PHYSICAL REVIEW D **96,** 104016 (2017)

Metric for two equal Kerr black holes

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(Received 20 February 2017; published 13 November 2017)

We show that the exact solution of Einstein's equations describing a binary system of aligned identical Kerr black holes separated by a massless strut follows straightforwardly from the extended 2-soliton solution possessing equatorial symmetry. We give its concise analytic form in terms of physical parameters and then compare with our old solution of that problem obtained in canonical parametrization, demonstrating the equivalence of the two approaches. A surprising physical by-product of our analysis is the discovery that up to three different configurations of two corotating Kerr sources can have equal masses and equal angular momenta. We also introduce physical parametrization to the general six-parameter asymmetric configuration which permits to treat analytically the case of two nonequal corotating black holes.

DOI: 10.1103/PhysRevD.96.104016

I. INTRODUCTION

In our recent paper [1] we have considered in detail a vacuum specialization of the general 2-soliton electrovac metric [2] (henceforth referred to as the MMR solution) in application to the description of the exterior geometry of neutron stars [3,4]. Although we gave in that paper three different representations of the vacuum MMR solution, still we had one more representation of the latter solution that had been worked out by us some time ago for treating the two-body configurations of spinning black holes of Kerr's type [5], but we left its consideration for the future because of the specific objectives of the paper [1]. However, the appearance of a preprint on corotating Kerr sources [6] that follows the ideas of [7] has motivated us, for several reasons, to make public our results on the vacuum MMR solution not earlier included into the paper [1]. First, because the problem of corotating identical Kerr particles was already solved (but not published) by us long ago using the usual canonical parametrization [8], it would certainly be interesting and instructive to consider both approaches to the same problem in one place and compare them. Second, although the authors of the paper [6] solved correctly the axis condition for the binary system described by the solution [2], they still were unable to get a concise form of the resulting metric, and they omitted some important details of the derivation that might be interesting to the reader. Therefore, it is likely to work out a simple form of the three-parameter subfamily of the MMR solution describing the system of two equal Kerr black holes kept apart by a massless strut [9] that would improve the representation obtained in [6]. Moreover, it would be also desirable to consider possible extensions of the results obtained for identical constituents to the case of nonequal Kerr sources. To accomplish the first two goals, we will first rewrite, using the procedure we have developed in a series of papers devoted to the binary black-hole configurations [10–13], the vacuum MMR metric in terms of physical parameters by taking as a starting point the axis data of the extended equatorially symmetric 2-soliton solution in the form [14]

$$e(z) = \frac{z^2 - \bar{b}_1 z + \bar{b}_2}{z^2 + b_1 z + b_2}, \qquad (1)$$

where b_1 and b_2 are two arbitrary complex constants, and a bar over a symbol means complex conjugation. The particular three-parameter case of two separated Kerr black holes will arise by just imposing the axis condition in the general four-parameter metric. We will then show how this three-parameter solution is obtainable using the standard parametrization of the extended double-Kerr (EDK) spacetime, and this will enable us to compare the two approaches employed for the derivation of that solution. A remarkable output of our analysis will be demonstration that the configurations with equal masses and angular momenta are not unique—something that has never been reported before for the binary configurations. For treating the more general case of nonequal Kerr black holes we shall reparametrize the axis data in a manner similar to the case of identical constituents, allowing for a concise representation of the corresponding metric functions.

The paper is organized as follows. In the next section we perform a reparametrization of the data (1) in terms of the quantities M, a, σ and R related, respectively, to the masses of the sources, their angular momenta, the horizons' half lengths and the coordinate distance between the centers of the sources. The reparametrized axis data is then used for writing out the MMR solution in a new concise representation with the aid of the general formulas of Ref. [15]. In Sec. III we solve the axis condition for the MMR solution and analyze the resulting three-parameter configuration of corotating Kerr black holes. By expanding the expression of the interaction force in inverse powers of R, we show in particular that the leading spin-spin repulsion term has precisely the same form as was given earlier by Dietz and Hoenselaers [16] through the analysis of two limiting cases of spinning particles. An alternative derivation of the three-parameter metric for identical Kerr sources is performed in Sec. IV, where we also compare the two approaches to this problem and establish that two different configurations of that type can be characterized by the same masses and angular momenta. In Sec. V we give the reparametrized form of the general asymmetric 2-soliton metric suitable for treating the case of two nonequal Kerr black holes. Section V contains concluding remarks.

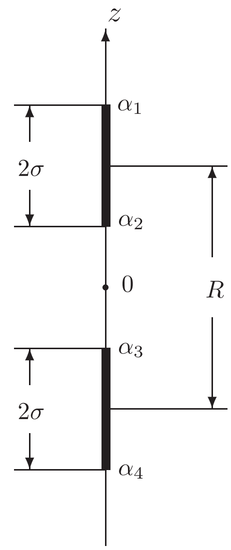

FIG. 1. Location of two identical Kerr black holes on the symmetry axis: $\alpha_4 = -\alpha_1$, $\alpha_3 = -\alpha_2$.

II. YET ANOTHER REPRESENTATION OF THE VACUUM MMR SOLUTION

We would like to recall that the extended vacuum soliton solutions [15] constructed with the aid of Sibgatullin's integral method [17] are written in terms of the parameters α_n and β_l, the former parameters taking real values or forming complex conjugate pairs (these determine the location of sources on the symmetry axis), and the latter being roots of the denominator in the axis data, hence taking arbitrary complex values.

In the 2-soliton case with the additional equatorial symmetry we have $\alpha_1 = -\alpha_4$, $\alpha_2 = -\alpha_3$, so that the α's can be parametrized as

$$\alpha_1 = \frac{1}{2}R + \sigma, \qquad \alpha_2 = \frac{1}{2}R - \sigma,$$
$$\alpha_3 = -\frac{1}{2}R + \sigma, \qquad \alpha_4 = -\frac{1}{2}R - \sigma, \tag{2}$$

or, inversely,

$$R = \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4),$$
$$\sigma = \frac{1}{2}(\alpha_1 - \alpha_2) = \frac{1}{2}(\alpha_3 - \alpha_4), \tag{3}$$

where R is the coordinate distance between the centers of black holes, and σ is the half length of the horizon of each black hole (see Fig. 1). Note that σ in the above formulas (2) and (3), as well as throughout this paper, can also take on pure imaginary values, in which case the solution would describe a pair of equal hyperextreme objects. However, except for some special occurrences, below we will restrict our analysis to the black-hole configurations only.

