Regular multihorizon black holes in modified gravity with nonlinear electrodynamics

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We investigated the regular multihorizon black holes in the Einstein gravity, $F(R)$ gravity, and the 5 dimensional Gauss-Bonnet gravity, all of them coupled with nonlinear electrodynamics. We presented several explicit examples of the actions which admit the solutions describing regular black hole space-time with multihorizons. Thermodynamics of the obtained black hole solutions is studied. The explicit expressions of the temperature, the entropy, the thermodynamical energy and the free energy are obtained. Although the temperature vanishes in the extremal limit where the radii of the two horizons coincide with each other as in the standard multihorizon black hole like the Reissner-Nordström black hole or the Kerr black hole, the larger temperature corresponds to the larger horizon radius. This is different from the standard black holes, where the larger temperature corresponds to the smaller horizon radius. We also found that the specific heat becomes positive for the large temperature, which is also different from the standard black holes, where the specific heat is negative. It should be also noted that the thermodynamical energy is not identical with the ADM mass. Furthermore in case of the Gauss-Bonnet gravity, it is demonstrated that the entropy can become negative.

I. INTRODUCTION

Usually a black hole has both curvature singularity and horizon(s). The singularity without horizon might be prohibited by the cosmic censorship [1]. Conversely, before the concept of the cosmic censorship, it has been found the space-time with horizon but without the curvature singularity can be realized by using the nonlinear electromagnetism [2]. This fact initiated the study of the regular black hole (with nonlinear electrodynamics): [3–21]. Recently in [20], the construction of the nonsingular black holes by using the nonlinear electrodynamics has been completely generalized. The interest in gravity with nonlinear electrodynamics is caused by several reasons. First of all, nonlinear electrodynamics models with gravity are predicted by low-energy string effective action. Second, such theories may appear as the effective action from quantum gravity coupled with matter. Third, such theories may be considered as simple enough models mimicking much more complicated fundamental proposals for nonperturbative quantum gravity with matter at the early universe. It is expected that nonlinear character of the theory may be manifested in quite unusual

aspects of black hole thermodynamics. Specifically, we consider multihorizon black holes where Nariai limit may be easily accessed. This is caused by the well-known fact that such black holes are supposed to be the primordial black holes which were created at the early universe. It is expected that some of primordial black holes which may have the unusual properties may survive at the current universe. If so, they may give us the important information about the properties of the early universe. For instance, as we discuss later on, unlike to standard one-horizon black holes, the multihorizon black holes may show up not only the Hawking evaporation but also anti-evaporation or related instability phenomenon.

The purpose of this work is to study the nonsingular black holes in modified gravity with nonlinear electrodynamics. By using the formulation of Ref. [20] we consider the problem of nonsingular black holes in $F(R)$ gravity which may be responsible for current acceleratingly expanding universe (see [22–24] and for the review, see [25–28]) and 5-dimensional Gauss-Bonnet gravity. We give some examples of the nonsingular black hole space-time with multihorizons and investigate their thermodynamical properties. A motivation why we consider the 5-dimensional gravity is related with the AdS/CFT correspondence [29,30], where the gravity theory in the D dimensional anti-de

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Sitter (AdS) space-time is equivalent to the conformal field theory (CFT) in the $D-1$ dimensional flat space-time, which can be identified with the boundary of the AdS space-time. We show, however, that our model does not correspond to any 4-dimensional relativistic field theory but the effective 3-dimensional theory or nonrelativistic theory. We also show that there are some regions of the parameter space where the entropy of 5-dimensional Einstein-Gauss-Bonnet gravity becomes negative.

It is interesting to discuss the relation between the regular black holes and the energy conditions [31,32]. In the formulation of [20], one can construct any type of spherically symmetric regular black hole and one can check the energy conditions explicitly. In this paper, we check the conditions for some simple cases.

We often consider the Nariai limit where the radii of two horizons coincide with each other. In some cases, such a black hole is stable due to the phenomena called anti-evaporation. It may have quite long life-time although with the radius being small. This is to compare with small black hole which evaporates in a short time. Therefore, the Nariai black hole may be considered as primordial one and might survive even in the present universe.

In the next section, we give a general formulation to construct the regular black holes, especially with multi-horizons, based on Ref. [20]. Some explicit examples of the black hole space-time with multihorizons in the Einstein gravity with nonlinear electrodynamics are constructed. In Sec. III, we generalize the formulation of Ref. [20] for $F(R)$ gravity and give the explicit examples of the multi-horizon regular black hole space-time in the R^2 gravity. It is also demonstrated that one cannot construct any spherically symmetric regular black hole solutions only by $F(R)$ gravity. The thermodynamics is also investigated in Sec. IV) and some expressions for the entropy $\mathcal{S}$ and the energy E are found. It is shown that the obtained expression for the energy E is often different from the ADM mass [33]. In Sec. V, we consider 5-dimensional Gauss-Bonnet gravity with nonlinear electrodynamics. We develop a formulation to construct the models which generate the general spherically symmetric space-time and give an example of the regular black hole with two horizons. We also investigate the thermodynamics and we find that the entropy can be negative. In addition to the entropy $\mathcal{S}$, the expression of the free energy F is found. It indicates that the five-dimensional gravity model could not correspond to any 4-dimensional relativistic field theory but eventually it corresponds to the effective 3 dimensional theory or nonrelativistic theory. Some summary and outlook are given in the last section.

II. REGULAR BLACK HOLE WITH NONLINEAR ELECTROMAGNETISM

In this section, we consider the regular black hole with multihorizons in the Einstein gravity coupled with nonlinear electromagnetic field.

A. General formulation of reconstruction

In this subsection, we extend the formulation of the reconstruction in [20] so that we could construct solutions describing the black hole without curvature singularity but with multi-horizons.

We start from the following action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R}{2} - \mathcal{L}(I) \right\}. \tag{1}$$

Here we have chosen the gravitational constant to be unity, $\kappa = 1$ and $I \equiv \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$, with the field strength $F_{\mu\nu}$ of the electromagnetic field A_μ, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Then the equations of motion have the following form,

$$G^\nu_\mu = \frac{d\mathcal{L}}{dI} F_{\alpha\mu} F^{\alpha\nu} + \mathcal{L} \delta^\nu_\mu, \tag{2}$$

$$0 = \nabla^\mu \left(F_{\mu\nu} \frac{d\mathcal{L}}{dI} \right). \tag{3}$$

We now assume a spherically symmetric static background as in the Schwarzschild space-time as follows,

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{-2\nu(r)} dr^2 + r^2 \sum_{i,j=1}^{2} \tilde{g}_{ij} dx^i dx^j, \tag{4}$$

and $A_i = 0$ ($i = 1, 2, 3$) and $A_0 = A_0(r)$. Then Eq. (3) has the following form,

$$0 = \frac{d}{dr} \left(r^2 F^{0r} \frac{d\mathcal{L}}{dI} \right). \tag{5}$$

By using $I = -\frac{1}{2}(F_{0r})^2$, Eq. (5) can be solved as

$$r^2 \frac{d\mathcal{L}}{dI} = \frac{Q}{\sqrt{-2I}}. \tag{6}$$

We now define a new variable X by

$$X \equiv Q\sqrt{-2I}. \tag{7}$$

Then in [20], the following equations have been obtained,

$$\frac{d}{dr} (r(e^{2\nu} - 1)) = -r^2 \rho, \tag{8}$$

$$\mathcal{L} = \frac{X}{r^2} - \rho, \tag{9}$$

$$X = -\frac{r^3}{2} \frac{d\rho}{dr}. \tag{10}$$

Here ρ is the energy-density of the nonlinear electromagnetic field obtained from the Lagrangian density $\mathcal{L}$. For the geometry expressed by $e^{2\nu(r)}$ in (4), by using (8), we find the explicit r-dependence of ρ. Then by using (10), one can express X in terms of r, which can be solved with respect to

r as $r = r(X)$. Substituting the obtained expression $r = r(X)$ into (9), we obtain the explicit form of $\mathcal{L}$, $\mathcal{L} = \mathcal{L}(X)$. In general, however, one cannot solve Eq. (10) explicitly with respect to r as a function of X and also there might not be the one-to-one correspondence between X and r. In this case, one cannot obtain the Lagrangian density (9). In order to avoid this problem, we may introduce the auxiliary fields B and C. It is straightforward to obtain the r-dependence of the energy-density ρ by using (8) for a given $e^{2\nu(r)}$, $\rho = \rho(r)$. Then by using the auxiliary fields B and C, we may consider the following Lagrangian density $\mathcal{L}_{BCX}$ instead of $\mathcal{L}$,

$$\mathcal{L} \to \mathcal{L}_{BCX} \equiv \frac{X}{C^2} - \rho(r = C) + B\left(X + \frac{C^3}{2}\frac{d\rho}{dr}\bigg|_{r=C}\right). \quad (11)$$

By the variation over B, we obtain (10) with $r = C$. In case that C can be solved with respect to X, by substituting the obtained expression into (11), one gets the Lagrangian density in (9).

B. General regular black hole with multihorizons

We now consider the general regular multihorizon black hole, where there is no curvature singularity. In the background (4), the scalar curvature is given by

$$R = e^{2\nu}\left[-2\nu'' - 4\nu'^2 - \frac{8\nu'}{r} + \frac{2e^{-2\nu} - 2}{r^2}\right]. \quad (12)$$

Therefore in order to avoid the curvature singularity, we find $e^{2\nu} \to 1$ in the limit of $r \to 0$. Furthermore because $\partial_\mu e^{2\nu}$ is a single-valued function, we also require $e^{2\nu} = 1 + \mathcal{O}(r^2)$ ar $r \to 0$. We also require ν'' does not diverge at $r = 0$.

