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Inflation in a closed universe

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To derive a power spectrum for energy density inhomogeneities in a closed universe, we study a spatially-closed inflation-modified hot big bang model whose evolutionary history is divided into three epochs: an early slowly-rolling scalar field inflation epoch and the usual radiation and nonrelativistic matter epochs. (For our purposes it is not necessary to consider a final dark energy dominated epoch.) We derive general solutions of the relativistic linear perturbation equations in each epoch. The constants of integration in the inflation epoch solutions are determined from de Sitter invariant quantum-mechanical initial conditions in the Lorentzian section of the inflating closed de Sitter space derived from Hawking's prescription that the quantum state of the universe only include field configurations that are regular on the Euclidean (de Sitter) sphere section. The constants of integration in the radiation and matter epoch solutions are determined from joining conditions derived by requiring that the linear perturbation equations remain nonsingular at the transitions between epochs. The matter epoch power spectrum of gauge-invariant energy density inhomogeneities is not a power law, and depends on spatial wave number in the way expected for a generalization to the closed model of the standard flat-space scale-invariant power spectrum. The power spectrum we derive appears to differ from a number of other closed inflation model power spectra derived assuming different (presumably non de Sitter invariant) initial conditions.

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I. INTRODUCTION

In the standard scenario, dark energy dominates the current cosmological energy budget and results in the observed accelerating cosmological expansion. Earlier on nonrelativistic (cold dark and baryonic) matter dominated, powering the decelerating cosmological expansion. In flat-ΛCDM [1], the current "standard" cosmological model, Einstein's cosmological constant Λ is the dark energy with nonrelativistic cold dark matter (CDM) being the second biggest contributor to the current energy budget and spatial hypersurfaces are assumed to be flat. See Refs. [2] for reviews of the dark energy picture as well as of the modified gravity scenario.

The standard scenario is supported by a number of different measurements, but these do not rule out mildly varying—in time and space—dark energy or mildly curved spatial hypersurfaces. These measurements include cosmic microwave background (CMB) anisotropy observations [3], baryon acoustic oscillation (BAO) data [4], Hubble parameter versus redshift measurements [5],[1] Type Ia supernova apparent magnitude observations [8], as well as the growth of structure as a function of redshift [9].

Other measurements, which are not as constraining, are also consistent with the ΛCDM model. These include HII galaxy apparent magnitude versus redshift data [10], galaxy cluster number counts [11], angular size as a function of redshift measurements [12], lookback time observations [13], gamma-ray burst data [14], and cluster gas mass fraction observations [15]. Near-future data will provide more restrictive and possibly very interesting constraints [16].

It is reassuring that most current measurements are not inconsistent with the standard flat-ΛCDM model, although they are also not inconsistent with weakly varying dark energy or a mild amount of space curvature. To be able to distinguish between the options and better pin down cosmological parameter values will require resolution of a number of issues. For instance, for over a decade and a half now, median statistics analyses of Huchra's growing compilation of Hubble constant H_0 measurements have been consistent with $H_0 = 68 \pm 2.8$ km s^{-1} Mpc^{-1} [17], in good agreement with the range of values recently estimated from CMB anisotropy data [3,18], BAO observations [4,19], Hubble parameter measurements [20], and from a compilation of recent cosmological data and the standard model of particle physics with only three light neutrino species [21]. Unfortunately, however, local measurements of the expansion rate favor a significantly larger value, $H_0 = 73.24 \pm 1.74$ km s^{-1} Mpc^{-1} [22], larger than what is favored by a number of other observations. Until this difference is understood and resolved, it is probably wiser to proceed cautiously about judging the viability of cosmological models.[2]

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[1]These $H(z)$ data [6] are particularly interesting as they span a large redshift range, to almost $z = 2.4$, and show evidence consistent with a transition from early nonrelativistic-matter-dominated deceleration to current dark-energy-powered acceleration [5,7], in agreement with what is expected in ΛCDM and other dark energy models.

[2]Of course, similar issues affect measurements of other parameters.

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That said, there have been a number of recent papers suggesting that the predictions of flat-ΛCDM might not be compatible with some $H(z)$ data [23], as well as with a combination of cosmological observations [24,25], and that dynamical dark energy provides a better fit to these measurements. If this is supported by more and better-quality data, it will be an important clue about the nature of the dark energy. On the other hand, it would be useful to check if these data were also in accord with a non-flat ΛCDM model or if they prefer dynamical dark energy over spatial curvature.

Compared to the time-independent cosmological constant, a time-varying dark energy density evolves in a manner closer to that of spatial curvature energy density and this can cause a complication. For instance, when CMB anisotropy measurements are studied in the context of the ΛCDM model, they indicate that spatial hypersurfaces are close to flat, although a mild amount of curvature is still allowed [3]. On the other hand, under the assumption of flat spatial geometry these measurements favor a time-independent dark energy density, although mild dark energy time evolution remains an option. However, if CMB anisotropy data are analyzed using a nonflat dynamical dark energy model, there is degeneracy between space curvature and the parameter that governs the dark energy density, resulting in weaker constraints on both parameters when compared to the case when only either dark energy density time variability or nonzero spatial curvature is assumed [26]. This is the case for other data also, see Refs. [5,27].[3]

The simplest physically-consistent dynamical dark energy model is ϕCDM [29,30].[4] Here dark energy is a scalar field ϕ with a potential energy density $V(\phi)$ that gradually decreases with increasing ϕ. The original ϕCDM model assumed flat spatial hypersurfaces.This was generalized to the nonflat case in Ref. [32]; the time-dependent attractor solution discovered in the spatially-flat case is also present in the non-flat case.

To complete this nonflat dynamical dark energy model requires a prescription for what happens at very early times in the model. This is provided by inflation, [33,34], which is easily generalized to the spatially-open case in the Gott open-bubble inflation model [35]. In this model a spatially-open bubble nucleates and then inflates only for a limited time so spatial curvature is not completely diluted. If necessary, an earlier epoch of less-limited inflation can be used to explain spatial homogeneity.[5]

In this initial hyperbolic (or open) de Sitter space of the open bubble, the standard requirement that the ground state energy of the (appropriately rescaled) scalar inflaton field spatial inhomogeneity not diverge in the scale factor $a \to 0$ limit provides the needed initial condition [36] and results in a late-time energy density inhomogeneity power spectrum [37,38] that is the generalization to the open case [39] of the scale-invariant spectrum of the flat model [40].

Perhaps the simplest model of inflation in a closed universe is that based on Hawking's prescription for the quantum state of the universe [41]. Hawking proposes including in the functional integral only those field configurations which are regular on the Euclidean section [41,42]. This may be viewed as the nucleation of a closed de Sitter Lanczos universe on the Lorentzian section, because the waist of the Lorentzian de Sitter Lanczos hyperboloid and the equator of the Euclidean (de Sitter Lanczos) sphere are identified [41,42]. For variants of this scenario see Refs. [43–45]. If the nucleation process is slow enough it might be possible to make the nucleated Lorentzian closed de Sitter space sufficiently spatially homogeneous. See Refs. [46] for discussions of homogeneity in a more conventional closed inflation model.

During the Lorentzian closed de Sitter expansion, quantum mechanical spatial inhomogeneities in the scalar inflaton field could provide the needed density inhomogeneities. A major advantage of the Hawking proposal is that it provides reasonable quantum mechanical initial conditions for these fluctuations. In the closed de Sitter model the $a \to 0$ limit does not lie in the Lorentzian section [42], unlike in the open and flat cases. Remarkably, Hawking's prescription of only including field configurations regular on the Euclidean section does in fact correspond to the ground state energy of the (appropriately rescaled) scalar field inhomogeneity not diverging as $a \to 0$, which in this case is either the north or south pole of the Euclidean section sphere (actually there are an infinite number of spheres, each connected to the next at the poles) [42], and in fact leads to a de Sitter invariant ground state scalar field two-point correlation function, [42]. It is likely that this is the unique initial condition with this property [42].

In this paper we use this initial condition to compute the energy density inhomogeneity power spectrum in a closed universe in terms of the potential of the inflaton and other parameters. The spatial wave number dependence of the late-time spectrum we find, using a simple inflation model, is the generalization of the scale-invariant spectrum in the spatially-flat case [40] to the closed universe [44,47,48].[6] There have also been a number of earlier computations of spectra in the closed model [45,50,51], using different initial conditions compared to what we have used here (also see Ref. [52]). We emphasize that the initial conditions we

[3]See Ref. [28] for potential constraints on space curvature from proposed experiments.

[4]While the XCDM parametrization is often used to model dynamical dark energy, it is an incomplete and inconsistent parameterization (as it cannot describe inhomogeneities). It also does not accurately model even the spatially homogeneous part of the ϕCDM model [31].

[5]Alternately, if the bubble nucleation process is slow enough it might be possible to arrange for the interior of the open bubble to be homogeneous enough.

[6]See Ref. [49] for a discussion of a massive scalar field inflaton closed inflation model.

use here results in a scalar field two-point function that is de Sitter invariant [42] and it is unclear how to interpret any other initial condition.

A proper analysis of CMB anisotropy data in a slightly closed model—which is consistent with current observations—will make use of the spectrum we have derived here. While all that is needed for such an analysis is the spectral shape of the power spectrum (not the overall amplitude), which was previously known, it is also important to show that a computation using Hawking's initial conditions in a consistent inflation model—as done here—does result in such a power spectrum. We have also established that the de Sitter invariant initial conditions [41,42] do result in the expected power spectrum [47] that differs from those found in Refs. [45,50,51].

It seems that flat-ΛCDM, which is consistent with most observations, predicts more large-angle (low multipole ℓ) CMB temperature anisotropy power than is observed [3]. In the context of inflation and the energy density inhomogeneity power spectrum derived here and in the open inflation model [35], going to a slightly nonflat (closed) ΛCDM (or dynamical dark energy) model might help reduce this low-ℓ discrepancy [53], also see Ref. [54].

In Sec. II we review the spatial geometry of the closed model and various properties of the eigenfunctions of the spatial Laplacian. Synchronous gauge linear perturbation equations, in both the scalar field inflation epoch and fluid (radiation and nonrelativistic matter) epochs, are derived in Sec. III, where we also list the scalar (under general coordinate transformations) parts of these equations in spatial momentum space. These are used to establish that the synchronous gauge linear perturbation equations of a fluid model with a specified spacetime-dependent "speed of sound" coincide with those of the scalar field model (a generalization of the flat model result of Ref. [55]). In Sec. III D, we examine how the (scalar) synchronous gauge variables transform under the remnants of general coordinate invariance, construct gauge-invariant combinations of these variables, and derive equations of motion for these gauge-invariant variables. In Sec. IV we solve the inflation epoch equations and determine the constants of integration in the general solution for the perturbations by using the Hawking initial conditions. Here we also list expressions for the gauge-invariant variables, and compute the scalar field and energy density perturbation two-point correlation functions. In Sec. V A we derive general solutions for the gauge-invariant variables in the radiation epoch; in Sec. V B we solve the synchronous gauge equations in the nonrelativistic matter epoch, and list expressions for the gauge-invariant variables in this epoch. The general solutions in the radiation and matter epochs depend on constants of integration which are determined from joining conditions derived by requiring that the equations of motion be nonsingular at the transitions; these are listed in Sec. VI A. The constants of integration are determined in Sec. VI B while in Sec. VI C we extract the large-scale contribution to these expressions for the constants. Nonrelativistic matter epoch theoretical expressions characterizing large-scale structure are most conveniently compared to observational data on a spatial hypersurface on which the time derivative of the trace of the metric perturbation has been set to zero—this is the instantaneously Newtonian spatial hypersurface. We construct these coordinates, and list expressions for the relevant power spectra, in Sec. VII, where we also record the gauge-invariant energy density inhomogeneity power spectrum. We conclude in Sec. VIII.

