

Second-order cosmological perturbations. I. Produced by scalar-scalar coupling in synchronous gauge

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(Received 4 January 2017; published 17 November 2017)

We present a systematic study of the 2nd-order scalar, vector, and tensor metric perturbations in the Einstein-de Sitter Universe in synchronous coordinates. For the scalar-scalar coupling between 1st-order perturbations, we decompose the 2nd-order perturbed Einstein equation into the respective field equations of 2nd-order scalar, vector, and tensor perturbations, and obtain their solutions with general initial conditions. In particular, the decaying modes of solution are included, the 2nd-order vector is generated even if the 1st-order vector is absent, and the solution of the 2nd-order tensor corrects that in literature. We perform general synchronous-to-synchronous gauge transformations up to 2nd order generated by a 1st-order vector field $\xi^{(1)\mu}$ and a 2nd-order $\xi^{(2)\mu}$. All the residual gauge modes of 2nd-order metric perturbations and density contrast are found, and their number is substantially reduced when the transformed 3-velocity of dust is set to zero. Moreover, we show that only $\xi^{(2)\mu}$ is effective in carrying out 2nd-order transformations that we consider, because $\xi^{(1)\mu}$ has been used in obtaining the 1st-order perturbations. Holding the 1st-order perturbations fixed, the transformations by $\xi^{(2)\mu}$ on the 2nd-order perturbations have the same structure as those by $\xi^{(1)\mu}$ on the 1st-order perturbations.

DOI: [10.1103/PhysRevD.96.103522](https://doi.org/10.1103/PhysRevD.96.103522)

I. INTRODUCTION

Studies of metric perturbations of Robertson-Walker spacetimes within general relativity constitute the theoretical foundation of cosmology. In the past the perturbations, both scalar and tensorial, have been extensively explored to linear order [1–6], which have been used in calculations of large scale structure [7], cosmic microwave background radiation (CMB) [8–18], and relic gravitational wave (RGW) [19–29]. In the era of precision cosmology, it is necessary to study the 2nd-order perturbation beyond the linear perturbation, to include nonlinear effects of gravity on CMB anisotropies and polarization [30,31], the non-Gaussianity of primordial perturbation [32], and on relic gravitational waves [33,34] etc. Recently LIGO collaboration announced its direct detections of gravitational waves emitted from binary black holes [35]. RGW is not detected yet and only constraints are given upon the background of GW, including RGW, in a band 10–2000 Hz [36]. The current observational constraint on RGW is given by CMB observations, which is given in terms of the tensor-scalar ratio $r < 0.1$ over very low frequencies $10^{-18} \sim 10^{-16}$ Hz [37,38]. Since both scalar and tensor metric perturbations are generated during inflation, one might expect that they should be of the same order of amplitude. In some class of scalar inflation models, the ratio is predicted to be $r = 16\epsilon$, where ϵ is the slow-roll parameter. We like to see other possible mechanisms than the inflation, which may change the tensor perturbation during

the cosmic evolution. We are mainly motivated by this issue, and want to examine how the evolution of RGW during expansion will be affected by the metric perturbation itself. This is encoded in the 2nd-order perturbed Einstein equation, in which all three irreducible parts of metric perturbations will appear: scalar, vector, and tensor. In literature the 2nd-order perturbation has been studied to certain extent. Tomita started a study of the 2nd-order perturbations in synchronous coordinates, in particular, and gave the general equations of 2nd-order perturbations [39], analyzed the 2nd-order density contrast in some special cases [40]. In the context of large scale structure, gravitational instability was studied in the 2nd-order perturbations [41–43]. Ref. [44] studies the gauge-invariant definition of the second order curvature perturbation. The 2nd-order perturbations were studied in the Arnowitt-Deser-Misner framework [45,46], and the equation of density was given with the source of squared RGW [47]. A gauge-invariant formulation of 2nd-order perturbations was developed in Ref. [48,49]. Matarrese *et al.* derived the field equations of 2nd scalar and tensor perturbations in the Einstein-de Sitter model in the case of scalar-scalar coupling [50,51], but the 2nd-order vector is not given, and the 2nd-order tensor is not complete. The 2nd-order vector due to scalar-scalar coupling is explored in Poisson gauge [31,52]. Ref. [52] calculated only the 2nd-order vector perturbation in Poisson gauge, and did not give the 2nd-order scalar and tensor perturbations. In Λ CDM framework, Ref. [53] calculated 2nd-order scalar and vector perturbations in the Poisson gauge.

