Large-distance lens uncertainties and time-delay measurements of H_0

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Given the tension between the values of the Hubble parameter H_0 inferred from the cosmic microwave background (CMB) and from supernovae, attention is turning to time delays of strongly lensed quasars. Current time-delay measurements indicate a value of H_0 closer to that from supernovae, with errors on the order of a few percent, and future measurements aim to bring the errors down to the subpercent level. Here we consider the uncertainties in the mass distribution in the outskirts of the lens. We show that these can lead to errors in the inferred H_0 on the order of a percent and, once accounted for, would correct H_0 upward (thus increasing slightly the tension with the CMB). Weak gravitational lensing and simulations may help to reduce these uncertainties.

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The Hubble tension may now well provide the greatest challenge to the canonical cosmological model [1,2]. The value of the Hubble parameter H_0 obtained from the cosmic microwave background (CMB), where H_0 affects the very precisely determined angular scale of the acoustic peaks in the CMB power spectrum [3–5], is $H_0^{\rm CMB} = 67.3 \pm 1.0\,{\rm km\,s^{-1}\,Mpc^{-1}}$ [6], which tightens to $H_0^{\rm CMB+gal} = 67.6 \pm 0.6\,{\rm km\,s^{-1}\,Mpc^{-1}}$ when supplemented by galaxy-survey data [7–10].

The Hubble parameter can also be obtained by comparing the brightnesses and redshifts of standard candles [11]. Recent supernova observations have determined the value of the Hubble parameter to be $H_0^{\rm SNe} = 73.2 \pm 1.7\,{\rm km\,s^{-1}\,Mpc^{-1}}$ [12], at roughly 3σ tension with the CMB-inferred value. Cosmological explanations of the discrepancy are not easy to come by; they typically involve some modifications to the cosmic expansion history that then introduces some other tension with the detailed structure of the CMB power spectrum [12–14]. Another possibility is that the discrepancy may arise from measurement biases in one or both observables [15–17], and so it is of paramount importance to obtain a third independent probe of H_0.

Attention is thus turning now to the value of H_0 inferred from time delays of strongly lensed quasars [18–21]. There has been tremendous recent progress in this endeavor, with the H0LiCOW program recently reporting $H_0^{\rm lens} = 71.9^{+2.4}_{-3.0}\,{\rm km\,s^{-1}\,Mpc^{-1}}$ [22] from three lensing systems. Additional lenses are expected to reduce the error bars on H_0 even further [23].

In order to reach not only percent-level precision, but also percent-level accuracy, the mass distribution of the lens must be carefully modeled. For example, uncertainties in the radial mass profile assumed for the lens have been shown to induce errors of several percent in H_0 [24–26], whereas microlensing of the quasar source can cause comparable uncertainties [27]. Here, we focus specifically on the mass distribution of the lens at large distances from the lens center of mass. We show that uncertainties in this large-distance mass distribution may lead to uncertainties in H_0 of a few percent. We also argue that current modeling may be biasing the value of H_0 down (implying greater tension with the CMB).

The mass distribution of lens galaxies at large radii remains, to a great extent, unknown. Galactic mass profiles must be truncated at no more than the $\sim$Mpc typical intergalactic spacing, and weak-lensing studies [28] suggest the mass distributions of galaxies that resemble typical strong lenses ought to be truncated at distances $\lesssim 500$ kpc. The time delay depends on the total mass projected along the line of sight [29], and so there may be artificial contributions to the expected time delay if the lens mass is not truncated. Although this effect was too small to be of concern in prior work, it introduces, if neglected, a $\mathcal{O}(1\%)$ bias in the inferred Hubble parameter and, if considered, still implies a residual uncertainty of comparable magnitude.

We also consider a subtle issue about how cosmological lenses are embedded in a Friedmann-Robertson-Walker (FRW) universe. The usual discussions of lensing surmise that the mass associated with a lens is *added* to an otherwise homogeneous FRW universe, giving rise to a potential perturbation that falls off as $1/r$ with distance r from the lens. In our Universe, however, the mass associated with any given lens arises from a local overdensity which is compensated elsewhere with an underdensity. As a result, the potential perturbation associated with any particular lens should fall off far more rapidly than $1/r$ at large distances. We show that a correct accounting of this effect biases the inferred Hubble parameter downwards, but only by $(\delta H_0/H_0) \sim 10^{-4}$.

