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Second-order cosmological perturbations. II. Produced by scalar-tensor and tensor-tensor couplings

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We study the second-order perturbations in the Einstein-de Sitter Universe in synchronous coordinates. We solve the second-order perturbed Einstein equation with scalar-tensor and tensor-tensor couplings between 1st-order perturbations, and obtain, for each coupling, the solutions of scalar, vector, and tensor metric perturbations, including both the growing and decaying modes for general initial conditions. We perform general synchronous-to-synchronous gauge transformations up to 2nd order, which are generated by a 1st-order vector field and a 2nd-order vector field, and obtain all the residual gauge modes of the 2nd-order metric perturbations in synchronous coordinates. We show that only the 2nd-order vector field is effective for the 2nd-order transformations that we consider because the 1st-order vector field was already fixed in obtaining the 1st-order perturbations. In particular, the 2nd-order tensor is invariant under 2nd-order gauge transformations using $\xi^{(2)\mu}$ only, just like the 1st-order tensor is invariant under 1st-order transformations.

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I. INTRODUCTION

Metric perturbations of Lemaitre-Robertson-Walker spacetimes within general relativity are the theoretical foundation of cosmology. In the past, the linear perturbations of scalar type [1–8] have been used in the calculation of cosmic microwave background (CMB) and well tested in the measurements of CMB anisotropies and polarization [9,10]. As predicted by generic inflation models, besides the scalar metric perturbation, the tensor perturbation is also generated during the inflationary stage [11–21]. However, the magnetic polarization C_l^{BB} induced by the tensor perturbations [22–28] has not been detected by the current CMB observations, and only some constraint in terms of the tensor-scalar ratio of metric perturbations is given as $r < 0.1$ over very low frequencies $10^{-18} \sim 10^{-16}$ Hz [9,10]. This constraint on the ratio has been inferred from CMB anisotropies formed at a redshift at $z \sim 1100$ in the matter era, which is in a rather late stage of the expanding Universe. Furthermore, it has been also based on the formulations of linear metric perturbations. On the other hand, recently LIGO collaboration announced its direct detections of gravitational waves emitted from binary black holes [29], but did not detect relic gravitational waves (RGW), only gave constraints on the spectral energy density of RGW in a band 10–2000 Hz [30], less stringent than that from the CMB measurements. By estimations [31], it is still possible for the current LIGO to detect RGW around frequencies $\sim 10^2$ Hz if the running spectral index of the primordial RGW is large. In regard to these observational constraints from CMB measurements and LIGO, one would like to explore other possibilities that might affect the tensor cosmological perturbation significantly during the course of cosmic expansion.

To the linear level, the wave equation of RGW depends upon the scale factor $a(\tau)$ only, and is homogeneous because the anisotropic stress as its source is negligibly small except for neutrino free-streaming during radiation era [32,33]. Thus, the other thing that will affect RGW is the nonlinear couplings of metric perturbations themselves. To explore their impacts upon RGW, one needs to study the cosmological perturbations up to 2nd order, to see how nonlinear gravity changes the tensor perturbation. As is known, in perturbation formulations, there are three types of metric perturbations: scalar, vector, and tensor. The 2nd-order Einstein equation contains the couplings of 1st-order metric perturbations serving as a part of the source for the 2nd-order perturbations. For the Einstein-de Sitter mode filled with irrotational dust, the 1st-order vector metric perturbation can be set to zero as it is a residual gauge mode. As a result, the couplings of 1st-order metric perturbations consist of scalar-scalar, scalar-tensor, and tensor-tensor. So far, the 2nd-order perturbations have found their applications in detailed calculations of CMB anisotropies and polarization [34,35], in the estimation of the non-Gaussianality of primordial perturbation [36], and in relic gravitational waves [37,38]. In the literature [39–48], the studies of 2nd-order metric perturbations have been mostly on the scalar-scalar coupling, whereas the couplings involving the 1st-order tensor have not been sufficiently investigated, such as the scalar-tensor and the

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tensor-tensor. Ref. [44] derived the equation of 2nd-order density perturbation with the tensor-tensor coupling. In our previous work [49], we have solved the 2nd-order perturbed Einstein equation with the scalar-scalar coupling in the Einstein-de Sitter model, and obtained all the solutions of the 2nd-order scalar, vector, and tensor perturbations, under general initial conditions.

In this paper, we shall extend the study to the cases of scalar-tensor and tensor-tensor couplings. We shall derive the corresponding solutions of 2nd-order scalar, vector, and tensor metric perturbations with general initial conditions. In addition, we shall perform 2nd-order gauge transformations, and identify the residual gauge modes of the 2nd-order metric perturbations in synchronous coordinates.

In Sec. II, we briefly review the necessary results of 1st-order perturbations, which are used in calculations of the 2nd order later.

In Sec. III, We split the 2nd-order perturbed Einstein equations as the set of equations of the energy constraint, momentum constraint, and evolution, each containing the scalar-tensor and tensor-tensor couplings, respectively.

In Sec. IV, we derive the solutions of 2nd-order metric perturbations with the scalar-tensor coupling.

In Sec. V we obtain the solutions of 2nd-order metric perturbations with the tensor-tensor coupling.

In Sec. VI we derive the 2nd-order gauge modes.

We work within the synchronous coordinates, and, for simple comparisons with literature, use notations mostly as in Refs. [47,49]. We use a unit in which the speed of light is $c = 1$.

II. FIRST-ORDER PERTURBATIONS

In this section, we introduce notations and outline the results of 1st-order perturbations, which will be used in later sections. We consider the universe filled with the irrotational, pressureless dust with the energy-momentum tensor $T^{\mu\nu} = \rho U^\mu U^\nu$, where ρ is the mass density, $U^\mu = (a^{-1}, 0, 0, 0)$ is 4-velocity such that $U^\mu U_\mu = -1$. As in paper I [49], we take the perturbations of velocity to be $U^{(1)\mu} = U^{(2)\mu} = 0$. The nonvanishing component is $T^{00} = a^{-2}\rho$ and $T_{00} = a^2\rho$, where ρ is written as

$$\rho = \rho^{(0)}\left(1 + \delta^{(1)} + \frac{1}{2}\delta^{(2)}\right), \tag{1}$$

where $\rho^{(0)}$ is the background density, $\delta^{(1)}$, $\delta^{(2)}$ are the 1st, 2nd-order density contrasts. The spatial flat Robertson-Walker (RW) metric in synchronous coordinates

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)[-d\tau^2 + \gamma_{ij}dx^i dx^j], \tag{2}$$

where τ is conformal time, $a(\tau) \propto \tau^2$ for the Einstein-de Sitter model, γ_{ij} is written as

$$\gamma_{ij} = \delta_{ij} + \gamma^{(1)}_{ij} + \frac{1}{2}\gamma^{(2)}_{ij}, \tag{3}$$

where $\gamma^{(1)}_{ij}$ and $\gamma^{(2)}_{ij}$ are the 1st- and 2nd-order metric perturbations, respectively. From (3), one has $g^{ij} = a^{-2}\gamma^{ij}$ with $\gamma^{ij} = \delta^{ij} - \gamma^{(1)ij} - \frac{1}{2}\gamma^{(2)ij} + \gamma^{(1)ik}\gamma^{(1)j}_k$, where δ^{ij} is used to raise the 3-dim spatial indices of perturbed metric, such as $\gamma^{(1)ik}$ and $\gamma^{(2)ik}$. We use the superscripts or subscripts μ, ν etc. to denote 0, 1, 2, 3, and i, j etc. to denote 1, 2, or 3. The perturbed Einstein equation is

$$G^{(A)}_{\mu\nu} = 8\pi G T^{(A)}_{\mu\nu}, \tag{4}$$

where $A = 1, 2$ denotes the perturbation order, and we shall study up to 2nd order. For each order of (4), the (00) component is the energy constraint, $(0i)$ components are the momentum constraints, and (ij) components contain the evolution equations. The set of (4) are complete to determine the dynamics of gravitational systems, and also imply $T^{(A)\mu\nu}{}_{;\nu} = 0$,; i.e, the conservation of energy and momentum of matter by the structure of general relativity.

The first-order metric perturbation $\gamma^{(1)}_{ij}$ can be written as

$$\gamma^{(1)}_{ij} = -2\phi^{(1)}\delta_{ij} + \chi^{(1)}_{ij}, \tag{5}$$

where $\phi^{(1)}$ is the trace part of scalar perturbation, and $\chi^{(1)}_{ij}$ is traceless and can be further decomposed into a scalar and a tensor

$$\chi^{(1)}_{ij} = D_{ij}\chi^{\parallel(1)} + \chi^{\top(1)}_{ij}, \tag{6}$$

where $\chi^{\parallel(1)}$ is a scalar function, $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$, and $D_{ij}\chi^{\parallel(1)}$ is the traceless part of the scalar perturbation, and $\chi^{\top(1)}_{ij}$ is the tensor part, satisfying the traceless and transverse conditions: $\chi^{\top(1)i}{}_i = 0$, $\partial^i\chi^{\top(1)}_{ij} = 0$. In this paper, we do not consider the 1st-order vector perturbation since the matter is an irrotational dust. However, as shall be seen later, the 2nd-order vector perturbation will appear. Thus, the 2nd-order perturbation is written as

$$\gamma^{(2)}_{ij} = -2\phi^{(2)}\delta_{ij} + \chi^{(2)}_{ij}, \tag{7}$$

with the traceless part

$$\chi^{(2)}_{ij} = D_{ij}\chi^{\parallel(2)} + \chi^{\perp(2)}_{ij} + \chi^{\top(2)}_{ij}, \tag{8}$$

where the vector mode satisfies a condition

$$\partial^i\partial^j\chi^{\perp(2)}_{ij} = 0, \tag{9}$$

which can be written in terms of a curl vector

$$\chi_{ij}^{\perp(2)} = 2A_{(i,j)} \equiv \partial_i A_j + \partial_j A_i, \qquad \partial^i A_i = 0. \tag{10}$$

Since the 3-vector A_i is divergenceless and has only two independent components, the vector metric perturbation $\chi_{ij}^{\perp(2)}$ has two independent polarization modes, correspondingly. We remark that the 2nd-order vector mode $\chi_{ij}^{\perp(2)}$ of the metric perturbation is inevitably produced from the interaction of the 1st-order perturbations even though the matter is irrotational dust.

The 1st-order perturbations are well known, and we have calculated the 1st-order perturbations in detail in our previous work of Ref. [49]. In this paper, we shall list the 1st-order results, and details can be seen in Ref. [49]. The 1st-order density contrast is

$$\delta^{(1)} = \frac{\tau^2}{6}\nabla^2\varphi + \frac{3X}{\tau^3}. \quad \text{with} \quad \nabla^2\varphi \equiv \frac{6}{\tau_0^2}\delta_{0g}^{(1)}, \tag{11}$$

where $\delta_{0g}^{(1)}$ is the initial value of the growing mode at time τ_0, φ is the corresponding gravitational potential. $\delta_0^{(1)} \equiv \frac{\tau_0^2}{6}\nabla^2\varphi + \frac{3X}{\tau_0^3}$ will denote the initial value of $\delta^{(1)}$. And the solutions of two scalar perturbations are

$$\phi^{(1)}(\mathbf{x},\tau) = \frac{5}{3}\varphi(\mathbf{x}) + \frac{\tau^2}{18}\nabla^2\varphi(\mathbf{x}) + \frac{X(\mathbf{x})}{\tau^3}, \tag{12}$$

$$D_{ij}\chi^{\parallel(1)}(\mathbf{x},\tau) = -\frac{\tau^2}{3}\left(\varphi(\mathbf{x})_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\varphi(\mathbf{x})\right) - \frac{6\nabla^{-2}D_{ij}X(\mathbf{x})}{\tau^3}. \tag{13}$$

The 1st-order gravitational wave equation is

$$\chi_{ij}^{\top(1)\prime\prime} + \frac{4}{\tau}\chi_{ij}^{\top(1)\prime} - \nabla^2\chi_{ij}^{\top(1)} = 0. \tag{14}$$

The solution is

$$\chi_{ij}^{\top(1)}(\mathbf{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{s=+,\times} \overset{s}{\epsilon}_{ij}(k)\overset{s}{h}_k(\tau), \qquad \mathbf{k} = k\hat{k}, \tag{15}$$

with two polarization tensors in Eq. (15) satisfying

$$\overset{s}{\epsilon}_{ij}(k)\delta_{ij} = 0, \qquad \overset{s}{\epsilon}_{ij}(k)k_i = 0, \qquad \overset{s}{\epsilon}_{ij}(k)\overset{s'}{\epsilon}_{ij}(k) = 2\delta_{ss'}.$$

During the matter dominant stage the mode is given by

$$\overset{s}{h}_k(\tau) = \frac{1}{a(\tau)}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\tau}{2}}[\overset{s}{d}_1(k)H_{\frac{3}{2}}^{(1)}(k\tau) + \overset{s}{d}_2(k)H_{\frac{3}{2}}^{(2)}(k\tau)], \tag{16}$$

where the coefficients $\overset{s}{d}_1$, $\overset{s}{d}_2$ are determined by the initial condition during inflation and by subsequent evolutions through the reheating, radiation dominant stages [11,21]. Here cosmic processes, such as neutrino free-streaming [32,33], QCD transition, and e^+e^- annihilation [50] only slightly modify the amplitude of RGW and will be neglected in this study. For RGW generated during inflation [11,14–17], the two modes $\overset{s}{h}_k(\tau)$ with $s = +,\times$ are usually assumed to be statistically equivalent, the superscript s can be dropped.