To identify the complex parameters β_1 and β_2, one has to introduce explicitly the axis data—the value of the Ernst complex potential [18] on the upper part of the symmetry axis. In our case such data is given by formula (1), and obviously can be cast into the equivalent form

$$e(z) = \frac{z^2 - 2(M + ia)z + c + id}{z^2 + 2(M - ia)z + c - id}, \tag{4}$$

involving four arbitrary real constants M, a, c and d. Since β_1 and β_2 are roots of the denominator on the right-hand side of (4), it is clear that these verify the relation $\beta_1 + \beta_2 = -2(M - ia)$ and $\beta_1\beta_2 = c - id$, while the denominator itself can be formally written as $(z - \beta_1)(z - \beta_2)$.

We must bear in mind that the parameters α_n in Sibgatullin's method satisfy the equation

$$e(z) + \bar{e}(z) = 0, \tag{5}$$

which means that if we want to introduce these α_n into the 2-soliton solution as arbitrary parameters in the form (2), then we have to solve the equation

$$e(z) + \bar{e}(z) = \frac{2(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)(z - \alpha_4)}{(z - \beta_1)(z - \beta_2)(z - \bar{\beta}_1)(z - \bar{\beta}_2)} \tag{6}$$

for the constants c and d by equating the coefficients at the same powers of z on both sides of (6). A simple algebra then yields

$$c = -\frac{1}{4}R^2 + 2M^2 - 2a^2 - \sigma^2,$$
$$d = \epsilon\sqrt{(R^2 - 4M^2 + 4a^2)(\sigma^2 - M^2 + a^2)}, \qquad \epsilon = \pm 1, \tag{7}$$

with which the axis data (4) finally takes the form

$$e(z) = \frac{z^2 - 2(M + ia)z - \frac{1}{4}R^2 + 2M^2 - 2a^2 - \sigma^2 + id}{z^2 + 2(M - ia)z - \frac{1}{4}R^2 + 2M^2 - 2a^2 - \sigma^2 - id}, \tag{8}$$

where the constant quantity d has been defined in (7).

Therefore, we have rewritten the axis data (4) containing the parameters M, a, c and d in the equivalent form (8) involving the desired set of the parameters M, a, R and σ. It is worth noting that while the physical meaning of the constants R and σ is transparent, the interpretation of the parameters M and a can be revealed by calculating the solution's total mass M_T and total angular momentum J_T from (8) with the help of the Fodor *et al.* procedure [19] for the evaluation of Geroch-Hansen multipole moments [20,21]. Thus we get

$$M_T = 2M, \qquad J_T = 4Ma - d, \tag{9}$$

whence it follows immediately that M is half the total mass of the configuration, whereas a is the rotational parameter. Observe that M does not coincide exactly with the mass of each black-hole constituent because the intermediate region $\{\rho = 0, |z| < \alpha_2\}$ in Fig. 1 may in principle carry some mass, positive or negative.

Once the axis data is worked out, the corresponding potential $\mathcal{E}$ satisfying the Ernst equation [18],

$$(\mathcal{E} + \bar{\mathcal{E}})\Delta\mathcal{E} = 2(\nabla\mathcal{E})^2, \tag{10}$$

can be obtained from the formula [15]

$$\mathcal{E} = \frac{E_+}{E_-}, \quad E_\pm = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ \pm 1 & \frac{r_1}{\alpha_1 - \beta_1} & \frac{r_2}{\alpha_2 - \beta_1} & \frac{r_3}{\alpha_3 - \beta_1} & \frac{r_4}{\alpha_4 - \beta_1} \\ \pm 1 & \frac{r_1}{\alpha_1 - \beta_2} & \frac{r_2}{\alpha_2 - \beta_2} & \frac{r_3}{\alpha_3 - \beta_2} & \frac{r_4}{\alpha_4 - \beta_2} \\ 0 & \frac{1}{\alpha_1 - \bar{\beta}_1} & \frac{1}{\alpha_2 - \bar{\beta}_1} & \frac{1}{\alpha_3 - \bar{\beta}_1} & \frac{1}{\alpha_4 - \bar{\beta}_1} \\ 0 & \frac{1}{\alpha_1 - \bar{\beta}_2} & \frac{1}{\alpha_2 - \bar{\beta}_2} & \frac{1}{\alpha_3 - \bar{\beta}_2} & \frac{1}{\alpha_4 - \bar{\beta}_2} \end{vmatrix}, \tag{11}$$

by just substituting the expressions of α's and β's determined by (2) and (8) into (11), and taking into account that the functions r_n, which depend on the coordinates ρ and z, have the form $r_n = \sqrt{\rho^2 + (z - \alpha_n)^2}$.

In the Ernst formalism [18], the knowledge of the potential $\mathcal{E}$ is sufficient for the construction of the corresponding metric functions f, γ and ω from the stationary axisymmetric line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{12}$$

and the explicit expressions for these functions defined by the potential (11) can be found in Ref. [15] both in the form of determinants and in the expanded form most suitable for concrete computations and presentation of the results. Our own evaluation of $\mathcal{E}$, f, γ and ω for the axis data (8) yields the following final formulas:

$$\begin{aligned} \mathcal{E} &= \frac{A - B}{A + B}, \qquad f = \frac{A\bar{A} - B\bar{B}}{(A + B)(\bar{A} + \bar{B})}, \\ e^{2\gamma} &= \frac{A\bar{A} - B\bar{B}}{K_0^2 R_+ R_- r_+ r_-}, \qquad \omega = 4a - \frac{2\mathrm{Im}[G(\bar{A} + \bar{B})]}{A\bar{A} - B\bar{B}}, \\ A &= R^2(R_+ - R_-)(r_+ - r_-) - 4\sigma^2(R_+ - r_+)(R_- - r_-), \\ B &= 2R\sigma[(R + 2\sigma)(R_- - r_+) - (R - 2\sigma)(R_+ - r_-)], \\ G &= -zB + R\sigma[2R(R_- r_- - R_+ r_+) + 4\sigma(R_+ R_- - r_+ r_-) \\ &\quad - (R^2 - 4\sigma^2)(R_+ - R_- - r_+ + r_-)], \end{aligned} \tag{13}$$

where

$$\begin{aligned} R_\pm &= \frac{-M(\pm 2\sigma + R) + id}{2M^2 + (R + 2ia)(\pm\sigma + ia)} \times \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \sigma\right)^2}, \\ r_\pm &= \frac{-M(\pm 2\sigma - R) + id}{2M^2 - (R - 2ia)(\pm\sigma + ia)} \times \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma\right)^2}, \end{aligned} \tag{14}$$

and

$$K_0 = \frac{4R^2\sigma^2(R^2 - 4\sigma^2)[(R^2 + 4a^2)(\sigma^2 + a^2) - 4M(M^3 + ad)]}{[M^2(R + 2\sigma)^2 + d^2][M^2(R - 2\sigma)^2 + d^2]}. \tag{15}$$