We begin by reviewing the lensing formalism. Given a mass density $\rho(\mathbf{r})$ of the lens, the mass distribution projected onto the lens plane at angular position $\boldsymbol{\theta}$ is obtained by integrating over the line-of-sight distance z,

$$\Sigma(\boldsymbol{\theta}) = \int dz \rho(D_L \boldsymbol{\theta}, z), \tag{1}$$

where D_L is the angular-diameter distance to the lens. We can divide Σ by the critical density $\Sigma_{\rm crit} \equiv c^2 D_S/(4\pi G D_L D_{LS})$ to separate strong from weak lensing, where D_S and D_{LS} are, respectively, the angular-diameter distances to the source and between the lens and the source, to obtain the convergence [30] $\kappa(\boldsymbol{\theta}) = \Sigma(\boldsymbol{\theta})/\Sigma_{\rm crit}$. The lensing potential is the projection of the gravitational potential ϕ, given by [31],

$$\psi(\boldsymbol{\theta}) = \frac{2D_{LS}}{c^2 D_L D_S} \int dz \phi(D_L \boldsymbol{\theta}, z), \tag{2}$$

which is related to the convergence through $\nabla^2_\theta \psi = 2\kappa$. This potential yields a deflection angle $\alpha = \boldsymbol{\nabla}_\theta \psi$, which defines where images are formed through the lens equation,

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta}), \tag{3}$$

where $\boldsymbol{\beta}$ is the impact parameter. The $\boldsymbol{\beta}$ in Eq. (3) is unknown *a priori*, and is obtained by fitting to the observed image positions $\boldsymbol{\theta}$, and the $\boldsymbol{\alpha}(\boldsymbol{\theta})$ is predicted by the lens model (i.e., mass distribution). Signals from the source will arrive as different images, at positions $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_j$, with a time delay given by [32,33]

$$\Delta t_{ij} = \frac{D_{\Delta t}}{c} \left(\frac{\boldsymbol{\alpha}^2(\boldsymbol{\theta}_i) - \boldsymbol{\alpha}^2(\boldsymbol{\theta}_j)}{2} - [\psi(\boldsymbol{\theta}_i) - \psi(\boldsymbol{\theta}_j)] \right), \tag{4}$$

where we have defined the time-delay distance,

$$D_{\Delta t} \equiv (1 + z_L) \frac{D_L D_S}{D_{LS}} \propto H_0^{-1}, \tag{5}$$

and z_L is the redshift of the lens. Given that $D_{\Delta t}$ is a ratio of distances, it is inversely proportional to the Hubble parameter H_0 and only weakly dependent on other cosmological parameters [23].

The usual procedure is to consider a parametrized family of convergences $\kappa(\boldsymbol{\theta}; \boldsymbol{\xi})$ with parameters $\boldsymbol{\xi}$. These parameters are obtained by fitting to the observed image positions $\{\boldsymbol{\theta}_i\}$. The Hubble parameter is then inferred by comparing the time delay expected from Eq. (4) with that observed.

One issue that arises, though, is the mass-sheet degeneracy, in which the effect of a constant additional surface-mass density on the observed image positions can be compensated by a change in the impact parameter. If the real convergence of the lens is $\kappa^{\rm real}$, but it is modeled as

$$\kappa^{\rm model} = (1 - \lambda)\kappa^{\rm real} + \lambda, \tag{6}$$

the observed image positions will be the same, granted that the impact parameter is changed as $\boldsymbol{\beta}^{\rm model} = (1 - \lambda)\boldsymbol{\beta}^{\rm real}$. However, the expected time delay is changed to $\Delta t_{ij}^{\rm model} = (1 - \lambda)\Delta t_{ij}^{\rm real}$, thus yielding a different value,

$$H_0^{\rm model} = (1 - \lambda) H_0^{\rm real}, \tag{7}$$

for the Hubble parameter. The mass-sheet degeneracy is not just a theoretical curiosity. It is expected that the large-scale structure along the line of sight causes light rays to focus and defocus, introducing an external convergence $\kappa_{\rm ext}$ [34,35].

There are two avenues to breaking this degeneracy. The first is to use dynamical measurements of stellar velocities in the lens, as the transformation in Eq. (6) also implies a change $(\sigma_{\rm vel}^{\rm model})^2 = (1 - \lambda)(\sigma_{\rm vel}^{\rm real})^2$ to stellar velocities [36]. In practice, however, uncertainties in the lens profile can induce errors when extrapolating the mass measurement at small radii to the larger Einstein radius [37]. Moreover, the possibility of anisotropy in the lens hampers translation from kinematic data to the lens mass [24].