Thus, the 1st-order metric perturbation is given by [47,49]:

$$\gamma_{ij}^{(1)} = -\frac{10}{3}\varphi\delta_{ij} - \frac{\tau^2}{3}\varphi_{,ij} - \frac{6}{\tau^3}\nabla^{-2}X_{,ij} + \chi_{ij}^{\top(1)}, \tag{17}$$

which will appear as the coupling terms in the equations of the second-order perturbation $\gamma_{ij}^{(2)}$.

III. THE SECOND-ORDER CONSTRAINTS AND EVOLUTION EQUATIONS

According to Ref. [49], by using the 2nd-order perturbed Einstein equation $G_{\mu\nu}^{(2)} = 8\pi G T_{\mu\nu}^{(2)}$, and the 2nd-order density contrast

$$\delta^{(2)} = \delta_0^{(2)} - \frac{1}{2}\gamma_i^{(2)i} + \frac{1}{2}\gamma_{0i}^{(2)i} + \frac{1}{4}(\gamma_i^{(1)i})^2 + \frac{1}{4}(\gamma_{0i}^{(1)i})^2 - \frac{1}{2}\gamma_i^{(1)i}\gamma_{0j}^{(1)j} + \frac{1}{2}\gamma^{(1)ij}\gamma_{ij}^{(1)} - \frac{1}{2}\gamma_0^{(1)ij}\gamma_{0ij}^{(1)} - \gamma_i^{(1)i}\delta_0^{(1)} + \gamma_{0i}^{(1)i}\delta_0^{(1)}, \tag{18}$$

following the conservation of energy $T^{0\mu}{}_{;\mu}$ with $\delta_0^{(2)}$, $\gamma_{0ij}^{(1)}$, $\gamma_{0ij}^{(2)}$ being the initial values at τ_0, one has the 2nd-order energy constraint involving the couplings $\varphi\chi_{ij}^{\top(1)}$ and $X\chi_{ij}^{\top(1)}$ as:

$$\frac{2}{\tau}\phi_{s(t)}^{(2)\prime} - \frac{1}{3}\nabla^2\phi_{s(t)}^{(2)} + \frac{6}{\tau^2}\phi_{s(t)}^{(2)} - \frac{1}{12}D^{ij}\chi_{s(t),ij}^{\parallel(2)} = E_{s(t)}, \tag{19}$$

and the momentum constraint:

$$2\phi_{s(t),j}^{(2)\prime} + \frac{1}{2}D_{ij}\chi_{s(t)}^{\parallel(2)\prime,i} + \frac{1}{2}\chi_{s(t)ij}^{\perp(2)\prime,i} = M_{s(t)j}, \tag{20}$$

and the evolution equation:

$$-\left(\phi^{(2)''}_{s(t)}+\frac{4}{\tau}\phi^{(2)'}_{s(t)}\right)\delta_{ij}+\phi^{(2)}_{s(t),ij}+\frac{1}{2}\left(D_{ij}\chi^{\parallel(2)''}_{s(t)}+\frac{4}{\tau}D_{ij}\chi^{\parallel(2)'}_{s(t)}\right)+\frac{1}{2}\left(\chi^{\perp(2)''}_{s(t)ij}+\frac{4}{\tau}\chi^{\perp(2)'}_{s(t)ij}\right)+\frac{1}{2}\left(\chi^{\top(2)''}_{s(t)ij}+\frac{4}{\tau}\chi^{\top(2)'}_{s(t)ij}-\nabla^2\chi^{\top(2)}_{s(t)ij}\right)$$
$$-\frac{1}{4}D_{kl}\chi^{\parallel(2),kl}_{s(t)}\delta_{ij}+\frac{2}{3}\nabla^2\chi^{\parallel(2)}_{s(t),ij}-\frac{1}{2}\nabla^2D_{ij}\chi^{\parallel(2)}_{s(t)}=S_{s(t)ij}, \tag{21}$$

where

$$E_{s(t)}\equiv\frac{5\tau}{18}\chi^{\top(1)'ij}\varphi_{,ij}+\frac{5}{9}\chi^{\top(1)ij}\varphi_{,ij}-\frac{\tau^2}{18}\varphi_{,ij}\nabla^2\chi^{\top(1)ij}-\frac{\tau^2}{36}\chi^{\top(1)ij,k}\varphi_{,ijk}-\frac{2\tau_0^2}{3\tau^2}\varphi^{,ij}\chi^{\top(1)}_{0ij}-\frac{2}{\tau^2}\delta^{(2)}_{s(t)0}+\frac{6}{\tau^2}\phi^{(2)}_{s(t)0}$$
$$+\frac{5}{2\tau^4}\chi^{\top(1)'}_{kl}\nabla^{-2}X^{,kl}-\frac{1}{\tau^3}\nabla^{-2}X^{,kl}\nabla^2\chi^{\top(1)}_{kl}-\frac{1}{2\tau^3}\chi^{\top(1)}_{km,l}\nabla^{-2}X^{,klm}-\frac{12}{\tau_0^3\tau^2}\chi^{\top(1)}_{0kl}\nabla^{-2}X^{,kl}, \tag{22}$$

$$M_{s(t)j}\equiv\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)'}_{kl,j}-\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)'}_{jk,l}+\frac{\tau^2}{6}\varphi^{,kl}_{,j}\chi^{\top(1)'}_{kl}-\frac{\tau^2}{6}\chi^{\top(1)'}_{kj}\nabla^2\varphi^{,k}+\frac{\tau}{3}\varphi^{,kl}\chi^{\top(1)}_{kl,j}+\frac{5}{3}\varphi^{,k}\chi^{\top(1)'}_{kj}$$
$$-\frac{9}{\tau^4}\chi^{\top(1)}_{kl,j}\nabla^{-2}X^{,kl}-\frac{6}{\tau^3}\chi^{\top(1)'}_{jk,l}\nabla^{-2}X^{,kl}+\frac{6}{\tau^3}\chi^{\top(1)'}_{kl,j}\nabla^{-2}X^{,kl}+\frac{3}{\tau^3}\chi^{\top(1)'}_{kl}\nabla^{-2}X^{,kl}_{,j}-\frac{3}{\tau^3}\chi^{\top(1)'}_{kj}X^{,k}, \tag{23}$$

$$S_{s(t)ij}\equiv-\frac{\tau^2}{6}\varphi^{,kl}\chi^{\top(1)''}_{kl}\delta_{ij}-\frac{2\tau}{3}\varphi^{,k}_{,i}\chi^{\top(1)'}_{kj}-\frac{2\tau}{3}\varphi^{,k}_{,j}\chi^{\top(1)'}_{ki}+\frac{\tau}{3}\chi^{\top(1)'}_{ij}\nabla^2\varphi-\frac{\tau}{2}\varphi^{,kl}\chi^{\top(1)'}_{kl}\delta_{ij}+\frac{10}{3}\chi^{\top(1)}_{ij}\nabla^2\varphi+\frac{5}{3}\varphi_{,kl}\chi^{\top(1)kl}\delta_{ij}$$
$$-\frac{10}{3}\varphi^{,k}_{,j}\chi^{\top(1)}_{ki}-\frac{10}{3}\varphi^{,k}_{,i}\chi^{\top(1)}_{kj}+\frac{10}{3}\varphi\nabla^2\chi^{\top(1)}_{ij}+\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{ij,kl}+\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kl,ij}-\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{li,jk}-\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{lj,ik}+5\varphi^{,k}\chi^{\top(1)}_{ij,k}$$
$$-\frac{5}{3}\varphi^{,k}\chi^{\top(1)}_{ki,j}-\frac{5}{3}\varphi^{,k}\chi^{\top(1)}_{kj,i}+\frac{\tau^2}{6}\chi^{\top(1)}_{ij,k}\nabla^2\varphi^{,k}-\frac{\tau^2}{6}\chi^{\top(1)}_{ki,j}\nabla^2\varphi^{,k}-\frac{\tau^2}{6}\chi^{\top(1)}_{kj,i}\nabla^2\varphi^{,k}+\frac{\tau^2}{6}\varphi^{,kl}_{,i}\chi^{\top(1)}_{kl,j}+\frac{\tau^2}{6}\varphi^{,kl}_{,j}\chi^{\top(1)}_{kl,i}$$
$$-\frac{\tau^2}{12}\varphi^{,kml}\chi^{\top(1)}_{km,l}\delta_{ij}-\frac{33}{2\tau^4}\chi^{\top(1)'}_{kl}\nabla^{-2}X^{,kl}\delta_{ij}-\frac{3}{2\tau^3}\chi^{\top(1)}_{kl,m}\nabla^{-2}X^{,klm}\delta_{ij}-\frac{3}{\tau^3}\chi^{\top(1)''}_{kl}\nabla^{-2}X^{,kl}\delta_{ij}-\frac{9}{\tau^4}X\chi^{\top(1)'}_{ij}$$
$$+\frac{18}{\tau^4}\chi^{\top(1)'}_{ki}\nabla^{-2}X^{,k}_{,j}+\frac{18}{\tau^4}\chi^{\top(1)'}_{kj}\nabla^{-2}X^{,k}_{,i}-\frac{3}{\tau^3}X^{,k}\chi^{\top(1)}_{kj,i}-\frac{3}{\tau^3}X^{,k}\chi^{\top(1)}_{ki,j}+\frac{3}{\tau^3}X^{,k}\chi^{\top(1)}_{ij,k}-\frac{6}{\tau^3}\chi^{\top(1)}_{lj,ik}\nabla^{-2}X^{,kl}-\frac{6}{\tau^3}\chi^{\top(1)}_{li,jk}\nabla^{-2}X^{,kl}$$
$$+\frac{6}{\tau^3}\chi^{\top(1)}_{ij,kl}\nabla^{-2}X^{,kl}+\frac{6}{\tau^3}\chi^{\top(1)}_{kl,ij}\nabla^{-2}X^{,kl}+\frac{3}{\tau^3}\chi^{\top(1)}_{kl,i}\nabla^{-2}X^{,kl}_{,j}+\frac{3}{\tau^3}\chi^{\top(1)}_{kl,j}\nabla^{-2}X^{,kl}_{,i}. \tag{24}$$

The subscript "$s(t)$" denotes those contributed by the scalar-tensor coupling. It is seen that $E_{s(t)}$ contains the initial values $\delta^{(2)}_{s(t)0}$, $\phi^{(2)}_{s(t)0}$, $\chi^{\top(1)}_{s(t)0ij}$ etc. at τ_0. Also we notice that neither the tensor $\chi^{\top(2)}_{s(t)ij}$ nor the vector $\chi^{\perp(2)}_{s(t)ij}$ appears in the energy constraint (19). We also observe that $M_{s(t)j}$ on the rhs of (20) has a nonvanishing curl, $\epsilon^{ikj}\partial_k M_{s(t)j}\neq 0$, and, to balance that, a vector perturbation $\chi^{\perp(2)}_{s(t)ij}$ must be introduced on the lhs of the equation. Note that, $S_{s(t)ij}$ plays a role of source of evolution, and the 2nd-order scalar, vector, and tensor perturbations, all appear in the evolution equation (21).