II. TECHNICAL PRELIMINARIES

The positive spatial curvature (closed) Friedman-Lemaître-Robertson-Walker (FLRW) model has the line element

$$\begin{aligned} ds^2 &= dt^2 - a^2(t)H_{ij}(\vec{x})dx^i dx^j \\ &= dt^2 - a^2(t)[d\chi^2 + \sin^2(\chi)\{d\theta^2 + \sin^2(\theta)d\phi^2\}], \end{aligned} \quad (1)$$

where $a(t)$ is the FLRW scale factor, $H_{ij}(\vec{x})$ the metric on the closed spatial hypersurface, the "radial" coordinate $0 \leq \chi < \pi$, and θ, ϕ are the usual angular coordinates on the two-sphere. The square of the distance between two points, (t,χ,θ,ϕ) and (t,χ',θ',ϕ'), is

$$\sigma^2 = 2a^2(t)[-1 + \cos(\gamma_3)], \quad (2)$$

$$\cos(\gamma_3) = \cos(\chi)\cos(\chi') + \sin(\chi)\sin(\chi')\cos(\gamma_2), \quad (3)$$

where γ_2 is the usual angle between the two points (θ,ϕ) and (θ',ϕ') on the two-sphere

$$\cos(\gamma_2) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi'). \quad (4)$$

The three-dimensional spatial covariant derivative of a spatial vector (or tensor) will be denoted by a $_{|}$, is defined in the usual way

$$\begin{aligned} A^i{}_{|j} &= A^i{}_{,j} + \Gamma^i{}_{jk}A^k, \\ A_{i|j} &= A_{i,j} - \Gamma^k{}_{ij}A_k, \end{aligned} \quad (5)$$

where the commas denote spatial differentiation, and obeys the usual relations of covariant differentiation. The three-dimensional spatial Christoffel symbol is

$$\Gamma^i{}_{jk} = \frac{1}{2}H^{il}(H_{lj,k} + H_{lk,j} - H_{jk,l}). \quad (6)$$

The $_{|}$ operator obeys the usual relations of covariant differentiation.

The three-dimensional spatial Laplacian for the metric of Eq. (1) is

$$L^2 = \frac{1}{\sin^2(\chi)}\frac{\partial}{\partial\chi}\left(\sin^2(\chi)\frac{\partial}{\partial\chi}\right) + \frac{1}{\sin^2(\chi)\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\chi)\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}. \quad (7)$$

The scalar eigenfunctions Y_{ABC} of L^2 obey, [42,56],

$$L^2 Y_{ABC}(\Omega) = H^{ij}(\Omega)[Y_{ABC}(\Omega)]_{|i|j} = -A(A+2)Y_{ABC}(\Omega), \quad (8)$$

where $\Omega = (\chi, \theta, \phi)$, integer $A = 0, 1, 2\cdots$, and the two 'magnetic' integral indices $B\epsilon[-A, A]$, and $C\epsilon[-B, B]$. *The $O(4)$ symmetry makes the spatial Laplacian eigenvalues independent of the two magnetic indices B and C, see discussion in Appendix B of Ref.* [42]. The orthonormal eigenfunctions are, [42,56],

$$Y_{ABC}(\Omega) = \sqrt{\frac{(A+1)\Gamma(A+B+2)}{\Gamma(A-B+1)}} \times [\sin(\chi)]^{-1/2} P^{-B-1/2}_{A-1/2}(\cos(\chi)) Y_{BC}(\theta,\phi), \quad (9)$$

where Y_{BC} is the standard two-dimensional spherical harmonic, Γ is the gamma function, and P^{μ}_{ν} is the associated Legendre function of the first kind (Chap. 3 of Ref. [57] or Chap. 8 of Ref. [58]). The orthonormality relation is

$$\int_0^{\pi} d\chi \sin^2(\chi) \int_{S^2} d\Omega_2 Y_{ABC}(\Omega)[Y_{A'B'C'}(\Omega)]^* = \delta_{A,A'}\delta_{B,B'}\delta_{C,C'}, \quad (10)$$

where S^2 is the two-dimensional unit sphere with volume element $d\Omega_2$, and $\delta_{A,A'}$, $\delta_{B,B'}$, and $\delta_{C,C'}$ are Kronecker deltas. The addition theorem is, [42],

$$P^{-1/2}_{A+1/2}(\cos(\gamma_3)) = \frac{(2\pi)^{3/2}}{(A+1)^2}[\sin(\gamma_3)]^{1/2}\sum_{B,C} Y_{ABC}(\Omega)[Y_{ABC}(\Omega')]^*, \quad (11)$$

where γ_3 is in Eq. (3).

We shall have need for the following relations, which may be derived by using standard manipulations (see the first of Refs. [56]),

$$Y_{|i|j} = Y_{|j|i}, \quad (12)$$

$$H^{jk}Y_{|k|i|j} = -(A^2+2A-2)Y_{|i}, \quad (13)$$

$$H^{kl}Y_{|l|j|i|k} = -(A^2+2A-5)Y_{|i|j} + A(A+2)YH_{ij}, \quad (14)$$

$$H^{kl}Y_{|i|j|k|l} = -(A^2+2A-6)Y_{|i|j} + 2A(A+2)YH_{ij}, \quad (15)$$

where we have suppressed the spatial momentum indices on the scalar (under the spatial reparameterization remnants of general coordinate transformations in synchronous gauge) spatial harmonic $Y_{ABC}(\Omega)$.

Also, the Ricci tensor on the spatial hypersurface is

$${}^{(3)}R_{ij} = \Gamma^k{}_{ij,k} - \Gamma^k{}_{ki,j} + \Gamma^k{}_{kl}\Gamma^l{}_{ij} - \Gamma^k{}_{lj}\Gamma^l{}_{ki}, \quad (16)$$

and it may be shown that for the spatial metric given in Eq. (1),

$${}^{(3)}R_{ij} = 2H_{ij}. \quad (17)$$

III. EQUATIONS OF MOTION

In this section we derive the general, closed FLRW model, position space, synchronous gauge, linear perturbation theory equations of motion, for both the homogeneous background fields and for the spatial irregularities, in the early time scalar field inflation epoch and in the late time ideal fluid (radiation or matter) epochs. (The current dark energy dominated epoch is not as analytically tractable and so is ignored here; our matter epoch results suffice for our purposes.) We then extract the scalar (under general coordinate transformations) parts of these equations (i.e., we ignore transverse peculiar velocity perturbations and gravitational wave perturbations), and record their spatial momentum space form.

For later use, we establish that the synchronous gauge linear perturbation theory equations of a fluid model which allows for a specified spacetime-dependent "speed of sound" are identical to the scalar field model synchronous gauge linear perturbation equations.

We also examine how the (scalar) synchronous gauge spatial irregularity variables of interest transform under the remnants of general coordinate invariance in synchronous gauge, write down combinations of these variables that are invariant under these transformations, and derive the equations of motion for these gauge-invariant variables.

A. Einstein-scalar-field model equations of motion

The Einstein-scalar field action, for the metric tensor $g_{\mu\nu}$ and inflaton scalar field Φ, is

$$S = \frac{{m_p}^2}{16\pi}\int dt d^3x\sqrt{-g}\left[-R + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}V(\Phi)\right], \quad (18)$$

where $m_p = G^{-1/2}$ is the Planck mass. Varying, we find the equations of motion,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi)+\frac{1}{2}V'(\Phi)=0, \tag{19}$$

$$R_{\mu\nu}=\frac{8\pi}{m_p{}^2}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right), \tag{20}$$

where prime denotes a derivative with respect to Φ and T is the trace of the stress-energy tensor,

$$T_{\mu\nu}=\frac{m_p{}^2}{16\pi}\left[\partial_\mu\Phi\partial_\nu\Phi-\frac{1}{2}g_{\mu\nu}\{g^{\lambda\rho}\partial_\lambda\Phi\partial_\rho\Phi-V(\Phi)\}\right]. \tag{21}$$

To derive the equations of motion for the spatially homogeneous background fields and for the spatial irregularities, we linearize Eqs. (19)–(21) about a closed FLRW model and a spatially homogeneous scalar field. We work in synchronous gauge, with line element

$$ds^2=dt^2-a^2(t)[H_{ij}(\vec{x})-h_{ij}(t,\vec{x})]dx^idx^j, \tag{22}$$

where the background metric on the closed spatial hypersurfaces, H_{ij}, is given in Eq. (1), and the metric perturbations are denoted by h_{ij}. The expansion for the scalar field is

$$\Phi(t,\vec{x})=\Phi_b(t)+\phi(t,\vec{x}), \tag{23}$$

where Φ_b and ϕ are the spatially homogeneous and inhomogeneous parts of the scalar field (the scalar field perturbation ϕ should not be confused with the angular variable ϕ of Sec. II). The linearized stress-energy tensor components are

$$T_{00}=\frac{m_p{}^2}{32\pi}[\dot{\Phi}_b{}^2+V(\Phi_b)]+\frac{m_p{}^2}{16\pi}\left[\dot{\Phi}_b\dot{\phi}+\frac{1}{2}V'(\Phi_b)\phi\right]+\cdots, \tag{24}$$

$$T_{0i}=\frac{m_p{}^2}{16\pi}\dot{\Phi}_b\partial_i\phi+\cdots, \tag{25}$$

$$\begin{aligned}T_{ij}=&\frac{m_p{}^2}{32\pi}a^2H_{ij}[\dot{\Phi}_b{}^2-V(\Phi_b)]\\&+\frac{m_p{}^2}{16\pi}a^2\left[H_{ij}\left\{\dot{\Phi}_b\dot{\phi}-\frac{1}{2}V'(\Phi_b)\phi\right\}\right.\\&\left.-\frac{1}{2}h_{ij}\{\dot{\Phi}_b{}^2-V(\Phi_b)\}\right]+\cdots,\end{aligned} \tag{26}$$

where the ellipses denote terms of second and higher order in the perturbations.