If the background is the Minkowski space-time or the anti-de Sitter space-time, one may express $e^{2\nu(r)}$ as follows,

$$e^{2\nu(r)} = \frac{(r - r_1)(r - r_2)\cdots(r - r_{2N})}{h(r)}. \quad (13)$$

Here N is a positive integer, $h(r)$ does not vanish and $h(r) \to r_1 r_2 \cdots r_{2N}(1 - (\sum_{i=1}^{2N}\frac{1}{r_i})r + \mathcal{O}(r^2))$ in the limit of $r \to 0$. Therefore $e^{2\nu(r)}$ behaves as $e^{2\nu(r)} \to 1 + \mathcal{O}(r^2)$ and therefore the space-time is regular at $r = 0$. On the other hand, in the limit of $r \to \infty$, $h(r)$ behaves as $h(r) \to r^{2N}$ for the Minkowski background or $h(r) \to \frac{r^{2N+2}}{l^2}$ with a length parameter l for the anti-de Sitter background. In (13), there are $2N$ horizons at $r = r_1, r_2, \ldots$, and r_{2N}. We may assume $r_1 \leq r_2 \leq \cdots \leq r_{2N}$. On the other hand, in the de Sitter background, $h(r)$ could be given by

$$e^{2\nu(r)} = \frac{(r - r_1)(r - r_2)\cdots(r - r_{2N-1})}{h(r)}. \quad (14)$$

Here, again, N is a positive integer, $h(r)$ does not vanish. In the limit of $r \to 0$, we require $h(r) \to -r_1 r_2 \cdots r_{2N-1}(1 - (\sum_{i=1}^{2N-1}\frac{1}{r_i})r + \mathcal{O}(r^2))$ and therefore e2$\nu(r)$ behaves as $e^{2\nu(r)} \to 1 + \mathcal{O}(r^2)$, again and the space-time is regular at $r = 0$. On the other hand, in the limit of $r \to \infty$, we also require $h(r)$ behaves as $h(r) \to -\frac{r^{2N+1}}{l^2}$ with a length parameter l. In (14), there are $2N - 1$ horizons at $r = r_1, r_2, \ldots$, and r_{2N-1} and if we assume $r_1 \leq r_2 \leq \cdots \leq r_{2N-1}$, $r = r_{2N-1}$ corresponds to the cosmological horizon. This may show that the regular black hole in the anti-de Sitter space-time or the Minkowski space-time has even-number of horizons. On the other hand, the regular black hole in de Sitter space-time has odd number of horizons, in which the largest horizon corresponds to the cosmological horizon.

C. Example in the Minkowski background

We consider the model

$$e^{2\nu} = 1 - \frac{\alpha r^2}{\beta + r^3} = \frac{(r + r_0)(r - r_1)(r - r_2)}{\beta + r^3},$$

$$\alpha = \frac{r_1^2 + r_2^2 + r_1 r_2}{r_1 + r_2}, \quad \beta = \frac{r_1^2 r_2^2}{r_1 + r_2}, \quad r_0 = \frac{r_1 r_2}{r_1 + r_2}. \quad (15)$$

Then we find

$$\rho = \frac{3\alpha\beta}{(\beta + r^3)^2} > 0,$$

$$X = \frac{9\alpha\beta r^5}{(\beta + r^3)^3}, \quad \mathcal{L} = \frac{3\alpha\beta(2r(X)^3 - \beta)}{(\beta + r(X)^3)^3}. \quad (16)$$

By using the auxiliary field B and C, one gets

$$\mathcal{L}_{BCX} = \frac{3\alpha\beta(2C^3 - \beta)}{(\beta + C^3)^3} + B\left(X - \frac{9\alpha\beta C^5}{(\beta + C^3)^3}\right). \quad (17)$$

Then redefining $C \to \beta^{\frac{1}{3}}C$, $A_\mu \to Q^{-1}\beta^{\frac{8}{3}}A_\mu$ $(X \to \beta^{\frac{8}{3}}X)$, $B \to \beta^{-\frac{8}{3}}$, and $\alpha \to \beta\tilde{\alpha}$, we can rewrite the Lagrangian density (17) as follows,

$$\mathcal{L}_{BCX} = \frac{3\tilde{\alpha}(2C^3 - 1)}{(1 + C^3)^3} + B\left(X - \frac{9\tilde{\alpha}C^5}{(1 + C^3)^3}\right). \quad (18)$$

Therefore the only parameter in the theory is

$$\tilde{\alpha} = \frac{\alpha}{\beta} = \frac{r_1^2 + r_2^2 + r_1 r_2}{r_1^2 r_2^2}. \quad (19)$$

Then we can choose the extremal limit where

$$r_1, r_2 \to \sqrt{\frac{3}{\tilde{\alpha}}}. \tag{20}$$

One may consider the energy conditions,

(i) Null Energy Condition (NEC): $\rho + p_r \geq 0$ and $\rho + p_T \geq 0$

(ii) Weak Energy Condition (WEC): $\rho \geq 0$, $\rho + p_r \geq 0$, $\rho + p_T \geq 0$

(iii) Strong Energy Condition (SEC): $\rho + p_r + 2p_T \geq 0$, $\rho + p_r \geq 0$, and $\rho + p_T \geq 0$

(iv) Dominant Energy Condition (DEC): $\rho \geq 0$, $\rho \pm p_r \geq 0$, and $\rho \pm p_T \geq 0$.

Here p_r is that the pressure for the radial direction and p_T is the pressure for the angle direction.

$$\rho = \frac{X}{r^2} - \mathcal{L} = \frac{3\alpha\beta}{(\beta + r^3)^2} > 0, \qquad p_r = \frac{X}{r^2} + \mathcal{L},$$

$$p_T = \mathcal{L} = \frac{3\alpha\beta(2r^3 - \beta)}{(\beta + r^3)^3}, \qquad X = \frac{9\alpha\beta r^5}{(\beta + r^3)^3} > 0. \tag{21}$$

Then we find,

$$\rho + p_r = \frac{2X}{r^2} > 0, \qquad \rho + p_T = \frac{X}{r^2} > 0. \tag{22}$$

Therefore the null energy condition and the weak energy condition are satisfied. On the other hand, because

$$\rho + p_r + 2p_T = \frac{2X}{r^2} + 2\mathcal{L} = \frac{6\alpha\beta(5r^3 - \beta)}{(\beta + r^3)^3}, \tag{23}$$

the strong energy condition is not satisfied in the region

$$r < \left(\frac{\beta}{5}\right)^{\frac{1}{3}} = \left(\frac{r_1^2 r_2^2}{5(r_1 + r_2)}\right)^{\frac{1}{3}}. \tag{24}$$

We also find

$$\rho - p_r = -2\mathcal{L} = \frac{6\alpha\beta(-2r^3 + \beta)}{(\beta + r^3)^3},$$

$$\rho - p_T = \frac{3\alpha\beta(-r^3 + 2\beta)}{(\beta + r^3)^3}. \tag{25}$$

Therefore the dominant energy condition is broken in the region

$$r < (2\beta)^{\frac{1}{3}} = \left(\frac{2r_1^2 r_2^2}{r_1 + r_2}\right)^{\frac{1}{3}}. \tag{26}$$

Thus, we demonstrated that black hole configuration under consideration may fulfil the energy conditions.

D. Example in the anti-de Sitter background

As an example of the regular black hole where there appear two horizons in the anti-de Sitter background, we consider the following one,

$$e^{2\nu(r)} = \frac{(r^2 - r_1^2)(r^2 - r_2^2)}{r_1 r_2 (r^2 + r_1 r_2)} = \frac{(r - r_1)(r - r_2)}{h(r)},$$

$$h(r) = \frac{r_1 r_2 (r^2 + r_1 r_2)}{(r + r_1)(r + r_2)}. \tag{27}$$

Then the length parameter in the anti-de Sitter space-time is given by $l^2 = r_1 r_2$. By using Eqs. (8) and (10), we find

$$\rho = -\frac{2\{3r^4 + (4r_1 r_2 - r_1^2 - r_2^2)r^2 - 3r_1^2 r_2^2 - 3r_1^3 r_2 - 3r_1 r_2^3\}}{r_1 r_2 (r^2 + r_1 r_2)^2}, \tag{28}$$

$$X = -\frac{2r^4\{-8r^6 + (-4r_1 r_2 + r_1^2 + r_2^2)r^4 + (6r_1^2 r_2^2 + 5r_1^3 r_2 + 5r_1 r_2^3)r^2 + 10r_1^3 r_2^3 + 5r_1^4 r_2^2 + 5r_1^2 r_2^4\}}{r_1 r_2 (r^2 + r_1 r_2)^3}. \tag{29}$$

Because we cannot solve Eq. (29) with respect to r, we use the Lagrangian density in (11),

$$\mathcal{L}_{BCX} = \frac{X}{C^2} - \frac{2\{3C^4 + (4r_1 r_2 - r_1^2 - r_2^2)C^2 - 3r_1^2 r_2^2 - 3r_1^3 r_2 - 3r_1 r_2^3\}}{r_1 r_2 (C^2 + r_1 r_2)^2}$$
$$+ B\left[X + \frac{2C^4\{-8C^6 + (-4r_1 r_2 + r_1^2 + r_2^2)C^4 + (6r_1^2 r_2^2 + 5r_1^3 r_2 + 5r_1 r_2^3)C^2 + 10r_1^3 r_2^3 + 5r_1^4 r_2^2 + 5r_1^2 r_2^4\}}{r_1 r_2 (C^2 + r_1 r_2)^3}\right]. \tag{30}$$