Similarly, by using the subscript "T" to denote the 2nd-order terms contributed by the tensor-tensor coupling, one has the energy constraint:

$$\frac{2}{\tau}\phi^{(2)'}_T-\frac{1}{3}\nabla^2\phi^{(2)}_T+\frac{6}{\tau^2}\phi^{(2)}_T-\frac{1}{12}D^{ij}\chi^{\parallel(2)}_{T,ij}=E_T, \tag{25}$$

the momentum constraint:

$$2\phi^{(2)'}_{T,j}+\frac{1}{2}D_{ij}\chi^{\parallel(2)',i}_T+\frac{1}{2}\chi^{\perp(2)',i}_{Tij}=M_{Tj}, \tag{26}$$

and the evolution equation:

$$-\left(\phi^{(2)''}_T+\frac{4}{\tau}\phi^{(2)'}_T\right)\delta_{ij}+\phi^{(2)}_{T,ij}+\frac{1}{2}\left(D_{ij}\chi^{\parallel(2)''}_T+\frac{4}{\tau}D_{ij}\chi^{\parallel(2)'}_T\right)+\frac{1}{2}\left(\chi^{\perp(2)''}_{Tij}+\frac{4}{\tau}\chi^{\perp(2)'}_{Tij}\right)+\frac{1}{2}\left(\chi^{\top(2)''}_{Tij}+\frac{4}{\tau}\chi^{\top(2)'}_{Tij}-\nabla^2\chi^{\top(2)}_{Tij}\right)$$
$$-\frac{1}{4}D_{kl}\chi^{\parallel(2),kl}_T\delta_{ij}+\frac{2}{3}\nabla^2\chi^{\parallel(2)}_{T,ij}-\frac{1}{2}\nabla^2D_{ij}\chi^{\parallel(2)}_T=S_{Tij}, \tag{27}$$

where

$$E_T \equiv -\frac{1}{24}\chi^{\top(1)\prime ij}\chi^{\top(1)\prime}_{ij} - \frac{2}{3\tau}\chi^{\top(1)\prime ij}\chi^{\top(1)}_{ij} + \frac{1}{6}\chi^{\top(1)ij}\nabla^2\chi^{\top(1)}_{ij} + \frac{1}{8}\chi^{\top(1)ij,k}\chi^{\top(1)}_{ij,k} - \frac{1}{12}\chi^{\top(1)ij,k}\chi^{\top(1)}_{kj,i} - \frac{1}{\tau^2}\chi^{\top(1)ij}\chi^{\top(1)}_{ij} + \frac{1}{\tau^2}\chi^{\top(1)ij}_{0}\chi^{\top(1)}_{0ij} - \frac{2}{\tau^2}\delta^{(2)}_{T0} + \frac{6}{\tau^2}\phi^{(2)}_{T0}, \tag{28}$$

$$M_{Tj} \equiv \chi^{\top(1)ik}(\chi^{\top(1)\prime}_{kj,i} - \chi^{\top(1)\prime}_{ki,j}) - \frac{1}{2}\chi^{\top(1)ik}_{,j}\chi^{\top(1)\prime}_{ik}, \tag{29}$$

$$S_{Tij} \equiv \chi^{\top(1)\prime k}_{i}\chi^{\top(1)\prime}_{kj} - \frac{1}{8}\chi^{\top(1)\prime kl}\chi^{\top(1)\prime}_{kl}\delta_{ij} + \chi^{\top(1)kl}\chi^{\top(1)}_{li,jk} + \chi^{\top(1)kl}\chi^{\top(1)}_{lj,ik} - \chi^{\top(1)kl}\chi^{\top(1)}_{kl,ij} - \chi^{\top(1)kl}\chi^{\top(1)}_{ij,kl} + \chi^{\top(1),k}_{li}\chi^{\top(1),l}_{kj} - \chi^{\top(1)k}_{i,l}\chi^{\top(1),l}_{jk} - \frac{1}{2}\chi^{\top(1)kl}_{,i}\chi^{\top(1)}_{kl,j} + \frac{3}{8}\chi^{\top(1)kl,m}\chi^{\top(1)}_{kl,m}\delta_{ij} - \frac{1}{4}\chi^{\top(1)}_{ml,k}\chi^{\top(1)mk,l}\delta_{ij} + \frac{1}{2}\chi^{\top(1)kl}\chi^{\top(1)\prime\prime}_{kl}\delta_{ij} + \frac{2}{\tau}\chi^{\top(1)kl}\chi^{\top(1)\prime}_{kl}\delta_{ij}, \tag{30}$$

where E_T contains the initial values $\delta^{(2)}_{T0}$, $\phi^{(2)}_{T0}$, $\chi^{\top(1)}_{T0ij}$ etc. at τ_0.

In the following, we shall solve the set of equations with scalar-tensor couplings and tensor-tensor couplings respectively.

IV. 2ND-ORDER PERTURBATIONS WITH THE SOURCE $\varphi\chi^{\top(1)}_{ij}$

A. Scalar perturbation $\phi^{(2)}_{s(t)}$

Combing the constraint equations [Eq. (19) $+\frac{1}{6}\partial^j\int_{\tau_0}^{\tau}d\tau'$ Eq. (20)] gives

$$\frac{2}{\tau}\phi^{(2)\prime}_{s(t)} + \frac{6}{\tau^2}\phi^{(2)}_{s(t)} = E_{s(t)} + \frac{1}{6}\int_{\tau_0}^{\tau}d\tau' M^{,j}_{s(t)j} + \frac{1}{3}\nabla^2\phi^{(2)}_{s(t)0} + \frac{1}{18}\nabla^2\nabla^2\chi^{\parallel(2)}_{s(t)0}.$$

Substituting the known $E_{s(t)}$ and $M^{,j}_{s(t)j}$ into the above, using the 1st-order GW equation (14) to replace $\nabla^2\chi^{\top(1)}_{ij}$ contained in $M^{,j}_{s(t)j}$, one has the first-order differential equation of $\phi^{(2)}_{s(t)}$ as the following:

$$\phi^{(2)\prime}_{s(t)} + \frac{3}{\tau}\phi^{(2)}_{s(t)} = \frac{\tau^2}{9}\varphi^{,kl}\chi^{\top(1)\prime}_{kl} + \frac{\tau}{3}\chi^{\top(1)}_{kl}\varphi^{,kl} + \frac{1}{\tau}\left(3\phi^{(2)}_{s(t)0} - \delta^{(2)}_{s(t)0} - \frac{\tau_0^2}{3}\varphi^{,ij}\chi^{\top(1)}_{0ij}\right) - \frac{\tau}{12}C + \frac{5}{4\tau^3}\chi^{\top(1)\prime}_{kl}\nabla^{-2}X^{,kl} - \frac{6}{\tau_0^3\tau}\chi^{\top(1)}_{0kl}\nabla^{-2}X^{,kl} + \frac{3\tau}{4}[\nabla^{-2}X^{,kl}]\int_{\tau_0}^{\tau}\frac{1}{\tau'^4}\nabla^2\chi^{\top(1)}_{kl}d\tau', \tag{31}$$

where the constant

$$C \equiv \frac{\tau_0^2}{3}\varphi^{,kl}\nabla^2\chi^{\top(1)}_{0kl} + \frac{\tau_0^2}{6}\chi^{\top(1)}_{0kl,m}\varphi^{,klm} - \frac{\tau_0}{3}\varphi^{,kl}\chi^{\top(1)\prime}_{0kl} + \frac{2}{3}\varphi^{,kl}\chi^{\top(1)}_{0kl} - 2\nabla^2\phi^{(2)}_{s(t)0} - \frac{1}{3}\nabla^2\nabla^2\chi^{\parallel(2)}_{s(t)0} + \frac{6}{\tau_0^3}\nabla^2\chi^{\top(1)}_{0kl}\nabla^{-2}X^{,kl} + \frac{3}{\tau_0^3}\chi^{\top(1)}_{0kl,m}\nabla^{-2}X^{,klm}, \tag{32}$$

depending on the initial values of metric perturbations at τ_0. The solution of Eq. (31) is

$$\phi^{(2)}_{s(t)} = \left(\phi^{(2)}_{s(t)0} - \frac{1}{3}\delta^{(2)}_{s(t)0} - \frac{\tau_0^2}{9}\varphi^{,ij}\chi^{\top(1)}_{0ij}\right) - \frac{\tau^2}{60}C + \frac{\tau^2}{9}\varphi^{,kl}\chi^{\top(1)}_{kl} - \frac{2}{9\tau^3}\varphi^{,kl}\int_{\tau_0}^{\tau}\tau'^4\chi^{\top(1)}_{kl}d\tau' + \frac{3\tau^2}{20}(\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\frac{1}{\tau'^4}\nabla^2\chi^{\top(1)}_{kl}d\tau' - \frac{3}{20\tau^3}(\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\tau'\nabla^2\chi^{\top(1)}_{kl}d\tau' + \frac{5}{4\tau^3}\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl} - \frac{2}{\tau_0^3}\chi^{\top(1)}_{0kl}\nabla^{-2}X^{,kl} + \frac{W(\mathbf{x})}{\tau^3}, \tag{33}$$

where integration by parts has been used, and $W(\mathbf{x})$ is a time-independent function. By letting $\phi^{(2)}_{s(t)}(\tau_0) = \phi^{(2)}_{s(t)0}$ at $\tau = \tau_0$ in (33), $W(\mathbf{x})$ is fixed as following

$$W(\mathbf{x}) = \frac{3}{4}\chi^{\top(1)}_{0kl}\nabla^{-2}X^{,kl} + \frac{\tau_0^3}{3}\delta^{(2)}_{s(t)0} + \frac{\tau_0^5}{60}C. \tag{34}$$

As we have checked, the solution (33) can be also derived by the trace part of the evolution equation (21) together with the energy constraint (19).

B. Scalar perturbation $\chi^{\|(2)}_{s(t)}$

The expression $\partial^j \int_{\tau_0}^{\tau} d\tau'$ Eq. (20) gives

$$2\nabla^2\phi^{(2)}_{s(t)} + \frac{1}{2}D_{ij}\chi^{\|(2),ij}_{s(t)} = \int_{\tau_0}^{\tau} d\tau' M^{,j}_{s(t)j} + 2\nabla^2\phi^{(2)}_{s(t)0} + \frac{1}{2}D_{ij}\chi^{\|(2),ij}_{s(t)0}. \tag{35}$$

Substituting $M_{s(t)j}$ of (23) and $\phi^{(2)}_{s(t)}$ of Eq. (33) into the above yields

$$\begin{aligned}
\chi^{\|(2)}_{s(t)} = {} & Z + \frac{\tau^2}{10}\nabla^{-2}C + \nabla^{-2}\left[-\frac{2\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kl} + \frac{4}{3\tau^3}\varphi^{,kl}\int_{\tau_0}^{\tau}\tau'^4\chi^{\top(1)}_{kl}d\tau'\right] \\
& + \nabla^{-2}\nabla^{-2}\left[\tau^2\varphi^{,kl}\nabla^2\chi^{\top(1)}_{kl} + \frac{\tau^2}{2}\chi^{\top(1)}_{kl,m}\varphi^{,klm} - \tau\varphi^{,kl}\chi^{\top(1)'}_{kl} + 2\varphi^{,kl}\chi^{\top(1)}_{kl}\right] \\
& + \nabla^{-2}\left[-\frac{15}{2\tau^3}\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl} - \frac{9\tau^2}{10}(\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\frac{1}{\tau'^4}\nabla^2\chi^{\top(1)}_{kl}d\tau' + \frac{9}{10\tau^3}(\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\tau'\nabla^2\chi^{\top(1)}_{kl}d\tau'\right] \\
& + \nabla^{-2}\nabla^{-2}\left[\frac{36}{\tau^3}\nabla^2\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl} + \frac{18}{\tau^3}\chi^{\top(1)}_{kl,m}\nabla^{-2}X^{,klm} + (\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\frac{54}{\tau'^4}\nabla^2\chi^{\top(1)}_{kl}d\tau'\right] - \frac{6}{\tau^3}\nabla^{-2}W,
\end{aligned} \tag{36}$$