The equations of motion for the spatially homogeneous parts of the fields are

$$\ddot{\Phi}_b+3\frac{\dot{a}}{a}\dot{\Phi}_b+\frac{1}{2}V'(\Phi_b)=0, \tag{27}$$

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{1}{12}[\dot{\Phi}_b{}^2+V(\Phi_b)]-\frac{1}{a^2}, \tag{28}$$

$$\frac{\ddot{a}}{a}=-\frac{1}{6}\dot{\Phi}_b{}^2+\frac{1}{12}V(\Phi_b), \tag{29}$$

where an overdot denotes a derivative with respect to time. The only change, relative to the equations for the flat model (Sec. VII of Ref. [30] and Sec. II of Ref. [55]), is the new term $(1/a^2)$ on the right-hand side of Eq. (28). The first order perturbation equations are

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}-\frac{L^2}{a^2}\phi+\frac{1}{2}V''(\Phi_b)\phi=\frac{1}{2}\dot{h}\dot{\Phi}_b, \tag{30}$$

$$\ddot{h}+2\frac{\dot{a}}{a}\dot{h}=2\dot{\Phi}_b\dot{\phi}-\frac{1}{2}V'(\Phi_b)\phi, \tag{31}$$

$$\dot{h}_{|i}-(H^{jk}\dot{h}_{ki})_{|j}=\dot{\Phi}_b\phi_{|i}, \tag{32}$$

$$\begin{aligned}&\ddot{h}_{ij}+3\frac{\dot{a}}{a}\dot{h}_{ij}+\frac{\dot{a}}{a}H_{ij}\dot{h}-\frac{1}{a^2}h_{|i|j}\\&\quad+\frac{1}{a^2}[H^{kl}(h_{li|j}+h_{lj|i}-h_{ij|l})]_{|k}-\frac{4}{a^2}h_{ij}\\&=-\frac{1}{2}H_{ij}V'(\Phi_b)\phi,\end{aligned} \tag{33}$$

where the trace of the metric perturbation is denoted by $h(=H^{ij}h_{ij})$ and spatial indices are raised and lowered with the background metric H_{ij}. Equation (30) governs the evolution of the scalar field perturbation, Eq. (31) that of the trace of the metric perturbation, and Eqs. (32) and (33) that of the remaining part of the metric perturbation. Besides the expected change, relative to the equations of the flat model (Sec. VII of Ref. [30] and Sec. II of Ref. [55]), of all spatial derivatives being replaced by spatial covariant derivatives, the only other change is the new last term on the left-hand side of Eq. (33), $4h_{ij}/a^2$.

To extract the scalar parts of Eqs. (30)–(33) in spatial momentum space we focus on a mode with spatial momentum characterized by the indices (A,B,C), [56],

$$\phi(\Omega,t)=\phi(A,B,C,t)Y(\Omega), \tag{34}$$

$$\begin{aligned}h_{ij}(\Omega,t)=&\frac{1}{3}h(A,B,C,t)H_{ij}(\Omega)Y(\Omega)\\&+\mathcal{H}(A,B,C,t)\left[\frac{Y_{|i|j}(\Omega)}{A(A+2)}+\frac{1}{3}H_{ij}(\Omega)Y(\Omega)\right],\end{aligned} \tag{35}$$

where $h(A,B,C,t)$ is the trace of the metric perturbation (the perturbation to the size of the proper volume element)

and $\mathcal{H}(A,B,C,t)$ is the trace-free part (the shearing perturbation of the volume element). Equation (35) is the most general decomposition of the scalar part of the metric perturbation (we have ignored gravitational wave perturbations). The scalar parts of Eqs. (30)–(33) for a given mode in spatial momentum space are

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}+\frac{A(A+2)}{a^2}\phi+\frac{1}{2}V''(\Phi_b)\phi=\frac{1}{2}\dot{h}\dot{\Phi}_b, \quad (36)$$

$$\ddot{h}+2\frac{\dot{a}}{a}\dot{h}=2\dot{\Phi}_b\dot{\phi}-\frac{1}{2}V'(\Phi_b)\phi, \quad (37)$$

$$\dot{\mathcal{H}}=\frac{A(A+2)}{(A-1)(A+3)}\left[\frac{3}{2}\dot{\Phi}_b\phi-\dot{h}\right], \quad (38)$$

$$\ddot{h}+6\frac{\dot{a}}{a}\dot{h}+\frac{(A^2+2A-4)}{a^2}h+\ddot{\mathcal{H}}+3\frac{\dot{a}}{a}\dot{\mathcal{H}}+\frac{(A^2+2A-4)}{a^2}\mathcal{H}=-\frac{3}{2}V'(\Phi_b)\phi, \quad (39)$$

$$\ddot{\mathcal{H}}+3\frac{\dot{a}}{a}\dot{\mathcal{H}}-\frac{A(A+2)}{3a^2}\mathcal{H}-\frac{A(A+2)}{3a^2}h=0. \quad (40)$$

B. Einstein-fluid model equations of motion

The fluid model equations of motion are covariant conservation of stress-energy

$$T_{\alpha}{}^{\beta}{}_{;\beta}=0, \quad (41)$$

and the Einstein equations, Eq. (20), where the stress-energy tensor for the fluid is

$$T^{\mu\nu}=(\rho+p)u^{\mu}u^{\nu}-g^{\mu\nu}p, \quad (42)$$

where ρ and p are the fluid energy density and pressure and u^{μ} is the fluid coordinate peculiar velocity.

To derive the equations of motion for the spatially homogeneous background fields and for the spatial irregularities, we linearize Eqs. (41), (20), and (42) about a spatially closed FLRW model and a spatially homogeneous background fluid. We work in synchronous gauge, with the line element of Eq. (22). The expansions for the fluid variables are

$$\rho(t,\vec{x})=\rho_b(t)[1+\delta(t,\vec{x})], \quad (43)$$

$$p(t,\vec{x})=p_b(t)+c_s{}^2\rho_b(t)\delta(t,\vec{x}), \quad (44)$$

$$u^0(t,\vec{x})=1, \quad (45)$$

$$u^i(t,\vec{x})=0+u^i(t,\vec{x}), \quad (46)$$

i.e., u^i is taken to be of the same order as the fractional perturbation in the fluid energy density, δ. Here ρ_b and p_b are the homogeneous background fluid energy density and pressure and the background equation of state is taken to be

$$p_b(t)=\nu\rho_b(t), \quad (47)$$

where ν is a constant. The speed of propagation of "acoustic" waves is

$$c_s{}^2=\frac{dp}{d\rho}, \quad (48)$$

and, for the present, will be allowed to be a function of the spacetime coordinates. Expanding the fluid stress-energy tensor, Eq. (42), we find the components

$$T_{00}=\rho_b+\rho_b\delta+\cdots, \quad (49)$$

$$T_{0i}=-a^2(\rho_b+p_b)H_{ij}u^j+\cdots, \quad (50)$$

$$T_{ij}=a^2H_{ij}p_b+a^2(c_s{}^2\rho_b\delta H_{ij}-p_bh_{ij})+\cdots, \quad (51)$$

where the ellipses denote terms of quadratic and higher order in the perturbations.

The equations of motion for the spatially homogeneous parts of the fields are

$$\dot{\rho}_b=-3\frac{\dot{a}}{a}(\rho_b+p_b), \quad (52)$$

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi}{3m_p{}^2}\rho_b-\frac{1}{a^2}, \quad (53)$$

$$\frac{\ddot{a}}{a}=-\frac{4\pi}{3m_p{}^2}(\rho_b+3p_b). \quad (54)$$

The only change, relative to the equations of the flat model (Secs. 82 and 85 of Ref. [59] and Sec. I of Ref. [60]), is the new term $(1/a^2)$ on the right-hand side of Eq. (53). The first order perturbation equations are

$$\rho_b\dot{\delta}-(\rho_b+p_b)\left(\frac{1}{2}\dot{h}-u^i{}_{|i}\right)=3\frac{\dot{a}}{a}(p_b-c_s{}^2\rho_b)\delta, \quad (55)$$

$$\ddot{h}+2\frac{\dot{a}}{a}\dot{h}=\frac{8\pi}{m_p{}^2}(1+3c_s{}^2)\rho_b\delta, \quad (56)$$

$$[a^5(\rho_b+p_b)H_{kl}u^l]_{,0}=-a^3(c_s{}^2\rho_b\delta)_{|k}, \quad (57)$$

$$\dot{h}_{|i}-(H^{jk}\dot{h}_{ki})_{|j}=-\frac{16\pi}{m_p{}^2}a^2(\rho_b+p_b)H_{ij}u^j, \quad (58)$$

$$\ddot{h}_{ij}+3\frac{\dot{a}}{a}\dot{h}_{ij}+\frac{\dot{a}}{a}H_{ij}\dot{h}-\frac{1}{a^2}h_{|i|j}+\frac{1}{a^2}[H^{kl}(h_{li|j}+h_{lj|i}-h_{ij|l})]_{|k}$$
$$-\frac{4}{a^2}h_{ij}=-\frac{8\pi}{{m_p}^2}H_{ij}(1-{c_s}^2)\rho_b\delta. \quad (59)$$

Equation (55) governs the evolution of the fractional energy density perturbation, Eq. (57) that of the peculiar velocity perturbation, Eq. (56) that of the trace of the metric perturbation, and Eqs. (58) and (59) that of the remaining part of the metric perturbation. Besides the expected change relative to the equations of the flat model (Sec. II of Ref. [55]), of all spatial derivatives being replaced by spatial covariant derivatives, the only other change is the new last term on the left-hand side of Eq. (59), $4h_{ij}/a^2$.

To extract the scalar parts of Eqs. (55)–(59) in spatial momentum space we focus on a mode with spatial momentum characterized by the indices (A, B, C) and write

$$\delta(\Omega,t)=\delta(A,B,C,t)Y(\Omega), \quad (60)$$

$$u_i(\Omega,t)=-\frac{1}{A(A+2)}u(A,B,C,t)Y_{|i}(\Omega); \quad (61)$$

we also use the metric perturbation decomposition of Eq. (35). Equation (61) only accounts for longitudinal peculiar velocity perturbations (we ignore the transverse peculiar velocity). The scalar parts of Eqs. (55)–(59), for a given mode, are

$$\rho_b\dot{\delta}-(\rho_b+p_b)\left(\frac{1}{2}\dot{h}-u\right)=3\frac{\dot{a}}{a}(p_b-{c_s}^2\rho_b)\delta, \quad (62)$$

$$\ddot{h}+2\frac{\dot{a}}{a}\dot{h}=\frac{8\pi}{{m_p}^2}(1+3{c_s}^2)\rho_b\delta, \quad (63)$$

$$[a^5(\rho_b+p_b)u]_{,0}=A(A+2)a^3{c_s}^2\rho_b\delta, \quad (64)$$

$$\dot{\mathcal{H}}=\frac{A(A+2)}{(A-1)(A+3)}\left[\frac{24\pi}{{m_p}^2}\frac{a^2(\rho_b+p_b)u}{A(A+2)}-\dot{h}\right], \quad (65)$$

$$\ddot{h}+6\frac{\dot{a}}{a}\dot{h}+\frac{(A^2+2A-4)}{a^2}h+\ddot{\mathcal{H}}+3\frac{\dot{a}}{a}\dot{\mathcal{H}}$$
$$+\frac{(A^2+2A-4)}{a^2}\mathcal{H}+\frac{24\pi}{{m_p}^2}(1-{c_s}^2)\rho_b\delta=0, \quad (66)$$

$$\ddot{\mathcal{H}}+3\frac{\dot{a}}{a}\dot{\mathcal{H}}-\frac{A(A+2)}{3a^2}\mathcal{H}-\frac{A(A+2)}{3a^2}h=0. \quad (67)$$

C. Scalar field as spacetime-dependent speed of sound fluid

We have shown that in the spatially flat and spatially open models the synchronous gauge linear perturbation equations of a fluid model with a given spacetime-dependent speed of propagation of acoustic disturbances are identical to those of a scalar field model, Sec. II of Ref. [55] and Sec. III. C of Ref. [38]. Here we show that this result also holds in the closed model.