Equations (13)–(15) and (7) fully determine the desired representation of the four-parameter vacuum MMR solution which, as will be seen in the next section, is very suitable for treating the case of two separated Kerr black holes. One can check by direct calculation that on the upper part of the symmetry axis $\{\rho = 0, z > \frac{1}{2}R + \sigma\}$ the potential $\mathcal{E}$ in (13) reduces to the axis data (8).

III. TWO IDENTICAL KERR BLACK HOLES SEPARATED BY A STRUT

The MMR solution discussed in the previous section can be interpreted as describing a pair of corotating Kerr black holes after subjecting its parameters to the constraint

$$\omega = 0 \quad \text{for } \rho = 0, \qquad |z| < \frac{1}{2}R - \sigma, \tag{16}$$

which is known as the axis condition; this being satisfied, converts the region $\{\rho = 0, |z| < \frac{1}{2}R - \sigma\}$ into a massless conical singularity, a strut [9], which separates the two black-hole constituents and prevents them from falling onto each other. In this special case, the parameter M becomes equal to the Komar mass [22] of each constituent exactly, while the individual angular momentum J of each black hole becomes equal to $J_T/2$ because the strut does not make contribution into the mass and angular momentum of the configuration.

On the symmetry axis, the metric function ω of the 2-soliton metric takes constant values generically [23], so that from the condition (16) we get a (complicated) algebraic equation for the parameters M, a, σ and R, which nonetheless factorizes and eventually leads to the quadratic equation for σ,

$$(R^2 + 2MR + 4a^2)^2\sigma^2 - M^2R^2(R + 2M)^2 + a^2(R^2 - 4M^2 + 4a^2)(R^2 + 4MR - 4M^2 + 4a^2) = 0, \tag{17}$$

with the positive root

$$\sigma = \sqrt{M^2 - a^2 + \frac{4M^2a^2(R^2 - 4M^2 + 4a^2)}{(R^2 + 2MR + 4a^2)^2}}, \tag{18}$$

which coincides with the expression for σ obtained in Ref. [6].

Taking into account (18), the constant quantity d from (7) assumes the form

$$d = \frac{2Ma(R^2 - 4M^2 + 4a^2)}{R^2 + 2MR + 4a^2}, \tag{19}$$

and this is exactly the quantity δ from the paper [6]. The constant K_0 from (15) rewrites, with account of (18) and (19), as

$$K_0 = \frac{4\sigma^2[(R^2 + 2MR + 4a^2)^2 - 16M^2a^2]}{M^2[(R + 2M)^2 + 4a^2]}. \tag{20}$$

We mention that the above expression for d can be also used for writing σ in a slightly simpler form

$$\sigma = \sqrt{M^2 - a^2 + d^2(R^2 - 4M^2 + 4a^2)^{-1}}. \tag{21}$$

Therefore, the three-parameter specialization of the MMR solution describing two equal corotating Kerr black holes separated by a strut is defined concisely by formulas (13), (14) and (18)–(20). Apparently, our expressions for the Ernst potential and for all metric functions defining this subfamily are a good deal simpler than the ones obtained in Ref. [6].

On the horizons (the null hypersurfaces $\rho = 0, -\sigma < z - \frac{1}{2}R < \sigma$ and $\rho = 0, -\sigma < z + \frac{1}{2}R < \sigma$—two thick rods in Fig. 1), the black-hole constituents of this binary configuration are expected to verify the well-known Smarr mass formula [24]

$$M = \frac{1}{4\pi}\kappa S + 2\Omega J, \tag{22}$$

where κ is the surface gravity, S the area of the horizon, Ω the horizon's angular velocity and J the Komar angular momentum of a black hole. Apparently, because of the equatorial symmetry of the problem, the relation (22) should be checked only for one of the constituents, say, for the upper one. Since the black holes are corotating, their Komar masses and angular momenta are both halves of the respective total values, M_T and J_T, determined by (40); hence, the mass of each black hole is M, while the corresponding individual angular momentum J is given, as it follows from (40) and (19), by the expression [6]

$$J = \frac{Ma[(R + 2M)^2 + 4a^2]}{R^2 + 2MR + 4a^2}, \tag{23}$$

and one can see that the inverse dependence $a(J)$ is defined by a cubic equation.

For the calculation of the quantities κ, Ω and S, the following formulas should be used [25]:

$$\kappa = \sqrt{-\omega_H^{-2}e^{-2\gamma_H}}, \qquad \Omega = \omega_H^{-1}, \qquad S = 4\pi\sigma\kappa^{-1}, \tag{24}$$

where ω_H and γ_H denote the values of the metric functions ω and γ on the horizon. The straightforward calculations carried out for the upper black hole yield the following expression for the horizon's angular velocity:

$$\Omega = \frac{(M - \sigma)(R^2 + 2MR + 4a^2)}{2Ma[(R + 2M)^2 + 4a^2]}, \tag{25}$$

while the quantities S and κ are defined by the formula [6]

$$S = \frac{4\pi\sigma}{\kappa} = \frac{8\pi M[(R + 2M)^2 + 4a^2][(R + 2M)(M + \sigma) - 4a^2]}{(R + 2\sigma)(R^2 + 2MR + 4a^2)}. \tag{26}$$

Then it is easy to see that Smarr's relation (22) is indeed verified by virtue of (23), (25) and (26).