We now redefine,

$$C^2 \to r_1 r_2 C^2, \qquad X \to r_1 r_2 X, \qquad B \to \frac{B}{r_1 r_2}, \qquad \alpha \equiv \frac{r_1^2 + r_2^2}{r_1 r_2}, \qquad l^2 = r_1 r_2. \tag{31}$$

Then the Lagrangian density (30) can be rewritten as

$$\mathcal{L}_{BCX} = \frac{X}{C^2} - \frac{2\{3C^4 + (4 - \alpha)C^2 - 3 - 3\alpha\}}{l^2(C^2 + 1)^2} + B\left[X + \frac{2C^4\{-8C^6 + (-4 + \alpha)C^4 + (6 + 5\alpha)C^2 + 10 + 5\alpha\}}{(C^2 + 1)^3}\right]. \tag{32}$$

The Lagrangian density (32) still includes a parameter Q, which appears as a solution in (6) but as clear in (7), the parameter Q can be always absorbed into the redefinition of A_μ as $A_\mu \to Q^2 A_\mu$. Therefore we can find that this model given by the Lagrangian density (32) has only two coupling constants α and l^2. In terms of α and l^2, the horizon radii r_1 and r_2 are given by

$$r_1 = l\sqrt{\frac{\alpha - \sqrt{\alpha^2 - 4}}{2}}, \qquad r_2 = l\sqrt{\frac{\alpha + \sqrt{\alpha^2 - 4}}{2}}. \quad (33)$$

This tells that by keeping the coupling constant α and l^2 to be constant, we cannot consider the Nariai limit, where $r_1 \to r_2$, or, $\alpha \to 2$. In order to solve this problem, we make the parameter α a dynamical field and we add the following Lagrangian density,

$$\mathcal{L}_{B\alpha} = H^\mu \partial_\mu \alpha. \quad (34)$$

Then by the variation of H^μ, we obtain the equation,

$$0 = \partial_\mu \alpha, \quad (35)$$

and therefore α becomes a constant, whose value could be dynamically determined. Due to Eq. (35), the energy momentum tensor coming from the Lagrangian (34) vanishes and therefore the Lagrangian does not contribute to the geometry. Then we can consider the Nariai limit by making $\alpha \to 2$,

$$r_1 = l - \epsilon, \quad r_2 = l + \epsilon, \quad r = l + \epsilon \sin\theta, \quad t = \frac{\tau}{\epsilon}, \quad (36)$$

and taking the limit of $\epsilon \to 0$. By the definition of (36), $e^{2\nu(r)}$ in (27) behaves as

$$e^{2\nu(r)} \to -\frac{2\epsilon^2}{l^2} \cos^2\theta, \quad (37)$$

and therefore the metric is given by

$$ds^2 = \frac{2\cos^2\theta}{l^2} d\tau^2 - \frac{l^2}{2} d\theta^2 + l^2 \sum_{i,j=1}^{2} \tilde{g}_{ij} dx^i dx^j. \quad (38)$$

Further redefining

$$\cos\theta = \frac{1}{\cosh\frac{2\rho}{l^2}}, \quad \text{that is } d\theta = \frac{2}{l^2 \cosh\frac{2\rho}{l^2}} d\rho, \quad (39)$$

we obtain the following metric

$$ds^2 = \frac{2}{l^2 \cosh^2\frac{\rho}{l^2}} (d\tau^2 - d\rho^2) + l^2 \sum_{i,j=1}^{2} \tilde{g}_{ij} dx^i dx^j, \quad (40)$$

which is similar to the metric in the Nariai space-time.

The energy density (28) includes the contribution from the cosmological constant because we are considering the asymptotically anti-de Sitter space-time. By subtracting the contribution from the cosmological constant, we find

$$\rho_{EM} = \rho + \frac{6}{r_1 r_2} = \frac{2\{(2r_1 r_2 + r_1^2 + r_2^2)r^2 - 3(r_1^3 r_2 + r_1 r_2^3)\}}{r_1 r_2 (r^2 + r_1 r_2)^2}. \quad (41)$$

Therefore in the region

$$r < \sqrt{\frac{3(r_1^3 r_2 + r_1 r_2^3)}{2r_1 r_2 + r_1^2 + r_2^2}}, \quad (42)$$

the energy density ρ_{EM} becomes negative and therefore the weak and dominant energy conditions are not satisfied although the null energy condition is always satisfied, what is clear from (22). Although we need to subtract the contributions from the cosmological constant, there could be a region where the strong energy condition is not satisfied.

E. Regular black hole with three horizons

We may also consider another example,

$$e^{2\nu(r)} = -\frac{(r^2 - r_1^2)(r^2 - r_2^2)(r^2 - r_3^2)}{l^2(r^2 + \frac{r_1 r_2 r_3}{l})^2}. \quad (43)$$

In the limit $r \to \infty$, we find $e^{2\nu(r)} \to -\frac{r^2}{l^2}$. Therefore the space-time is the de Sitter space-time. Then by using (8), we find

$$\rho = -\frac{2}{l^2 r^2 (r^2 + \frac{r_1 r_2 r_3}{l})^3} \left[-3r^8 + \left(r_1^2 + r_2^2 + r_3^2 + l^2 - \frac{7r_1 r_2 r_3}{l} \right) r^6 \right.$$
$$+ \left(r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2 + 3l r_1 r_2 r_3 + \frac{5r_1 r_2 r_3}{l}(r_1^2 + r_2^2 + r_3^2) \right) r^4$$
$$\left. - \frac{3r_1 r_2 r_3}{l}(r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2) r^2 + \frac{2r_1^3 r_2^3 r_3^3}{l} \right]. \quad (44)$$

Now Eq. (10) has the following form,

$$X = \frac{2}{l^2 r^2 (r^2 + \frac{r_1 r_2 r_3}{l})^4} \left[\left(-l^2 - r_1^2 - r_2^2 - r_3^2 - \frac{2r_1 r_2 r_3}{l} \right) r^8 \right.$$

$$+ \left(-2(r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2) - \frac{8r_1 r_2 r_3}{l} (r_1^2 + r_2^2 + r_3^2) - 4r_1 r_2 r_3 l - \frac{14 r_1^3 r_2^3 r_3^3}{l^2} \right) r^6$$

$$\left. + \left(\frac{5r_1^2 r_2^2 r_3^2}{l^2} (r_1^2 + r_2^2 + r_3^2) + \frac{10 r_1 r_2 r_3}{l} (r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2) + 3r_1^2 r_2^2 r_3^2 \right) r^4 - \frac{8r_1^3 r_2^3 r_3^3}{l} r^2 - \frac{2r_1^4 r_2^4 r_3^4}{l^2} \right]. \tag{45}$$

In order to obtain the Lagrangian density $\mathcal{L}$, we need to solve Eq. (45) with respect to r but the equation is the 5th order algebraic equation with respect to r, which cannot be solved algebraically. Furthermore the one-to-one correspondence between X and r is not satisfied in general and therefore we cannot construct the Lagrangian density (9) in an explicit form. Then one may use the Lagrangian density in (11), and obtain

$$\mathcal{L}_{BCX} = \frac{X}{C^2} + \frac{2}{l^2 C^2 (C^2 + \frac{r_1 r_2 r_3}{l})^3} \left[-3C^8 + \left(r_1^2 + r_2^2 + r_3^2 + l^2 - \frac{7r_1 r_2 r_3}{l} \right) C^6 \right.$$

$$\left. + \left(r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2 + 3l r_1 r_2 r_3 + \frac{5r_1 r_2 r_3}{l} (r_1^2 + r_2^2 + r_3^2) \right) C^4 - \frac{3r_1 r_2 r_3}{l} (r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2) C^2 + \frac{2r_1^3 r_2^3 r_3^3}{l} \right]$$

$$+ B \left[X - \frac{2}{l^2 C^2 (C^2 + \frac{r_1 r_2 r_3}{l})^4} \left\{ \left(-l^2 - r_1^2 - r_2^2 - r_3^2 - \frac{2r_1 r_2 r_3}{l} \right) C^8 \right. \right.$$

$$+ \left(-2(r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2) - \frac{8r_1 r_2 r_3}{l} (r_1^2 + r_2^2 + r_3^2) - 4r_1 r_2 r_3 l - \frac{14 r_1^3 r_2^3 r_3^3}{l^2} \right) C^6$$

$$\left. \left. + \left(\frac{5r_1^2 r_2^2 r_3^2}{l^2} (r_1^2 + r_2^2 + r_3^2) + \frac{10 r_1 r_2 r_3}{l} (r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2) + 3r_1^2 r_2^2 r_3^2 \right) C^4 - \frac{8r_1^3 r_2^3 r_3^3}{l} C^2 - \frac{2r_1^4 r_2^4 r_3^4}{l^2} \right\} \right]. \tag{46}$$

We now define parameters α, β, and γ as follows,

$$C^2 \to \frac{r_1 r_2 r_3}{l} C^2, \qquad X \to \frac{r_1 r_2 r_3}{l} X, \qquad B \to \frac{l}{r_1 r_2 r_3} B,$$

$$\alpha \equiv \frac{l(r_1^2 + r_2^2 + r_3^2 + l^2)}{r_1 r_2 r_3}, \qquad \beta \equiv \frac{l^2 (r_2^2 r_3^2 + r_3^2 r_1^2 + r_1^2 r_2^2)}{r_1^2 r_2^2 r_3^2}, \qquad \gamma \equiv \frac{l^3}{r_1 r_2 r_3}. \tag{47}$$