Defining the background energy density and pressure of the scalar field

$$\rho_{b\Phi}=\frac{{m_p}^2}{32\pi}[\dot{\Phi}_b{}^2+V(\Phi_b)], \quad (68)$$

$$p_{b\Phi}=\frac{{m_p}^2}{32\pi}[\dot{\Phi}_b{}^2-V(\Phi_b)], \quad (69)$$

and the fractional energy density, peculiar velocity, and speed of sound of the scalar field perturbation,

$$\rho_{b\Phi}\delta_\Phi=\frac{{m_p}^2}{16\pi}\left[\dot{\Phi}_b\dot{\phi}+\frac{1}{2}V'(\Phi_b)\phi\right], \quad (70)$$

$$a^2(\rho_{b\Phi}+p_{b\Phi})H_{ij}u^j_\Phi=-\frac{{m_p}^2}{16\pi}\dot{\Phi}_b\partial_i\phi, \quad (71)$$

$${c_s}_\Phi{}^2\rho_{b\Phi}\delta_\Phi=\frac{{m_p}^2}{16\pi}\left[\dot{\Phi}_b\dot{\phi}-\frac{1}{2}V'(\Phi_b)\phi\right] \quad (72)$$

we see that the fluid stress-energy tensor, Eqs. (49)–(51), coincides with the scalar field stress-energy tensor, Eqs. (24)–(26). It is straightforward to show that when Eqs. (68) and (69) are used in Eqs. (52)–(54) these homogeneous fluid equations coincide with the homogeneous scalar field equations, Eqs. (27)–(29). Using the definitions of Eqs. (70)–(72) in the fluid spatial irregularity equations (56), (58), and (59), we find that they reproduce the scalar field spatial irregularity equations (31)–(33). It may also be shown that when the definitions of Eqs. (71) and (72) are used in Eq. (57) this equation reduces to an identity [if the equation for the spatially homogeneous part of the scalar field, Eq. (27), is satisfied]. It is only a little bit more involved to show that the definitions (68)–(72) imply that Eq. (55) reduces to Eq. (30) (the manipulations are very similar to those outlined at the end of Sec. II of Ref. [55]).

D. Gauge-invariant variables

Choosing synchronous gauge does not completely fix general coordinate invariance—there are four remaining time-independent gauge symmetries. Their effect on the metric perturbation is

$$\delta h_{ij}(\Omega,t)=-({f^0}_{|i|j}(\Omega)+{f^0}_{|j|i}(\Omega))\int^t\frac{dt'}{a^2(t')}-\omega_{i|j}(\Omega)$$
$$-\omega_{j|i}(\Omega)-2\frac{\dot{a}}{a}f^0(\Omega)H_{ij}(\Omega), \quad (73)$$

where the general coordinate transformation parameters f^0 and ω_i are time independent. The scalar field perturbation and the variables derived from it transform according to

$$\delta\phi(\Omega,t)=\dot{\Phi}_b f^0(\Omega), \tag{74}$$

$$\delta[\delta_\Phi(\Omega,t)]=\frac{\dot{\rho}_{b\Phi}}{\rho_{b\Phi}}f^0(\Omega), \tag{75}$$

$$\delta u^i_\Phi(\Omega,t)=-\frac{1}{a^2}H^{ij}f^0{}_{|j}(\Omega), \tag{76}$$

$$\delta[\{{c_s}_\Phi{}^2\delta_\Phi\}(\Omega,t)]=\frac{\dot{p}_{b\Phi}}{\rho_{b\Phi}}f^0(\Omega), \tag{77}$$

while the fluid variables transform, as expected, according to

$$\delta[\delta(\Omega,t)]=\frac{\dot{\rho}_b}{\rho_b}f^0(\Omega), \tag{78}$$

$$\delta u^i(\Omega,t)=-\frac{1}{a^2}H^{ij}f^0{}_{|j}(\Omega), \tag{79}$$

$$\delta[\{{c_s}^2\delta\}(\Omega,t)]=\frac{\dot{p}_b}{\rho_b}f^0(\Omega). \tag{80}$$

In spatial momentum space the scalar parts of the fields transform as

$$\delta\mathcal{H}(A,B,C,t)=-2A(A+2)f^0(A,B,C)\int^t\frac{dt'}{a^2(t')}+2\omega(A,B,C), \tag{81}$$

$$\delta h(A,B,C,t)=2A(A+2)f^0(A,B,C)\int^t\frac{dt'}{a^2(t')}-2\omega(A,B,C)-6\frac{\dot{a}}{a}f^0(A,B,C), \tag{82}$$

$$\delta\phi(A,B,C,t)=\dot{\Phi}_b f^0(A,B,C), \tag{83}$$

$$\delta[\delta(A,B,C,t)]=\frac{\dot{\rho}_b}{\rho_b}f^0(A,B,C), \tag{84}$$

$$\delta[u(A,B,C,t)]=\frac{A(A+2)}{a^2}f^0(A,B,C), \tag{85}$$

$$\delta[\{{c_s}^2\delta\}(A,B,C,t)]=\frac{\dot{p}_b}{\rho_b}f^0(A,B,C), \tag{86}$$

where ω_i and ω obey a relation like Eq. (61).

Following Ref. [55], it may be shown that all gauge-invariant information about the scalar part of the fluid perturbations is encoded in the gauge-invariant combinations

$$\Delta(A,B,C,t)=\delta(A,B,C,t)+3\frac{\dot{a}}{a}\left(\frac{\rho_b+p_b}{\rho_b}\right)\frac{a^2u(A,B,C,t)}{A(A+2)}, \tag{87}$$

$$A(A,B,C,t)=\delta(A,B,C,t)-\frac{\rho_b+p_b}{2\rho_b}[h(A,B,C,t)+\mathcal{H}(A,B,C,t)] \tag{88}$$

(the variable A should not be confused with the spatial momentum A). In the scalar field model Eqs. (87) and (88) may be rewritten, using Eqs. (68)–(71), as

$$\Delta_\Phi=\frac{1}{\dot{\Phi}_b{}^2+V(\Phi_b)}[2\dot{\Phi}_b\dot{\phi}+V'(\Phi_b)\phi+6\frac{\dot{a}}{a}\dot{\Phi}_b\phi], \tag{89}$$

$$A_\Phi=\frac{1}{\dot{\Phi}_b{}^2+V(\Phi_b)}[2\dot{\Phi}_b\dot{\phi}+V'(\Phi_b)\phi-\dot{\Phi}_b{}^2(h+\mathcal{H})]. \tag{90}$$

We now record the equations of motion for the fluid gauge-invariant variables, Δ and A. We have need only for the equations in the ideal fluid model, so we set

$${c_s}^2=\nu, \tag{91}$$

where ν is a numerical constant defined in Eq. (47). It is convenient to work with

$$D=A/(\rho_b+p_b), \tag{92}$$

instead of the variable A of Eq. (88). Using the fluid equations of motion, Eqs. (52)–(54) and (62)–(67), we find that Δ and D obey

$$\dot{\Delta}+\left[\frac{3}{2}(1-\nu)\frac{\dot{a}}{a}+\left\{\frac{(1+3\nu)}{2}+\frac{A(A+2)}{3}\right\}\frac{1}{a\dot{a}}\right]\Delta=\frac{1}{3}(A-1)(A+3)(1+\nu)\frac{\rho_b}{a\dot{a}}D, \tag{93}$$

$$\dot{D}-\left[\frac{A(A+2)}{3a\dot{a}}+3(1+\nu)\frac{\dot{a}}{a}\right]D=-\frac{A(A+2)}{(1+\nu)\rho_b}\left[\left\{\frac{1}{3}+\frac{3}{2}\frac{(1+\nu)}{(A-1)(A+3)}\right\}\frac{1}{a\dot{a}}+\frac{3}{2}\frac{(1+\nu)}{(A-1)(A+3)}\frac{\dot{a}}{a}\right]\Delta. \tag{94}$$

These equations may be combined to yield

$$\ddot{\Delta}+(2-3\nu)\frac{\dot{a}}{a}\dot{\Delta}+\left[-\frac{3}{2}(1-\nu)(1+3\nu)\left(\frac{\dot{a}}{a}\right)^2+\left\{-\frac{3}{2}(1-\nu)(1+3\nu)+\nu A(A+2)\right\}\frac{1}{a^2}\right]\Delta=0; \tag{95}$$

a similar second order equation may be derived for the variable D—since we have no need for it we do not record it here.

IV. INFLATION EPOCH

In this section we solve the synchronous gauge equations of motion to derive expressions for the spatially homogeneous and inhomogeneous fields in the inflation epoch.

The potential energy density for the scalar field Φ which drives inflation is taken to be

$$V(\Phi) = 12h^2[1 - \epsilon\Phi], \tag{96}$$

where h^2 is a numerical parameter related to the inflation epoch cosmological constant (the parameter h should not be confused with the trace of the metric perturbation h) and ϵ is a small numerical parameter. (These two free parameters will be constrained by comparing our predictions to observational data.) The first term, $12h^2$, is large and is responsible for driving the expansion of the universe during inflation, and the term proportional to $\epsilon\Phi$ is small and is responsible for forcing the scalar field down the slope. This form of potential energy density is chosen so that the leading term acts like a cosmological constant and results in closed de Sitter inflation while the subleading term powers a very slowly rolling inflaton field.

Besides the standard expansion in spatial irregularity (or the Newtonian gravitational constant) used to derived the usual equations of synchronous gauge relativistic linear perturbation theory, we shall also make use of an expansion in the parameter ϵ to simplify the computation, [38,61]. This second expansion assumes that ϵ is small; we shall have to check that this is a consistent assumption by comparing our predictions to observational data and verifying that the needed numerical value of ϵ is indeed small.

A. Spatially homogeneous background fields

We wish to determine the solutions of Eqs. (27)–(29) for the model with the scalar field potential energy density of Eq. (96). Our ansatz for the homogeneous fields is

$$\Phi_b(t) = \Phi_{b0}(t) + \epsilon\Phi_{b1}(t), \tag{97}$$

$$a(t) = a_0(t)[1 + \epsilon f(t)], \tag{98}$$

where $\Phi_{b0}(t)$, $\Phi_{b1}(t)$, $a_0(t)$, and $f(t)$ are independent of ϵ and will be determined below.