Let us briefly comment on the possibility of the equilibrium without a strut between two corotating Kerr sources. If we denote by γ_0 the constant value of the metric function γ on the strut, then the interaction force in our binary system can be found by means of the formula $\mathcal{F} = (e^{-\gamma_0} - 1)/4$ [9,26], thus yielding [6]

$$\mathcal{F} = \frac{M^2[(R+2M)^2 - 4a^2]}{(R^2 - 4M^2 + 4a^2)[(R+2M)^2 + 4a^2]}. \tag{27}$$

This force becomes zero at infinite separation of the constituents, and also when $|a| = (R+2M)/2$. In the latter case, σ becomes a pure imaginary quantity, which means that balance at finite separation is only possible between two hyperextreme Kerr sources; the value of the angular momentum leading to the equilibrium is $|J| = M(R+2M)^2/(R+M)$, being characteristic of the Dietz-Hoenselaers equilibrium configuration [16].

In order to have a somewhat better idea about the interaction force in the generic case, it seems plausible to resort to some approximations in (27) for introducing the angular momentum J explicitly. Then we readily get from (23) and (27) the following approximate formula for $\mathcal{F}$ as $R \to \infty$:

$$\mathcal{F} \simeq \frac{M^2}{R^2} + \frac{4M^4 - 12J^2}{R^4} + \frac{80MJ^2}{R^5} + O\left(\frac{1}{R^6}\right). \tag{28}$$

The form of the leading term in (28) responsible for the spin-spin interaction of corotating Kerr sources coincides with the one already given by Dietz and Hoenselaers [16] through the analysis of two limiting cases of spinning particles in the double-Kerr solution [27].

IV. RESOLUTION OF THE PROBLEM OF COROTATING IDENTICAL KERR SOURCES IN CANONICAL PARAMETRIZATION

In this section we shall present our old solution of the problem of identical Kerr sources separated by a massless strut obtained by us already in 2009 using canonical representation of the EDK solution, but not published to date due to various circumstances. It will be shown that the binary configurations obtainable in this way are fully equivalent to the two-body systems considered in the previous section.

We remind that the Ernst potential of the EDK solution, after the expansion of the determinants in formula (11), takes the form [15,28]

$$\begin{aligned}
&\mathcal{E} = E_+/E_-, \qquad E_\pm = \Lambda \pm \Gamma, \qquad \Lambda = \sum_{1\le i<j\le 4} \lambda_{ij} r_i r_j, \qquad \Gamma = \sum_{i=1}^{4} \gamma_i r_i,\\
&\lambda_{ij} = (-1)^{i+j}(\alpha_i - \alpha_j)(\alpha_{i'} - \alpha_{j'})X_i X_j, \qquad (i', j' \neq i, j; i' < j')\\
&\gamma_i = (-1)^i (\alpha_{i'} - \alpha_{j'})(\alpha_{i'} - \alpha_{k'})(\alpha_{j'} - \alpha_{k'})X_i, \qquad (i', j', k' \neq i; i' < j' < k')\\
&X_n = \frac{(\alpha_n - \bar\beta_1)(\alpha_n - \bar\beta_2)}{(\alpha_n - \beta_1)(\alpha_n - \beta_2)}, \qquad r_i = \sqrt{\rho^2 + (z - \alpha_i)^2},
\end{aligned} \tag{29}$$

while the corresponding metric functions are defined by the expressions

$$\begin{aligned}
&f = \frac{E_+\bar E_- + \bar E_+ E_-}{2E_-\bar E_-}, \qquad e^{2\gamma} = \frac{E_+\bar E_- + \bar E_+ E_-}{2\lambda_0\bar\lambda_0 r_1 r_2 r_3 r_4}, \qquad \omega = \frac{2i(G\bar E_- - \bar G E_-)}{E_+\bar E_- + \bar E_+ E_-},\\
&G = -\sigma_0\Lambda + \bar\sigma_0\Gamma + z\Gamma + \sum_{1\le i<j\le 4}(\alpha_i + \alpha_j)\lambda_{ij} r_i r_j - \sum_{i=1}^{4}(\alpha_{i'} + \alpha_{j'} + \alpha_{k'})\gamma_i r_i, \qquad (i', j', k' \neq i; i' < j' < k')\\
&\lambda_0 = \sum_{1\le i<j\le 4}\lambda_{ij}, \qquad \sigma_0 \equiv \beta_1 + \beta_2 = \frac{1}{\lambda_0}\left[\gamma_0 + \sum_{1\le i<j\le 4}(\alpha_i + \alpha_j)\lambda_{ij}\right], \qquad \gamma_0 = \sum_{i=1}^{4}\gamma_i.
\end{aligned} \tag{30}$$

Here the parameters α_n are the same as used in the previous sections, i.e., they determine the location of sources on the symmetry axis and can take on real values or occur in complex conjugate pairs; β_l are two arbitrary complex constants, while the four constant objects X_n are such that

$$\begin{aligned}
&X_i\bar X_i = 1 \quad \text{for real } \alpha_i,\\
&X_i\bar X_j = 1 \quad \text{for } \alpha_i = \bar\alpha_j,
\end{aligned} \tag{31}$$

and these X_n can be used as independent parameters instead of the constants β_l.

Assuming the usual ordering of the parameters α_n, namely, $\mathrm{Re}(\alpha_1) \geq \mathrm{Re}(\alpha_2) > \mathrm{Re}(\alpha_3) \geq \mathrm{Re}(\alpha_4)$, in the equatorially symmetric case [29,30] the above α_n and X_n must verify the relations [28]

$$\alpha_4 = -\alpha_1, \qquad \alpha_3 = -\alpha_2, \\ X_4 = -1/X_1, \qquad X_3 = -1/X_2 \tag{32}$$

independently of whether α's are real or complex valued. Therefore, we can choose α_1, α_2, X_1 and X_2 as arbitrary parameters defining this equatorially symmetric subfamily of the EDK solution. Note that Eqs. (29)–(32) are equivalent to the vacuum MMR metric.