Then the Lagrangian density (46) has the following form

$$\mathcal{L}_{BCX} = \frac{X}{C^2} + \frac{2\{-3C^8 + (\alpha - 7)C^6 + (5\alpha + \beta - 2\gamma)C^4 - 3\beta C^2 + 2\}}{l^2 C^2 (C^2 + 1)^3}$$

$$+ B \left[X - \frac{2\{(-\beta - 2)C^8 + (-8\alpha - 2\beta + 4\gamma - 14)C^6 + (5\alpha + 10\beta - 2\gamma)C^4 - 8C^2 - 2\}}{l^2 C^2 (C^2 + 1)^4} \right]. \tag{48}$$

This tells us that this model has four independent parameters l, α, β, and γ besides the gravitational constant. Therefore when we consider the Nariai limit, we need to keep these parameters constant. By assuming r_2, $r_3 \to r_0$, we find

$$\alpha = \frac{l(r_1^2 + 2r_0^2 + l^2)}{r_1 r_0^2}, \qquad \beta = \frac{l^2 (r_0^2 + 2r_1^2)}{r_1^2 r_0^2}, \qquad \gamma = \frac{l^3}{r_1 r_0^2}. \tag{49}$$

Then by deleting r_1, we obtain the following equations,

$$\alpha = \frac{\gamma}{l^2} \left(\frac{l^6}{\gamma^2 r_0^4} + 2r_0^2 + l^2 \right), \qquad \beta = \frac{\gamma^2}{l^4} r_0^4 + \frac{2l^2}{r_0^2}. \tag{50}$$

Because the above two equations are independent from each other, there is no solution for r_0 in general. Therefore one cannot get the Nariai limit in general. Then we may use the formulation as in (34). Instead of using the formulation, we may consider a special case,

$$\alpha = \gamma + 3\gamma^{\frac{1}{3}}, \qquad \beta = 3\gamma^{\frac{2}{3}}. \tag{51}$$

Then one has a solution,

$$r_0 = r_1 = \gamma^{-\frac{1}{3}}l, \tag{52}$$

that is, the radii of three horizons coincide with each other. We may also consider the case that the parameters α, β, and γ in (49) are given by two parameters ξ_0 and ξ_1 as follows,

$$\alpha = \frac{\xi_0^2}{\xi_1} + 2\xi_1 + \xi_0^2\xi_1, \quad \beta = \xi_1^2 + 2\xi_0^2, \quad \gamma = \xi_0^2\xi_1. \tag{53}$$

Then the solution for r_0 and r_1 is given by

$$r_0 = \frac{l}{\xi_0}, \qquad r_1 = \frac{l}{\xi_1}. \tag{54}$$

Hence even if we consider the infinitesimal shift from $r_2 = r_3 = r_0$,

$$r_2 = r_0 - \epsilon, \qquad r_3 = r_0 + \epsilon, \tag{55}$$

the parameters α, β, and γ do not change or more exactly the variations of the parameters are $\mathcal{O}(\epsilon^2)$. Then as in (36), by redefining

$$r = r_0 + \epsilon \sin\theta, \qquad t = \frac{\tau}{\epsilon}, \tag{56}$$

we obtain the Nariai limit,

$$e^{2\nu(r)} \rightarrow \frac{4r_0^2(r_0^2 - r_1^2)\epsilon^2 \cos^2\theta}{l^2(1 + \frac{r_1}{l})^2}, \tag{57}$$

and the metric is given by

$$ds^2 = -\frac{4r_0^2(r_0^2 - r_1^2)\cos^2\theta}{l^2(1 + \frac{r_1}{l})^2}d\tau^2 + \frac{l^2(1 + \frac{r_1}{l})^2}{4r_0^2(r_0^2 - r_1^2)}d\theta^2$$

$$+ r_0^2 \sum_{i,j=1}^{2} \tilde{g}_{ij}dx^i dx^j. \tag{58}$$

III. $F(R)$ GRAVITY WITH NONLINEAR ELECTROMAGNETISM

In this section, we consider $F(R)$ theory instead of the Einstein gravity. We show that the extension of above formulation is rather straightforward. Note that the black holes in $F(R)$ gravity have been well-studied [34–40] including the solution of the black hole with multihorizons in $F(R)$-gravity, (see [41]). It is interesting that for such $F(R)$ black holes as was shown in Refs. [41–43], the anti-evaporation phenomena in the Nariai space-time [44–46] may occur. In this section, we also prove that within only $F(R)$ gravity, we cannot construct the regular black hole solution of the Schwarzschild type (4).

A. Regular black hole in $F(R)$ gravity with nonlinear electromagnetism

We start with the action of $F(R)$ gravity coupled with matter:

$$S = \int d^4x \sqrt{-g}\{F(R) + L_m\}. \tag{59}$$

Here $F(R)$ is a function of the scalar curvature and L_m is a matter Lagrangian. The equation of the motion is given by

$$0 = -\frac{1}{2}e^{2\nu}F(R) - e^{4\nu}\left\{\nu'' + 2\nu'^2 + \frac{2\nu'}{r}\right\}F'(R) - e^{4\nu}\nu'\frac{dF'(R)}{dr} + \frac{e^{2\nu}}{r^2}\frac{d}{dr}\left(e^{2\nu}r^2\frac{dF'(R)}{dr}\right) - \frac{1}{2}\rho e^{2\nu}, \tag{60}$$

$$0 = \frac{1}{2}e^{-2\nu}F(R) + \left\{\nu'' + 2\nu'^2 + \frac{2\nu'}{r}\right\}F'(R) + \frac{d^2F'(R)}{dr^2} + \nu'\frac{dF'(R)}{dr} - \frac{e^{-2\nu}}{r^2}\frac{d}{dr}\left(e^{2\nu}r^2\frac{dF'(R)}{dr}\right) + \frac{1}{2}pe^{-2\nu}, \tag{61}$$

$$0 = \frac{r^2}{2}F(R) - \{1 + (-1 - 2r\nu')e^{2\nu}\}F'(R) + e^{2\nu}r\frac{dF'(R)}{dr} - \frac{d}{dr}\left(e^{2\nu}r^2\frac{dF'(R)}{dr}\right) + \frac{r^2}{2}p_T. \tag{62}$$

Even in $F(R)$ gravity theory, we have Eqs. (9) and (10), again but Eq. (8) is replaced by (60). For the geometry given by $e^{2\nu(r)}$ in (4), Eq. (60) determines the r-dependence of ρ, $\rho = \rho(r)$. By solving (10), we find the r-dependence of X, which can be solved with respect to r, $X = X(r)$. Substituting the expression $X(r)$ into (9), we obtain the Lagrangian density $\mathcal{L}$. Even if one cannot solve (10), we can use the Lagrangian density (11) by using the auxiliary fields B and C.

For the example (27), one finds

$$R = \frac{2\{-6r^6 + (r_1^2 + r_2^2 - 16r_1r_2)r^4 + (3r_1^3r_2 + 3r_1r_2^3 - 12r_1^2r_2^2)r^2 + 6r_1^4r_2^2 + 6r_1^2r_2^4 + 6r_1^3r_2^3\}}{r_1r_2(r^2 + r_1r_2)^3}. \tag{63}$$

Let us consider the following $F(R)$ gravity,

$$F(R) = \frac{R}{2} + \frac{c}{2}R^2, \tag{64}$$

with a constant c. Using (60), we find

$$\rho = \frac{2\{3r^4 + (4r_1 r_2 - r_1^2 - r_2^2)r^2 - 3r_1^3 r_2^3 - 3r_1^3 r_2 - 3r_1 r_2^3\}}{r_1 r_2 (r^2 + r_1 r_2)^2} + \frac{4c}{r_1^2 r_2^2 (r^2 + r_1 r_2)^6}[72r^{12} + (408 r_1 r_2 - 12 r_1^2 - 12 r_2^2)r^{10}$$

$$+ (-68 r_1^3 r_2 - 68 r_1 r_2^3 - 3r_1^4 - 3r_2^4 - 950 r_1^2 r_2^2)r^8 + (-18 r_1^5 r_2 - 18 r_1 r_2^5 + 1308 r_1^3 r_2^3 - 84 r_1^4 r_2^2 - 84 r_1^2 r_2^4)r^6$$

$$+ (-207 r_1^6 r_2^2 - 207 r_1^2 r_2^4 - 1056 r_1^5 r_2^3 - 1056 r_1^3 r_2^5 - 618 r_1^4 r_2^4)r^4 + (192 r_1^7 r_2^3 + 192 r_1^3 r_2^7 + 184 r_1^6 r_2^4 + 184 r_1^4 r_2^6 + 506 r_1^5 r_2^5)r^2$$

$$+ 72 r_1^8 r_2^4 + 72 r_1^4 r_2^8 - 36 r_1^7 r_2^5 - 36 r_1^5 r_2^7 - 144 r_1^6 r_2^6]. \tag{65}$$

Then by using (10), one gets

$$X = -\frac{2r^4\{-8r^6 + (-4r_1 r_2 + r_1^2 + r_2^2)r^4 + (6r_1^2 r_2^2 + 5r_1^3 r_2 + 5r_1 r_2^3)r^2 + 10 r_1^3 r_2^3 + 5r_1^4 r_2^2 + 5r_1^2 r_2^4\}}{r_1 r_2 (r^2 + r_1 r_2)^3}$$

$$+ \frac{8c(r_1 + r_2)^2 r^4}{r_1^2 r_2^2 (r^2 + r_1 r_2)}[-6r^{10} + (-3r_1^2 - 3r_2^2 - 32 r_1 r_2)r^8 + (-21 r_1^3 r_2 - 21 r_1 r_2^3 + 52 r_1^2 r_2^2)r^6$$

$$+ (-387 r_1^4 r_2^2 - 387 r_1^2 r_2^4 - 1212 r_1^3 r_2^3)r^4 + (687 r_1^5 r_2^3 + 687 r_1^3 r_2^5 + 142 r_1^4 r_2^4)r^2 + 120 r_1^6 r_2^4 + 120 r_1^4 r_2^6 - 440 r_1^5 r_2^5]. \tag{66}$$