To lowest order in ϵ Eqs. (27)–(29) are

$$\ddot{\Phi}_{b0} + 3\frac{\dot{a}_0}{a_0}\dot{\Phi}_{b0} = 0, \tag{99}$$

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 - \frac{1}{12}\dot{\Phi}_{b0}{}^2 - h^2 + \frac{1}{a_0{}^2} = 0, \tag{100}$$

$$\frac{\ddot{a}_0}{a_0} + \frac{1}{6}\dot{\Phi}_{b0}{}^2 - h^2 = 0. \tag{101}$$

The first integral of Eq. (99) is

$$\dot{\Phi}_{b0}(t) = \dot{\Phi}_{b0i}\left(\frac{a_{0i}}{a_0(t)}\right)^3, \tag{102}$$

where $\dot{\Phi}_{b0i}(a_{0i})^3$ is a constant of integration. This solution decreases with time, because of Hubble damping, and we choose the constant to be

$$\dot{\Phi}_{b0i} = 0. \tag{103}$$

The lowest order solution for the scalar field is then

$$\Phi_{b0}(t) = \Phi_{b0i}, \tag{104}$$

where Φ_{b0i} is a constant of integration. The lowest order solution for the scale factor is

$$a_0(t) = h^{-1}\cosh(ht). \tag{105}$$

The first order in ϵ parts of Eqs. (27)–(29) are

$$\ddot{\Phi}_{b1} + 3\frac{\dot{a}_0}{a_0}\dot{\Phi}_{b1} - 6h^2 = 0, \tag{106}$$

$$2\frac{\dot{a}_0}{a_0}\dot{f} - \frac{2f}{a_0{}^2} + h^2\Phi_{b0i} = 0, \tag{107}$$

$$\ddot{f} + 2\frac{\dot{a}_0}{a_0}\dot{f} + h^2\Phi_{b0i} = 0 \tag{108}$$

After some work, it may be shown that the solutions of these equations are

$$\Phi_{b1}(t) = \bar{c}_0 + \frac{\bar{c}_1}{2h}\left[\frac{\sinh(ht)}{\cosh^2(ht)} + \tan^{-1}\{\sinh(ht)\}\right] + 2\left[\ln\{\cosh(ht)\} - \frac{1}{\cosh^2(ht)}\right], \tag{109}$$

$$f(t) = \frac{1}{2}\Phi_{b0i} - \left[\bar{c}_3h^2 + \frac{1}{2}\Phi_{b0i}ht\right]\tanh(ht), \tag{110}$$

where $\bar{c}_0$, $\bar{c}_1$, and $\bar{c}_3$ are constants of integration.

B. Spatial irregularities

We shall only have need for the order ϵ^0 part of ϕ. To this order Eq. (36) is

$$\ddot{\phi}_0 + 3h\tanh(ht)\dot{\phi}_0 + \frac{A(A+2)h^2}{\cosh^2(ht)}\phi_0 = 0. \tag{111}$$

The solution of this equation is

$$\phi_0(A,B,C,t)=\frac{h}{\cosh(ht)}[c_+\{\sinh(ht)-i(A+1)\}e^{-i(A+1)\tan^{-1}\{\sinh(ht)\}} + c_-\{\sinh(ht)+i(A+1)\}e^{i(A+1)\tan^{-1}\{\sinh(ht)\}}], \quad (112)$$

where $c_\pm$ are A dependent constants of integration which will be determined from quantum mechanical initial conditions. We note that, to leading order in ϵ, the two solutions in this equation are gauge invariant.

We shall have need for the fractional energy density and peculiar velocity perturbations during the inflation epoch, Eqs. (70) and (71). Using Eqs. (60), (61), and (112), and the expressions of Sec. IV A, we have, to lowest order in ϵ,

$$\delta_\Phi(A,B,C,t)=-\frac{\epsilon}{6\cosh^5(ht)}[c_+e^{-i(A+1)\tan^{-1}\{\sinh(ht)\}}[A(A+2)\{\bar{c}_1+2h\sinh(ht)[\cosh^2(ht)+2]\} + 6h\cosh^4(ht)[\sinh(ht)-i(A+1)]]+c_-e^{i(A+1)\tan^{-1}\{\sinh(ht)\}} \times[A(A+2)\{\bar{c}_1+2h\sinh(ht)[\cosh^2(ht)+2]\}+6h\cosh^4(ht)[\sinh(ht)+i(A+1)]]], \quad (113)$$

$$u_\Phi(A,B,C,t)=\frac{A(A+2)h^3}{\epsilon[\bar{c}_1+2h\sinh(ht)\{\cosh^2(ht)+2\}]}[\{\sinh(ht)-i(A+1)\}c_+e^{-i(A+1)\tan^{-1}\{\sinh(ht)\}} + \{\sinh(ht)+i(A+1)\}c_-e^{i(A+1)\tan^{-1}\{\sinh(ht)\}}] \quad (114)$$

We shall also have need for expressions for the gauge-invariant variables Δ_Φ and A_Φ during inflation. We find, to leading order in ϵ,

$$\Delta_\Phi(A,B,C,t)=\frac{\epsilon}{6\cosh^5(ht)}[c_+e^{-i(A+1)\tan^{-1}\{\sinh(ht)\}}[3\sinh(ht)\{\bar{c}_1+2h\sinh(ht)[\cosh^2(ht)+2]\}[\sinh(ht)-i(A+1)] - A(A+2)\{\bar{c}_1+2h\sinh(ht)[\cosh^2(ht)+2]\}-6h\cosh^4(ht)[\sinh(ht)-i(A+1)]] + c_-e^{i(A+1)\tan^{-1}\{\sinh(ht)\}}[3\sinh(ht)\{\bar{c}_1+2h\sinh(ht)[\cosh^2(ht)+2]\}[\sinh(ht)+i(A+1)] - A(A+2)\{\bar{c}_1+2h\sinh(ht)[\cosh^2(ht)+2]\}-6h\cosh^4(ht)[\sinh(ht)+i(A+1)]]] \quad (115)$$

$$A_\Phi(A,B,C,t)=\delta_\Phi(A,B,C,t). \quad (116)$$

where δ_Φ is given in Eq. (113).

C. Initial conditions and two-point correlation functions

Conformal time $\tilde{t}$ is related to t through

$$\tan\tilde{t}=\sinh(ht). \quad (117)$$

In Eq. (112), defining the constants $\tilde{c}_\pm$,

$$c_\pm=\pm\frac{i}{\sqrt{2A(A+1)(A+2)}}\left(\frac{16\pi}{m_p{}^2}\right)^{1/2}\tilde{c}_\pm, \quad (118)$$

the initial conditions, Sec. VII of Ref. [42], require that we choose (up to an irrelevant phase)

$$\tilde{c}_+=1 \quad \text{and} \quad \tilde{c}_-=0. \quad (119)$$

This is equivalent to Hawking's prescription of including only regular Euclidean field configurations [41], and is de Sitter invariant, see Secs. VI–IX of Ref. [42].

In the closed de Sitter model the $a\to 0$ limit does not lie in the Lorentzian section [42], unlike in the open and flat cases. Hawking's prescription [41] of only including field configurations regular on the Euclidean section does in fact correspond to the ground state energy of the rescaled scalar field inhomogeneity not diverging as $a\to 0$, which in this case is a pole of the Euclidean section sphere [42] and in fact leads to a de Sitter invariant ground state scalar field two-point correlation function, [42].

With this choice we find that the equal-time scalar field perturbation two-point correlation function is

$$\langle\phi_0(A,B,C,t)\phi_0^*(A',B',C',t)\rangle = |\phi_0(A,B,C,t)|^2\delta_{A,A'}\delta_{B,B'}\delta_{C,C'}, \quad (120)$$

$$|\phi_0(A,B,C,t)|^2=\frac{16\pi}{m_p{}^2}\frac{1}{2(A+1)a^2}\left[1+\frac{h^2a^2}{A(A+2)}\right]. \quad (121)$$

This result coincides with Eq. (7.13) of Ref. [42]. We note that at late time the right-hand side of Eq. (121) becomes

time independent, as does the corresponding two-point correlation function in flat (exponentially expanding) de Sitter spacetime (Sec. V of Ref. [42]) as well as in open de Sitter spacetime (Sec. IV of Ref. [36] and Sec. IV 3 of Ref. [38]), however, the dependence on spatial momentum in the long wavelength limit are quite different in the nonflat and flat cases.

This difference in the infrared behavior is also seen in the fractional energy density perturbation two-point correlation function. We find

$$\langle \delta_\Phi(A,B,C,t)\delta_\Phi^*(A',B',C',t)\rangle = |\delta_\Phi(A,B,C,t)|^2\delta_{A,A'}\delta_{B,B'}\delta_{C,C'}, \tag{122}$$

where the fractional energy density perturbation power spectrum is

$$|\delta_\Phi(A,B,C,t)|^2 = \epsilon^2\frac{16\pi}{{m_p}^2}\frac{1}{2A(A+1)(A+2)a^2} \times\left[(A+1)^2+\left[\sqrt{h^2a^2-1}+\frac{A(A+2)}{6h^5a^4} \times\{\bar{c}_1+2h\sqrt{h^2a^2-1}(h^2a^2+2)\}\right]^2\right], \tag{123}$$

where $\bar{c}_1$ is the real constant of integration in the expression in Eq. (109). In the short wavelength limit the last term in the inner square parentheses dominates, and at late times

$$|\delta_\Phi|^2 \propto A/a^4, \tag{124}$$

which is what one finds in the flat de Sitter case (Eq. (3.56) of Ref. [62], also see Ref. [63]); this is the scale-invariant spectrum, [40]. In the long wavelength limit the first term in the inner square parentheses dominates at late time

$$|\delta_\Phi|^2 \propto 1/A; \tag{125}$$

this suggests that in the closed model the large-scale energy density power spectrum will break away from the scale-invariant form and will instead behave like an $n=-1$ spectrum, like in the open case, see Eq. (4.31) of Ref. [38].

V. THE RADIATION AND MATTER EPOCHS

In this section we solve the equations of motion to derive expressions for the spatially homogeneous and inhomogeneous fields in the radiation and matter epochs.

A. The radiation epoch

In this epoch $\nu=1/3={c_s}^2$ and from Eq. (52) $\rho_{bR}\propto a^{-4}$, or

$$\rho_{bR}(t) = \frac{3{m_p}^2}{8\pi}\frac{{h_R}^2}{a^4(t)}, \tag{126}$$

where h_R is a constant of integration determined below. We shall not have need for the explicit expression for $a(t)$.