By construction, the metric functions γ and ω of the four-parameter solution (29)–(32) take zero values on the upper and lower parts of the symmetry z-axis defined, respectively, by $z > \mathrm{Re}(\alpha_1)$ and $z < -\mathrm{Re}(\alpha_1)$. Hence, in order to interpret this configuration as a system of two separated identical Kerr black holes or hyperextreme objects it is only necessary to solve the axis condition on the part $-\mathrm{Re}(\alpha_2) \leq z \leq \mathrm{Re}(\alpha_2)$ of the symmetry axis separating the two constituents:

$$\omega(\rho = 0, z) = \omega^{(0)} = 0, \tag{33}$$

where $\omega^{(0)}$ denotes the constant value taken by the function ω in the intermediate region (see Fig. 1). The latter constant value $\omega^{(0)}$ was calculated for the general EDK solution in the paper [28], where it was found to have the form $\omega^{(0)} = C_\omega^{(0)}/C_f^{(0)}$, the constant objects $C_\omega^{(0)}$ and $C_f^{(0)}$ being defined by formula (7) of [28]. In our particular equatorially symmetric case the use of the latter formulas leads, after a convenient parametrization of α_1 and α_2 as

$$\alpha_1 = s\left(\frac{1}{2} + \sigma\right), \qquad \alpha_2 = s\left(\frac{1}{2} - \sigma\right), \tag{34}$$

where s is the separation distance fully analogous to R in (2), and σ is the dimensionless analog of σ in (2), to the following explicit form of the axis condition (33):

$$\frac{8i\sigma s N}{D} = 0,$$
$$N = 2\sigma^2(X_1X_2 + 1)[X_1 + X_2 + X_1X_2(X_1 + X_2 + 4)] - \sigma(X_1 - X_2)(X_1 + 1)(X_2 + 1)(X_1X_2 - 1) + \frac{1}{2}(X_1 - X_2)^2(X_1X_2 + 1),$$
$$D = [2\sigma(X_1X_2 - 1) + X_1 - X_2][4\sigma^2(X_1X_2 + 1)^2 + (X_1 - X_2)^2]. \tag{35}$$

It is easy to see that after the redefinition $\sigma = (X_1 - X_2)\kappa$, the factor $(X_1 - X_2)$ cancels out in (35), and both the numerator and denominator of (35) become functions of the product X_1X_2 and the sum $X_1 + X_2$ only. The product X_1X_2 is a complex unitary number, so it can be defined as $X_1X_2 = \phi^2$, $\phi\bar{\phi} = 1$, and the sum can be chosen in the form $X_1 + X_2 = 2\phi\mu$, where μ, as is not difficult to show, is some real function, so that X_1 and X_2 are roots of the quadratic equation $X^2 - 2\phi\mu X + \phi^2 = 0$ whose solution is

$$X_1 = \phi\left(\mu + \sqrt{\mu^2 - 1}\right), \qquad X_2 = \phi\left(\mu - \sqrt{\mu^2 - 1}\right). \tag{36}$$

Expressions (36) yield for X_1 and X_2 the unitary complex values when $|\mu| < 1$, in which case α_1 and α_2 are real valued, thus determining a pair of subextreme constituents; but if $|\mu| > 1$, then $X_1\bar{X}_2 = 1$ and we have the case of hyperextreme constituents. Accounting for (36), the redefined σ assumes the form $\sigma = 2\phi\kappa\sqrt{\mu^2 - 1}$, and, observing that the product $\phi\kappa$ is a pure imaginary quantity for any choice of α_1 and α_2, it is advantageous to further redefine κ as $\kappa = -i/(2\phi\nu)$, ν being a real quantity, with which σ assumes the new form

$$\sigma = -\frac{i\sqrt{\mu^2 - 1}}{\nu}, \tag{37}$$

whence it follows in particular that $\nu > 0$.

After the substitution of (36) and (37) into (35), the axis condition finally rewrites as

$$\frac{2s\phi\{2\mu[\nu\phi(\phi^2 - 1) + i(\phi^2 + 1)^2] + (\phi^2 + 1)[\nu(\phi^2 - i\nu\phi - 1) + 4i\phi]\}}{(\phi^2 + i\nu\phi - 1)[\phi^4 + \phi^2(\nu^2 - 2) - 1]} = 0, \tag{38}$$

with the apparent solution

$$\mu = \frac{(\phi^2 + 1)[\phi(\nu^2 - 4) + i\nu(\phi^2 - 1)]}{2[(\phi^2 + 1)^2 - i\nu\phi(\phi^2 - 1)]}. \tag{39}$$

Therefore, we have solved the axis condition and arrived at the following important conclusion: formulas (36), (37) and (39) define completely the subfamily of the EDK solution for two identical corotating Kerr sources. The three parameters it involves are ϕ, ν and s.

To answer the question of whether in this particular solution it is possible to use the mass and angular momentum as arbitrary parameters instead of ϕ and ν, we shall use the dimensionless quantities m and a in terms of which the total mass M and total angular momentum J of the system have the form

$$M = 2ms, \qquad J = 2mas^2, \tag{40}$$

so that m and a represent the individual mass and angular momentum per unit mass of each constituent. The procedure [19] then readily gives in our case the expressions of M and J, and consequently of m and a, in terms of ϕ and ν:

$$m = -\frac{2\phi^2}{(\phi^2+1)^2 - i\nu\phi(\phi^2-1)},$$
$$a = \frac{(\phi^2+1)(\phi^2-1-i\nu\phi)^2}{2\nu\phi[(\phi^2+1)^2 - i\nu\phi(\phi^2-1)]}, \tag{41}$$

and despite the aspect of m and a in (41), these are the real quantities because of the unitarity of ϕ and reality of ν. Solving now the first equation in (41) for ν and substituting the result into the expression of a, we get

$$\nu = -\frac{i[m(\phi^2+1)^2 + 2\phi^2]}{m\phi(\phi^2-1)},$$
$$a = -\frac{i\phi^2(2m+1)^2(\phi^2+1)}{(\phi^2-1)[m(\phi^2+1)^2+2\phi^2]}, \tag{42}$$

while the expression (39) for μ rewrites as

$$\mu = \frac{(\phi^2+1)[4m^2(\phi^4+1) + m(\phi^4+6\phi^2+1) + 2\phi^2]}{2m\phi(\phi^2-1)^2}. \tag{43}$$

It follows from (42) that finding ϕ as a function of m and a reduces to the solution of the cubic equation

$$Z^3 - pZ^2 + \bar{p}Z - 1 = 0,$$
$$p = -\frac{m+2}{m} - \frac{i(2m+1)^2}{ma}, \qquad Z \equiv \phi^2, \tag{44}$$

which to some extent is analogous to the cubic Eq. (23) of the previous section. The need to solve the above equation for introducing explicitly the genuine angular momentum explicitly into the solution under consideration indicates first of all that, like in the case of the parametrization used in the previous section, such introduction would only highly complicate the form of the solution and hence does not look to be a practical goal to pursue. On the other hand, the cubic Eq. (44), unlike Eq. (23), turns out to be very suitable for the analysis of such an important issue as the uniqueness of the configurations with equal masses and equal angular momenta, and below we shall demonstrate that, quite unexpectedly, the latter two physical characteristics do not always define a two-Kerr system uniquely.