where the constant

$$\begin{aligned}
Z \equiv {} & \chi^{\|(2)}_{s(t)0} + \nabla^{-2}\left(2\delta^{(2)}_{s(t)0} + \frac{2\tau_0^2}{3}\varphi^{,ij}\chi^{\top(1)}_{0ij}\right) + \nabla^{-2}\nabla^{-2}\left(-\tau_0^2\varphi^{,kl}\nabla^2\chi^{\top(1)}_{0kl} - \frac{\tau_0^2}{2}\chi^{\top(1)}_{0kl,m}\varphi^{,klm} + \tau_0\varphi^{,kl}\chi^{\top(1)'}_{0kl} - 2\varphi^{,kl}\chi^{\top(1)}_{0kl}\right) \\
& + \nabla^{-2}\left[\frac{12}{\tau_0^3}\chi^{\top(1)}_{0kl}\nabla^{-2}X^{,kl}\right] + \nabla^{-2}\nabla^{-2}\left[-\frac{36}{\tau_0^3}\nabla^2\chi^{\top(1)}_{0kl}\nabla^{-2}X^{,kl} - \frac{18}{\tau_0^3}\chi^{\top(1)}_{0kl,m}\nabla^{-2}X^{,klm}\right],
\end{aligned} \tag{37}$$

depending on the initial values of metric perturbations at τ_0. Thus, the scalar perturbation $D_{ij}\chi^{\|(2)}_{s(t)}$ is obtained:

$$\begin{aligned}
D_{ij}\chi^{\|(2)}_{s(t)} = {} & D_{ij}Z + \frac{\tau^2}{10}D_{ij}\nabla^{-2}C + D_{ij}\nabla^{-2}\left[-\frac{2\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kl} + \frac{4}{3\tau^3}\varphi^{,kl}\int_{\tau_0}^{\tau}\tau'^4\chi^{\top(1)}_{kl}d\tau'\right] \\
& + D_{ij}\nabla^{-2}\nabla^{-2}\left[\tau^2\varphi^{,kl}\nabla^2\chi^{\top(1)}_{kl} + \frac{\tau^2}{2}\chi^{\top(1)}_{kl,m}\varphi^{,klm} - \tau\varphi^{,kl}\chi^{\top(1)'}_{kl} + 2\varphi^{,kl}\chi^{\top(1)}_{kl}\right] \\
& + D_{ij}\nabla^{-2}\left[-\frac{15}{2\tau^3}\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl} - \frac{9\tau^2}{10}(\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\frac{1}{\tau'^4}\nabla^2\chi^{\top(1)}_{kl}d\tau' + \frac{9}{10\tau^3}(\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\tau'\nabla^2\chi^{\top(1)}_{kl}d\tau'\right] \\
& + D_{ij}\nabla^{-2}\nabla^{-2}\left[\frac{36}{\tau^3}\nabla^2\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl} + \frac{18}{\tau^3}\chi^{\top(1)}_{kl,m}\nabla^{-2}X^{,klm} + (\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\frac{54}{\tau'^4}\nabla^2\chi^{\top(1)}_{kl}d\tau'\right] - \frac{6}{\tau^3}D_{ij}\nabla^{-2}W.
\end{aligned} \tag{38}$$

We remark that the solution (38) can be also obtained by the traceless part of the evolution equation (21) together with the momentum constraint (20). Our result (38) contains the nonzero initial values (through τ_0) and decaying modes, and applies to general situations.

C. Vector perturbation $\chi^{\perp(2)}_{s(t)\vec{i}\vec{j}}$

The time integral of the momentum constraint (20) from τ_0 to τ is

$$2\phi^{(2)}_{s(t),j}+\frac{1}{3}\nabla^2\chi^{\parallel(2)}_{s(t),j}+\frac{1}{2}\chi^{\perp(2),i}_{s(t)ij}=\int_{\tau_0}^{\tau}d\tau' M_{s(t)j}+2\phi^{(2)}_{s(t)0,j}+\frac{1}{3}\nabla^2\chi^{\parallel(2)}_{s(t)0,j}+\frac{1}{2}\chi^{\perp(2),i}_{s(t)0ij}. \tag{39}$$

Using $M_{s(t)j}$ of (23) and Eq. (14), one has

$$\begin{aligned}\int_{\tau_0}^{\tau}d\tau' M_{s(t)j}=&\int_{\tau_0}^{\tau}\left[-\frac{\tau'}{3}\partial_j(\varphi^{,kl}\chi^{\top(1)}_{kl})+\frac{\tau'}{3}\nabla^2(\varphi^{,k}\chi^{\top(1)}_{kj})\right]d\tau'+\frac{\tau^2}{6}\partial_j(\varphi^{,kl}\chi^{\top(1)}_{kl})-\frac{\tau^2}{6}\chi^{\top(1)}_{kj}\nabla^2\varphi^{,k}-\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kj,l}\\&+\frac{\tau^2}{6}\varphi^{,kl}\chi^{\top(1)}_{kl,j}-\frac{\tau}{3}\varphi^{,k}\chi^{\top(1)'}_{kj}+\frac{2}{3}\varphi^{,k}\chi^{\top(1)}_{kj}+\left[-\frac{\tau_0^2}{6}\partial_j(\varphi^{,kl}\chi^{\top(1)}_{0kl})+\frac{\tau_0^2}{6}\chi^{\top(1)}_{0kj}\nabla^2\varphi^{,k}+\frac{\tau_0^2}{3}\varphi^{,kl}\chi^{\top(1)}_{0kj,l}-\frac{\tau_0^2}{6}\varphi^{,kl}\chi^{\top(1)}_{0kl,j}\right.\\&\left.+\frac{\tau_0}{3}\varphi^{,k}\chi^{\top(1)'}_{0kj}-\frac{2}{3}\varphi^{,k}\chi^{\top(1)}_{0kj}\right]+\int_{\tau_0}^{\tau}\left[\partial_j\left(\frac{3}{\tau'^3}\chi^{\top(1)'}_{kl}\nabla^{-2}X^{,kl}\right)-\frac{6}{\tau'^3}\chi^{\top(1)'}_{jk,l}\nabla^{-2}X^{,kl}-\frac{3}{\tau'^3}\chi^{\top(1)'}_{kj}X^{,k}\right]d\tau'\\&+\frac{3}{\tau^3}\chi^{\top(1)}_{kl,j}\nabla^{-2}X^{,kl}-\frac{3}{\tau_0^3}\chi^{\top(1)}_{0kl,j}\nabla^{-2}X^{,kl}.\end{aligned} \tag{40}$$

Plugging the solutions of $\phi^{(2)}_{s(t)}$ of (33) and $\chi^{\parallel(2)}_{s(t)}$ of (38) into (39), we directly read $\chi^{\perp(2),i}_{s(t)ij}$ as

$$\begin{aligned}\chi^{\perp(2),i}_{s(t)ij}=&Q_j+\int_{\tau_0}^{\tau}\left[-\frac{2\tau'}{3}\partial_j(\varphi^{,kl}\chi^{\top(1)}_{kl})+\frac{2\tau'}{3}\nabla^2(\varphi^{,k}\chi^{\top(1)}_{kj})\right]d\tau'+\frac{\tau^2}{3}\partial_j(\varphi^{,kl}\chi^{\top(1)}_{kl})-\frac{\tau^2}{3}\chi^{\top(1)}_{kj}\nabla^2\varphi^{,k}-\frac{2\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kj,l}\\&+\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kl,j}-\frac{2\tau}{3}\varphi^{,k}\chi^{\top(1)'}_{kj}+\frac{4}{3}\varphi^{,k}\chi^{\top(1)}_{kj}+\nabla^{-2}\partial_j\left[-\frac{2\tau^2}{3}\varphi^{,kl}\nabla^2\chi^{\top(1)}_{kl}-\frac{\tau^2}{3}\chi^{\top(1)}_{kl,m}\varphi^{,klm}+\frac{2\tau}{3}\varphi^{,kl}\chi^{\top(1)'}_{kl}-\frac{4}{3}\varphi^{,kl}\chi^{\top(1)}_{kl}\right]\\&+\int_{\tau_0}^{\tau}\left[-\frac{12}{\tau'^3}\chi^{\top(1)'}_{jk,l}\nabla^{-2}X^{,kl}-\frac{6}{\tau'^3}\chi^{\top(1)'}_{kj}X^{,k}+\partial_j\nabla^{-2}\left(\frac{12}{\tau'^3}\chi^{\top(1)'}_{kl,m}\nabla^{-2}X^{,klm}+\frac{6}{\tau'^3}\chi^{\top(1)'}_{kl}X^{,kl}\right)\right]d\tau'+\frac{6}{\tau^3}\chi^{\top(1)}_{kl,j}\nabla^{-2}X^{,kl}\\&-\partial_j\nabla^{-2}\left(\frac{6}{\tau^3}\nabla^2\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl}+\frac{6}{\tau^3}\chi^{\top(1)}_{kl,m}\nabla^{-2}X^{,klm}\right),\end{aligned} \tag{41}$$

where the constant vector

$$\begin{aligned}Q_j\equiv&\chi^{\perp(2),i}_{s(t)0ij}-\frac{\tau_0^2}{3}\partial_j(\varphi^{,kl}\chi^{\top(1)}_{0kl})+\frac{\tau_0^2}{3}\chi^{\top(1)}_{0kj}\nabla^2\varphi^{,k}+\frac{2\tau_0^2}{3}\varphi^{,kl}\chi^{\top(1)}_{0kj,l}-\frac{\tau_0^2}{3}\varphi^{,kl}\chi^{\top(1)}_{0kl,j}+\frac{2\tau_0}{3}\varphi^{,k}\chi^{\top(1)'}_{0kj}-\frac{4}{3}\varphi^{,k}\chi^{\top(1)}_{0kj}\\&+\nabla^{-2}\partial_j\left(\frac{2\tau_0^2}{3}\varphi^{,kl}\nabla^2\chi^{\top(1)}_{0kl}+\frac{\tau_0^2}{3}\chi^{\top(1)}_{0kl,m}\varphi^{,klm}-\frac{2\tau_0}{3}\varphi^{,kl}\chi^{\top(1)'}_{0kl}+\frac{4}{3}\varphi^{,kl}\chi^{\top(1)}_{0kl}\right)\\&-\frac{6}{\tau_0^3}\chi^{\top(1)}_{0kl,j}\nabla^{-2}X^{,kl}+\partial_j\nabla^{-2}\left(\frac{6}{\tau_0^3}\nabla^2\chi^{\top(1)}_{0kl}\nabla^{-2}X^{,kl}+\frac{6}{\tau_0^3}\chi^{\top(1)}_{0kl,m}\nabla^{-2}X^{,klm}\right),\end{aligned} \tag{42}$$

depending on the initial values at τ_0. To get $\chi^{\perp(2)}_{s(t)ij}$ from Eq. (41), one has to remove ∂^j as follows.

Writing $\chi^{\perp(2)}_{s(t)ij} = A_{s(t)i,j} + A_{s(t)j,i}$ in terms of a 3-vector $A_{s(t)i}$ as Eqs. (10), (41) yields

$$
\begin{aligned}
A_{s(t)j} = {} & \nabla^{-2}Q_j + \nabla^{-2}\int_{\tau_0}^{\tau}\left[-\frac{2\tau'}{3}\partial_j(\varphi^{,kl}\chi^{\top(1)}_{kl}) + \frac{2\tau'}{3}\nabla^2(\varphi^{,k}\chi^{\top(1)}_{kj})\right]d\tau' \\
& + \nabla^{-2}\left[\frac{\tau^2}{3}\partial_j(\varphi^{,kl}\chi^{\top(1)}_{kl}) - \frac{\tau^2}{3}\chi^{\top(1)}_{kj}\nabla^2\varphi^{,k} - \frac{2\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kj,l} + \frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kl,j} - \frac{2\tau}{3}\varphi^{,k}\chi^{\top(1)'}_{kj} + \frac{4}{3}\varphi^{,k}\chi^{\top(1)}_{kj}\right] \\
& + \nabla^{-2}\nabla^{-2}\partial_j\left[-\frac{2\tau^2}{3}\varphi^{,kl}\nabla^2\chi^{\top(1)}_{kl} - \frac{\tau^2}{3}\chi^{\top(1)}_{kl,m}\varphi^{,klm} + \frac{2\tau}{3}\varphi^{,kl}\chi^{\top(1)'}_{kl} - \frac{4}{3}\varphi^{,kl}\chi^{\top(1)}_{kl}\right] \\
& + \nabla^{-2}\int_{\tau_0}^{\tau}\left[-\frac{12}{\tau'^3}\chi^{\top(1)'}_{jk,l}\nabla^{-2}X^{,kl} - \frac{6}{\tau'^3}\chi^{\top(1)'}_{kj}X^{,k} + \partial_j\nabla^{-2}\left(\frac{12}{\tau'^3}\chi^{\top(1)'}_{kl,m}\nabla^{-2}X^{,klm} + \frac{6}{\tau'^3}\chi^{\top(1)'}_{kl}X^{,kl}\right)\right]d\tau' \\
& + \nabla^{-2}\left[\frac{6}{\tau^3}\chi^{\top(1)}_{kl,j}\nabla^{-2}X^{,kl} - \partial_j\nabla^{-2}\left(\frac{6}{\tau^3}\nabla^2\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl} + \frac{6}{\tau^3}\chi^{\top(1)}_{kl,m}\nabla^{-2}X^{,klm}\right)\right].
\end{aligned}
\tag{43}
$$