It suffices to derive expressions for the gauge-invariant variables Δ_R and A_R. Defining

$$x = a/h_R, \tag{127}$$

and using Eq. (53) to rewrite Eq. (95) in the radiation epoch we have

$$x^2(1-x^2)\Delta_R''-x^3\Delta_R'+\left[-2+\frac{1}{3}A(A+2)x^2\right]\Delta_R=0; \tag{128}$$

here a prime denotes a derivative with respect to x. The solution of this equation is

$$\Delta_R(x)=c_1^{(R)}x^2F(1+b,1-b;5/2;x^2) +c_2^{(R)}x^{-1}F(-1/2+b,-1/2-b;-1/2;x^2); \tag{129}$$

$c_\pm^{(R)}$ are spatial momentum dependent constants of integration, determined below, the F's are hypergeometric functions (Chap. 15 of Ref. [58]), and

$$b=\frac{1}{2}\left(\frac{A(A+2)}{3}\right)^{1/2}. \tag{130}$$

From Eqs. (92) and (93) we have

$$A_R(x)=\frac{3}{(A-1)(A+3)}\frac{(1-x^2)}{x}\Delta_R' +\frac{3}{(A-1)(A+3)}\left[\frac{1}{x^2}+\frac{A(A+2)}{3}\right]\Delta_R, \tag{131}$$

so from Eq. (129) we find

$$\begin{aligned}(A-1)(A+3)A_R(x) = 3c_1^{(R)}\Big[&-\frac{4}{5}(b^2-1)x^2(1-x^2)F(2+b,2-b;7/2;x^2)\\ &+\{3+(4b^2-2)x^2\}F(1+b,1-b;5/2;x^2)\Big]\\ &+3c_2^{(R)}[(4b^2-1)(1-x^2)x^{-1}F(1/2+b,1/2-b;1/2;x^2)\\ &+(4b^2+1)x^{-1}F(-1/2+b,-1/2-b;-1/2;x^2)].\end{aligned} \tag{132}$$

B. The matter epoch

In this epoch $\nu=0=c_s{}^2$ and from Eq. (52) $\rho_{bM}\propto a^{-3}$, or

$$\rho_{bM}(t)=\frac{3m_p{}^2}{8\pi}\frac{h_M{}^2}{a^3(t)}, \tag{133}$$

where h_M is a constant of integration determined below. We shall not have need for the explicit expression for $a(t)$.

In the matter epoch Eq. (64) reduces to

$$\frac{d}{dt}[a^5\rho_{bM}u_M]=0, \tag{134}$$

and we find

$$u_M(t)=\frac{c_8^{(M)}}{a^2(t)}, \tag{135}$$

where $c_8^{(M)}$ is a constant of integration. In this epoch Eqs. (62) and (63) reduce to

$$\dot\delta_M-\frac{1}{2}\dot h^{(M)}+u_M=0, \tag{136}$$

$$\ddot h^{(M)}+2\frac{\dot a}{a}\dot h^{(M)}=\frac{8\pi}{m_p{}^2}\rho_{bM}\delta_M, \tag{137}$$

where $h^{(M)}$ is the trace of the metric perturbation in the matter epoch. Differentiating Eq. (136) with respect to time, adding this result to Eq. (136) multiplied by $2\dot a/a$, and using Eqs. (135) and (137) we find

$$\ddot\delta_M+2\frac{\dot a}{a}\dot\delta_M-\frac{4\pi}{m_p{}^2}\rho_{bM}\delta_M=0. \tag{138}$$

Introducing the variable x, Sec. 11.C of Ref. [59],

$$x=a/h_M{}^2, \tag{139}$$

we find that Eq. (138) becomes

$$2x^2(1-x)\delta_M''+x(3-4x)\delta_M'-3\delta_M=0, \tag{140}$$

where a prime denotes a derivative with respect to x. The solution of this equation is, [59],

$$\delta_M(x)=c_2^{(M)}\frac{\sqrt{1-x}}{x^{3/2}}+c^{(M)}\left[-1+\frac{3}{x}-\frac{3\sqrt{1-x}}{x^{3/2}}\tan^{-1}\sqrt{\frac{x}{1-x}}\right], \tag{141}$$

where $c_2^{(M)}$ and $c^{(M)}$ are spatial momentum dependent constants of integration. In terms of the variable x Eq. (136) is

$$h^{(M)\prime}=2\delta_M'+2h_M{}^2\sqrt{\frac{x}{1-x}}u_M. \tag{142}$$

The solution of this equation is

$$h^{(M)}(x)=c_1^{(M)}+2\delta_M(x)-\frac{4c_8^{(M)}}{h_M{}^2}\sqrt{\frac{1-x}{x}}, \tag{143}$$

where $c_1^{(M)}$ is a spatial momentum dependent constant of integration and $\delta_M(x)$ is given in Eq. (141). It is straightforward to verify that the solutions of Eqs. (141) and (143) satisfy Eq. (137). Using Eq. (135), Eq. (65) reduces to:

$$\mathcal{H}^{(M)\prime}=-\frac{A(A+2)}{(A-1)(A+3)}h^{(M)\prime}+\frac{9}{(A-1)(A+3)h_M{}^2}\frac{c_8^{(M)}}{x^{5/2}\sqrt{1-x}}. \tag{144}$$

The solution of this equation is

$$\mathcal{H}^{(M)}(x)=c_9^{(M)}-\frac{A(A+2)}{(A-1)(A+3)}h^{(M)}(x)-\frac{6c_8^{(M)}}{(A-1)(A+3)h_M{}^2}\frac{\sqrt{1-x}(2x+1)}{x^{3/2}}, \tag{145}$$

where $c_9^{(M)}$ is a spatial momentum dependent constant of integration. In the matter epoch Eqs. (63), (66), and (67) may be combined to give

$$\frac{\dot a}{a}\dot h^{(M)}+\frac{(A-1)(A+3)}{3a^2}(h^{(M)}+\mathcal{H}^{(M)})+\frac{8\pi}{m_p{}^2}\rho_{bM}\delta_M=0. \tag{146}$$

Using Eqs. (141), (143), and (145), we find that this equation results in

$$c_9^{(M)}=\frac{3}{(A-1)(A+3)}[c_1^{(M)}-2c^{(M)}]. \tag{147}$$

It may be verified that this result with Eqs. (143) and (145) satisfies Eq. (67).

The matter epoch gauge-invariant variables, Δ_M and A_M, Eqs. (87) and (88), are

$$\Delta_M(x)=\left\{c_2^{(M)}+\frac{3c_8^{(M)}}{A(A+2)h_M{}^2}\right\}\frac{\sqrt{1-x}}{x^{3/2}}+c^{(M)}\left[-1+\frac{3}{x}-\frac{3\sqrt{1-x}}{x^{3/2}}\tan^{-1}\sqrt{\frac{x}{1-x}}\right], \tag{148}$$

$$A_M(x) = \frac{A(A+2)}{(A-1)(A+3)}\left\{c_2^{(M)} + \frac{3c_8^{(M)}}{A(A+2){h_M}^2}\right\}\frac{\sqrt{1-x}}{x^{3/2}} - c^{(M)}\left[1 - \frac{3A(A+2)}{(A-1)(A+3)}\left\{\frac{1}{x} - \frac{\sqrt{1-x}}{x^{3/2}}\tan^{-1}\sqrt{\frac{x}{1-x}}\right\}\right]. \quad (149)$$

VI. JOINING CONDITIONS AND EXPRESSIONS FOR THE INTEGRATION CONSTANTS

In the previous section we have derived expressions for the spatially homogeneous and spatially inhomogeneous fields in the radiation and matter epochs. These solutions depend on constants of integration, and in this section we list the equations that determine these constants of integration and compute them. We then approximate these expressions for the constants of integration by discarding the contribution from perturbations that were inside the Hubble radius at the reheating and radiation-matter transitions (since we have ignored physical processes that are relevant on these small length scales).

As in the models of Refs. [38,62,64], the constants of integration, in the radiation and matter epochs in the model at hand, are determined by joining conditions at the inflation-radiation (or reheating) transition and the radiation-matter transition. We make use of the spatially homogeneous local energy density spatial hypersurface transition model (discussed in Refs. [38,55,62]), generalized to the closed FLRW model, to derive the needed joining conditions. The resulting joining conditions are identical to those in Sec. VI A of Ref. [38], as they must be.

A. Joining conditions

In linear theory, the scalar field is identical to a spacetime-dependent speed of sound fluid (Sec. III C), so we treat both the reheating and radiation-matter transitions as special cases of an equation of state transition between two spacetime-dependent speed of sound fluid epochs. In the transition model we consider, it occurs instantaneously when the local energy density drops to a critical value (at different values of synchronous gauge time (t) in different parts of space). At the transition spatial hypersurface we require that the equation of state and speed of sound change discontinuously from the value appropriate to the pretransition fluid to that appropriate to the posttransition fluid. We consider a transition at $t = t_{MR}$ from an R fluid characterized by the variables ρ_{bR}, p_{bR}, ${c_{sR}}^2$, to an M fluid characterized by the variables ρ_{bM}, p_{bM}, ${c_{sM}}^2$, with a jump in the pressure at the transition.

Since spatial gradients in the local energy density are of first order in the perturbations, the spatially homogeneous local energy density spatial hypersurfaces and the synchronous gauge constant time hypersurfaces coincide at lowest order. We may therefore match the scale factor and the spatially homogeneous part of the energy density at the corresponding synchronous gauge constant time spatial hypersurface,

$$a_M(t_{MR}) = a_R(t_{MR}), \quad (150)$$

$$\rho_{bM}(t_{MR}) = \rho_{bR}(t_{MR}). \quad (151)$$

Joining conditions for the inhomogeneities are derived in Sec. VI A of Ref. [38]. For our purposes here we only need

$$\Delta_M(t_{MR}) = \Delta_R(t_{MR}), \quad (152)$$

$$\left(\frac{A_M}{\rho_{bM} + p_{bM}}\right)(t_{MR}) = \left(\frac{A_R}{\rho_{bR} + p_{bR}}\right)(t_{MR}). \quad (153)$$

B. Determining the constants of integration

Using the joining conditions for the scale factor and the background energy density, Eqs. (150) and (151), at the two transitions, we have, from Eqs. (68), (96), (126), and (133), to leading order in ϵ,

$$h = h_R/{a_{R\Phi}}^2, \quad (154)$$

$$h_R = h_M\sqrt{a_{MR}}, \quad (155)$$

where $a_{R\Phi}$ and a_{MR} are the values of the scale factor at the reheating and radiation-matter transitions and h, h_R, and h_M are the constants in Eqs. (96), (126), and (133). We note that at the reheating transition the radiation epoch variable x_R, Eq. (127), is given by

$$x_R(t_{R\Phi}) = \frac{a_{R\Phi}}{h_R} = \frac{1}{ha_{R\Phi}}, \quad (156)$$

while at the radiation-matter transition the matter epoch variable x_M, Eq. (139), is

$$x_M(t_{MR}) = \frac{a_{MR}}{{h_M}^2} = \frac{{a_{MR}}^2}{{h_R}^2} = {x_R}^2(t_{MR}). \quad (157)$$