In this paper we consider the Einstein-de Sitter model filled with irrotational matter, the 1st-order vector

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perturbation can be dropped as a gauge mode. However, the 2nd-order vector perturbation will inevitably appear along with the 2nd-order scalar and tensor perturbations. Moreover, these metric perturbations are coupled. In the 2nd-order perturbed Einstein equation, there are three kinds of coupling terms: scalar-scalar, scalar-tensor, and tensor-tensor, which are products of 1st-order metric perturbations. The (00) component of the equation is the energy constraint and the (0i) component is the momentum constraint; neither contain second-order time derivatives. The (ij) components are evolution equations and involve second-order time derivatives of metric perturbations, which need to be decomposed into equations of scalar, vector, and tensor, respectively. This will involve lengthy calculations. We shall write the 2nd-order perturbed Einstein equations into three sets of equations, each having, respectively, the coupling of scalar-scalar, scalar-tensor, and tensor-tensor. For the scalar-scalar coupling, we present a complete decomposition, derive the equations of 2nd-order scalar, vector, and tensor perturbation, respectively, and obtain their solutions with general initial conditions. Moreover, perturbations contain residual gauge modes in synchronous coordinates. We shall also calculate 2nd-order residual gauge transformations from synchronous to synchronous, and identify the residual gauge modes of the 2nd-order scalar, vector, tensor metric perturbations.

In Sec. II, we briefly review the results of 1st-order perturbations, which will be used later.

In Sec. III, we write the 2nd-order Einstein equations into three sets, according to scalar-scalar, scalar-tensor, and tensor-tensor couplings, respectively.

In Sec. IV, for the scalar-scalar coupling, we derive the solutions of the 2nd-order scalar, vector, tensor perturbations.

In Sec. V, we derive the residual gauge modes in the 2nd-order metric, and density perturbations in synchronous coordinates.

Section VI is the conclusions and discussion.

In the Appendixes, we attach the technical details of some formulas used in the paper. Appendix A lists the 1st-order metric perturbations and the density contrast. Appendix B gives the 2nd-order perturbed Ricci tensors. Appendix C gives the synchronous-to-synchronous gauge transformations up to 2nd order for a Robertson-Walker (RW) spacetime. Appendix D lists the 2nd-order synchronous-to-Poisson gauge transformation. We work with the synchronous coordinates, adopt mostly the notation in Ref. [51] for comparison, and take the speed of light $c = 1$.

II. FIRST-ORDER PERTURBATIONS

In this section, we introduce notations and outline the results of 1st-order perturbations, which will be used in later sections. We consider the Universe filled with the irrotational, pressureless dust with the energy-momentum tensor $T^{\mu\nu} = \rho U^\mu U^\nu$, where ρ is the mass density, $U^\mu = (a^{-1}, 0, 0, 0)$ is 4-velocity such that $U^\mu U_\mu = -1$. We take

the perturbations of velocity to be $U^{(1)\mu} = U^{(2)\mu} = 0$ [see relevant discussions in the paragraph around Eq. (C41) in Appendix C], but the density with perturbations to be

$$\rho = \rho^{(0)} \left(1 + \delta^{(1)} + \frac{1}{2} \delta^{(2)} \right), \quad (1)$$

where $\rho^{(0)}$ is the background density, $\delta^{(1)}$, $\delta^{(2)}$ are the 1st, 2nd-order density contrasts. The nonvanishing components are $T_{00} = a^2 \rho$ and $T^{00} = a^{-2} \rho$.