Thus, the vector perturbation is obtained:

$$
\begin{aligned}
\chi^{\perp(2)}_{s(t)ij} = {} & \nabla^{-2}Q_{i,j} + \int_{\tau_0}^{\tau}\frac{2\tau'}{3}[\partial_i(\varphi^{,k}\chi^{\top(1)}_{kj}) - \nabla^{-2}\partial_i\partial_j(\varphi^{,kl}\chi^{\top(1)}_{kl})]d\tau' + \nabla^{-2}\left[\frac{\tau^2}{3}\partial_i\partial_j(\varphi^{,kl}\chi^{\top(1)}_{kl})\right. \\
& \left. + \partial_i\left(\frac{\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kl,j} - \frac{\tau^2}{3}\chi^{\top(1)}_{kj}\nabla^2\varphi^{,k} - \frac{2\tau^2}{3}\varphi^{,kl}\chi^{\top(1)}_{kj,l} - \frac{2\tau}{3}\varphi^{,k}\chi^{\top(1)'}_{kj} + \frac{4}{3}\varphi^{,k}\chi^{\top(1)}_{kj}\right)\right] + \nabla^{-2}\nabla^{-2}\partial_i\partial_j\left[\frac{2\tau}{3}\varphi^{,kl}\chi^{\top(1)'}_{kl}\right. \\
& \left. - \frac{2\tau^2}{3}\varphi^{,kl}\nabla^2\chi^{\top(1)}_{kl} - \frac{\tau^2}{3}\chi^{\top(1)}_{kl,m}\varphi^{,klm} - \frac{4}{3}\varphi^{,kl}\chi^{\top(1)}_{kl}\right] + \nabla^{-2}\int_{\tau_0}^{\tau}\left[\partial_j\left(-\frac{12}{\tau'^3}\chi^{\top(1)'}_{ki,l}\nabla^{-2}X^{,kl} - \frac{6}{\tau'^3}\chi^{\top(1)'}_{ki}X^{,k}\right)\right. \\
& \left. + \partial_i\partial_j\nabla^{-2}\left(\frac{12}{\tau'^3}\chi^{\top(1)'}_{kl,m}\nabla^{-2}X^{,klm} + \frac{6}{\tau'^3}\chi^{\top(1)'}_{kl}X^{,kl}\right)\right]d\tau' + \nabla^{-2}\left[\partial_j\left(\frac{6}{\tau^3}\chi^{\top(1)}_{kl,i}\nabla^{-2}X^{,kl}\right)\right. \\
& \left. - \partial_i\partial_j\nabla^{-2}\left(\frac{6}{\tau^3}\nabla^2\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl} + \frac{6}{\tau^3}\chi^{\top(1)}_{kl,m}\nabla^{-2}X^{,klm}\right)\right] + (i \leftrightarrow j).
\end{aligned}
\tag{44}
$$

Actually, this vector mode $\chi^{\perp(2)}_{s(t)ij}$ can be also derived from the curl portion of the momentum constraint (20) itself without explicitly using the solutions $\phi^{(2)}_{s(t)}$ of (33) and $\chi^{\parallel(2)}_{s(t)}$ of (36). The result (44) explicitly demonstrates that the 2nd-order vector perturbation exists, whose effective source is the coupling of 1st-order perturbations, even though the matter source for the vector mode is zero in the synchronous gage, $T_{0i} = 0$, $T_{ij} = 0$.

D. Tensor perturbation $\chi^{\top(2)}_{s(t)ij}$

Next consider the traceless part of the evolution equation (21):

$$
\chi^{\top(2)''}_{s(t)ij} + \frac{4}{\tau}\chi^{\top(2)'}_{s(t)ij} - \nabla^2\chi^{\top(2)}_{s(t)ij} = 2\bar{S}_{s(t)ij} - \left(2D_{ij}\phi^{(2)}_{s(t)} + \frac{1}{3}\nabla^2 D_{ij}\chi^{\parallel(2)}_{s(t)}\right) - \left(D_{ij}\chi^{\parallel(2)''}_{s(t)} + \frac{4}{\tau}D_{ij}\chi^{\parallel(2)'}_{s(t)}\right) - \left(\chi^{\perp(2)''}_{s(t)ij} + \frac{4}{\tau}\chi^{\perp(2)'}_{s(t)ij}\right),
\tag{45}
$$

where $\bar{S}_{s(t)ij}$ is the traceless part of $S_{s(t)ij}$ as the following:

$$\bar{S}_{s(t)ij} = -\frac{2\tau}{3}\varphi_{,i}^{,k}\chi_{kj}^{\top(1)\prime} - \frac{2\tau}{3}\varphi_{,j}^{,k}\chi_{ki}^{\top(1)\prime} + \frac{4\tau}{9}\varphi^{,kl}\chi_{kl}^{\top(1)\prime}\delta_{ij} + \frac{\tau}{3}\chi_{ij}^{\top(1)\prime}\nabla^2\varphi + \frac{10}{3}\chi_{ij}^{\top(1)}\nabla^2\varphi + \frac{20}{9}\varphi^{,kl}\chi_{kl}^{\top(1)}\delta_{ij} - \frac{10}{3}\varphi_{,j}^{,k}\chi_{ki}^{\top(1)}$$
$$- \frac{10}{3}\varphi_{,i}^{,k}\chi_{kj}^{\top(1)} + \frac{10}{3}\varphi\nabla^2\chi_{ij}^{\top(1)} + \frac{\tau^2}{3}\varphi^{,kl}\chi_{ij,kl}^{\top(1)} + \frac{\tau^2}{3}\varphi^{,kl}\chi_{kl,ij}^{\top(1)} - \frac{\tau^2}{9}\varphi^{,kl}\nabla^2\chi_{kl}^{\top(1)}\delta_{ij} - \frac{\tau^2}{3}\varphi^{,kl}\chi_{li,jk}^{\top(1)} - \frac{\tau^2}{3}\varphi^{,kl}\chi_{lj,ik}^{\top(1)}$$
$$+ 5\varphi^{,k}\chi_{ij,k}^{\top(1)} - \frac{5}{3}\varphi^{,k}\chi_{ki,j}^{\top(1)} - \frac{5}{3}\varphi^{,k}\chi_{kj,i}^{\top(1)} + \frac{\tau^2}{6}\chi_{ij,k}^{\top(1)}\nabla^2\varphi^{,k} - \frac{\tau^2}{6}\chi_{ki,j}^{\top(1)}\nabla^2\varphi^{,k} - \frac{\tau^2}{6}\chi_{kj,i}^{\top(1)}\nabla^2\varphi^{,k} + \frac{\tau^2}{6}\varphi_{,i}^{,kl}\chi_{kl,j}^{\top(1)} + \frac{\tau^2}{6}\varphi_{,j}^{,kl}\chi_{kl,i}^{\top(1)}$$
$$- \frac{\tau^2}{9}\varphi^{,klm}\chi_{kl,m}^{\top(1)}\delta_{ij} - \frac{12}{\tau^4}\chi_{kl}^{\top(1)\prime}\nabla^{-2}X^{,kl}\delta_{ij} - \frac{2}{\tau^3}\chi_{kl,m}^{\top(1)}\nabla^{-2}X^{,klm}\delta_{ij} - \frac{2}{\tau^3}\nabla^2\chi_{kl}^{\top(1)}\nabla^{-2}X^{,kl}\delta_{ij} - \frac{9}{\tau^4}X\chi_{ij}^{\top(1)\prime}$$
$$+ \frac{18}{\tau^4}\chi_{ki}^{\top(1)\prime}\nabla^{-2}X_{,j}^{,k} + \frac{18}{\tau^4}\chi_{kj}^{\top(1)\prime}\nabla^{-2}X_{,i}^{,k} - \frac{3}{\tau^3}X^{,k}\chi_{kj,i}^{\top(1)} - \frac{3}{\tau^3}X^{,k}\chi_{ki,j}^{\top(1)} + \frac{3}{\tau^3}X^{,k}\chi_{ij,k}^{\top(1)} - \frac{6}{\tau^3}\chi_{lj,ik}^{\top(1)}\nabla^{-2}X^{,kl} - \frac{6}{\tau^3}\chi_{li,jk}^{\top(1)}\nabla^{-2}X^{,kl}$$
$$+ \frac{6}{\tau^3}\chi_{ij,kl}^{\top(1)}\nabla^{-2}X^{,kl} + \frac{6}{\tau^3}\chi_{kl,ij}^{\top(1)}\nabla^{-2}X^{,kl} + \frac{3}{\tau^3}\chi_{kl,i}^{\top(1)}\nabla^{-2}X_{,j}^{,kl} + \frac{3}{\tau^3}\chi_{kl,j}^{\top(1)}\nabla^{-2}X_{,i}^{,kl}. \tag{46}$$

One can substitute the known $\phi_{s(t)}^{(2)}$, $D_{ij}\chi_{s(t)}^{\|(2)}$, $\chi_{s(t)ij}^{\perp(2)}$ into Eq. (45), and solve for $\chi_{s(t)ij}^{\top(2)}$. But the following calculation is simpler and will yield the same result. Applying $\partial^i\partial^j$ to (45) yields:

$$-\left(2D_{ij}\phi_{s(t)}^{(2)} + \frac{1}{3}\nabla^2 D_{ij}\chi_{s(t)}^{\|(2)}\right) - \left(D_{ij}\chi_{s(t)}^{\|(2)\prime\prime} + \frac{4}{\tau}D_{ij}\chi_{s(t)}^{\|(2)\prime}\right) = -3D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}_{s(t)kl}^{,kl}. \tag{47}$$

Substituting Eq. (47) into the rhs of Eq. (45), one has

$$\chi_{s(t)ij}^{\top(2)\prime\prime} + \frac{4}{\tau}\chi_{s(t)ij}^{\top(2)\prime} - \nabla^2\chi_{s(t)ij}^{\top(2)} = 2\bar{S}_{s(t)ij} - 3D_{ij}\nabla^{-2}\nabla^{-2}\bar{S}_{s(t)kl}^{,kl} - \left(\chi_{s(t)ij}^{\perp(2)\prime\prime} + \frac{4}{\tau}\chi_{s(t)ij}^{\perp(2)\prime}\right). \tag{48}$$

Applying ∂^j to (48) and together with Eq. (10) leads to an equation of $A_{s(t)i}$:

$$0 = 2\bar{S}_{s(t)ij}^{,j} - 2\nabla^{-2}\bar{S}_{s(t)kl,i}^{,kl} - \nabla^2\left(A_{s(t)i}'' + \frac{4}{\tau}A_{s(t)i}'\right). \tag{49}$$

Thus, from Eqs. (10) and (49), one has

$$-\left(\chi_{s(t)ij}^{\perp(2)\prime\prime} + \frac{4}{\tau}\chi_{s(t)ij}^{\perp(2)\prime}\right) = -\partial_j\left(A_{s(t)i}'' + \frac{4}{\tau}A_{s(t)i}'\right) - \partial_i\left(A_{s(t)j}'' + \frac{4}{\tau}A_{s(t)j}'\right) = -2\nabla^{-2}\bar{S}_{s(t)ki,j}^{,k} - 2\nabla^{-2}\bar{S}_{s(t)kj,i}^{,k} + 4\nabla^{-2}\nabla^{-2}\bar{S}_{s(t)kl,ij}^{,kl}. \tag{50}$$

Substituting (50) into the rhs of Eq. (48), we obtain the equation for $\chi_{s(t)ij}^{\top(2)}$:

$$\chi_{s(t)ij}^{\top(2)\prime\prime} + \frac{4}{\tau}\chi_{s(t)ij}^{\top(2)\prime} - \nabla^2\chi_{s(t)ij}^{\top(2)} = J_{s(t)ij}(\mathbf{x},\tau), \tag{51}$$

with the source

$$J_{s(t)ij}(\mathbf{x},\tau) \equiv 2\bar{S}_{s(t)ij} + \nabla^{-2}\nabla^{-2}\bar{S}_{s(t)kl,ij}^{,kl} + \delta_{ij}\nabla^{-2}\bar{S}_{s(t)kl}^{,kl} - 2\nabla^{-2}\bar{S}_{s(t)ki,j}^{,k} - 2\nabla^{-2}\bar{S}_{s(t)kj,i}^{,k}, \tag{52}$$

where the known symmetric and traceless $\bar{S}_{s(t)ij}$ is given by (46). It is checked that $J_{s(t)ij}(\mathbf{x},\tau)$ is traceless and transverse.