Using the joining conditions of Eqs. (152) and (153) at the reheating transition, we find, from Eqs. (115), (116), (129), and (132), that to leading order in ϵ the radiation epoch constants of integration are given by

$$c_1^{(R)} = \frac{4i}{9\epsilon}\left(\frac{16\pi}{{m_p}^2}\right)^{1/2}\frac{(A-1)(A+3)}{\sqrt{2A(A+1)(A+2)}}\frac{CDE}{{h_R}^3{x_R}^6(t_{R\Phi})\{1-{x_R}^2(t_{R\Phi})\}}$$
$$\times F(-1/2+b,-1/2-b;-1/2;{x_R}^2(t_{R\Phi}))\exp\left\{-i(A+1)\tan^{-1}\sqrt{\frac{1-{x_R}^2(t_{R\Phi})}{{x_R}^2(t_{R\Phi})}}\right\}, \quad (158)$$

$$c_2^{(R)} = -\frac{4i}{9\epsilon}\left(\frac{16\pi}{{m_p}^2}\right)^{1/2}\frac{(A-1)(A+3)}{\sqrt{2A(A+1)(A+2)}}\frac{CDE}{{h_R}^3{x_R}^3(t_{R\Phi})\{1-{x_R}^2(t_{R\Phi})\}}$$
$$\times F(1+b,1-b;5/2;{x_R}^2(t_{R\Phi}))\exp\left\{-i(A+1)\tan^{-1}\sqrt{\frac{1-{x_R}^2(t_{R\Phi})}{{x_R}^2(t_{R\Phi})}}\right\}, \quad (159)$$

where

$$C^{-1} = (4b^2-1){x_R}^2(t_{R\Phi})F(1+b,1-b;5/2;{x_R}^2(t_{R\Phi}))F(1/2+b,1/2-b;1/2;{x_R}^2(t_{R\Phi}))$$
$$-3F(1+b,1-b;5/2;{x_R}^2(t_{R\Phi}))F(-1/2+b,-1/2-b;-1/2;{x_R}^2(t_{R\Phi}))$$
$$+\frac{4}{5}(b^2-1){x_R}^2(t_{R\Phi})F(-1/2+b,-1/2-b;-1/2;{x_R}^2(t_{R\Phi}))F(2+b,2-b;7/2;{x_R}^2(t_{R\Phi})), \quad (160)$$

$$D^{-1/2} = \bar{c}_1{x_R}^3(t_{R\Phi}) + 2h\sqrt{1-{x_R}^2(t_{R\Phi})}\{1+2{x_R}^2(t_{R\Phi})\}, \quad (161)$$

and

$$E = A(A+2)\bar{c}_1h^{-1}{x_R}^5(t_{R\Phi}) + 2A(A+2){x_R}^2(t_{R\Phi})\sqrt{1-{x_R}^2(t_{R\Phi})}\{1+2{x_R}^2(t_{R\Phi})\}$$
$$+6\{\sqrt{1-{x_R}^2(t_{R\Phi})} - i(A+1)x_R(t_{R\Phi})\}, \quad (162)$$

where b is defined in Eq. (130) and $\bar{c}_1$ in Eq. (109).

Using the joining conditions of Eqs. (152) and (153) at the radiation-matter transition, we find, from Eqs. (129), (132), (148), and (149), that the matter epoch constants of integration $c^{(M)}$ and

$$\hat{c}^{(M)} \equiv c_2^{(M)} + \frac{3c_8^{(M)}}{A(A+2){h_M}^2}, \quad (163)$$

are given by

$$c^{(M)} = c_1^{(R)}\left[-\frac{3}{5}(b^2-1)x_M(t_{MR})\{1-x_M(t_{MR})\}F(2+b,2-b;7/2;x_M(t_{MR}))\right.$$
$$\left.+\frac{1}{12}\{27-(A^2+2A+18)x_M(t_{MR})\}F(1+b,1-b;5/2;x_M(t_{MR}))\right]$$
$$+c_2^{(R)}\left[\frac{3}{4}(4b^2-1){x_M}^{-1/2}(t_{MR})\{1-x_M(t_{MR})\}F(1/2+b,1/2-b;1/2;x_M(t_{MR}))\right.$$
$$\left.-\frac{1}{12}(A^2+2A-9){x_M}^{-1/2}(t_{MR})F(-1/2+b,-1/2-b;-1/2;x_M(t_{MR}))\right], \quad (164)$$

$$\hat{c}^{(M)}\frac{4\sqrt{1-x_M(t_{MR})}}{3{x_M}^{3/2}(t_{MR})} = c_1^{(R)}\left[\frac{4}{5}(b^2-1)\{1-x_M(t_{MR})\}\left\{3-x_M(t_{MR})-3\sqrt{\frac{1-x_M(t_{MR})}{x_M(t_{MR})}}\tan^{-1}\sqrt{\frac{x_M(t_{MR})}{1-x_M(t_{MR})}}\right\}\right.$$
$$\times F(2+b,2-b;7/2;x_M(t_{MR})) - \left\{9/x_M(t_{MR})-(A^2+2A+27)+(A^2+2A+6)x_M(t_{MR})/9\right.$$
$$\left.-\{9/x_M(t_{MR})-(A^2+2A+18)/3\}\sqrt{\frac{1-x_M(t_{MR})}{x_M(t_{MR})}}\tan^{-1}\sqrt{\frac{x_M(t_{MR})}{1-x_M(t_{MR})}}\right\}$$
$$\left.\times F(1+b,1-b;5/2;x_M(t_{MR}))\right]$$
$$+c_2^{(R)}\left[(4b^2-1){x_M}^{-3/2}(t_{MR})\{1-x_M(t_{MR})\}\left\{-3+x_M(t_{MR})\right.\right.$$
$$\left.+3\sqrt{\frac{1-x_M(t_{MR})}{x_M(t_{MR})}}\tan^{-1}\sqrt{\frac{x_M(t_{MR})}{1-x_M(t_{MR})}}\right\}F(1/2+b,1/2-b;1/2;x_M(t_{MR}))$$
$$+\frac{1}{9{x_M}^{3/2}(t_{MR})}\left\{3(A^2+2A-9)-(A^2+2A-21)x_M(t_{MR})\right.$$
$$\left.\left.-3(A^2+2A-9)\sqrt{\frac{1-x_M(t_{MR})}{x_M(t_{MR})}}\tan^{-1}\sqrt{\frac{x_M(t_{MR})}{1-x_M(t_{MR})}}\right\}F(-1/2+b,-1/2-b;-1/2;x_M(t_{MR}))\right]. \tag{165}$$

C. Large-scale approximation

We have ignored small-scale processes like the production of entropy at reheating. Our expressions are therefore only relevant for large-scale perturbations. From Eq. (8) we see that the ratio of the Hubble length to a length scale which characterizes the perturbations is $\sqrt{A(A+2)}/(aH)$; small-scale perturbations are those for which this ratio is $\gg 1$. In this subsection we approximate the expressions for the constants of integration by discarding the contribution from small-scale perturbations at the reheating and radiation-matter transitions.

At reheating we have, from Eq. (127),

$$\frac{\sqrt{A(A+2)}}{a(t_{R\Phi})H(t_{R\Phi})} = x_R(t_{R\Phi})\sqrt{\frac{A(A+2)}{1-{x_R}^2(t_{R\Phi})}}, \tag{166}$$

so large-scale perturbations at reheating correspond to small $x_R(t_{R\Phi})$. Expanding Eqs. (158) and (159) in this limit we find for the radiation epoch constants of integration

$$c_1^{(R)} = -\frac{2i}{9\epsilon}\left(\frac{16\pi}{{m_p}^2}\right)^{1/2}\frac{(A-1)(A+3)}{\sqrt{2A(A+1)(A+2)}}\frac{e^{-i(A+1)\pi/2}}{h^2{h_R}^3{x_R}^6(t_{R\Phi})}$$
$$\times\left[1+(A-1)(A+3){x_R}^2(t_{R\Phi})+\left\{\frac{2i}{3}A(A+1)(A+2)-\frac{\bar{c}_1}{h}\right\}{x_R}^3(t_{R\Phi})+\cdots\right], \tag{167}$$

$$c_2^{(R)} = \frac{2i}{9\epsilon}\left(\frac{16\pi}{{m_p}^2}\right)^{1/2}\frac{(A-1)(A+3)}{\sqrt{2A(A+1)(A+2)}}\frac{e^{-i(A+1)\pi/2}}{h^2{h_R}^3{x_R}^3(t_{R\Phi})}$$
$$\times\left[1+\left\{\frac{4}{5}A(A+2)-\frac{21}{10}\right\}{x_R}^2(t_{R\Phi})+\left\{\frac{2i}{3}A(A+1)(A+2)-\frac{\bar{c}_1}{h}\right\}{x_R}^3(t_{R\Phi})+\cdots\right]; \tag{168}$$

we note that the $\bar{c}_1$ dependent contribution to these expressions are a subleading term.

At the radiation-matter transition the relevant ratio of length scales is, from Eq. (139),

$$\frac{\sqrt{A(A+2)}}{a(t_{MR})H(t_{MR})}=\sqrt{\frac{x_M(t_{MR})A(A+2)}{1-x_M(t_{MR})}}, \quad (169)$$

so large-scale perturbations at this transition correspond to small $x_M(t_{MR})$. Expanding Eqs. (164) and (165), and using Eqs. (167) and (168) as well as the relation $a(t_{MR})\gg a(t_{R\Phi})$, we find for the matter epoch constants of integration

$$c^{(M)}=-\frac{i}{2\epsilon}\left(\frac{16\pi}{{m_p}^2}\right)^{1/2}\frac{(A-1)(A+3)}{\sqrt{2A(A+1)(A+2)}} \times\frac{e^{-i(A+1)\pi/2}}{h^2{h_R}^3{x_R}^6(t_{R\Phi})}+\cdots, \quad (170)$$

$$\hat{c}^{(M)}=\frac{2}{45}{x_M}^{5/2}(t_{MR})c^{(M)}+\cdots. \quad (171)$$

VII. MATTER EPOCH "NEWTONIAN" SPATIAL HYPERSURFACE AND POWER SPECTRA

Often, theoretical expressions characterizing large-scale structure (for instance, the fractional mass perturbation and the peculiar velocity perturbation power spectra) are given in the coordinate system in which the time derivative of the trace of the metric perturbation has been removed on a given "observational" hypersurface; this is what is known as the instantaneously Newtonian synchronous coordinate system, Sec. V of Ref. [62]. In this section we construct this instantaneously Newtonian coordinate system (this is a generalization to the closed model of the flat model construction of Sec. V D of Ref. [62] so we can be brief; also see Sec. VII A of Ref. [38]), and record the power spectra of fractional energy density and peculiar velocity perturbations in this coordinate system. In this section we also record the matter epoch gauge-invariant fractional energy density power spectrum.