By using the Lagrangian density in (11), we find

$$\mathcal{L}_{BCX} = \frac{X}{C^2} - \frac{2\{3C^4 + (4r_1 r_2 - r_1^2 - r_2^2)C^2 - 3r_1^3 r_2^3 - 3r_1^3 r_2 - 3r_1 r_2^3\}}{r_1 r_2 (C^2 + r_1 r_2)^2}$$

$$- \frac{4c}{r_1^2 r_2^2 (C^2 + r_1 r_2)^6}[72C^{12} + (408 r_1 r_2 - 12 r_1^2 - 12 r_2^2)C^{10} + (-68 r_1^3 r_2 - 68 r_1 r_2^3 - 3r_1^4 - 3r_2^4 - 950 r_1^2 r_2^2)C^8$$

$$+ (-18 r_1^5 r_2 - 18 r_1 r_2^5 + 1308 r_1^3 r_2^3 - 84 r_1^4 r_2^2 - 84 r_1^2 r_2^4)C^6 + (-207 r_1^6 r_2^2 - 207 r_1^2 r_2^4 - 1056 r_1^5 r_2^3 - 1056 r_1^3 r_2^5 - 618 r_1^4 r_2^4)C^4$$

$$+ (192 r_1^7 r_2^3 + 192 r_1^3 r_2^7 + 184 r_1^6 r_2^4 + 184 r_1^4 r_2^6 + 506 r_1^5 r_2^5)C^2 + 72 r_1^8 r_2^4 + 72 r_1^4 r_2^8 - 36 r_1^7 r_2^5 - 36 r_1^5 r_2^7 - 144 r_1^6 r_2^6]$$

$$+ B\left[X + \frac{2C^4\{-8C^6 + (-4r_1 r_2 + r_1^2 + r_2^2)C^4 + (6r_1^2 r_2^2 + 5r_1^3 r_2 + 5r_1 r_2^3)C^2 + 10 r_1^3 r_2^3 + 5r_1^4 r_2^2 + 5r_1^2 r_2^4\}}{r_1 r_2 (C^2 + r_1 r_2)^3}\right.$$

$$- \frac{8c(r_1 + r_2)^2 C^4}{r_1^2 r_2^2 (C^2 + r_1 r_2)}[-6r^{10} + (-3r_1^2 - 3r_2^2 - 32 r_1 r_2)C^8 + (-21 r_1^3 r_2 - 21 r_1 r_2^3 + 52 r_1^2 r_2^2)C^6$$

$$\left. + (-387 r_1^4 r_2^2 - 387 r_1^2 r_2^4 - 1212 r_1^3 r_2^3)C^4 + (687 r_1^5 r_2^3 + 687 r_1^3 r_2^5 + 142 r_1^4 r_2^4)C^2 + 120 r_1^6 r_2^4 + 120 r_1^4 r_2^6 - 440 r_1^5 r_2^5]\right]. \tag{67}$$

Similarly, we can calculate the Lagrange density corresponding to the geometry (43) although the corresponding expression is very complicated.

With several horizons, one can consider the limit where the radius of one horizon coincides with that of another horizon, which is called the Nariai limit and the obtained space-time is the Nariai space-time. Usually by the Hawking radiation, the radius of the horizon decreases but in case of the Nariai space-time, the radius can increase by including the quantum effects, which is called as the anti-evaporation [44–46]. In [41–43], the anti-evaporation in $F(R)$ gravity was investigated and it was shown that the anti-evaporation can occur even at the classical level. The corresponding study for the above regular black hole will be done elsewhere.

B. A no-go theorem in $F(R)$ gravity

It is found above that the solutions describing the nonsingular black hole with multihorizons can be obtained in $F(R)$ gravity coupled with the nonlinear electromagnetic field. It is interesting to understand if we can realize the nonsingular black holes only by $F(R)$ gravity without the account of nonlinear electrodynamics. For this purpose, we consider the case of vacuum, where $\rho = p = p_T = 0$. Combining Eqs. (60) and (61) we find

$$0 = \frac{d^2 F'(R)}{dr^2}, \tag{68}$$

that is,

$$F'(R) = f_0 + f_1 r, \tag{69}$$

with constants f_0 and f_1. Hence, one can rewrite the equations in (60) as follows,

$$0 = \frac{1}{2}F(R) + \left\{ \nu'' + 2\nu'^2 + \frac{2\nu'}{r} \right\} e^{2\nu}(f_0 + f_1 r) - f_1 \nu' e^{2\nu} - \frac{2f_1 e^{2\nu}}{r},$$

$$0 = \frac{1}{2}F(R) - \frac{1}{r^2}\{1 + (-1 - 2r\nu')e^{2\nu}\}(f_0 + f_1 r) - \frac{f_1 e^{2\nu}}{r} - 2f_1 \nu' e^{2\nu}. \tag{70}$$

Deleting $F(R)$ in (70), we obtain,

$$0 = \left\{ \left(\nu'' + 2\nu'^2 - \frac{1}{r^2} \right) e^{2\nu} + \frac{1}{r^2} \right\}(f_0 + f_1 r) + f_1 \nu' e^{2\nu} - \frac{f_1 e^{2\nu}}{r}, \tag{71}$$

which is a differential equation for ν. By defining $N \equiv e^{2\nu}$, we rewrite Eq. (71) as follows,

$$0 = \frac{f_0 + f_1 r}{2} N'' + \frac{f_1}{2} N' - \left(\frac{f_0}{r^2} + \frac{2f_1}{r} \right) N + \frac{f_0 + f_1 r}{r^2}. \tag{72}$$

As a special case that $f_1 = 0$, the solution of (72) is given by

$$N = N^{(0)}(r) \equiv 1 + Ar^2 + \frac{B}{r}, \tag{73}$$

with constants A and B. This is nothing but the (A)dS-Schwarzschild solution. On the other hand, in case that $f_0 = 0$, we find

$$N = N^{(1)}(r) \equiv \frac{1}{2} + C^+ r^2 + \frac{C^-}{r^2}. \tag{74}$$

For (74), we get the following scalar curvature

$$R = e^{2\nu}\left[-\frac{d^2 e^{2\nu}}{dr^2} - \frac{4}{r}\frac{de^{2\nu}}{dr} + \frac{2e^{-2\nu} - 2}{r^2} \right] = \left(\frac{1}{2} + C^+ r^2 + \frac{C^-}{r^2} \right)\left(-8C^+ - \frac{4C^-}{r^4} - \frac{1}{r^2} \right), \tag{75}$$

which is singular at $r = 0$. In order to solve Eq. (72) for general case, we first consider the following homogeneous differential equation,

$$0 = \frac{f_0 + f_1 r}{2} \tilde{N}'' + \frac{f_1}{2} \tilde{N}' - \left(\frac{f_0}{r^2} + \frac{2f_1}{r} \right)\tilde{N}, \tag{76}$$

whose trivial solution is $\tilde{N} = r^2$. Then by assuming $\tilde{N} = A(r)r^2$, one finds the following equation,

$$0 = A'' + \left(\frac{2}{r} + \frac{2f_1}{f_0 + f_1 r} \right)A', \tag{77}$$

whose solution is given by

$$A' = \frac{C_1}{r^2(\frac{f_0}{f_1} + r)^2} = \frac{C_1 f_1^2}{f_0^2}\left(\frac{1}{r} - \frac{1}{\frac{f_0}{f_1} + r} \right)^2 = \frac{C_1 f_1^2}{f_0^2}\left(\frac{1}{r^2} - \frac{1}{(\frac{f_0}{f_1} + r)^2} - \frac{2f_1}{f_0}\left(\frac{1}{r} - \frac{1}{\frac{f_0}{f_1} + r} \right) \right), \tag{78}$$

with a constant C_1 of integration. The general solution of $\tilde{N}$ is given by

$$\tilde{N} = C_1 \left(-\frac{f_1}{f_0} + \frac{1}{\frac{f_0}{f_1} + r} + \frac{2f_1^3 r^2}{f_0^3}\ln\left(\frac{f_0}{f_1 r} + 1 \right) \right) + C_2 r^2, \tag{79}$$

with another constant C_2 of the integration. This tells the expression of the general solution of N as follows,

$$N = C_1 \left(-\frac{f_1}{f_0} + \frac{1}{\frac{f_0}{f_1} + r} + \frac{2f_1^3 r^2}{f_0^3} \ln \left(\frac{f_0}{f_1 r} + 1 \right) \right) + C_2 r^2$$

$$- \frac{1}{2} + \frac{2f_1 r}{f_0} + \frac{2f_1^2 r^2}{f_0^2} \ln \left(\frac{f_0}{f_1} + r \right) + \frac{2f_1 r}{f_0} \ln r$$

$$+ \frac{2f_1 r}{f_0} - \frac{2f_1 r}{f_0 (\frac{f_0}{f_1} + r)} \ln r - \frac{2f_1^2 r^2}{f_0^2} \left(\ln r - \ln \left(\frac{f_0}{f_1} + r \right) \right)$$

$$+ \frac{4f_1^2 r^2}{f_0^2} \int^r dr' \ln r' \left(\frac{1}{r'} - \frac{1}{\frac{f_0}{f_1} + r'} \right). \tag{80}$$

In order to avoid the singularity at $r = 0$, we require $N = e^{2\nu} = 1 + \mathcal{O}(r^2)$ but the expression in (80) is singular if $C_1 \neq 0$. Even if $C_1 = 0$, we find $N = e^{2\nu} \to \frac{1}{2}$ in the limit of $r \to 0$ and therefore there remains a singularity. Therefore, we cannot realize the nonsingular black hole if we assume the metric is given in the form of the Schwarzschild type (4).