The differential equation (51) is inhomogeneous, and its solution is given by

$$\chi_{s(t)ij}^{\top(2)}(\mathbf{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\left(\bar{I}_{s(t)ij}(s) + \frac{b_{1ij}}{s^{3/2}}H_{\frac{1}{2}}^{(1)}(s) + \frac{b_{2ij}}{s^{3/2}}H_{\frac{3}{2}}^{(2)}(s)\right), \tag{53}$$

where $s \equiv k\tau$,

$$\bar{I}_{s(t)ij}(s) \equiv \frac{1}{s^2}\left(\cos s - \frac{\sin s}{s}\right)\int_1^s dy y^2\left(\sin y + \frac{\cos y}{y}\right)\bar{J}_{s(t)ij}(y) - \frac{1}{s^2}\left(\sin s + \frac{\cos s}{s}\right)\int_1^s dy y^2\left(\cos y - \frac{\sin y}{y}\right)\bar{J}_{s(t)ij}(y), \tag{54}$$

with $\bar{J}_{s(t)ij}$ being the Fourier transformation of the source $J_{s(t)ij}$. In (53) the two terms associated with b_{1ij} and b_{2ij} are of the same form as the 1st-order solution $\chi_{ij}^{\top(1)}(\mathbf{x},\tau)$ in (15) and (16) and correspond to the homogeneous solution of (51). These two terms are kept in order to allow for a general initial condition at time τ_0. In particular, the coefficients b_{1ij} and b_{2ij} are to be determined by the 2nd-order tensor modes of precedent Radiation Dominated stage.

Thus, all the 2nd-order metric perturbations due to scalar-tensor coupling have been obtained. By (18), the corresponding 2nd-order density contrast is

$$\delta^{(2)}_{s(t)} = \delta^{(2)}_{s(t)0} + 3\left(\phi^{(2)}_{s(t)} - \phi^{(2)}_{s(t)0}\right) + \left(\chi^{\top(1)ij} D_{ij}\chi^{\parallel(1)} - \chi_0^{\top(1)ij} D_{ij}\chi_0^{\parallel(1)}\right), \quad (55)$$

which can be expressed as

$$\delta^{(2)}_{s(t)} = -\frac{\tau^2}{20}C - \frac{2}{3\tau^3}\varphi^{,kl}\int_{\tau_0}^{\tau}\tau'^4\chi^{\top(1)}_{kl}d\tau' - \frac{9}{20\tau^3}(\nabla^{-2}X^{,kl}) \times \int_{\tau_0}^{\tau}\tau'\nabla^2\chi^{\top(1)}_{kl}d\tau' + \frac{3}{\tau^3}W - \frac{9}{4\tau^3}\chi^{\top(1)}_{kl}\nabla^{-2}X^{,kl} + \frac{9\tau^2}{20}(\nabla^{-2}X^{,kl})\int_{\tau_0}^{\tau}\frac{1}{\tau'^4}\nabla^2\chi^{\top(1)}_{kl}d\tau', \quad (56)$$

after using the given $\phi^{(2)}_{s(t)}$, $D_{ij}\chi^{\parallel(1)}$, $\chi^{\top(1)ij}$.

V. 2ND-ORDER PERTURBATIONS WITH THE SOURCE $\chi^{\top(1)}_{kl}\chi^{\top(1)}_{ij}$

Now we turn to the set of Eqs. (25)–(27) with the source of the form of $\chi^{\top(1)}_{kl}\chi^{\top(1)}_{ij}$, and derive the solution of the second-order perturbations. The procedures involved are similar to those in Sec. IV.

A. Scalar perturbation $\phi^{(2)}_T$

Combing the constraint equations, i.e, (25) $+\frac{1}{6}\partial^j\int_{\tau_0}^{\tau}d\tau$ (26), using the 1st-order GW equation (14), one has the following differential equation of $\phi^{(2)}_T$:

$$\phi^{(2)\prime}_T + \frac{3}{\tau}\phi^{(2)}_T = -\frac{1}{3}\chi^{\top(1)\prime ij}\chi^{\top(1)}_{ij} - \frac{1}{2\tau}\chi^{\top(1)ij}\chi^{\top(1)}_{ij} + \frac{\tau}{6}\int_{\tau_0}^{\tau}\frac{\chi^{\top(1)\prime}_{kl}\chi^{\top(1)\prime kl}}{\tau'}d\tau' + \frac{1}{\tau}\left(3\phi^{(2)}_{T0} - \delta^{(2)}_{T0} + \frac{1}{2}\chi_0^{\top(1)ij}\chi^{\top(1)}_{0ij}\right) - \frac{\tau}{12}K, \quad (57)$$

where the constant

$$K \equiv -2\nabla^2\phi^{(2)}_{T0} - \frac{1}{3}\nabla^2\nabla^2\chi^{\parallel(2)}_{T0} + \frac{1}{2}\chi_0^{\top(1)kl,m}\chi^{\top(1)}_{0km,l} - \frac{3}{4}\chi_0^{\top(1)kl,m}\chi^{\top(1)}_{0kl,m} - \chi_0^{\top(1)kl}\nabla^2\chi^{\top(1)}_{0kl} + \frac{1}{4}\chi^{\top(1)\prime}_{0kl}\chi_0^{\top(1)\prime kl}, \quad (58)$$

depends on the initial metric perturbations at τ_0. The solution of Eq. (57) is

$$\phi^{(2)}_T = \left(\phi^{(2)}_{T0} - \frac{1}{3}\delta^{(2)}_{T0} + \frac{1}{6}\chi_0^{\top(1)ij}\chi^{\top(1)}_{0ij}\right) - \frac{\tau^2}{60}K + \frac{B(\mathbf{x})}{\tau^3} - \frac{1}{6}\chi^{\top(1)ij}\chi^{\top(1)}_{ij} + \frac{\tau^2}{30}\int_{\tau_0}^{\tau}\frac{\chi^{\top(1)\prime}_{kl}\chi^{\top(1)\prime kl}}{\tau'}d\tau' - \frac{1}{30\tau^3}\int_{\tau_0}^{\tau}\tau'^4\chi^{\top(1)\prime}_{kl}\chi^{\top(1)\prime kl}d\tau', \quad (59)$$

where integration by parts has been used, and $B(\mathbf{x})$ is fixed by setting $\tau = \tau_0$ and $\phi^{(2)}_T(\tau_0) = \phi^{(2)}_{T0}$ as

$$B(\mathbf{x}) = \frac{\tau_0^3}{3}\delta^{(2)}_{T0} + \frac{\tau_0^5}{60}K. \quad (60)$$

Notice that the solution (59) can be also derived by the trace part of the evolution equation (27) together with the energy constraint (25).

B. Scalar perturbation $\chi^{\parallel(2)}_T$

The expression $\partial^j\int_{\tau_0}^{\tau}d\tau'$ Eq. (26) gives the following equation:

$$2\nabla^2\phi^{(2)}_T + \frac{1}{2}D_{ij}\chi^{\parallel(2),ij}_T = \int_{\tau_0}^{\tau}d\tau' M^{,j}_{Tj} + 2\nabla^2\phi^{(2)}_{T0} + \frac{1}{3}\nabla^2\nabla^2\chi^{\parallel(2)}_{T0}. \quad (61)$$

Substituting M_{Tj} of (29) and $\phi^{(2)}_T$ of Eq. (59) into Eq. (61) yields:

$$\chi^{\parallel(2)}_T = Y + \frac{\tau^2}{10}\nabla^{-2}K - \frac{6}{\tau^3}\nabla^{-2}B + \nabla^{-2}\left[-\frac{1}{2}\chi^{\top(1)kl}\chi^{\top(1)}_{kl} - \frac{\tau^2}{5}\int_{\tau_0}^{\tau}\frac{\chi^{\top(1)\prime}_{kl}\chi^{\top(1)\prime kl}}{\tau'}d\tau' + \frac{1}{5\tau^3}\int_{\tau_0}^{\tau}\tau'^4\chi^{\top(1)\prime}_{kl}\chi^{\top(1)\prime kl}d\tau'\right] + \nabla^{-2}\nabla^{-2}\left[\frac{3}{2}\chi^{\top(1)kl,m}\chi^{\top(1)}_{km,l} + \frac{3}{4}\chi^{\top(1)kl,m}\chi^{\top(1)}_{kl,m} + \frac{3}{4}\chi^{\top(1)\prime}_{kl}\chi^{\top(1)\prime kl} + 6\int_{\tau_0}^{\tau}\frac{1}{\tau'}\chi^{\top(1)\prime}_{kl}\chi^{\top(1)\prime kl}d\tau'\right], \quad (62)$$

where the constant

$$Y \equiv \chi_{T0}^{\parallel(2)} + \nabla^{-2}(2\delta_{T0}^{(2)} - \chi_0^{\top(1)ij}\chi_{0ij}^{\top(1)}) + \nabla^{-2}\nabla^{-2}\left(-\frac{3}{2}\chi_0^{\top(1)kl,m}\chi_{0km,l}^{\top(1)} + \frac{9}{4}\chi_0^{\top(1)kl,m}\chi_{0kl,m}^{\top(1)} + 3\chi_0^{\top(1)kl}\nabla^2\chi_{0kl}^{\top(1)} - \frac{3}{4}\chi_{0kl}^{\top(1)\prime}\chi_0^{\top(1)\prime kl}\right), \tag{63}$$

depending on the initial values of metric perturbations at τ_0. Thus, the scalar perturbation $D_{ij}\chi_T^{\parallel(2)}$ is determined,

$$\begin{aligned} D_{ij}\chi_T^{\parallel(2)} = {} & D_{ij}Y + \frac{\tau^2}{10}D_{ij}\nabla^{-2}K - \frac{6}{\tau^3}D_{ij}\nabla^{-2}B + D_{ij}\nabla^{-2}\left[-\frac{1}{2}\chi^{\top(1)kl}\chi_{kl}^{\top(1)} - \frac{\tau^2}{5}\int_{\tau_0}^{\tau}\frac{1}{\tau'}\chi_{kl}^{\top(1)\prime}\chi^{\top(1)\prime kl}d\tau'\right. \\ & \left. + \frac{1}{5\tau^3}\int_{\tau_0}^{\tau}\tau'^4\chi_{kl}^{\top(1)\prime}\chi^{\top(1)\prime kl}d\tau'\right] + D_{ij}\nabla^{-2}\nabla^{-2}\left[\frac{3}{2}\chi^{\top(1)kl,m}\chi_{km,l}^{\top(1)} + \frac{3}{4}\chi^{\top(1)kl,m}\chi_{kl,m}^{\top(1)} + \frac{3}{4}\chi_{kl}^{\top(1)\prime}\chi^{\top(1)\prime kl}\right. \\ & \left. + 6\int_{\tau_0}^{\tau}\frac{1}{\tau'}\chi_{kl}^{\top(1)\prime}\chi^{\top(1)\prime kl}d\tau'\right]. \end{aligned} \tag{64}$$

We have checked that when $B(\mathbf{x})$ satisfies Eq. (60), at the initial time $\tau = \tau_0$, one has $\chi_T^{\parallel(2)} = \chi_{T0}^{\parallel(2)}$. Notice that the solution (64) can also be derived by the traceless part of the evolution equation (27) together with the momentum constraint (26).