The spatial flat RW metric in synchronous coordinates

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\tau) [-d\tau^2 + \gamma_{ij} dx^i dx^j], \quad (2)$$

where τ is conformal time, $a(\tau) \propto \tau^2$ for the Einstein-de Sitter model, γ_{ij} is written as

$$\gamma_{ij} = \delta_{ij} + \gamma_{ij}^{(1)} + \frac{1}{2} \gamma_{ij}^{(2)}, \quad (3)$$

where $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(2)}$ are the 1st- and 2nd-order metric perturbations, respectively. From (3), one has $g^{ij} = a^{-2} \gamma^{ij}$ with $\gamma^{ij} = \delta^{ij} - \gamma^{(1)ij} - \frac{1}{2} \gamma^{(2)ij} + \gamma^{(1)ik} \gamma_k^{(1)j}$. In this paper, we use the same notations as in Ref. [51] for simple comparisons, and use the indices $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, 3$. The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \quad (4)$$

The 0th order Einstein equation is

$$\left(\frac{a'}{a} \right)^2 = \frac{8\pi G}{3} a^2 \rho^{(0)}, \quad 2 \frac{a''}{a} - \left(\frac{a'}{a} \right)^2 = 0, \quad (5)$$

which also imply the continuity equation $\rho^{(0)'} + 3 \frac{a'}{a} \rho^{(0)} = 0$, where the prime denotes time derivative with respect to τ . The perturbed Einstein equation is

$$G_{\mu\nu}^{(A)} = 8\pi G T_{\mu\nu}^{(A)}, \quad A = 1, 2, \quad (6)$$

where we shall study up to 2nd order. For each order of (6), the (00) component is the energy constraint, (0i) components are the momentum constraints, and (ij) components contain the evolution equations. The set of equations (6) determines the dynamics of gravitational systems, and also implies conservation of energy and momentum of matter, $T^{(A)\mu\nu}{}_{;\nu} = 0$.

The 1st-order metric perturbation $\gamma_{ij}^{(1)}$ can be written as

$$\gamma_{ij}^{(1)} = -2\phi^{(1)} \delta_{ij} + \chi_{ij}^{(1)}, \quad (7)$$

where $\phi^{(1)}$ is the trace part of the scalar perturbation, and $\chi_{ij}^{(1)}$ is traceless and can be further decomposed into a scalar and a tensor

$$\chi_{ij}^{(1)} = D_{ij} \chi^{||1} + \chi_{ij}^{\top(1)}, \quad (8)$$

where $D_{ij} \equiv \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2$ is a traceless operator $D^i_i = 0$, $\chi^{\parallel(1)}$ is a scalar function, and $D_{ij} \chi^{\parallel(1)}$ is the traceless part of the scalar perturbation, and $\chi_{ij}^{\top(1)}$ is the tensor part (relic gravitational wave), satisfying the traceless and transverse conditions: $\chi^{\top(1)i}_i = 0$, $\partial^i \chi_{ij}^{\top(1)} = 0$. Thus the scalar perturbations have two modes: $\phi^{(1)}$ and $D_{ij} \chi^{\parallel(1)}$. In this paper, we do not consider the 1st-order vector perturbation which is a residual gauge mode and can be set to 0 since the matter is an irrotational dust. See the paragraph below (C22) in Appendix C for an explanation. However, as shall be seen later, the 2nd-order vector perturbation will inevitably appear due to couplings of the 1st-order perturbations. Thus, the 2nd-order perturbation is written as

