfortunately, not only did it not work out properly, but even the very existence and stability of cosmological regimes was rather hard to achieve in massive gravity [8].

It sparkled some interest towards extensions of massive gravity since what we have is a potentially healthy tensorial extension of general relativity which is very nontrivial to achieve and can have interesting implications for cosmology. Popular options include bimetric gravity in which the fiducial metric is made dynamical, and also scalar-tensor extensions of two types: variable mass and quasidilaton [9]. The quasidilaton model features an additional dynamical scalar field σ, and the elementary symmetric polynomials are calculated for the matrix $e^{\sigma/M_{\mathrm{Pl}}} \cdot \sqrt{g^{-1}f}$. Unfortunately, the cosmological perturbations turned out problematic [10], and an extended version of this extension has been proposed [11] which also shifts the fiducial metric by term proportional to $e^{-2\sigma/M_{\mathrm{Pl}}}\partial_\mu\sigma\partial_\nu\sigma$, see below.

The issue of stability in the full regime (as opposed to the so called late time limit) of extended quasidilaton presents a conundrum. In the first version of Shinji Mukohyama's preprint [12] the absence of the Boulware-Deser (BD) ghost has been proven with a particular gauge choice, which was however claimed to be not a good gauge [13]. In the work of Lavinia Heisenberg [14] indications were given for stability of cosmological perturbations, at least in the far ultraviolet limit. Then, in the recent paper [15] it was claimed that cosmological perturbations have the BD ghost mode in the infrared limit contrary to previous believes. And the second version of the Mukohyama's preprint [16] proves that the BD ghost is present in the model.

We also had independent calculations of cosmological perturbations in quasidilaton which agree with the conclusions of Refs. [15,16]. In Sec. II we briefly remind the formulation of extended quasidilaton model and its cosmological solutions. In Sec. III we present perturbative calculations in the language of cosmological variables *without* assuming infrared or ultraviolet limits. In Sec. IV we arrive at the same results using the Stückelberg fields. Finally, in Sec. V we discuss and conclude.

II. COSMOLOGICAL SOLUTIONS WITH EXTENDED QUASIDILATON

Let us briefly set the stage for perturbative analysis of extended quasidilaton cosmologies. The action of extended quasidilaton model is given by

$$S = \frac{M_{\mathrm{Pl}}^2}{2}\int d^4x\sqrt{-g}\bigg(R[g] - \frac{\omega}{M_{\mathrm{Pl}}^2}\partial_\mu\sigma\partial^\mu\sigma + 2m^2(\mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4)\bigg) \tag{1}$$

where we have the usual massive gravity potential with elementary symmetric polynomials

*agolovnev@yandex.ru
†alxtry@gmail.com

$$\mathcal{U}_2[K] = \frac{1}{2}([K]^2 - [K^2]), \tag{2a}$$

$$\mathcal{U}_3[K] = \frac{1}{6}([K]^3 - 3[K][K^2] + 2[K^3]), \tag{2b}$$

$$\mathcal{U}_4[K] = \frac{1}{24}([K]^4 - 6[K]^2[K^2] + 3[K^2]^2 + 8[K][K^3] - 6[K^4]) \tag{2c}$$

of the eigenvalues of the matrix

$$K^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - e^{\sigma/M_{\mathrm{Pl}}}\left(\sqrt{g^{-1}\tilde{f}}\right)^{\mu}{}_{\nu} \tag{3}$$

with the new fiducial metric given by

$$\tilde{f}_{\mu\nu} = f_{\mu\nu} - \frac{a_\sigma}{M_{\mathrm{Pl}}^2 m^2} e^{-2\sigma/M_{\mathrm{Pl}}} \partial_\mu\sigma\partial_\nu\sigma. \tag{4}$$

Setting the quasidilaton field to 0 gives the usual massive gravity. Vanishing of the new coupling constant a_σ while keeping the σ field arbitrary brings back the simple quasidilaton. Note that we neglect possible cosmological constant term in this action because it does not change our analysis, and anyway, one of the basic aims of those models is to obtain accelerated expansion *without* explicitly introducing a cosmological constant.

The fiducial metric is usually taken to be Minkowski: $f_{\mu\nu} = \eta_{\mu\nu}$. Note though that one can restore the diffeomorphism invariance by introducing the Stückelberg fields ϕ^a

$$f_{\mu\nu} = \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b \tag{5}$$

which describe the change of coordinates from those in which the fiducial metric is Minkowski to arbitrary ones.