C. Vector perturbation $\chi_{Tij}^{\perp(2)}$

The time integral of the momentum constraint (26) from τ_0 to τ is

$$\begin{aligned} & 2\phi_{T,j}^{(2)} + \frac{1}{3}\nabla^2\chi_{T,j}^{\parallel(2)} + \frac{1}{2}\chi_{Tij}^{\perp(2),i} \\ & = \int_{\tau_0}^{\tau}d\tau' M_{Tj} + 2\phi_{T0,j}^{(2)} + \frac{1}{3}\nabla^2\chi_{T0,j}^{\parallel(2)} + \frac{1}{2}\chi_{T0ij}^{\perp(2),i}. \end{aligned} \tag{65}$$

Using M_{Tj} in (29), one has

$$\begin{aligned} \int_{\tau_0}^{\tau}d\tau' M_{Tj} = {} & \frac{1}{2}\int_{\tau_0}^{\tau}P_j(\mathbf{x},\tau')d\tau' - \chi^{\top(1)kl}\chi_{kl,j}^{\top(1)} \\ & + \chi_0^{\top(1)kl}\chi_{0kl,j}^{\top(1)}, \end{aligned} \tag{66}$$

where

$$P_j(\mathbf{x},\tau) \equiv 2\chi^{\top(1)kl}\chi_{jk,l}^{\top(1)\prime} + \chi_{,j}^{\top(1)kl}\chi_{kl}^{\top(1)\prime}. \tag{67}$$

Plugging the solutions $\phi_T^{(2)}$ of (59) and $\chi_T^{\parallel(2)}$ of (64) into (65), after calculations similar to Sec. IV C, the vector perturbation is obtained:

$$\begin{aligned} \chi_{Tij}^{\perp(2)} = {} & \nabla^{-2}(N_{i,j} + N_{j,i}) + \nabla^{-2}\int_{\tau_0}^{\tau}[\partial_i P_j + \partial_j P_i]d\tau' \\ & - \partial_i\partial_j\nabla^{-2}\nabla^{-2}\left[2\chi^{\top(1)kl,m}\chi_{km,l}^{\top(1)} + \chi^{\top(1)kl,m}\chi_{kl,m}^{\top(1)}\right. \\ & \left. + \chi_{kl}^{\top(1)\prime}\chi^{\top(1)\prime kl} + 8\int_{\tau_0}^{\tau}\frac{\chi_{kl}^{\top(1)\prime}\chi^{\top(1)\prime kl}}{\tau'}d\tau'\right], \end{aligned} \tag{68}$$

where the constant 3-vector

$$\begin{aligned} N_j \equiv {} & \chi_{T0ij}^{\perp(2),i} + \partial_j\nabla^{-2}\left(\chi_0^{\top(1)kl,m}\chi_{0km,l}^{\top(1)} + \frac{1}{2}\chi_0^{\top(1)kl,m}\chi_{0kl,m}^{\top(1)}\right. \\ & \left. + \frac{1}{2}\chi_{0kl}^{\top(1)\prime}\chi_0^{\top(1)\prime kl}\right), \end{aligned} \tag{69}$$

depending on the initial values at τ_0. Notice that the solution (68) can be also derived by the traceless part of the evolution equation (27) together with the momentum constraint (26).

D. Tensor perturbation $\chi_{Tij}^{\top(2)}$

Next consider the traceless part of the evolution equation (27):

$$\begin{aligned} & \chi_{Tij}^{\top(2)\prime\prime} + \frac{4}{\tau}\chi_{Tij}^{\top(2)\prime} - \nabla^2\chi_{Tij}^{\top(2)} \\ & = 2\bar{S}_{Tij} - \left(2D_{ij}\phi_T^{(2)} + \frac{1}{3}\nabla^2 D_{ij}\chi_T^{\parallel(2)}\right) \\ & \quad - \left(D_{ij}\chi_T^{\parallel(2)\prime\prime} + \frac{4}{\tau}D_{ij}\chi_T^{\parallel(2)\prime}\right) - \left(\chi_{Tij}^{\perp(2)\prime\prime} + \frac{4}{\tau}\chi_{Tij}^{\perp(2)\prime}\right), \end{aligned} \tag{70}$$

where

$$\begin{aligned}\bar{S}_{Tij} &\equiv S_{Tij} - \frac{1}{3}\delta_{ij}S^k_{Tk} \\ &= \chi_i^{\top(1)'k}\chi_{kj}^{\top(1)'} - \frac{1}{3}\chi^{\top(1)'kl}\chi_{kl}^{\top(1)'}\delta_{ij} + \chi^{\top(1)kl}\chi^{\top(1)}_{li,jk} \\ &\quad + \chi^{\top(1)kl}\chi^{\top(1)}_{lj,ik} - \chi^{\top(1)kl}\chi^{\top(1)}_{kl,ij} - \chi^{\top(1)kl}\chi^{\top(1)}_{ij,kl} \\ &\quad + \chi^{\top(1),k}_{li}\chi^{\top(1),l}_{kj} - \chi^{\top(1)k}_{i,l}\chi^{\top(1),l}_{jk} - \frac{1}{2}\chi^{\top(1)kl}_{,i}\chi^{\top(1)}_{kl,j} \\ &\quad + \frac{1}{2}\chi^{\top(1)kl,m}\chi^{\top(1)}_{kl,m}\delta_{ij} - \frac{1}{3}\chi^{\top(1)}_{ml,k}\chi^{\top(1)mk,l}\delta_{ij} \\ &\quad + \frac{1}{3}\chi^{\top(1)kl}\nabla^2\chi^{\top(1)}_{kl}\delta_{ij}. \end{aligned} \tag{71}$$

By calculations similar to Sec. IV.D, Eq. (70) is written as

$$\chi^{\top(2)''}_{Tij} + \frac{4}{\tau}\chi^{\top(2)'}_{Tij} - \nabla^2\chi^{\top(2)}_{Tij} = J_{Tij}(\mathbf{x},\tau), \tag{72}$$

where the source

$$\begin{aligned}J_{Tij}(\mathbf{x},\tau) &\equiv 2\bar{S}_{Tij} + \nabla^{-2}\nabla^{-2}\bar{S}^{,kl}_{Tkl,ij} + \delta_{ij}\nabla^{-2}\bar{S}^{,kl}_{Tkl} \\ &\quad - 2\nabla^{-2}\bar{S}^{,k}_{Tki,j} - 2\nabla^{-2}\bar{S}^{,k}_{Tkj,i}. \end{aligned} \tag{73}$$

The differential equation (72) is inhomogeneous, and its solution is given by

$$\begin{aligned}\chi^{\top(2)}_{Tij}(\mathbf{x},\tau) &= \frac{1}{(2\pi)^{3/2}}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\left(\bar{I}_{Tij}(s) + \frac{c_{1ij}}{s^{3/2}}H^{(1)}_{\frac{3}{2}}(s)\right. \\ &\quad \left. + \frac{c_{2ij}}{s^{3/2}}H^{(2)}_{\frac{3}{2}}(s)\right), \end{aligned} \tag{74}$$

where $s \equiv k\tau$,

$$\begin{aligned}\bar{I}_{Tij}(s) &\equiv \frac{1}{s^2}\left(\cos s - \frac{\sin s}{s}\right)\int_1^s dy y^2\left(\sin y + \frac{\cos y}{y}\right)\bar{J}_{Tij}(y) \\ &\quad - \frac{1}{s^2}\left(\sin s + \frac{\cos s}{s}\right)\int_1^s dy y^2\left(\cos y - \frac{\sin y}{y}\right)\bar{J}_{Tij}(y), \end{aligned} \tag{75}$$

with $\bar{J}_{Tij}$ being the Fourier transformation of the source J_{Tij}. In (74) c_{1ij} and c_{2ij} terms represent a homogeneous solution, which should be determined by the initial condition at τ_0.

Thus, all the 2nd-order metric perturbations produced by tensor-tensor coupling have been obtained. Consequently, by (18), the corresponding 2nd-order density contrast

$$\delta^{(2)} = \delta^{(2)}_{T0} + 3(\phi^{(2)}_T - \phi^{(2)}_{T0}) + \frac{1}{2}(\chi^{\top(1)ij}\chi^{\top(1)}_{ij} - \chi_0^{\top(1)ij}\chi^{\top(1)}_{0ij}), \tag{76}$$

which can be written as

$$\begin{aligned}\delta^{(2)}_T &= -\frac{\tau^2}{20}K + \frac{3}{\tau^3}B - \frac{1}{10\tau^3}\int_{\tau_0}^{\tau}\tau'^4\chi_{kl}^{\top(1)'}\chi^{\top(1)'kl}d\tau' \\ &\quad + \frac{\tau^2}{10}\int_{\tau_0}^{\tau}\frac{\chi_{kl}^{\top(1)'}\chi^{\top(1)'kl}}{\tau'}d\tau'. \end{aligned} \tag{77}$$

So far, the 2nd-order metric perturbations have been obtained using the scalar-scalar, scalar-tensor, and tensor-tensor couplings. We can qualitatively assess which coupling is dominant during MD stage. By the solution (20) in the paper I [49] of the 1st order of perturbations, the scalar is $\propto\tau^2$, increasing with time, the tensor by (19) in the paper I [49] is $\propto\tau^{-\frac{3}{2}}H^{(1)}_{\frac{3}{2}}(k\tau), \tau^{-\frac{3}{2}}H^{(2)}_{\frac{3}{2}}(k\tau)$, whose amplitude is decreasing with time. So, the scalar-scalar terms are increasing as τ^4, the tensor-tensor terms are decreasing quickly, and the tensor-scalar terms behave as $\propto\tau^{1/2}H^{(1)}_{\frac{3}{2}}(k\tau), \tau^{1/2}H^{(2)}_{\frac{3}{2}}(k\tau)$, which are decreasing over the whole range at a slower rate than the tensor-tensor terms. Thus, qualitatively speaking, the scalar-scalar terms are dominant over the tensor terms during evolution. Therefore, the corresponding solutions of metric perturbations also share these generic features.

In applications, one has to deal with the second-order degrees of gauge freedom in these solutions, which is discussed in the latter section.

VI. THE 2ND-ORDER GAUGE TRANSFORMATIONS

Consider the coordinate transformation up to 2nd order [47,49]:

$$x^\mu \to \bar{x}^\mu = x^\mu + \xi^{(1)\mu} + \frac{1}{2}\xi^{(1)\mu}_{,\alpha}\xi^{(1)\alpha} + \frac{1}{2}\xi^{(2)\mu}, \tag{78}$$

where $\xi^{(1)}$ is a 1st-order vector field, $\xi^{(2)}$ is a 2nd-order vector field, and can be written in terms of their respective parameters

$$\xi^{(A)0} = \alpha^{(A)}, \tag{79}$$

$$\xi^{(A)i} = \partial^i\beta^{(A)} + d^{(A)i}, \tag{80}$$

with $A = 1, 2$ and a constraint $\partial_i d^{(A)i} = 0$. The 1st order gauge transformations between two synchronous coordinate systems for the Einstein-de Sitter model with $a(\tau) \propto \tau^2$ are listed in (179)–(183) in Ref. [49]. The general 2nd-order synchronous-to-synchronous gauge transformations of metric perturbations are given by (201), (207)–(209) of Ref. [49], which are valid for general cosmic expansion stages. In the following we apply them to the case of $a(\tau) \propto \tau^2$. So far in our paper, the perturbations of 4-velocity of dust have been taken to be $U^{(1)\mu} = U^{(2)\mu} = 0$.