$$\gamma_{ij}^{(2)} = -2\phi^{(2)}\delta_{ij} + \chi_{ij}^{(2)}, \quad (9)$$

with the traceless part

$$\chi_{ij}^{(2)} = D_{ij} \chi^{\parallel(2)} + \chi_{ij}^{\perp(2)} + \chi_{ij}^{\top(2)}, \quad (10)$$

where the vector mode satisfies a condition

$$\partial^i \partial^j \chi_{ij}^{\perp(2)} = 0, \quad (11)$$

which can be written in terms of a curl vector

$$\chi_{ij}^{\perp(2)} = 2A_{(i,j)} \equiv \partial_i A_j + \partial_j A_i, \quad \partial^i A_i = 0. \quad (12)$$

Since the 3-vector A_i is divergenceless and has only two independent components, the vector metric perturbation $\chi_{ij}^{\perp(2)}$ has two independent polarization modes, correspondingly.

The 1st-order perturbations are well known (see Appendix A). We list the solutions that will appear in the 2nd-order equations as the source. From Eq. (A26) in Appendix A, the solution of the 1st-order matter density contrast is

$$\delta^{(1)} = \delta_0^{(1)} - 3(\phi^{(1)} - \phi_0^{(1)}), \quad (13)$$

where $\delta_0^{(1)}$ is the initial value of the 1st-order density contrast and $\phi_0^{(1)}$ is the initial value of scalar perturbation $\phi^{(1)}$. The 1st-order perturbed Einstein equation consists of $G_{00}^{(1)} = 8\pi G a^2 \rho^{(0)} \delta^{(1)}$, $G_{0i}^{(1)} = 0$, $G_{ij}^{(1)} = 0$, where $T_{\mu}^{(1)} = 0$ for the dust. The solutions of scalar are

$$\phi^{(1)}(\mathbf{x}, \tau) = \frac{5}{3} \varphi(\mathbf{x}) + \frac{\tau^2}{18} \nabla^2 \varphi(\mathbf{x}) + \frac{X(\mathbf{x})}{\tau^3}, \quad (14)$$

$$D_{ij} \chi^{\parallel(1)}(\mathbf{x}, \tau) = -\frac{\tau^2}{3} \left(\varphi(\mathbf{x})_{,ij} - \frac{1}{3} \delta_{ij} \nabla^2 \varphi(\mathbf{x}) \right) - \frac{6\nabla^2 D_{ij} X(\mathbf{x})}{\tau^3}; \quad (15)$$

the 1st-order density contrast is

$$\delta^{(1)} = \frac{1}{6} \tau^2 \nabla^2 \varphi + \frac{3X}{\tau^3}. \quad (16)$$

In the above, φ is the gravitational potential at time τ_0

$$\nabla^2 \varphi = \frac{6}{\tau_0^2} \delta_{0g}^{(1)}, \quad (17)$$

determined by the initial density of the growing mode, and X represents the decaying mode of perturbations such that $\frac{3X}{\tau_0^3}$ is the initial density contrast of decaying mode. The reason to keep the decaying mode is the following: in principle, for a full treatment, the initial condition at τ_0 , i.e. at the radiation-matter equality ($z \sim 3500$), should be determined by the connection of physical quantities such as the energy density etc., which generally leads to continuous connection of perturbations and their time derivatives. For such a connection to be made consistently, both growing and decaying modes in solutions of perturbations should be kept. This is true for the 1st-order, as well as the 2nd-order perturbations, respectively.