We are interested in the spatially flat Friedmann–Lemaitre–Robertson–Walker solutions

$$\mathrm{d}s_g^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{6}$$

$$\mathrm{d}s_{\tilde{f}}^2 = -n^2(t)dt^2 + \delta_{ij}dx^i dx^j, \tag{7}$$

with $\sigma = \sigma(t)$. They correspond to

$$\phi_0 = \phi_0(t), \qquad \phi_i = x_i, \tag{8}$$

$$n^2 = \dot{\phi}_0^2 + \frac{a_\sigma}{M_{\mathrm{Pl}}^2 m^2} e^{-2\sigma/M_{\mathrm{Pl}}}\dot{\sigma}^2. \tag{9}$$

in the Stückelberg language.

Background equations are easily found to be (see, e.g. [14])

$$3H^2 = \Lambda_A + \frac{\omega\dot{\sigma}^2}{2M_{\mathrm{Pl}}^2 N^2}, \tag{10}$$

$$\frac{2\dot{H}}{N} = \frac{(1-r)\dot{\Lambda}_A}{3HN - 3\dot{\sigma}/M_{\mathrm{Pl}}} - \frac{\omega\dot{\sigma}^2}{M_{\mathrm{Pl}}^2 N^2}, \tag{11}$$

$$\partial_t\left(\frac{1}{n} m^2 M_{\mathrm{Pl}}^2 a^4 J(A-1)A\dot{\phi}_0\right) = 0, \tag{12}$$

$$\begin{aligned} &m^2 M_{\mathrm{Pl}} N^3 A(3(r-1)A(-2+\alpha_3(A-1)) \\ &\quad + J(-3 + r(-1+4A))) = \omega(3HN^2\dot{\sigma} + N\ddot{\sigma} - \dot{N}\dot{\sigma}) \end{aligned} \tag{13}$$

where we have introduced the following standard notations

$$A(t) \equiv \frac{e^{\sigma/M_{\mathrm{Pl}}}}{a}, \tag{14a}$$

$$H(t) \equiv \frac{\dot{a}}{aN}, \tag{14b}$$

$$r(t) \equiv \frac{na}{N}, \tag{14c}$$

$$J(t) \equiv 3 + 3(1-A)\alpha_3 + (1-A)^2\alpha_4, \tag{14d}$$

$$\Lambda_A \equiv m^2(A-1)[J + (A-1)(\alpha_3(A-1) - 3)]. \tag{14e}$$

III. THE GHOST MODE IN COSMOLOGICAL PERTURBATIONS

Let us now study the cosmological perturbations. Since the BD ghost affects the scalar sector, we ignore vector and tensor perturbations. We use the following parametrization of the scalar metric perturbations

$$\delta g_{00} = -2N^2\frac{\Phi}{M_{\mathrm{Pl}}}, \tag{15a}$$

$$\delta g_{0i} = Na\partial_i\frac{B}{M_{\mathrm{Pl}}}, \tag{15b}$$

$$\delta g_{ij} = a^2\left[2\delta_{ij}\frac{\psi}{M_{\mathrm{Pl}}} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\partial^k\partial_k\right)\frac{E}{M_{\mathrm{Pl}}}\right]. \tag{15c}$$

We consider quadratic expansion of the action around a cosmological solution. Then we go to the Fourier space with spatial momentum k, and get the quadratic action with the following relevant parts (for calculating the Hessian of the kinetic term):

$$\mathcal{L}^{(2)} = -\frac{k^2 a^3 A B^2 Q}{2(-1+A)^2(1+r)} - 2k^2 a^2 B H \Phi - a^3 \Lambda_A \Phi^2 + \frac{1}{2} a^3 \left(\omega + \frac{a^2(-1+A) J a_\sigma \dot{\phi}_0^2}{A r^3} \right) \delta\dot{\sigma}^2 - \frac{1}{3} k^4 a^2 B \dot{E} + \frac{1}{12} k^4 a^3 \dot{E}^2 - \omega a^3 \Phi \delta\dot{\sigma}\, \dot{\sigma} + (2k^2 a^2 B + 6a^3 H \Phi)\dot{\psi} - 3a^3 \dot{\psi}^2 + \cdots \tag{16}$$

where

$$Q = -m^2 J(A-1) + (\Lambda_A + m^2(A-1)^2)A, \tag{17}$$

and, for the sake of simplicity, we have set the lapse to unity, $N = 1$, and omitted the Planck mass $M_{\rm Pl}$. Ellipsis at the end of Eq. (16) shows that only those terms are written here which either are quadratic in velocities of dynamical variables or contain Φ or B fields.