The demonstration is not sophisticated and is based on the assertion that Eq. (44) admits at least one unitary solution for any p, and that there are values of p for which all three solutions of (44) are unitary. Indeed, if $\{Z_1, Z_2, Z_3\}$ are solutions of the above equation, then obviously $\{\bar{Z}_1, \bar{Z}_2, \bar{Z}_3\}$ are solutions of the complex conjugate equation $Z^3 - \bar{p}Z^2 + pZ - 1 = 0$ whose solutions, as can be easily verified, are also $\{1/Z_1, 1/Z_2, 1/Z_3\}$. The correspondence between the latter two sets of solutions naturally leads to the following two possibilities:

$$\bar{Z}_1 = 1/Z_1, \qquad \bar{Z}_2 = 1/Z_2, \qquad \bar{Z}_3 = 1/Z_3, \tag{45}$$

with three unitary solutions, and

$$\bar{Z}_1 = 1/Z_1, \qquad \bar{Z}_2 = 1/Z_3, \qquad \bar{Z}_3 = 1/Z_2, \tag{46}$$

where only one of the solutions (the first one) is unitary, and this completes the argument.

To answer the question about the conditions under which Eq. (44) has one or three unitary roots, we note that when all three solutions are unitary they can be written in the form

$$Z_1 = e^{i\psi_1}, \qquad Z_2 = e^{i\psi_2}, \qquad Z_3 = e^{i\psi_3}, \tag{47}$$

where ψ_1, ψ_2 and ψ_3 are arbitrary real constants, while in the case of only one unitary solution, the roots of Eq. (44) can be written as

$$Z_1 = e^{i\psi_1}, \qquad Z_2 = qe^{i\psi_2}, \qquad Z_3 = e^{i\psi_2}/q, \tag{48}$$

where q is a real constant. Moreover, assuming that the dependence of the solutions on p in (44) is reasonably continuous, the passage from the set (47) to the set (48) of solutions occurs when there appears a double unitary root corresponding to $q = 1$ in (48):

$$Z_1 = e^{i\psi_1}, \qquad Z_2 = Z_3 = e^{i\psi_2}. \tag{49}$$

In any of the three cases (47), (48) and (49) the symmetric functions associated with the cubic Eq. (44) permit to express p in terms of the roots of this equation by means of the relations

$$Z_1Z_2Z_3 = 1, \qquad \bar{p} = Z_1Z_2 + Z_1Z_3 + Z_2Z_3,$$
$$p = Z_1 + Z_2 + Z_3, \tag{50}$$

but the result is especially interesting in the boundary case (49) for which the system (50) takes the form

$$e^{i(\psi_1+2\psi_2)} = 1, \qquad \bar{p} = 2e^{i(\psi_1+\psi_2)} + e^{2i\psi_2},$$
$$p = e^{i\psi_1} + 2e^{i\psi_2}. \tag{51}$$

From (51) we get

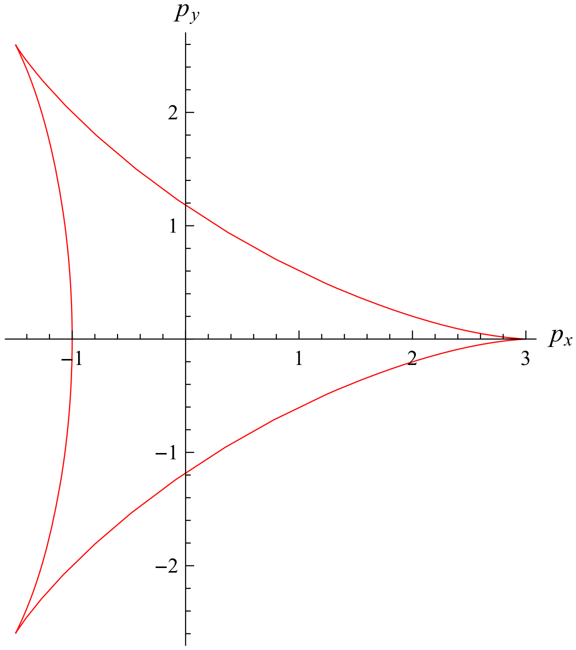

FIG. 2. Plot of boundary values of $p = p_x + ip_y$ separating the case of unique configurations with particular m and a from the case of three different configurations with the same m and a. The vertices of the curved triangle correspond to $p = 3, \frac{3}{2}(-1 \pm i\sqrt{3})$.

$$\psi_1 = -2\psi_2 + 2\pi n, \qquad (n = 0, \pm 1, \pm 2, \ldots)$$
$$p = 2\cos\psi_2 + \cos 2\psi_2 + i(2\sin\psi_2 - \sin 2\psi_2), \tag{52}$$

and the plot of this p defining the case of two unitary roots of Eq. (44) has a curious form shown in Fig. 2 which we dub a "tanga" curve. Apparently, this curved triangle represents a boundary which separates the cubic equations with one unitary solution from those having three such solutions. Note that the latter case is determined by the values of p lying inside the tanga curve.

As can be seen from (44), each value of p determines uniquely a pair of physical parameters m and a via the formulas

$$m = -\frac{2}{p_x + 1}, \qquad a = \frac{(p_x - 3)^2}{2p_y(p_x + 1)} \tag{53}$$

(p_x and p_y denote, respectively, the real and imaginary parts of p) whence it follows that the sign of m depends exclusively on the value of p_x, being positive for $p_x < -1$. Taking into account that there exist values of p giving rise to three possible values of Z, which in turn define six values of $\phi = \pm\sqrt{Z}$, reducible to only three acceptable values after a correct choice of sign ensuring $\nu > 0$, we finally arrive at the conclusion that each particular value of p from the tanga zone defines three possible values of ϕ for the same values of m and a, or in other words, there exist three different solutions for corotating Kerr sources with the same values of mass and angular momentum.