The solution of tensor is

$$\chi_{ij}^{\top(1)}(\mathbf{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{s=+,\times} \hat{\epsilon}_{ij}(k) \hat{h}_k^s(\tau), \quad (18)$$

$$\mathbf{k} = k \hat{k},$$

with two polarization tensors satisfying

$$\hat{\epsilon}_{ij}(k) \delta_{ij} = 0, \quad \hat{\epsilon}_{ij}(k) k_i = 0, \quad \hat{\epsilon}_{ij}(k) \hat{\epsilon}'_{ij}(k) = 2\delta_{ss'}.$$

During the matter dominant stage the mode is given by

$$\hat{h}_k^s(\tau) = \frac{1}{a(\tau)} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\tau}{2}} [d_1^s(k) H_{\frac{3}{2}}^{(1)}(k\tau) + d_2^s(k) H_{\frac{3}{2}}^{(2)}(k\tau)], \quad (19)$$

where the coefficients d_1^s, d_2^s are determined by the initial condition during inflation and by subsequent evolutions through the reheating, radiation dominant stages [19,28]. Here cosmic processes, such as neutrino free-streaming [54,55], QCD transition, and e^+e^- annihilation [56] only slightly modify the amplitude of RGW and will be neglected in this study. For relic gravitational waves (RGW) generated during inflation [19,22–25], the two modes $\hat{h}_k^s(\tau)$ with $s = +, \times$ are usually assumed to be statistically equivalent, the superscript s can be dropped.

Put together, the solution of 1st-order metric perturbation is

$$\gamma_{ij}^{(1)} = -\frac{10}{3}\varphi\delta_{ij} - \frac{\tau^2}{3}\varphi_{,ij} - \frac{6}{\tau^3}\nabla^{-2}X_{,ij} + \chi_{ij}^{\top(1)}, \quad (20)$$

which will be used later. We remark that the evolution equations [see Eqs. (A13) and (A14) in Appendix A] of scalars $\phi^{(1)}$ and $\chi^{\parallel(1)}$ are not wave equations, in contrast to Eq. (A11) for tensor $\chi_{ij}^{\top(1)}$. Thus, the scalar perturbations do not propagate at the speed of light, but just follow where the density perturbation is distributed.

III. THE SECOND-ORDER PERTURBED EINSTEIN EQUATIONS

The (00) component of the 2nd-order perturbed Einstein equation is

$$G_{00}^{(2)} \equiv R_{00}^{(2)} - \frac{1}{2}g_{00}^{(0)}R^{(2)} = 4\pi Ga^2\rho^{(0)}\delta^{(2)}, \quad (21)$$

where $\delta^{(2)}$ is the 2nd-order density contrast. Calculations give

$$\begin{aligned} \nabla^2\phi^{(2)} - \frac{3a'}{a}\phi^{(2)'} + \frac{1}{6}\nabla^2\nabla^2\chi^{\parallel(2)} &= 4\pi Ga^2\rho^{(0)}\delta^{(2)} + 12\frac{a'}{a}\phi^{(1)'}\phi^{(1)} - 3\phi^{(1)'}\phi^{(1)'} - 3\phi_{,k}^{(1)}\phi^{(1),k} - 8\phi^{(1)}\nabla^2\phi^{(1)} \\ &\quad - 2\phi^{(1)}D^{kl}\chi_{,kl}^{\parallel(1)} + \phi_{,kl}^{(1)}D^{kl}\chi^{\parallel(1)} + \frac{1}{8}D^{kl}\chi^{\parallel(1)'}D_{kl}\chi^{\parallel(1)'} + \frac{a'}{a}D^{kl}\chi^{\parallel(1)}D_{kl}\chi^{\parallel(1)'} \\ &\quad + D_{m\ell}\chi_{,k}^{\parallel(1),l}D^{km}\chi^{\parallel(1)} - \frac{1}{2}D^{km}\chi^{\parallel(1)}\nabla^2D_{km}\chi^{\parallel(1)} + \frac{1}{2}D^{km}\chi_{,k}^{\parallel(1)}D_{m\ell}\chi^{\parallel(1),l} \\ &\quad - \frac{3}{8}D^{km}\chi_{,l}^{\parallel(1)}D_{km}\chi^{\parallel(1),l} + \frac{1}{4}D^{km}\chi_{,l}^{\parallel(1)}D_{k\ell}^l\chi^{\parallel(1),m} + \chi^{\top(1)kl}\phi_{,kl}^{(1)} + \frac{1}{4}\chi_{kl}^{\top(1)'}D^{kl}\chi^{\parallel(1)'} \\ &\quad + \frac{a'}{a}\chi^{\top(1)kl}D_{kl}\chi^{\parallel(1)'} + \frac{a'}{a}\chi_{kl}^{\top(1)'}D^{kl}\chi^{\parallel(1)} + \chi^{\top(1)km}D_{m\ell}\chi_{,k}^{\parallel(1),l} - \frac{1}{2}\chi^{\top(1)km}\nabla^2D_{km}\chi^{\parallel(1)} \\ &\quad - \frac{1}{2}D^{km}\chi^{\parallel(1)}\nabla^2\chi_{km}^{\top(1)} - \frac{3}{4}\chi_{km}^{\top(1),l}D^{km}\chi_{,l}^{\parallel(1)} + \frac{1}{2}\chi_{k,m}^{\top(1)l}D^{km}\chi_{,l}^{\parallel(1)} + \frac{1}{8}\chi^{\top(1)kl}\chi_{kl}^{\top(1)'} \\ &\quad + \frac{a'}{a}\chi^{\top(1)kl}\chi_{kl}^{\top(1)'} - \frac{1}{2}\chi^{\top(1)km}\nabla^2\chi_{km}^{\top(1)} - \frac{3}{8}\chi_{,l}^{\top(1)km}\chi_{km}^{\top(1),l} + \frac{1}{4}\chi_{,l}^{\top(1)km}\chi_{k,m}^{\top(1)l}, \end{aligned} \quad (22)$$