It is proper to require also the transformed 3-velocity perturbations

$$\bar{U}^{(1)i} = 0, \tag{81}$$

$$\bar{U}^{(2)i} = 0, \tag{82}$$

in the new synchronous coordinate [41,49]. Under the constraints (81), the 1st-order vector field $\xi^{(1)\mu}$ is [49]

$$\alpha^{(1)} = \frac{A^{(1)}}{\tau^2}, \tag{83}$$

$$\beta^{(1)}_{,i} = C^{\|(1)}(\mathbf{x})_{,i}, \tag{84}$$

$$d^{(1)}_i = 0. \tag{85}$$

In the above, $A^{(1)}$ is an arbitrary constant, and $C^{\|(1)}(\mathbf{x})$ is an arbitrary function. The 1st-order residual gauge transformations are [49]

$$\bar{\phi}^{(1)} = \phi^{(1)} + 2\frac{A^{(1)}}{\tau^3} + \frac{1}{3}\nabla^2 C^{\|(1)}(\mathbf{x}), \tag{86}$$

$$\bar{D}_{ij}\chi^{\|(1)} = D_{ij}\chi^{\|(1)} - 2D_{ij}C^{\|(1)}(\mathbf{x}), \tag{87}$$

$$\bar{\chi}^{\top(1)}_{ij} = \chi^{\top(1)}_{ij}. \tag{88}$$

We shall first give the 2nd-order gauge transformations for the scalar-tensor coupling. From the general formulas (194), (199), and (200) of Ref. [49], keeping only the $\chi^{\top(1)}_{ij}$-linear- dependent terms and using the conditions (81) and (82), the 2nd-order vector field $\xi^{(2)\mu}$ is given as the following:

$$\alpha^{(2)} = \frac{A^{(2)}}{\tau^2}, \tag{89}$$

$$\beta^{(2)}_{,i} = C^{\|(2)}(\mathbf{x})_{,i}, \tag{90}$$

$$d^{(2)}_i = C^{\perp(2)}_i(\mathbf{x}), \tag{91}$$

where $A^{(2)}$ is an arbitrary constant, $C^{\|(2)}(\mathbf{x})$ is an arbitrary function, $C^{\perp(2)}_i(\mathbf{x})$ is an arbitrary curl vector; all of them shall be linearly depending on $\chi^{\top(1)}_{ij}$ at some fixed time. Accordingly, by the general formulas (201), (207)–(209) in Ref. [49] of the 2nd-order residual gauge transformation of metric perturbations, keeping only the scalar-tensor terms, one obtains:

$$\bar{\phi}^{(2)}_{s(t)} = \phi^{(2)}_{s(t)} + \frac{2}{3}C^{\|(1),kl}\chi^{\top(1)}_{kl} + \frac{2}{\tau^3}A^{(2)} + \frac{1}{3}\nabla^2 C^{\|(2)}, \tag{92}$$

$$\begin{aligned} D_{ij}\bar{\chi}^{\|(2)}_{s(t)} = {} & D_{ij}\chi^{\|(2)}_{s(t)} - D_{ij}\nabla^{-2}\nabla^{-2}[9C^{\|(1),klm}\chi^{\top(1)}_{kl,m} \\ & + 6\chi^{\top(1)}_{kl}\nabla^2 C^{\|(1),kl}] + 2D_{ij}\nabla^{-2}[C^{\|(1),kl}\chi^{\top(1)}_{kl}] \\ & - 2D_{ij}C^{\|(2)}, \end{aligned} \tag{93}$$

$$\begin{aligned} \bar{\chi}^{\perp(2)}_{s(t)ij} = {} & \chi^{\perp(2)}_{s(t)ij} - 2\partial_i\nabla^{-2}(\chi^{\top(1)}_{kj}\nabla^2 C^{\|(1),k} + 2\chi^{\top(1)}_{kj,l}C^{\|(1),kl} \\ & + C^{\|(1),kl}_{,j}\chi^{\top(1)}_{kl}) + \partial_i\partial_j\nabla^{-2}\nabla^{-2}(4\chi^{\top(1)}_{kl}\nabla^2 C^{\|(1),kl} \\ & + 6C^{\|(1),klm}\chi^{\top(1)}_{kl,m}) - C^{\perp(2)}_{i,j} + (i \leftrightarrow j), \end{aligned} \tag{94}$$

$$\begin{aligned} \bar{\chi}^{\top(2)}_{s(t)ij} = {} & \chi^{\top(2)}_{s(t)ij} - \left[\frac{8}{\tau^3}A^{(1)}\chi^{\top(1)}_{ij} + \frac{2}{\tau^2}A^{(1)}\chi^{\top(1)\prime}_{ij} + 2C^{\|(1),k}\chi^{\top(1)}_{ij,k}\right] + \nabla^{-2}[-2C^{\|(1),k}_{,i}\nabla^2\chi^{\top(1)}_{kj} - 2C^{\|(1),k}_{,j}\nabla^2\chi^{\top(1)}_{ki} \\ & + 2\chi^{\top(1)}_{ki,j}\nabla^2 C^{\|(1),k} + 2\chi^{\top(1)}_{kj,i}\nabla^2 C^{\|(1),k} + 4C^{\|(1),kl}\chi^{\top(1)}_{ki,jl} + 4C^{\|(1),kl}\chi^{\top(1)}_{kj,il} + 2C^{\|(1),kl}_{,ij}\chi^{\top(1)}_{kl} \\ & - 2C^{\|(1),kl}\chi^{\top(1)}_{kl,ij} + 2C^{\|(1),kl}\nabla^2\chi^{\top(1)}_{kl}\delta_{ij} + C^{\|(1),klm}\chi^{\top(1)}_{kl,m}\delta_{ij}] - \partial_i\partial_j\nabla^{-2}\nabla^{-2}[2\chi^{\top(1)}_{kl}\nabla^2 C^{\|(1),kl} + 3C^{\|(1),klm}\chi^{\top(1)}_{kl,m}], \end{aligned} \tag{95}$$

where the constants $A^{(1)}$, $C^{\|(1)}(\mathbf{x})$, $C^{\perp(1)}_i(\mathbf{x})$ are all independent of the tensor $\chi^{\top(1)}_{ij}$. As Eq. (95) tells, the transformation of 2nd-order tensor involves only the vector field $\xi^{(1)}$, independent of $\xi^{(2)}$.

It should be pointed out that the roles of $\xi^{(1)}$ and $\xi^{(2)}$ are different. When one sets $\xi^{(2)} = 0$ in Eq. (78) [49,51], only $\xi^{(1)}$ remains, which ensures $\bar{g}^{(1)}_{00} = 0$ and $\bar{g}^{(1)}_{0i} = 0$ in a new synchronous coordinate. Nevertheless, now one has no freedom to make $\bar{g}^{(2)}_{00} = 0$ and $\bar{g}^{(2)}_{0i} = 0$, since $\xi^{(1)}$ has already been used in obtaining $\bar{g}^{(1)}_{00} = 0$ and $\bar{g}^{(1)}_{0i} = 0$. Thus, 2nd-order transformations from synchronous to synchronous can not effectively be made when one sets $\xi^{(2)} = 0$. On the other hand, if one does not transform the 1st order, but only transforms the 2nd-order metric perturbations [52], one simply sets $\xi^{(1)} = 0$ but $\xi^{(2)} \neq 0$. Then (92)–(95) reduce to

$$\bar{\phi}^{(2)}_{s(t)} = \phi^{(2)}_{s(t)} + \frac{2}{\tau^3}A^{(2)} + \frac{1}{3}\nabla^2 C^{||(2)}, \tag{96}$$

$$D_{ij}\bar{\chi}^{||(2)}_{s(t)} = D_{ij}\chi^{||(2)}_{s(t)} - 2D_{ij}C^{||(2)}, \tag{97}$$

$$\bar{\chi}^{\perp(2)}_{s(t)ij} = \chi^{\perp(2)}_{s(t)ij} - (C^{\perp(2)}_{i,j} + C^{\perp(2)}_{j,i}), \tag{98}$$

$$\bar{\chi}^{\top(2)}_{s(t)ij} = \chi^{\top(2)}_{s(t)ij}. \tag{99}$$

From the transformation formula $\bar{\rho}^{(2)}_{s(t)} = \rho^{(2)}_{s(t)} - \mathcal{L}_{\xi^{(2)}}\rho^{(0)}$ [49], the transformation of the 2nd-order density perturbation is

$$\bar{\rho}^{(2)}_{s(t)} = \rho^{(2)}_{s(t)} + 6\frac{A^{(2)}}{\tau^3}\rho^{(0)}, \tag{100}$$

by which the transformation of the 2nd-order density contrast is given by

$$\bar{\delta}^{(2)}_{s(t)} = \delta^{(2)}_{s(t)} + 6\frac{A^{(2)}}{\tau^3}, \tag{101}$$

where $A^{(2)}$ shall be linear depending on $\chi^{\top(1)}_{ij}$ at some fixed time. These transformations by $\xi^{(2)}$ have the same structure as the 1st-order gauge transformations [49].

By this result, we can identify the residual gauge modes in the 2nd-order solutions for the scalar-tensor coupling. In the solution of scalar $\phi^{(2)}_{s(t)}$ in (33), the constant terms $(\phi^{(2)}_{s(t)0} - \frac{1}{3}\delta^{(2)}_{s(t)0} - \frac{\tau_0^2}{9}\varphi^{,ij}\chi^{\top(1)}_{0ij})$ are a residual gauge mode, which can be changed by a choice of $C^{||(2)}$. In the solution of scalar $D_{ij}\chi^{\|(2)}_{s(t)}$ in (38), the constant term $D_{ij}Z$ is also a gauge term that will be changed by $C^{||(2)}$ accordingly. Similarly, in the solution of vector $\chi^{\perp(2)}_{s(t)ij}$ in (44), the constant term $\nabla^{-2}(Q_{i,j} + Q_{j,i})$ is a residual gauge mode and can be removed by a choice of $(C^{\perp(2)}_{i,j} + C^{\perp(2)}_{j,i})$, but other time-dependent terms in (44) are not gauge modes. In contrast, the 2nd-order tensor is invariant under the transformation by $\xi^{(2)}$ as demonstrated by Eq. (99), and the solution of tensor in Eq. (53) thus contains no gauge mode.

Next consider the case of tensor-tensor coupling, the analysis is similar to the above paragraphs. In particular, the 2nd-order residual gauge transformation is effectively implemented only by the 2nd-order vector field $\xi^{(2)}$ even given nonzero $\xi^{(1)}$, and the gauge transformations are similar to (96)–(99) and (101)

$$\bar{\phi}^{(2)}_{T} = \phi^{(2)}_{T} + \frac{2}{\tau^3}A^{(2)} + \frac{1}{3}\nabla^2 C^{||(2)}(\mathbf{x}), \tag{102}$$

$$D_{ij}\bar{\chi}^{||(2)}_{T} = D_{ij}\chi^{||(2)}_{T} - 2D_{ij}C^{||(2)}(\mathbf{x}), \tag{103}$$

$$\bar{\chi}^{\perp(2)}_{Tij} = \chi^{\perp(2)}_{Tij} - (C^{\perp(2)}_{i,j}(\mathbf{x}) + C^{\perp(2)}_{j,i}(\mathbf{x})), \tag{104}$$

$$\bar{\chi}^{\top(2)}_{Tij} = \chi^{\top(2)}_{Tij}, \tag{105}$$

$$\bar{\delta}^{(2)}_{s(t)} = \delta^{(2)}_{s(t)} + 6\frac{A^{(2)}}{\tau^3}, \tag{106}$$

where $A^{(2)}$, $C^{||(2)}(\mathbf{x})$, $C^{\perp(2)}_{i}(\mathbf{x})$ shall depend on tensor-tensor terms such as $\chi^{\top(1)}_{ij}\chi^{\top(1)}_{kl}$ at some fixed time. By (102)–(105), the residual gauge modes in the solutions of 2nd-order metric perturbations for the tensor-tensor coupling can be identified similarly. For instance, the constant terms in the solutions (59), (64), and (68) are residual gauge modes, and can be changed by choices of $C^{||(2)}$ and $C^{\perp(2)}_{i}$ respectively. Furthermore, Eq. (105) shows that $\chi^{\top(2)}_{Tij}$ generated by the tensor-tensor coupling is gauge invariant, so that the solution of (74) in synchronous coordinates contains no gauge mode.

VII. CONCLUSION

We have studied the 2nd-order cosmological perturbations in the Einstein-de Sitter Universe in synchronous coordinates. The scalar-tensor and tensor-tensor types of couplings of 1st-order metric perturbations serve as a part of effective source for the 2nd-order metric perturbations. For each coupling, respectively, the 2nd-order perturbed Einstein equation has been solved with general initial conditions, and the explicit solutions of scalar, vector, and tensor 2nd-order metric perturbations have been obtained.

We have also performed general 2nd-order synchronous-to-synchronous gauge transformations, which are generated by a 1st-order vector field and a 2nd-order vector field. For the scalar-tensor and tensor-tensor couplings, respectively, we have identified all the residual gauge modes of the 2nd-order metric perturbations in synchronous coordinates. By analysis, we point out that, holding the 1st-order solutions fixed, only the 2nd-order transformation vector field is effective in carrying out the 2nd-order transformations. This is because of the fact that the 1st-order vector field has been already determined in the 1st-order transformations. In particular, the 2nd-order tensor is found to be invariant under 2nd-order gauge transformations just like the 1st-order tensor is invariant under the 1st-order transformations.

Thus, together with the case of scalar-scalar couplings in our previous work, we have obtained the full solution of the 2nd-order cosmological perturbations and all their residual gauge modes of the Einstein-de Sitter Universe in synchronous coordinates, where all the couplings of 1st-order perturbations are included. As a possible application of the

results of 2nd-order perturbations to CMB, one can use the derived expressions $\gamma_{ij}^{(1)} + \frac{1}{2}\gamma_{ij}^{(2)}$ into the Sachs-Wolfe term of the Boltzmann equation of photon gas. The corresponding spectra C_l^{XX} of on CMB anisotropies and polarization will contain the contributions from the 2nd-order effects of $\gamma_{ij}^{(2)}$.

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