The second method is to simulate fields of view in cosmological N-body simulations to obtain a probability distribution function (PDF) for $\kappa_{\rm ext}$ [38–40]. In an FRW universe this PDF has mean $\langle \kappa_{\rm ext} \rangle = 0$, but the finite width $\langle \kappa_{\rm ext}^2 \rangle \neq 0$ is one of the limiting factors in current time-delay H_0 measurements [39]. There has been great development in the study of this PDF; for instance, we have learned that multiply imaged quasars, as biased tracers of the underlying matter distribution, live preferentially in overdense regions, which causes a percent-level bias on $\langle \kappa_{\rm ext} \rangle$ and thus on the inferred H_0 [41]. An example of this bias is found in the lens system RXJ1131-1231, which resides in a line of sight with $\sim$40% more galaxies than average [39], which causes the expectation value of the external convergence to be $\langle \kappa_{\rm ext} \rangle \approx 0.1$. In an effort to find the PDF of $\kappa_{\rm ext}$ for each individual system, instead of the average PDF of an FRW universe, both the average number counts of galaxies in the field [42], as well as the external shear $\gamma_{\rm ext}$ [39], have been used as ancillary data.

The aforementioned N-body studies quantify the contributions of independent structures, along the line of sight, that are at large (cosmological) physical distances from the lens. What we will consider now, though, is the mass distribution *in the lensing system*, but at physical distances (e.g., $\sim$100 s kpc) large compared with the Einstein radius and impact parameter (e.g., $\sim$10 kpc). We will first show that this can be approximated as a mass-sheet transformation.

We focus on the spherically symmetric power-law models, with mass density given by

$$\rho_{\rm model}(r) = \rho_0 (r/r_0)^{-\gamma'}, \tag{8}$$

that are usually used in lens modeling. This density profile gives rise to a projected surface-mass density at a distance $b (= D_L \theta)$ from the center of the lens of

$$\Sigma_{\rm model}(b) = \sqrt{\pi} \rho_0 r_0^{\gamma'} b^{1-\gamma'} \Gamma\left(\frac{\gamma' - 1}{2}\right) \left[\Gamma\left(\frac{\gamma'}{2}\right)\right]^{-1}, \tag{9}$$

where Γ is the gamma function. We now compute the critical density through $\pi R_E^2 \Sigma_{\rm crit} = M_{\rm los}$, where $M_{\rm los}$ is the line-of-sight mass contained within a cylinder of radius R_E, to find [42]

$$\Sigma_{\rm crit} = -\rho_0 r_0^{\gamma'} R_E^{1-\gamma'} \sqrt{\pi}\Gamma\left(\frac{\gamma'-3}{2}\right)\left[\Gamma\left(\frac{\gamma'}{2}\right)\right]^{-1}. \tag{10}$$

We then obtain the convergence [36,43] $\kappa_{\rm model}(\theta) = (3-\gamma')(\theta_E/\theta)^{\gamma'-1}/2$. The parameters of the model are the power-law index γ', and the Einstein angle $\theta_E \equiv R_E/D_L$. Using this model, augmented with an ellipticity parameter to account for the noncircularity of the lens, the authors of Ref. [39] inferred a Hubble constant $H_0 = 78.7^{+4.3}_{-4.5}$ km s^{-1} Mpc^{-1} from the RXJ1131-1231 system.

The issue we first address is the uncertainty associated with the assumption of a power-law mass distribution that extends to infinite radius. It is clear that the mass distribution cannot extend to infinity (and that the total mass cannot be infinite, as the power-law mass profile implies). Still, the contribution to the convergence, and thus the observables, is small enough to be neglected in prior work. As we move to subpercent precision/accuracy, though, the effects of the truncation radius become significant. To see this, we truncate the mass density from Eq. (8) at a finite radius by adding the negative-mass distribution $\rho_t = -\rho_0 (r/r_0)^{-\gamma'}\Theta(r-r_t)$, to the model, where $\Theta(r)$ is the Heaviside step function, and r_t is the truncation radius. This distribution gives rise to a projected surface mass density $\Sigma_t = -2\rho_0 r_0^{\gamma'} r_t^{1-\gamma'}/(1-\gamma')$, neglecting terms of $\mathcal{O}[(b/r_t)^3]$ or larger. Again dividing by the critical density[1] from Eq. (10), we find the convergence due to this negative-mass distribution to be

$$\kappa_t = \frac{2\Gamma(\frac{\gamma'}{2})(\gamma'-1)}{\sqrt{\pi}\Gamma(\frac{\gamma'-3}{2})}\left(\frac{R_E}{r_t}\right)^{\gamma'-1} < 0. \tag{11}$$

This large-radius negative-mass distribution thus modifies the convergence to

$$\kappa_{\rm real}(\theta) = \frac{3-\gamma'}{2}\left(\frac{\theta'_E}{\theta}\right)^{\gamma'-1} + \kappa_t, \tag{12}$$

independently of θ, and is then equivalent to a mass-sheet transformation with $\lambda_t = -\kappa_t$. We thus use Eq. (7) to relate the real H_0 to the one inferred by the nontruncated model,

$$H_0^{\rm real} = \frac{H_0^{\rm model}}{(1+\kappa_t)} \approx (1-\kappa_t) H_0^{\rm model} > H_0^{\rm model}. \tag{13}$$

Thus, time-delay measurements of H_0 are biased low if the finite extent of the lensing mass distribution is not taken into account.