where the coupling terms of 1st-order perturbations are moved to the rhs of the equation, and they serve as an effective source of 2nd-order metric perturbations besides $4\pi Ga^2\rho^{(0)}\delta^{(2)}$. By comparison, the structure of Eq. (22) is similar to Eq. (A6) of the 1st order except these coupling terms. Using $\delta^{(2)}$ of (A29) in Appendix A and Eqs. (14) and (15) to express the 1st-order perturbations in terms of the potential φ , Eq. (22) is written as the 2nd-order energy constraint [51]:

$$\frac{2}{\tau}\phi^{(2)'} + \frac{6}{\tau^2}\phi^{(2)} - \frac{1}{3}\nabla^2\phi^{(2)} - \frac{1}{12}D^{ij}\chi_{,ij}^{\parallel(2)} = E_S + E_{s(t)} + E_T, \quad (23)$$

where

$$\begin{aligned} E_S &\equiv \left(\frac{100}{27} + \frac{20\tau_0^2}{9\tau^2}\right)\varphi\nabla^2\varphi + \frac{25}{9}\varphi_{,i}\varphi^{,i} + \left(-\frac{5\tau^2}{54} + \frac{\tau_0^4}{9\tau^2}\right)\varphi_{,ij}\varphi^{,ij} + \frac{5\tau^2}{27}\varphi_{,i}\nabla^2\varphi^{,i} + \left(\frac{4\tau^2}{27} + \frac{\tau_0^4}{18\tau^2}\right)(\nabla^2\varphi)^2 - \frac{\tau^4}{216}\nabla^2\varphi^{,i}\nabla^2\varphi_{,i} \\ &\quad + \frac{\tau^4}{216}\varphi^{,ijk}\varphi_{,ijk} - \frac{2}{\tau^2}\delta_{s0}^{(2)} + \frac{6}{\tau^2}\phi_{s0}^{(2)} - \frac{9}{2\tau^8}X^2 + \frac{45}{2\tau^8}\nabla^{-2}X_{,kl}\nabla^{-2}X^{,kl} - \frac{3}{2\tau^6}X^{,k}X_{,k} + \frac{3}{2\tau^6}\nabla^{-2}X^{,kml}\nabla^{-2}X_{,kml} \\ &\quad + \frac{1}{3\tau^3}X\nabla^2\varphi + \frac{10}{3\tau^3}X^{,k}\varphi_{,k} + \frac{5}{3\tau^3}\varphi_{,kl}\nabla^{-2}X^{,kl} + \frac{36}{\tau_0^6\tau^2}\nabla^{-2}X^{,ij}\nabla^{-2}X_{,ij} + \frac{18}{\tau_0^6\tau^2}X^2 + \frac{40}{\tau_0^3\tau^2}\varphi X + \frac{4}{\tau_0\tau^2}\varphi_{,ij}\nabla^{-2}X^{,ij} \\ &\quad + \frac{2}{\tau_0\tau^2}X\nabla^2\varphi - \frac{1}{6\tau}X_{,k}\nabla^2\varphi^{,k} + \frac{1}{6\tau}\varphi^{,kml}\nabla^{-2}X_{,kml}, \end{aligned} \quad (24)$$