IV. THERMODYNAMICS OF REGULAR MULTIHORIZON BLACK HOLES

Let us study the thermodynamics of the obtained regular black hole. The Hawking temperature T is now defined by

$$T = \frac{1}{4\pi} \frac{de^{2\nu}}{dr} \bigg|_{r=r_H}. \tag{81}$$

Here r_H is the radius of the horizon.

First we consider the metric (15). Then when $r_H = r_2 > r_1$ ($r_H = r_1$), we find the temperature $T = T_2$ (T_1),

$$4\pi T_2 = \frac{(r_2 + r_0)(r_2 - r_1)}{\beta + r_2^3} \left(4\pi T_1 = \frac{(r_1 + r_0)(r_1 - r_2)}{\beta + r_1^3} < 0 \right). \tag{82}$$

By solving (19) with respect to r_1, one gets

$$r_1 = \frac{(-1 \pm \sqrt{4\tilde{\alpha} r_2^2 - 3}) r_2}{2(1 - \tilde{\alpha} r_2^2)}. \tag{83}$$

As given in (20), we are assuming $\tilde{\alpha} r_2^2 > 3$ and therefore the expression (83) is real and because $r_1 > 0$, we find

$$r_1 = \frac{(1 + \sqrt{4\tilde{\alpha} r_2^2 - 3}) r_2}{2(\tilde{\alpha} r_2^2 - 1)}. \tag{84}$$

Therefore Eq. (16) shows

$$r_0 = \frac{(1 + \sqrt{4\tilde{\alpha} r_2^2 - 3}) r_2}{2\tilde{\alpha} r_2^2 - 1 + \sqrt{4\tilde{\alpha} r_2^2 - 3}},$$

$$\beta = \frac{(1 + \sqrt{4\tilde{\alpha} r_2^2 - 3})^2 r_2^3}{2(\tilde{\alpha} r_2^2 - 1)(2\tilde{\alpha} r_2^2 - 1 + \sqrt{4\tilde{\alpha} r_2^2 - 3})}, \tag{85}$$

and

$$4\pi T_2 = \frac{2(\tilde{\alpha} r_2^2 + \sqrt{4\tilde{\alpha} r_2^2 - 3})(2\tilde{\alpha} r_2^2 - 3 - \sqrt{4\tilde{\alpha} r_2^2 - 3})}{(4\tilde{\alpha}^2 r_2^4 - 1 + 2\tilde{\alpha} r_2^2 \sqrt{4\tilde{\alpha} r_2^2 - 3}) r_2}. \tag{86}$$

In the extremal limit (20), the temperature T_2 vanishes. On the other hand, for large r_2, T_2 behaves as

$$4\pi T_2 \sim \frac{1}{r_2}. \tag{87}$$

As in the usual black hole, the large black hole has low temperature. For the large black hole, the entropy $\mathcal{S}$ is given by

$$\mathcal{S} = \frac{A}{4} = \pi r_2^2 \sim \frac{1}{16\pi T_2^2}. \tag{88}$$

By using the thermodynamical relation, $dE = T d\mathcal{S}$, we may estimate the thermodynamical energy.

$$E \sim \frac{1}{8\pi T_2} = \frac{r_2}{2} + E_0, \tag{89}$$

which coincides with the ADM mass $M = \frac{r_2}{2}$ if we choose the constant E_0 of the integration to vanish.

Then for the metric (27), the temperature is given by

$$T = T_2 = \frac{r_2 - r_1}{2\pi r_1 r_2} = \frac{r_2 - \frac{l^2}{r_2}}{2\pi l^2}. \tag{90}$$

Here we have evaluated the temperature at the outer horizon $r = r_2 > r_1$. The temperature T vanishes in the extremal limit where $r_2 \to r_1 > 0$ as for the standard singular black holes with multihorizons like the Reissner-Nordström black hole. At the inner horizon at $r = r_1$, we obtain the expression by exchanging r_1 and r_2 although the obtained expression is negative,

$$T = T_1 = -\frac{r_2 - r_1}{2\pi r_1 r_2}. \tag{91}$$

The negativity could be understood if the temperature measured in the region $r < r_1$ could be given by replacing $\frac{d}{dr}$ with $-\frac{d}{dr}$ in the definition of (81), which is positive $T = -T_1 = T_2 > 0$. In case of Einstein gravity, the entropy $\mathcal{S}$ on the outer horizon is given by

$$\mathcal{S} = \frac{A}{4} = \pi r_2^2. \tag{92}$$

In case of $F(R)$ gravity (64), there is a correction,

$$\mathcal{S} = \frac{A}{4} F'(R)|_{r=r_2}$$

$$= \pi r_2^2 \left(1 + 4c \frac{-5r_2^4 - 13r_1 r_2^3 - 5r_1^2 r_2^2 + 9r_1^3 r_2 + 6r_1^4 r_2}{r_1(r_1 + r_2)} \right). \tag{93}$$

Here r_1 is given by

$$r_1 = \frac{l^2}{r_2}. \tag{94}$$

By definition in (92) or (93), the entropy $\mathcal{S}$ is always positive for $F(R)$ gravity as long as we do not consider the anti-gravity region, where $F'(R) < 0$ and therefore this region is unphysical.

We now consider the thermodynamics by fixing the parameter l^2 but varying the parameter α. Equation (90) can be solved with respect to r_2 as follows,

$$r_2 = \pi l^2 T \pm \sqrt{\pi^2 l^4 T^2 + l^2}, \tag{95}$$

Because $r_2 > 0$, we choose the plus sign in (95),

$$r_2 = \pi l^2 T + \sqrt{\pi^2 l^4 T^2 + l^2}, \tag{96}$$

We should note that when the temperature becomes large $T \to \infty$, the radius of the outer horizon becomes also large, whose situation is different from that in the standard black hole, where the large temperature corresponds to the small horizon radius. Then the entropy (92) is expressed as

$$\mathcal{S} = \frac{\pi}{4}(2\pi^2 l^4 T^2 + l^2 + 2\pi l^2 T \sqrt{\pi^2 l^4 T^2 + l^2}). \tag{97}$$

We should note that the entropy does not vanish even if we consider the limit of $T \to 0$, which corresponds to the Nariai limit $r_2 \to r_1$,

$$\mathcal{S} \to \frac{\pi l^2}{4}, \tag{98}$$

which is different from the standard thermodynamics. This may indicate that there remain some information in the regular part at the origin. In case of $F(R)$ gravity (93), the expression is rather complicated. By using the thermodynamical relation, $dE = T d\mathcal{S}$, we may estimate the thermodynamical energy.

$$E = \frac{\pi^3}{3}l^4 T^3 + \frac{1}{l^2}(\pi^2 l^4 T^2 + l^2)^{\frac{3}{2}} - \frac{1}{2}\sqrt{\pi^2 l^4 T^2 + l^2} + E_0. \tag{99}$$

Here E_0 is a constant of the integration. In case of the AdS-Schwarzschild black hole, we find

$$e^{2\nu} = \frac{r^2}{l^2} + 1 - \frac{2M}{r}, \tag{100}$$

Here M corresponds to the ADM mass of the black hole [33]. In case of Einstein gravity, the mass coincides with the thermodynamical energy E. On the other hand, in case of (27), in the limit of $r \to \infty$, we find

$$e^{2\nu} \to \frac{r^2}{r_1 r_2} - \frac{r_1^2 + r_2^2 + r_1 r_2}{r_1 r_2} + \mathcal{O}(r^{-2}). \tag{101}$$

Then there is no parameter corresponding to the mass. The discrepancy may occur due to the energy coming from the non-linear electromagnetic field. For large T, the thermo-dynamical energy E in (99) behaves as

$$E \sim \frac{4}{3}\pi^3 l^4 T^3. \tag{102}$$

Therefore the specific heat $C = \frac{dE}{dT}$ is positive, which is different from the case of the standard black hole, where the specific heat is negative. This could be related with the observation in (96), where the large temperature corresponds to the large horizon radius.

V. EINSTEIN-GAUSS-BONNET GRAVITY IN FIVE DIMENSIONS

Let us now consider the Einstein-Gauss-Bonnet gravity in five-dimensions. It is known that such theory is very useful in the study of the AdS/CFT correspondence.