We now consider the range of reasonable values for the truncation radius r_t. As the analysis above indicates, the image positions do not depend significantly on the mass distribution at large radii, a consequence of the fact that the light rays for observed images have trajectories with impact parameters comparable to the Einstein radius, which is much smaller than r_t [26]. Nonetheless, galaxies produce weak lensing at very wide angular separations, which can be detected by the shear created on background galaxies [44]. In Ref. [45] a study of the truncation radius of galaxies was performed, where the lens mass density was modeled as a dual pseudoisothermal elliptical mass distribution [46]

$$\rho(r) = \frac{\sigma_{\rm vel}^2 s^2}{4\pi G r^2 (r^2+s^2)}, \tag{14}$$

which is isothermal for $r \ll s$, and decays as r^{-4} for larger radii, effectively showing a cutoff at $r \sim s$. To first nonvanishing order in r/s the mass distribution in Eq. (14) yields a convergence

$$\kappa(r) = \frac{\theta_E}{2\theta}\left(1+\frac{R_E}{2s}\right) - \frac{R_E}{2s}, \tag{15}$$

which can be identified with the isothermal case ($\gamma' = 2$) of our Eq. (12), if $\kappa_t = -R_E/(2s)$. By comparing to Eq. (11), $\kappa_t = -R_E/(\pi r_t)$, we find $s = \pi r_t/2$. This also shows that for the purposes of strong lensing, where $r \sim R_E \ll r_t$, our sharp cutoff is a good approximation to the smooth truncation scheme in Eq. (14), while remaining valid for $\gamma' \neq 2$, thus fitting most lens models, which are not isothermal. The size of s has been estimated in Ref. [45] to be $s \gtrsim 100 h^{-1}$ kpc, whereas a more recent study in Ref. [47] found $s = 185^{+30}_{-28}\ h^{-1}$ kpc on average over an ensemble population of all galaxies. Furthermore, in Ref. [28] it was found that red galaxies, which tend to be early type and thus more likely to be strong lenses, have on average larger truncation radii, $s \approx 300\ h^{-1}$ kpc.

We thus find that time-delay Hubble-parameter measurements are biased low by

$$\frac{\delta H_0}{H_0} \approx -0.01\left(\frac{R_E}{10\ {\rm kpc}}\right)\left(\frac{r_t}{300\ {\rm kpc}}\right)^{-1}, \tag{16}$$

where for simplicity we have set $\gamma' = 2$. Although the precise bias will differ from lens to lens, the bias will survive even if Hubble parameters inferred from multiple systems are averaged, as it has the same sign for all lenses, and thus averages to some nonzero value $\bar{\kappa}_t$. The uncertainties in the values of r_t for each lens introduce moreover an accompanying error in the inferred value of H_0. If the κ_t for different lenses are distributed about the mean with a variance $\sigma_{\kappa_t}^2$, then there will still be an uncertainty in H_0 of $\sigma_{H_0}/H_0 \approx \sigma_{\kappa_t}/\sqrt{N}$, from N time-delay systems. Moreover, the average $\bar{\kappa}_t$ between the lenses can only be inferred with an error $\sigma_{\bar{\kappa}_t} \approx \sigma_{\kappa_t}/\sqrt{N}$. Therefore, subtracting our estimate of the average from the data yields a residual bias

[1] Which is accurate to first nonvanishing order in R_E/r_t, since otherwise $\Sigma_{\rm crit}$ from Eq. (10) would depend on r_t.

$\delta H_0/H_0 \sim \sigma_{\bar{\kappa}_t}$. Detailed studies of the lens-galaxy population are thus imperative to overcome these uncertainties.

We have chosen a simple power-law model to illustrate the effects of truncation of the mass distribution, although a similar uncertainty should be present in other models. For instance, in Refs. [48,49] truncated Navarro-Frenk-White models were presented. Furthermore, in Refs. [50,51] the lens systems RXJ1131-1231 and HE 04351223 were fit with both a power-law distribution and a composite model, which includes dark matter and baryons. The composite model presents an effective cutoff with respect to the power law, due to the faster decrease of the dark-matter density at large radii [24]. This is to be compared with our modeled cutoff in Eq. (12), from which we would expect a higher inferred value of H_0 for the composite model. Nevertheless, this effect—which we estimate to bias H_0 by one percent—is smaller than current measurement and modeling uncertainties.