The fields Φ and B can be eliminated by solving *algebraic* equations since their derivatives do not enter the Lagrangian [we have included all Φ- and B-terms in (16)]. After that we find

$$\det \mathcal{H} \equiv \begin{vmatrix} \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{\psi}^2} & \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{\psi} \partial \dot{E}} & \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{\psi} \partial (\delta \dot{\sigma})} \\ \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{E} \partial \dot{\psi}} & \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{E}^2} & \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{E} \partial (\delta \dot{\sigma})} \\ \frac{\partial^2 \mathcal{L}^{(2)}}{\partial (\delta \dot{\sigma}) \partial \dot{\psi}} & \frac{\partial^2 \mathcal{L}^{(2)}}{\partial (\delta \dot{\sigma}) \partial \dot{E}} & \frac{\partial^2 \mathcal{L}^{(2)}}{\partial (\delta \dot{\sigma})^2} \end{vmatrix} = -\frac{k^4 \omega a^{13} (-1+A) J \dot{\sigma}^2 Q a_\sigma \dot{\phi}_0^2}{2r^3 (2k^2(-1+A)^2 H^2 (1+r) - a^2 A Q \Lambda_A)}. \tag{18}$$

It is easy to see that generically $\det \mathcal{H} \neq 0$ unless $a_\sigma = 0$ (simple quasidilaton), or in the late time limit (the $J = 0$ branch, [11]). Therefore, there is the Boulware-Deser mode in the theory.

It should also be noted that $\dot{\phi}_0 = 0$ is a singular case which does not correspond to the model with fixed fiducial metric. Indeed, the Stückelberg fields describe the change of coordinates, and as such they must satisfy $\det \frac{\partial \phi^a}{\partial x^\mu} \neq 0$. Therefore, the $\dot{\phi}_0 = 0$ case from the Ref. [15] belongs to an unrestricted Stückelberg model but not to the initial theory which has been covariantized by Stückelbergs.

Our ghost result is in accordance with the claims made in [15]. Moreover, their expressions can be easily obtained in the $k \to 0$ limit of our quadratic action. In most parts our formulas are also similar to those from Ref. [14] which had the opposite conclusion. However, expressions from Ref. [14] do lack terms with a_σ, presumably due to considerations of deep UV limit,[1] and setting this coupling to zero reduces the model to simple quasidilaton which is indeed BD-ghost-free. Note that taking a deep UV limit is a shaky ground for calculating the number of degrees of freedom since the Hessian might be nondegenerate at any finite wave number, but its different eigenvalues can have different $k \to \infty$ asymptotics which could lead to degeneracy in a simplified UV analysis.

[1] Lavinia Heisenberg, private communication.

IV. TREATMENT WITH STÜCKELBERG FIELDS

We have also checked the conclusion by calculations in the Stückelberg picture. In this case we worked with slightly different formulation. In terms of β coefficients instead of α-s the theory takes the form

$$S = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} \times \left(R[g] - \frac{\omega}{M_{\rm Pl}^2} \partial_\mu \sigma \partial^\mu \sigma + 2m^2 \sum_{n=0}^{4} \beta_n \mathcal{U}_n[\mathbb{X}] \right) \tag{19}$$

with

$$\mathbb{X}^\mu{}_\nu = e^{\sigma/M_{\rm Pl}} \left(\sqrt{g^{-1} \tilde{f}} \right)^\mu{}_\nu \tag{20}$$

where $\mathcal{U}_0[\mathbb{X}] = 1$ and $\mathcal{U}_1[\mathbb{X}] = [\mathbb{X}]$.

This is an equivalent formulation since matrices $-\mathbb{X}$ and K differ only by addition of the unit matrix, and therefore their elementary symmetric polynomials are related to each other. In particular, the β_n coefficients should be taken as

$$\beta_0 = 6 + 4\alpha_3 + \alpha_4, \tag{21a}$$

$$\beta_1 = -3 - 3\alpha_3 - \alpha_4, \tag{21b}$$

$$\beta_2 = 1 + 2\alpha_3 + \alpha_4, \tag{21c}$$

$$\beta_3 = -\alpha_3 - \alpha_4, \tag{21d}$$

$$\beta_4 = \alpha_4 \tag{21e}$$

in terms of α_3, α_4 parameters from the previous section.