A. General formulation

The action of Einstein-Gauss-Bonnet gravity coupled with matter is expressed by

$$S = \int d^5 x \sqrt{-g}\left\{\frac{R}{2} + c(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}) + L_m\right\}. \tag{103}$$

The equations of motion have the form

$$0 = \frac{1}{2}g_{\mu\nu}\left\{\frac{R}{2} + c(R^2 - 4R_{\rho\sigma}R^{\rho\sigma} + R_{\rho\tau\xi\sigma}R^{\rho\tau\xi\sigma})\right\}$$
$$- \frac{1}{2}R_{\mu\nu} + c(-2RR_{\mu\nu} + 4R_{\mu\rho}R_\nu{}^\rho + 4R^\rho{}_\mu{}^\sigma{}_\nu R_{\rho\sigma}$$
$$- 2R_\mu{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}) + T_{\mu\nu}. \tag{104}$$

By assuming a spherically-symmetric static background as (4) but in 5 dimensions

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{-2\nu(r)}dr^2 + r^2\sum_{i,j=1}^{3}\tilde{g}_{ij}dx^i dx^j, \tag{105}$$

the $(\mu, \nu) = (t, t)$, (r, r) and (i, j) components of Eq. (104) have the following expressions,

$$0 = -\frac{e^{-2\nu}}{2}\left[-\frac{24ce^{2\nu}(1 - e^{2\nu})\nu'}{r^3} + \frac{3e^{2\nu}}{2}\left\{-\frac{2\nu'}{r} + \frac{2(1 - e^{2\nu})e^{-2\nu}}{r^2}\right\}\right] - \rho e^{-2\nu}, \tag{106}$$

$$0 = \frac{e^{2\nu}}{2}\left[-\frac{24ce^{2\nu}(1 - e^{2\nu})\nu'}{r^3} + \frac{3e^{2\nu}}{2}\left\{-\frac{2\nu'}{r} + \frac{2(1 - e^{2\nu})e^{-2\nu}}{r^2}\right\}\right] + pe^{2\nu}, \tag{107}$$

$$0 = \frac{1}{2r^2}\left[-ce^{2\nu}\left\{2(1-e^{2\nu})\left(\frac{4(\nu''+2\nu'^2)}{r^2}+\frac{16\nu'}{r^3}\right)-\frac{16\nu'^2}{r^2}\right\}\right.$$
$$\left.+\frac{e^{2\nu}}{2}\left\{-2\nu''-4\nu'^2-\frac{12\nu'}{r}+\frac{12(1-e^{2\nu})e^{-2\nu}}{r^2}\right\}\right]+\frac{p_T}{r^2}. \tag{108}$$

We now consider the nonlinear electromagnetic field as a matter as in (1). Then in five dimensions, instead of (5), we find

$$0 = \frac{d}{dr}\left(r^3 F^{0r}\frac{d\mathcal{L}}{dI}\right). \tag{109}$$

Using $I = -\frac{1}{2}(F_{0r})^2$ as in (6), we find $X \equiv Q\sqrt{-2I}$

$$r^3\frac{d\mathcal{L}}{dI} = \frac{Q}{\sqrt{-2I}}. \tag{110}$$

Furthermore one may define a variable X by (7) and rewrite Eq. (110),

$$-2I\frac{d\mathcal{L}}{dI} = \frac{X}{r^3}. \tag{111}$$

Because

$$\rho = -2I\frac{d\mathcal{L}}{dI} - \mathcal{L}, \tag{112}$$

instead of (9), we obtain

$$\mathcal{L} = \frac{X}{r^3} - \rho. \tag{113}$$

Furthermore due to

$$I\frac{d}{dI} = \frac{X}{2}\frac{d}{dX}, \tag{114}$$

Eq. (111) gives

$$-\frac{d\mathcal{L}}{dX} = \frac{1}{r^3}, \tag{115}$$

and therefore Eq. (9) gives

$$\frac{d\rho}{dr} = \frac{1}{r^3}\frac{dX}{dr} - \frac{3X}{r^4} - \frac{dX}{dr}\frac{d\mathcal{L}}{dX} = -\frac{3X}{r^4}, \tag{116}$$

that is,

$$X = -\frac{r^4}{3}\frac{d\rho}{dr}. \tag{117}$$

Therefore as in case of four dimensions, for the geometry expressed by $e^{2\nu(r)}$ in (4), by using (106), we find the explicit r-dependence of ρ. Then Eq. (117) makes to express X in terms of r, which could be solved with respect to r as $r = r(X)$. Substituting the obtained expression $r = r(X)$ into (113), we obtain the explicit form of $\mathcal{L}$, $\mathcal{L} = \mathcal{L}(X)$. In case that one cannot solve Eq. (117) explicitly and/or there could not be the one-to-one correspondence between X and r, we can use the Lagrangian density corresponding to (11),

$$\mathcal{L}_{BCX} \equiv \frac{X}{C^3} - \rho(r=C) + B\left(X + \frac{C^4}{3}\frac{d\rho}{dr}\bigg|_{r=C}\right). \tag{118}$$

B. An example

As in the case of four dimensions, in the background of the Minkowski space-time or the anti-de Sitter space-time, we may express $e^{2\nu(r)}$ as in (13) and we require $h(r) \to r_1 r_2 ... r_{2N}(1 - (\sum_{i=1}^{2N}\frac{1}{r_i})r + \mathcal{O}(r^2))$ in the limit of $r \to 0$ and in the limit of $r \to \infty$, $h(r)$ behaves as $h(r) \to r^{2N}$ for the Minkowski background or $h(r) \to \frac{r^{2N+2}}{l^2}$ with a length parameter l for the anti-de Sitter background. In the de Sitter background, $h(r)$ is expressed as in (14) and we require $h(r) \to -r_1 r_2 ... r_{2N-1}(1 - (\sum_{i=1}^{2N-1}\frac{1}{r_i})r + \mathcal{O}(r^2))$ in the limit of $r \to 0$ and $h(r) \to -\frac{r^{2N+1}}{l^2}$ with a length parameter l in the limit of $r \to \infty$. The regular black hole in the anti-de Sitter space-time or the Minkowski space-time has an even number of horizons. On the other hand, the regular black hole in the de Sitter space-time has an odd number of horizons, in which the largest horizon corresponds to the cosmological horizon.

As an example, we consider the regular black hole in (27), where there appear two horizons in the anti-de Sitter background. Then by using (106) and (117), we find

$$\rho = \frac{3\{-2r^4 + (r_2 - r_1)^2 r^2 + 2r_1 r_2(r_1^2 + r_2^2 + r_1 r_2)\}}{2r_1 r_2(r^2 + r_1 r_2)^2}$$
$$- \frac{12c\{r^6 + (-r_1^2 - r_2^2 + r_1 r_2)r^4 - 3r_1 r_2(r_1^2 + r_2^2 + r_1 r_2)r^2 + r_1 r_2(r_1^2 + r_2^2 + r_1 r_2)^2\}}{r_1^2 r_2^2(r^2 + r_1 r_2)^3}, \tag{119}$$

$$X = -r^5\left[\frac{(r_1 + r_2)^2(r^2 + 3r_1 r_2)}{r_1 r_2(r^2 + r_1 r_2)^3} + \frac{8c(r_1 + r_2)^2\{-r^4 - 4r_1 r_2 r^2 + 3r_1 r_2(r_1^2 + r_2^2 + r_1 r_2)\}}{r_1^2 r_2^2(r^2 + r_1 r_2)^4}\right]. \tag{120}$$

Then the Lagrangian density in (118) has the following form,

$$\mathcal{L}_{BCX} = \frac{X}{C^3} - \frac{3\{-2C^4 + (r_2 - r_1)^2 C^2 + 2r_1 r_2 (r_1^2 + r_2^2 + r_1 r_2)\}}{2r_1 r_2 (C^2 + r_1 r_2)^2}$$

$$- \frac{12c\{C^6 + (-r_1^2 - r_2^2 + r_1 r_2)C^4 - 3r_1 r_2 (r_1^2 + r_2^2 + r_1 r_2)C^2 + r_1 r_2 (r_1^2 + r_2^2 + r_1 r_2)^2\}}{r_1^2 r_2^2 (C^2 + r_1 r_2)^3}$$

$$+ B\left[X + C^5 \left\{ \frac{(r_1 + r_2)^2 (C^2 + 3r_1 r_2)}{r_1 r_2 (C^2 + r_1 r_2)^3} + \frac{8c(r_1 + r_2)^2 \{-C^4 - 4r_1 r_2 C^2 + 3r_1 r_2 (r_1^2 + r_2^2 + r_1 r_2)\}}{r_1^2 r_2^2 (C^2 + r_1 r_2)^4} \right\} \right]. \quad (121)$$

By using the redefinitions in (31), we rewrite the Lagrangian density (121) as follows,

$$\mathcal{L}_{BCX} = \frac{X}{C^3} - \frac{3\{-2C^4 + (\alpha - 2)C^2 + 2(\alpha + 1)\}}{2l^2 (C^2 + 1)^2} - \frac{12c\{C^6 + (-\alpha + 1)C^4 - 3(\alpha + 1)C^2 + (\alpha + 1)^2\}}{l^4 (C^2 + 1)^3}$$

$$+ B\left[X + lC^5 \left\{ \frac{(\alpha + 2)(C^2 + 3)}{(C^2 + 1)^3} + \frac{8c(\alpha + 2)\{-C^4 - 4C^2 + 3(\alpha + 1)\}}{(C^2 + 1)^4} \right\} \right]. \quad (122)$$

Then we find again that this model given by the Lagrangian density (122) has only two coupling constants α and l^2. In terms of α and l^2, the horizon radii r_1 and r_2 are given by (33). Therefore in order to consider the Nariai limit, where $r_1 \to r_2$, or, $\alpha \to 2$, we need to add the Lagrangian density in (34). The Nariai limit is given by making $\alpha \to 2$ and redefining (56) and taking the limit of $\epsilon \to 0$. By using the further redefinitions in (39), we obtain the metric corresponding to (40),

$$ds^2 = \frac{2}{l^2 \cosh^2 \frac{\rho}{l^2}} (d\tau^2 - d\rho^2) + l^2 \sum_{i,j=1}^3 \tilde{g}_{ij} dx^i dx^j. \quad (123)$$