There is another issue, of a more conceptual nature, which we now consider. In the usual discussions of lensing, a lensing mass distribution (e.g., from a galaxy or cluster) is *added* to an otherwise FRW universe, thus giving rise to potential perturbations that fall off as $1/r$ with the distance r from the lens. In our Universe, however, galaxies and clusters are formed from local overdensities, in an otherwise FRW universe, that are then compensated by underdensities elsewhere. Thus, if we go to distances large compared with the typical intergalactic separation, there will be no residual $1/r$ potential perturbation (somewhat analogously to Debye screening in a plasma). What we are considering here thus *differs* from prior work [52,53] in which the lens was embedded in a spacetime that asymptotes to FRW at large distance (the residual $1/r$ potential perturbation still arises there). Our analysis also differs from that of Ref. [54] in that we compensate the mass of the strong lens, instead of the weak perturbers along the line of sight.

To estimate the impact of this issue, we consider a lens of mass M that is surrounded by a spherical negative-mass shell (NMS) of the same total mass at some large radius R_f—i.e., we take the lens to be a spherical mass distribution of zero total mass. We take R_f to be the radius in a homogeneous universe of matter density ρ_m, at which an object of mass M_L dominates the gravitational potential, i.e., $R_f \sim [3M_L/(4\pi\rho_m)]^{1/3}$, which for a matter density of $\rho_m(z) \approx 5\times 10^{-8}(1+z)^3\ M_\odot/\mathrm{pc}^3$ is

$$R_f \sim \mathrm{Mpc}\left(\frac{M_L}{10^{11}\ M_\odot}\right)^{1/3}\frac{1}{1+z_L}. \tag{17}$$

The NMS has a mass distribution,

$$\rho_{\mathrm{NMS}} = -\frac{M_g}{4\pi R_f^2}\delta_D(r-R_f), \tag{18}$$

which gives rise to a convergence,

$$\kappa_{\mathrm{NMS}}(b) = -\frac{R_E^2}{2R_f\sqrt{R_f^2-b^2}}. \tag{19}$$

For $R_f \sim$ Mpc, and for an Einstein radius $R_E \sim 10$ kpc, we find $\kappa_{\mathrm{NMS}} \approx -(R_E^2/R_f^2)/2 \sim -10^{-4}$, which is, again, independent of angle to first nonvanishing order. The convergence thus resembles a negative-mass-sheet, since we are only observing it at distances $b \ll R_f$, where the curvature of the NMS is negligible. The magnitude of the bias and uncertainty introduced in H_0 measurements is only $\mathcal{O}(10^{-4})$ and thus not significant for current or forthcoming measurements of H_0.

We now return to the bias and error in H_0 introduced by the uncertainty in the large-radius mass distribution, and now consider what is known about the truncation radius and what more might be learned about it in the future. Weak-lensing measurements are already beginning to provide some constraints on the average value of r_t, but r_t varies amongst different types of galaxies [28]. It will thus be important to extend such measurements further restricting the population of lens galaxies to those that more closely resemble strong-lensing systems. The challenge here will be statistics with the reduced number of systems and then beyond that, separating the effects, in galaxy-galaxy lensing, of the lens potential, from those of large-scale clustering.

Even if a lens-like population of galaxies can be well characterized, one might want to measure the truncation radius for an individual lens. This will be difficult with traditional weak-lensing measurements, given the relatively small masses of the lens galaxies and the finite number of background sources to be lensed. Still, in the longer term, radio arrays may provide measurements of the 21-cm line during the dark ages to arcsecond resolution. This would allow studies of weak lensing around individual objects, characterizing their environment to great accuracy [55].

Although simulations might not shed light on the large-distance mass distribution of every individual lens, there is more that can be done to determine the PDF of the effective convergence associated with a family of lenses. The procedure should be analogous to that used to infer the PDF of the external convergence due to line-of-sight objects, (e. g., ray-tracing through simulations,) albeit applied to the outskirts of lens-like galaxies with the necessary resolution.

As yet, the effects we have considered here have been subdominant compared with other uncertainties associated with modeling the lens mass distribution. As we move forward, though, to subpercent precision, there will need to be more focus on the mass distribution in the outskirts of the lens, work that can be pushed forward with weak lensing and simulations, to enable a precise and unbiased subpercent-level measurement of H_0.

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