In the Stückelberg picture we consider only perturbations of the quasidilton and of the Stückelberg fields. The relevant terms in the quadratic action are those that include $\dot{\delta\sigma}$, $\delta\dot{\phi}_a$ combinations (they do not kinetically mix with the physical metric)

$$\mathcal{L}^{(2)} = \frac{e^{-3\sigma/M_{\rm Pl}}}{2a^3m^2M_{\rm Pl}^2n^3N^2(an+N)}[-2e^{\frac{2\sigma}{M_{\rm Pl}}}m^2M_{\rm Pl}N(an+N)\Theta\delta\dot{\sigma}\,\dot{\sigma}\,\delta\dot{\phi}_0\,\dot{\phi}_0\,a_\sigma$$
$$+(an+N)\delta\dot{\sigma}^2(a^3e^{\frac{3\sigma}{M_{\rm Pl}}}m^2M_{\rm Pl}^2n^3\omega + e^{\frac{2\sigma}{M_{\rm Pl}}}m^2M_{\rm Pl}^2n^2N\Theta a_\sigma - N\Theta\dot{\sigma}^2a_\sigma^2)$$
$$+e^{\frac{4\sigma}{M_{\rm Pl}}}m^4M_{\rm Pl}^2N((an+N)\delta\dot{\phi}_0{}^2(n^2-\dot{\phi}_0{}^2)\Theta - an^3\delta\dot{\phi}_i{}^2\Xi)] \tag{22}$$

with

$$\Theta = a^3\beta_1 + 3a^2e^{\frac{\sigma}{M_{\rm Pl}}}\beta_2 + 3ae^{\frac{2\sigma}{M_{\rm Pl}}}\beta_3 + e^{\frac{3\sigma}{M_{\rm Pl}}}\beta_4, \tag{23a}$$

$$\Xi = a^3\beta_1 + 2a^2e^{\frac{\sigma}{M_{\rm Pl}}}\beta_2 + ae^{\frac{2\sigma}{M_{\rm Pl}}}\beta_3 \tag{23b}$$

where we have switched off the physical metric perturbations since now we are in the framework with restored diffeomorphism invariance, and the danger comes from the "matter" sector represented by Stückelberg fields.

In the previous notations (14), (17) the final expression for $\mathcal{L}^{(2)}$ is

$$\mathcal{L}^{(2)} = \frac{1}{2N^2}\left(\omega\delta\dot{\sigma}^2 + \frac{m^2N}{a^3A^3}\left(\frac{a^5A^4Q\delta\dot{\phi}_i{}^2}{m^2(A-1)^2(an+N)} + \frac{a^4(-1+A)A^4J\delta\dot{\phi}_0{}^2(n^2-\dot{\phi}_0{}^2)}{n^3} - \frac{2a^2(-1+A)A^2J\delta\dot{\sigma}\,\dot{\sigma}\,\delta\dot{\phi}_0\,\dot{\phi}_0\,a_\sigma}{m^2M_{\rm Pl}n^3} + \frac{(-1+A)J(\delta\dot{\sigma})^2a_\sigma(a^2A^2m^2M_{\rm Pl}^2n^2-\dot{\sigma}^2a_\sigma)}{m^4M_{\rm Pl}^2n^3}\right)\right) \tag{24}$$

The corresponding Hessian is

$$\det\frac{\partial\mathcal{L}^{(2)}}{\partial\{\delta\dot{\sigma},\delta\dot{\phi}_a\}} = \frac{\omega a_\sigma a^5A^2JQ^3\dot{\sigma}^2}{(-1+A)^5M_{\rm Pl}^2n^3N^6(an+N)^3} \neq 0, \tag{25}$$

from which the presence of the B-D ghost is apparent.

V. DISCUSSION AND CONCLUSIONS

We confirm the presence of BD ghost in extended quasidilaton massive gravity which has been observed in Refs. [15,16]. A comment on the ghost-freedom "proof" of the paper [12] is in order. There the gauge choice of $\phi^0 = -e^{-\sigma}$ has been used. It was noted in Refs. [13,16] that it is not a correct gauge choice. We do not find a proper explanation there which is however quite simple. A gauge choice would amount to a condition which can be ensured by a coordinate transformation without restricting the physical variables. Obviously, with two arbitrary fields $\phi^0(t,x)$ and $\sigma(t,x)$ which can have different constant value surfaces in the spacetime, it is not possible to make one field a function of the other with simply a coordinate choice.

Now it seems firmly established that the model with extended quasidilaton is not ghost-free, and therefore it is not a viable option for massive cosmology. Though extremely compelling from the theoretical viewpoint, massive gravity is not doing as good for phenomenology (if not to play with bimetric regimes close to GR [17]). The quest for solving fundamental cosmological puzzles is as open as ever before.

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