A. Example: Two solutions with the same mass and angular momentum

To illustrate the above analysis of nonuniqueness, let us take for simplicity some value of p belonging to the curved triangle, in which case we will have two different solutions with the same m and a. If we choose for instance $\psi_2 = 5\pi/6$ in (52), then we get from (52) and (53) that

$$p = \frac{1}{2} + i - \left(1 - \frac{i}{2}\right)\sqrt{3}, \qquad \psi_1 = 2\pi - 2\psi_2 = \frac{\pi}{3},$$
$$m = 4 + \frac{8}{\sqrt{3}}, \qquad a = -10 - \frac{37}{2\sqrt{3}}, \tag{54}$$

the roots of the cubic Eq. (44) being

$$Z_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \qquad Z_2 = Z_3 = -\frac{\sqrt{3}}{2} + \frac{i}{2}. \tag{55}$$

The two values of ϕ corresponding to Z_1 are $\pm(\sqrt{3} + i)/2$, and we have to choose the minus sign in order to have positive ν in (42):

$$\phi = -\frac{\sqrt{3}}{2} - \frac{i}{2}, \qquad \nu = \frac{3}{2} + \sqrt{3}. \tag{56}$$

Similarly, in the case of the second root Z_2 we get for ϕ and ν

$$\phi = -\frac{(\sqrt{3} + i)(1 + i)}{2\sqrt{2}}, \qquad \nu = \frac{1}{2}\sqrt{2 - \sqrt{3}}. \tag{57}$$

Furthermore, calculation of μ and σ by means of formulas (37) and (43) reveals that both solutions describe the hyperextreme objects, giving in the first case

$$\mu = \frac{1}{4}(70 + 43\sqrt{3}), \qquad \sigma = -\frac{i}{2}\sqrt{259 + \frac{416}{\sqrt{3}}}, \tag{58}$$

and in the second case yielding

$$\mu = -\frac{1}{4}\sqrt{122 + 65\sqrt{3}}, \qquad \sigma = -\frac{i}{2}\sqrt{407 + 236\sqrt{3}}. \tag{59}$$

For the quantities X_1 and X_2, formula (36) gives us

$$X_{1,2} = -\frac{1}{8}(\sqrt{3} + i)\left(70 + 43\sqrt{3} \pm \sqrt{10431 + 6020\sqrt{3}}\right) \tag{60}$$

(the first solution), and

$$X_{1,2} = \frac{1}{8\sqrt{2}}(\sqrt{3}+i)(1+i) \times \left(\sqrt{122+65\sqrt{3}} \mp \sqrt{106+65\sqrt{3}}\right) \qquad (61)$$

(the second solution).

Therefore, we have identified two different configurations of corotating Kerr sources possessing the same masses and angular momenta. An additional check that the solutions are intrinsically different provides the values of their nondimensional quadrupole moments $k = Q/ms^3$ which have been found to be

$$k^{(1)} = -\frac{1297}{3} - 244\sqrt{3} \simeq -854.954,$$
$$k^{(2)} = -\frac{1325}{3} - \frac{764}{\sqrt{3}} \simeq -882.762. \qquad (62)$$

It would be instructive to conclude this section by showing that the parametrization introduced in the previous two sections is congruent with the canonical approach discussed in the present section. For this purpose we rewrite formulas (18) and (23) in the dimensionless form via the substitutions $R \to s$, $\sigma \to \sigma/s$, $M \to ms$, $J \to js^2$, $a \to qs$ (the latter a should not be confused with the a introduced in this section for the angular momentum per unit mass), resulting in

$$\sigma^2 = m^2 - q^2 - \frac{4m^2q^2(4m^2-4q^2-1)}{(2m+4q^2+1)^2},$$
$$j = \frac{2mq[(2m+1)^2+4q^2]}{2m+4q^2+1}. \qquad (63)$$

Since j in the second equation from (63) is equal to the product ma of dimensionless mass and angular momentum per unit mass, we can substitute the particular values of m and a from (54) into that cubic equation and solve it for q, thus getting

$$q^{(1)} = -8 - \frac{9\sqrt{3}}{2}, \qquad q^{(2)} = q^{(3)} = -1 - \frac{5\sqrt{3}}{6}, \qquad (64)$$

and the subsequent substitution of the above values of q into the first formula from (63) [with the particular m from (54)] then leads to the following two values of σ^2:

$$-\frac{259}{4} - \frac{104}{\sqrt{3}} \quad \text{and} \quad -\frac{407}{4} - 59\sqrt{3}, \qquad (65)$$

which coincide exactly with the squared σ's from (58) and (59). This means that both parametrizations are appropriate for treating the systems of two equal Kerr particles, though it seems that the cubic Eq. (44) still permits a more consistent analysis and classification of particular configurations than its counterpart (23).

V. TOWARDS THE DESCRIPTION OF TWO NONEQUAL KERR BLACK HOLES

We will now outline a possible approach to treating the general case of interacting nonequal Kerr black holes which is likely to provide new information in the future about the spin-spin repulsion force in binary systems of rotating bodies. This approach consists in reparametrizing the general extended 2-soliton solution in the manner similar to the one already applied to the equatorially symmetric case in the previous sections. The starting point of such a procedure is the axis data of the form

$$e(z) = \frac{z^2 + a_1 z + a_2}{z^2 + b_1 z + b_2}, \qquad (66)$$

where a_1, a_2, b_1 and b_2 are four arbitrary complex constants, together with the choice of the parameters α_n of the extended soliton solution in the form slightly different from (2) (see Fig. 3),

$$\alpha_1 = \frac{1}{2}R + \sigma_1, \qquad \alpha_2 = \frac{1}{2}R - \sigma_1,$$
$$\alpha_3 = -\frac{1}{2}R + \sigma_2, \qquad \alpha_4 = -\frac{1}{2}R - \sigma_2, \qquad (67)$$

σ_1 and σ_2 taking real or pure imaginary values (real σ's, as usual, define black holes, while pure imaginary σ's—the hyperextreme objects). The elimination of the angular momentum monopole parameter in (66) with the aid of the Fodor *et al.* method [19] and fixing the origin of coordinates by means of (67) reduces the number of

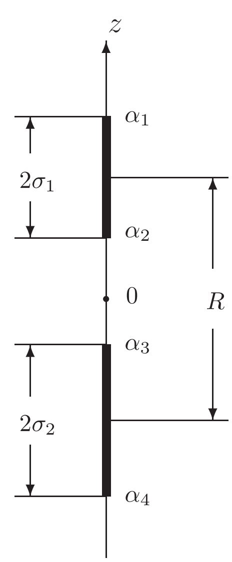

FIG. 3. Location of two nonequal Kerr black holes on the symmetry axis.