$$\begin{aligned}
E_{s(t)} \equiv & \frac{5\tau}{18}\chi^{\top(1)ij}\varphi_{,ij} + \frac{5}{9}\chi^{\top(1)ij}\varphi_{,ij} - \frac{\tau^2}{18}\varphi_{,ij}\nabla^2\chi^{\top(1)ij} - \frac{\tau^2}{36}\chi^{\top(1)ij,k}\varphi_{,ijk} - \frac{2\tau_0^2}{3\tau^2}\varphi^{ij}\chi_{0ij}^{\top(1)} - \frac{2}{\tau^2}\delta_{s(t)0}^{(2)} + \frac{6}{\tau^2}\phi_{s(t)0}^{(2)} \\
& + \frac{5}{2\tau^4}\chi_{kl}^{\top(1)'}\nabla^{-2}X^{,kl} - \frac{1}{\tau^3}\nabla^{-2}X^{,kl}\nabla^2\chi_{kl}^{\top(1)} - \frac{1}{2\tau^3}\chi_{km,l}^{\top(1)}\nabla^{-2}X^{,klm} - \frac{12}{\tau_0^3\tau^2}\chi_{0kl}^{\top(1)}\nabla^{-2}X^{,kl},
\end{aligned} \tag{25}$$

$$\begin{aligned}
E_T \equiv & -\frac{1}{24}\chi^{\top(1)ij}\chi_{ij}^{\top(1)'} - \frac{2}{3\tau}\chi^{\top(1)ij}\chi_{ij}^{\top(1)} + \frac{1}{6}\chi^{\top(1)ij}\nabla^2\chi_{ij}^{\top(1)} + \frac{1}{8}\chi^{\top(1)ij,k}\chi_{ij,k}^{\top(1)} - \frac{1}{12}\chi^{\top(1)ij,k}\chi_{kj,i}^{\top(1)} - \frac{1}{\tau^2}\chi^{\top(1)ij}\chi_{ij}^{\top(1)} \\
& + \frac{1}{\tau^2}\chi_0^{\top(1)ij}\chi_{0ij}^{\top(1)} - \frac{2}{\tau^2}\delta_{T0}^{(2)} + \frac{6}{\tau^2}\phi_{T0}^{(2)},
\end{aligned} \tag{26}$$

are the coupling terms of the 1st-order perturbations. All E_S , $E_{s(t)}$ and E_T contain the initial values $\delta_0^{(2)}$, $\phi_0^{(2)}$, $\chi_{0ij}^{\top(1)}$ etc. at τ_0 . The subscript “ S ” denotes those contributed by scalar-scalar couplings $\varphi\varphi$ and $X\varphi$ and XX , “ $s(t)$ ” by scalar-tensor $\varphi\chi_{ij}^{\top(1)}$ and $X\chi_{ij}^{\top(1)}$, and “ T ” by tensor-tensor $\chi_{ij}^{\top(1)}\chi_{kl}^{\top(1)}$. We notice that neither the tensor $\chi_{ij}^{\top(2)}$ nor the vector $\chi_{ij}^{\perp(2)}$ appears in the energy constraint (23).