A. Instantaneously "Newtonian" synchronous coordinates

The following derivation is a generalization of that of Sec. V D of Ref. [62] so we will omit technical details here. We choose coordinates $\hat{x}^\mu=(\hat{t},\hat{x}^i)$,

$$\hat{t}=t-\Delta t(t_N,\vec{x}), \quad (172)$$

$$\hat{x}^i=x^i-f^i(t,\vec{x}), \quad (173)$$

which are synchronous, and require that the time derivative of the trace of the metric perturbation, $\hat{\partial}_0\hat{h}(\hat{x})$, vanish on a spatial hypersurface at the observational time $\hat{t}=\hat{t}_N$. For the coordinates $\hat{x}^\mu$ to be synchronous we must require

$$f^i(t,\vec{x})=H^{ij}(\vec{x})\partial_j\Delta t(t_N,\vec{x})\int^t\frac{dt'}{a^2(t')}+\omega^i(\vec{x}); \quad (174)$$

in what follows we set $\omega^i=0$. The fields in the two coordinate systems are related by

$$\hat{\delta}(\hat{x})=\delta(x)+\frac{\dot{\rho}_b(t)}{\rho_b(t)}\Delta t(t_N,\vec{x}), \quad (175)$$

$$\hat{u}^i(\hat{x})=u^i(x)-\frac{1}{a^2(t)}H^{ij}(\vec{x})\partial_j\Delta t(t_N,\vec{x}), \quad (176)$$

$$\hat{h}_{ij}(\hat{x})=h_{ij}(x)-2\frac{\dot{a}}{a}\Delta t(t_N,\vec{x})H_{ij}(\vec{x}) -H_{ik}(\vec{x}){f^k}_{|j}(x)-H_{kj}(\vec{x}){f^k}_{|i}(x), \quad (177)$$

and from the last equation, and the matter epoch equations (53) and (54), we have

$$\hat{\partial}_0\hat{h}(\hat{x})=\dot{h}(x)+\left[\frac{24\pi}{{m_p}^2}\rho_b(t)-\frac{6}{a^2(t)}\right]\Delta t(t_N,\vec{x}) -\frac{2}{a^2}H^{ij}(\vec{x})\Delta t_{|i|j}(t_N,\vec{x}). \quad (178)$$

Using the matter epoch ($\nu=0={c_s}^2$) fluid equations of motion in the unbarred coordinates, Sec. III B, it is straightforwardly established that when $\hat{\partial}_0\hat{h}(\hat{t}_N,\hat{x}^k)=0$,

$$\hat{\partial}_0\hat{\delta}(\hat{t}_N,\hat{x}^k)+{\hat{u}^i}_{|i}(\hat{t}_N,\hat{x}^k)=0, \quad (179)$$

$$\hat{\partial}_0^2\hat{\delta}(\hat{x})+2\hat{H}(\hat{t})\hat{\partial}_0\hat{\delta}(\hat{x})=\frac{4\pi}{{m_p}^2}\hat{\rho}_b(\hat{t})\hat{\delta}(\hat{x}); \quad (180)$$

these are the Newtonian matter epoch equations of motion, Secs. IX B and X of Ref. [59]. Comparing the second one of these to the matter epoch version of Eq. (9.19) of Ref. [59], we find that the Newtonian gravitational potential in these coordinates, $\hat{\varphi}$, obeys

$$\frac{\hat{\nabla}^2\hat{\varphi}}{\hat{a}^2}=\frac{4\pi}{{m_p}^2}\hat{\rho}_b\hat{\delta}. \quad (181)$$

In spatial momentum space, the scalar parts of the above equations are

$$\hat{\delta}(A,B,C,\hat{t})=\delta(A,B,C,t)+\frac{\dot{\rho}_b(t)}{\rho_b(t)}\Delta t(A,B,C,t_N), \quad (182)$$

$$\hat{v}(A,B,C,\hat{t})=v(A,B,C,t)+\frac{A(A+2)}{a(t)}\Delta t(A,B,C,t_N) \quad (183)$$

(where $v=au$), and

$$\hat{\partial}_0\hat{h}(A,B,C,\hat{t}) = \dot{h}(A,B,C,t) + \left[\frac{24\pi}{{m_p}^2}\rho_b(t) + 2\frac{(A-1)(A+3)}{a^2(t)}\right]\Delta t(A,B,C,t_N). \tag{184}$$

Defining the Newtonian hypersurface by requiring

$$\hat{\partial}_0\hat{h}(A,B,C,\hat{t}_N) = 0, \tag{185}$$

we find, in the matter epoch, from Eq. (184),

$$\Delta t(A,B,C,t_N) = -\dot{h}^{(M)}(A,B,C,t_N)\left[\frac{24\pi}{{m_p}^2}\rho_{bM}(t_N) + \frac{2(A-1)(A+3)}{a^2(t_N)}\right]^{-1}. \tag{186}$$

Using the matter epoch solutions of Sec. V B we find

$$\Delta t(A,B,C,t_N) = {h_M}^2[9+2(A-1)(A+3)x_M(t_N)]^{-1}\left[c_2^{(M)}\{3-2x_M(t_N)\} - 2\frac{c_8^{(M)}}{{h_M}^2}x_M(t_N) + c^{(M)}\left\{9\sqrt{x_M(t_N)\{1-x_M(t_N)\}} - \{9-6x_M(t_N)\}\tan^{-1}\sqrt{\frac{x_M(t_N)}{1-x_M(t_N)}}\right\}\right], \tag{187}$$

where the variables and coefficients are defined in Sec. V B, and we have, for the Newtonian hypersurface fractional energy density and peculiar velocity,

$$\hat{\delta}_M(A,B,C,\hat{t}_N) = [9+2(A-1)(A+3)x_M(t_N)]^{-1}\left[2A(A+2)\hat{c}^{(M)}\sqrt{\frac{1-x_M(t_N)}{x_M(t_N)}} + c^{(M)}\left\{6A(A+2) - 2(A-1)(A+3)x_M(t_N) - 6A(A+2)\sqrt{\frac{1-x_M(t_N)}{x_M(t_N)}}\tan^{-1}\sqrt{\frac{x_M(t_N)}{1-x_M(t_N)}}\right\}\right], \tag{188}$$

$$\hat{v}_M(A,B,C,\hat{t}_N) = A(A+2)[9+2(A-1)(A+3)x_M(t_N)]^{-1}\left[\hat{c}^{(M)}\left(\frac{3-2x_M(t_N)}{x_M(t_N)}\right) + c^{(M)}\left\{9\sqrt{\frac{1-x_M(t_N)}{x_M(t_N)}} - \left(\frac{9-6x_M(t_N)}{x_M(t_N)}\right)\tan^{-1}\sqrt{\frac{x_M(t_N)}{1-x_M(t_N)}}\right\}\right], \tag{189}$$

where $\hat{c}^{(M)}$ is defined in Eq. (163) and the other expressions are defined in Sec. V B.

B. Power spectra

From Eqs. (53), (133), and (139) we find, in the matter epoch,

$$x_M(t) = \frac{\Omega_0 - 1}{\Omega_0(1+z)}. \tag{190}$$

The matter fractional energy density perturbation and peculiar velocity perturbation equal-time two-point correlation functions are

$$\langle\hat{\delta}_M(A,B,C,\hat{t}_N)\hat{\delta}^*_M(A',B',C',\hat{t}_N)\rangle = \hat{P}(A,\hat{t}_N)\delta_{A,A'}\delta_{B,B'}\delta_{C,C'}, \tag{191}$$

$$\langle\hat{v}_M(A,B,C,\hat{t}_N)\hat{v}^*_M(A',B',C',\hat{t}_N)\rangle = \hat{P}_v(A,\hat{t}_N)\delta_{A,A'}\delta_{B,B'}\delta_{C,C'}, \tag{192}$$

where, from the results of the previous subsection, the Newtonian hypersurface spectra are

$$\hat{P}(A,\hat{t}_N) = {W_5}^2\left(\frac{W_1}{c_1}\right)^2\frac{(A-1)^2(A+3)^2}{A(A+1)(A+2)}\left[\frac{A(A+2)+e_1}{A(A+2)+d_1}\right]^2, \tag{193}$$

$$\hat{P}_v(A,\hat{t}_N) = {W_5}^2\left(\frac{W_3}{c_1}\right)^2\frac{(A-1)^2A(A+2)(A+3)^2}{(A+1)[A(A+2)+d_1]^2}, \tag{194}$$

where we have defined

$$W_1 = \frac{4}{45}\sqrt{\frac{1+\Omega_0 z_N}{\Omega_0-1}}\left[\frac{\Omega_0-1}{\Omega_0(1+z_{MR})}\right]^{5/2} + 6 - 2\left[\frac{\Omega_0-1}{\Omega_0(1+z_N)}\right] - 6\sqrt{\frac{1+\Omega_0 z_N}{\Omega_0-1}}\tan^{-1}\sqrt{\frac{\Omega_0-1}{1+\Omega_0 z_N}}, \tag{195}$$

$$W_2 = 6\left[\frac{\Omega_0-1}{\Omega_0(1+z_N)}\right], \tag{196}$$

$$W_3 = \frac{2}{45}\frac{(2+\Omega_0+3\Omega_0 z_N)}{\Omega_0-1}\left[\frac{\Omega_0-1}{\Omega_0(1+z_{MR})}\right]^{5/2} + 9\sqrt{\frac{1+\Omega_0 z_N}{\Omega_0-1}} - 3\left[\frac{2+\Omega_0+3\Omega_0 z_N}{\Omega_0-1}\right]\tan^{-1}\sqrt{\frac{\Omega_0-1}{1+\Omega_0 z_N}}, \tag{197}$$

$$W_5 = \frac{1}{2\epsilon}\left(\frac{16\pi}{{m_p}^2}\right)^{1/2}\frac{(1+z_{R\Phi})^2}{a_0}\sqrt{\frac{\Omega_0}{2(\Omega_0-1)(1+z_{MR})}}, \tag{198}$$

$$e_1 = \frac{W_2}{W_1} = \frac{3c_1}{W_1}, \tag{199}$$

$$c_1 = \frac{2(\Omega_0-1)}{\Omega_0(1+z_N)}, \tag{200}$$

$$c_2 = 9 - 3c_1, \tag{201}$$

$$d_1 = \frac{c_2}{c_1} = \frac{(6+3\Omega_0+9\Omega_0 z_N)}{2(\Omega_0-1)}; \tag{202}$$

here z_N, z_{MR}, and $z_{R\Phi}$ are the redshifts of the Newtonian hypersurface, the radiation-matter transition, and the reheating transition. The terms dependent on z_{MR} in the expressions for W_1 and W_3 are the contributions of the decaying solution.

We note that the matter epoch power spectrum for the gauge-invariant variable Δ_M, Eq. (148), is

$$P_\Delta(A,t) = {W_5}^2\left(\frac{W_1}{c_1}\right)^2\frac{(A-1)^2(A+3)^2}{A(A+1)(A+2)}, \tag{203}$$

with z_N in the definitions of W_1 and c_1 above replaced by z. This is the generalization of the flat-space scale-invariant spectrum [40] to the closed model [47]. In the small-scale limit, which is the same as the flat-space limit, A is large and becomes the usual flat-space Fourier wave number k and this power spectrum reduces to $P_\Delta \propto k$, the standard scale-invariant expression [40]. The full closed-space power spectrum above is plotted in Fig. 1 of Ref. [53], where it is compared to an almost scale-invariant flat-space power spectrum.

VIII. CONCLUSION

Using Hawking's prescription for the quantum state of the universe as the initial conditions, we have shown that in a closed, inflating universe model the late-time power spectrum of gauge-invariant energy density inhomogeneities is not a power law. This power spectrum depends on wave number in the way expected for a generalization to the closed model of the standard flat-space scale-invariant power spectrum [47]. The power spectrum we derive appears to differ from a number of other closed inflation models power spectra derived assuming different (presumably non–de Sitter invariant) initial conditions.

Recent suggestions that dynamical dark energy might provide a better fit to the observations requires consideration of nonflat cosmological models. It is not yet clear if nonflat ΛCDM, without dynamical dark energy, is able to accommodate these data. Also, even if the universe is flat, to properly establish spatial flatness from the CMB anisotropy data requires use of a physically consistent nonflat cosmological model, such as that developed here for the positive curvature case. The power spectrum we have derived in this model will also be needed for a proper analysis of CMB anisotropy data in a mildly closed model, which not only remains observationally viable but might be in better accord with the low ℓ CMB anisotropy observations [53,54].

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