C. Thermodynamics

Let us investigate the thermodynamical properties of the obtained black hole solution (27). In order to estimate the entropy, we use the Wald formula [47–49]. In general D dimensional space-time, the formula is given by

$$\mathcal{S} = -\frac{1}{8G} \int_{\text{horizon}} d^{D-2}x \sqrt{h} \frac{\delta \mathcal{L}_{\text{gravity}}}{\delta R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon_{\rho\sigma}. \quad (124)$$

Here G is the Newton constant, which we now define as $8\pi G = 1$. $\epsilon_{\mu\nu}$ is the antisymmetric tensor in the two dimensional space-time perpendicular to the horizon, that is, the space-time is given by $\mu, \nu = t, r$ and we choose $\epsilon_{tr} = 1$. Furthermore $\mathcal{L}_{\text{gravity}}$ is the Lagrangian density of the gravity theory, which is now given by

$$\mathcal{L}_{\text{gravity}} = \frac{R}{2} + c(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma})$$

$$= \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma} + c(g^{\mu\rho} g^{\nu\sigma} g^{\mu'\rho'} g^{\nu'\sigma'}$$

$$- 4g^{\mu\mu'} g^{\rho\rho'} g^{\nu\sigma} g^{\nu'\sigma'} + g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'}) R_{\mu\nu\rho\sigma} R_{\mu'\nu'\rho'\sigma'}. \quad (125)$$

As the area of three-dimensional sphere is given by $2\pi^2 r^3$, we find,

$$\mathcal{S} = 2\pi^3 r_2^3 \{1 + 2c(2R - 4e^{-2\nu}R_{tt} + 4e^{2\nu}R_{rr} + 4R_{trtr})\}|_{r=r_2}. \quad (126)$$

Then by using

$$R_{trtr} = e^{2\nu}(\nu'' + 2\nu'^2) = \frac{r^6 + 3r_1 r_2 r^2 + r_1 r_2 (3r_1^2 + 3r_2^2 + 9r_1 r_2)r^2 - r_1^2 r_2^2 (r_1^2 + r_2^2 + r_1 r_2)}{r_1 r_2 (r^2 + r_1 r_2)^3},$$

$$R_{tt} = e^{4\nu}\left(\nu'' + 2\nu'^2 + \frac{3\nu'}{r}\right) = 4e^{2\nu}\left(\frac{r^6 + 3r_1 r_2 r^4 + 3r_1^2 r_2^2 r^2 - r_1^2 r_2^2 (r_1^2 + r_2^2 + r_1 r_2)}{r_1 r_2 (r^2 + r_1 r_2)^3}\right),$$

$$R_{rr} = -\left(\nu'' + 2\nu'^2 + \frac{3\nu'}{r}\right) = -4e^{-2\nu}\left(\frac{r^6 + 3r_1 r_2 r^4 + 3r_1^2 r_2^2 r^2 - r_1^2 r_2^2 (r_1^2 + r_2^2 + r_1 r_2)}{r_1 r_2 (r^2 + r_1 r_2)^3}\right),$$

$$R = e^{2\nu}\left(-2\nu'' - 4\nu'^2 - \frac{12\nu'}{r} + \frac{6e^{-2\nu} - 6}{r^2}\right)$$

$$= \frac{2\{-10r^6 + 3(r_1^2 + r_2^2 - 8r_1 r_2)r^4 + 3r_1 r_2 (3r_1^2 + 3r_2^2 - 4r_1 r_2)r^2 + 10r_1^2 r_2^2 (r_1^2 + r_2^2 + r_1 r_2)\}}{r_1 r_2 (r^2 + r_1 r_2)^3}, \quad (127)$$

one gets

$$S = 2\pi^3 r_2^3 \left[1 + \frac{8c(17r_1^2 - 5r_1 r_2 - 14r_2^2)}{r_1 r_2^2 (r_1 + r_2)} \right]. \quad (128)$$

With the help of (94) and (95),

$$S = 2\pi^3 \left(2\pi^2 l^4 T^2 + l^2 + 2\pi l^2 T \sqrt{\pi^2 l^4 T^2 + l^2} \right)$$
$$\times \left\{ 1 - \frac{8c(l^2 - 3\pi^2 l^4 T^2 + 31\pi l^2 T \sqrt{\pi^2 l^4 T^2 + l^2})}{l^2 (\pi l^2 T + \sqrt{\pi^2 l^4 T^2 + l^2}) \sqrt{\pi^2 l^4 T^2 + l^2}} \right\}. \quad (129)$$

Then if $T^2 < (3\pi l^2)^{-1}$ and c is positive and large enough, or even if $T^2 > (3\pi l^2)^{-1}$, if c is negative and large enough, the entropy becomes negative. The negative entropy in the 5-dimensional Gauss-Bonnet gravity was first observed in Ref. [50] (for the study of black hole thermodynamics in this case, see [51–56]. When the negative entropy appears, the corresponding black hole solution becomes unstable or there is an ambiguity in the definition of the entropy in general [51,52] related with possible AdS/dS transition [52]. Because the internal energy E is given by $dE = T dS$ and the free energy is also defined by $F = E - TS$, we find

$$\frac{dF}{dT} = -S, \quad (130)$$

and therefore

$$F = -\int dT S$$
$$= -2\pi^3 \left[l^2 T + \frac{2}{3}\pi^2 l^4 T^3 + \frac{2}{3} \left(\frac{1}{\pi} + \pi l^2 T^2 \right) \sqrt{\pi^2 l^4 T^2 + l^2} \right.$$
$$- 8c \left\{ T + \frac{28}{3}\pi^2 l^2 T^3 + \frac{2}{3} \left(\frac{20}{\pi l^2} + 14\pi T^2 \right) \right.$$
$$\left. \left. \times \sqrt{\pi^2 l^4 T^2 + l^2} \right\} \right]. \quad (131)$$

From the viewpoint of AdS/CFT correspondence, the free energy should correspond to that of the field theory on the boundary of the anti-de Sitter space-time. In the standard AdS/CFT correspondence [29,30], the conformal field theory (CFT) is super Yang-Mills (SYM) theory. In the conformal field theory, all the particles are massless and also for the SYM theory, we are usually assuming that all the particles are massless and therefore the free energy F should be proportional to the 4th power of the temperature T, that is $F \propto T^4$. For example, in case of $\mathcal{N} = 4$ $U(N)$ SYM model, the free energy is given by [57]

$$F = - \left(\frac{3}{4} + \mathcal{O}(N^{-\frac{3}{2}}) \right) \frac{\pi^2}{6} N^2 V_0 T^4. \quad (132)$$

On the other hand, the free energy in (131) is not proportional to T^4. In the high energy limit, the usual particles behave as massless particles but for the large T, the free energy F in (131) behaves as

$$F \sim -2\pi^3 \left(\frac{4}{3}\pi^2 l^4 - \frac{448}{3} c\pi^2 l^2 \right) T^3, \quad (133)$$

that is, F is proportional to the 3d power of T, which may suggest the effectively 3 dimensional field theory or non-relativistic theory on the AdS boundary.

Therefore one may conjecture the correspondence of our 5d nonsingular black hole with some system of the condensed matter, for example. For large T, the entropy in (129) behaves as

$$S \sim 8\pi^5 l^4 \left(1 - \frac{224c}{l^2} \right) T^2, \quad (134)$$

and therefore the thermodynamical energy $E = F + TS$ behaves as

$$E \sim \frac{16\pi^5 l^4}{3} \left(1 - \frac{280c}{l^2} \right) T^3. \quad (135)$$

Therefore if $1 - \frac{280c}{l^2} > 0$, the specific heat $C = \frac{dE}{dT}$ becomes positive again, which is different from the case of the standard black hole. This requests further deep investigation of nonsingular black holes thermodynamics in nonlinear theories.

VI. DISCUSSION

In summary, we investigated the regular black holes with multihorizons in modified gravity with nonlinear electromagnetism. We presented several explicit examples of the actions which give the solutions describing the nonsingular black hole space-time with multihorizons in the Einstein gravity, the $F(R)$ gravity, and the 5 dimensional Gauss-Bonnet gravity when coupling with nonlinear electromagnetism is present. We also studied the thermodynamics of the obtained nonsingular black hole solutions and found the explicit expressions for the temperature, the entropy, the thermodynamical energy, and the free energy. Although the temperature vanishes in the extremal limit where the radii of the two horizons coincide as for the standard black hole with multihorizons, the larger temperature corresponds to the larger horizon radius at least, for some examples. This is different from the standard black holes thermodynamics, where the larger temperature corresponds to the smaller horizon radius. In relation with the above observation, we also found that the specific heat often becomes positive for the large temperature, which is also different from the standard black holes, where the specific heat is negative. We should also note that the thermodynamical energy is not identical with the ADM mass. Furthermore in the case of the Gauss-Bonnet gravity, the entropy can become negative what

may indicate to the instability of the corresponding parameters region.

Note that there is big interest to the study of black holes thermodynamics in $F(R)$ gravity and five-dimensional Einstein-Gauss-Bonnet gravity coupled with (non)linear electromagnetism (see Refs. [49,51,58–74]). It is clear that these works may be generalized and more complicated nonsingular black holes may be obtained in this case too. One can also investigate the anti-evaporation phenomena for such black holes. This will be done elsewhere.

Also, there exists some interest in the study of relation between the regular black holes and the energy conditions [31,32]. As one can construct general type of regular black holes by using the formulation in [20], it could be interesting to clarify this relation in most general case.

ACKNOWLEDGMENTS

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