arbitrary real parameters in the data (66) to six overall, and the procedure of introducing the parameters α_n into the axis data described in Sec. I then leads to the following expression for the reparametrized data (66):

$$e(z) = \frac{z^2 - (M + ia)z + c + id}{z^2 + (M - ia)z + g + ih}, \tag{68}$$

where M is the total mass, a is the rotational parameter, while the constant quantities c, d, g and h are defined as follows:

$$c = s - \mu, \qquad g = s + \mu,$$
$$d = \frac{1}{4a}(\tau + \delta), \qquad h = \frac{1}{4a}(\tau - \delta), \tag{69}$$

with

$$s = -\frac{1}{4}[R^2 + 2(\sigma_1^2 + \sigma_2^2 - M^2 + a^2)],$$
$$\delta = \epsilon\sqrt{\tau^2 - \kappa}, \qquad \epsilon = \pm 1,$$
$$\tau = 2R(\sigma_1^2 - \sigma_2^2) - 4M\mu,$$
$$\kappa = a^2[16(\mu^2 - s^2) + (R^2 - 4\sigma_1^2)(R^2 - 4\sigma_2^2)]. \tag{70}$$

The six arbitrary real parameters involved in the axis data (68) are hence M, a, R, σ_1, σ_2, μ, and one can see that in the particular case $\mu = 0$, $\sigma_1 = \sigma_2 = \sigma$ the data (68) reduces to the equatorially symmetric data (8), albeit a formal redefinition $M \to 2M$, $a \to 2a$.

Using the general formulas of the paper [15], we have worked out the Ernst potential and the whole metric determined by the axis data (68) in the following concise form:

$$\mathcal{E} = \frac{A - B}{A + B}, \qquad f = \frac{A\bar{A} - B\bar{B}}{(A + B)(\bar{A} + \bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{\mathcal{K}_0 R_+ R_- r_+ r_-}, \qquad \omega = 2a - \frac{2\mathrm{Im}[G(\bar{A} + \bar{B})]}{A\bar{A} - B\bar{B}},$$
$$A = [R^2 - (\sigma_1 + \sigma_2)^2](R_+ - R_-)(r_+ - r_-) - 4\sigma_1\sigma_2(R_+ - r_-)(R_- - r_+),$$
$$B = 2\sigma_1(R^2 - \sigma_1^2 + \sigma_2^2)(R_- - R_+) + 2\sigma_2(R^2 + \sigma_1^2 - \sigma_2^2)(r_- - r_+) + 4R\sigma_1\sigma_2(R_+ + R_- - r_+ - r_-),$$
$$G = -zB + \sigma_1(R^2 - \sigma_1^2 + \sigma_2^2)(R_- - R_+)(r_+ + r_- + R) + \sigma_2(R^2 + \sigma_1^2 - \sigma_2^2)(r_- - r_+)(R_+ + R_- - R)$$
$$- 2\sigma_1\sigma_2\{2R[r_+ r_- - R_+ R_- - \sigma_1(r_- - r_+) + \sigma_2(R_- - R_+)] + (\sigma_1^2 - \sigma_2^2)(r_+ + r_- - R_+ - R_-)\}, \tag{71}$$

where the functions $R_\pm$ and $r_\pm$ are given by the expressions

$$R_\pm = \frac{\delta + 2ia[M(\pm 2\sigma_2 + R) - 2\mu]}{\tau - ia[(\pm 2\sigma_2 + R)(\pm 2\sigma_2 + R + 2ia) + 4s]} \times \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \sigma_2\right)^2},$$
$$r_\pm = \frac{\delta + 2ia[M(\pm 2\sigma_1 - R) - 2\mu]}{\tau - ia[(\pm 2\sigma_1 - R)(\pm 2\sigma_1 - R + 2ia) + 4s]} \times \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma_1\right)^2}, \tag{72}$$

and the choice of the constant $\mathcal{K}_0$ in the formula for γ must preserve the asymptotic flatness of the solution.

In order to interpret the metric (71) as describing two unequal Kerr black holes, it is necessary to solve the condition $\omega = 0$ on the part $\{\rho = 0, -\frac{1}{2}R + \sigma_2 < z < \frac{1}{2}R - \sigma_1\}$ of the z-axis. However, the bad thing is that, compared to the equatorially symmetric case, the resulting explicit form of the axis condition in the general case is extremely cumbersome, so that really very powerful computers are needed to be able to perform the required calculations in the analytical form. In spite of that, the numerical analysis of the axis condition suggests that the analytical treatment of the general case is still possible in principle because this condition leads to the quartic algebraic equation for the parameter μ. A particular configuration of two nonequal black holes obtainable in this way is the following:

$$M = 2, \qquad a = 1, \qquad R = 6, \qquad \sigma_1 = 1/2,$$
$$\sigma_2 = 1/4, \qquad \mu \simeq -2.1876. \tag{73}$$

We do not exclude that some clever redefinitions of the parameters or fortunate substitutions might cause the factorization of the axis condition and the eventual resolution of the problem in a relatively compact form on the basis of the metric (71). But the accomplishment of this technically very complicated mission still remains an interesting task for the future.

VI. CONCLUSION

Therefore, we have shown that the vacuum MMR solution is very fit for the analytical description and study of the binary configuration of corotating identical Kerr black holes, for which we have worked out a concise representation that improves the one obtained in Ref. [6]. We have also presented our old solution of the problem of corotating Kerr particles in the canonical parametrization and demonstrated that configurations with the same masses and angular momenta can be not unique. We have restricted our consideration exclusively to the case of nonextreme constituents because the extreme case of two equal or nonequal Kerr black holes is described by a subclass of the well-known Kinnersley-Chitre solution [31] which was already identified and discussed in our earlier work [32].

We are convinced that in order to get a better insight into the nature of the spin-spin interaction, future research should be more concentrated on the configurations of nonequal spinning bodies because, apparently, the cases of identical constituents can be considered as degenerations of the respective generic cases and hence could in principle hide some important information about the real strength of the spin-spin repulsion or attraction. In this respect, a good understanding of the systems of identical spinning bodies is certainly necessary and brings us closer to the description of more sophisticated binary configurations that arise, for instance, within the framework of the general 2-soliton spacetime (71).

ACKNOWLEDGMENTS

We thank the referee for useful suggestions. This work was partially supported by CONACYT of Mexico, and by Project No. FIS2015-65140-P (MINECO/FEDER) of Spain.

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