The (0i) component of the 2nd-order perturbed Einstein equation is

$$G_{0i}^{(2)} = R_{0i}^{(2)} = 0, \tag{27}$$

with $T_{0i}^{(2)} = 0$. Using $R_{0i}^{(2)}$ in Eq. (B2) in Appendix B, Eq. (27) leads to the 2nd-order momentum constraint [51]:

$$2\phi_{,j}^{(2)'} + \frac{1}{2}D_{ij}\chi^{\parallel(2)'}{}_{,i} + \frac{1}{2}\chi_{ij}^{\perp(2)'}{}_{,i} = M_{Sj} + M_{s(t)j} + M_{Tj}, \tag{28}$$

where

$$\begin{aligned}
M_{Sj} \equiv & -\frac{10\tau}{9}\varphi_{,j}\nabla^2\varphi + \frac{\tau^3}{9}\varphi_{,kj}\nabla^2\varphi^k - \frac{10\tau}{9}\varphi^k\varphi_{,kj} - \frac{\tau^3}{9}\varphi^{ik}\varphi_{,ijk} + \frac{54}{\tau^7}\nabla^{-2}X^{,kl}\nabla^{-2}X_{,klj} - \frac{54}{\tau^7}X^{,k}\nabla^{-2}X_{,jk} + \frac{30}{\tau^4}\varphi_{,j}X \\
& + \frac{30}{\tau^4}\varphi^k\nabla^{-2}X_{,jk} - \frac{3}{\tau^2}\nabla^{-2}X_{,jk}\nabla^2\varphi^k + \frac{2}{\tau^2}X^{,k}\varphi_{,jk} + \frac{3}{\tau^2}\varphi_{,klj}\nabla^{-2}X^{,kl} - \frac{2}{\tau^2}\varphi^{kl}\nabla^{-2}X_{,klj},
\end{aligned} \tag{29}$$

$$\begin{aligned}
M_{s(t)j} \equiv & \frac{\tau^2}{3}\left[\varphi_{,ik}(\chi_j^{\top(1)ik} - \chi_j^{\top(1)k,i}) + \frac{1}{2}\chi^{\top(1)ik}\varphi_{,ijk} - \frac{1}{2}\chi_{kj}^{\top(1)'}\nabla^2\varphi^k\right] + \frac{\tau}{3}\varphi_{,ik}\chi_j^{\top(1)ik} + \frac{5}{3}\varphi^k\chi_{kj}^{\top(1)'} - \frac{9}{\tau^4}\chi_{kl,j}^{\top(1)}\nabla^{-2}X^{,kl} \\
& - \frac{6}{\tau^3}\chi_{jk,l}^{\top(1)'}\nabla^{-2}X^{,kl} + \frac{6}{\tau^3}\chi_{kl,j}^{\top(1)'}\nabla^{-2}X^{,kl} + \frac{3}{\tau^3}\chi_{kl}^{\top(1)'}\nabla^{-2}X_{,j}^{,kl} - \frac{3}{\tau^3}\chi_{kj}^{\top(1)'}X^{,k},
\end{aligned} \tag{30}$$

$$M_{Tj} \equiv \chi^{\top(1)ik}(\chi_{kj,i}^{\top(1)'} - \chi_{ki,j}^{\top(1)'}) - \frac{1}{2}\chi_{,j}^{\top(1)ik}\chi_{ik}^{\top(1)'}, \tag{31}$$

are the couplings of the 1st-order perturbations, formed from various products of the gravitational potential φ and tensor $\chi_{ij}^{\top(1)}$. Eq. (28) is similar to Eq. (A2) for the 1st order, except for the vector on the lhs and the coupling terms on the rhs.

Notice that the tensor $\chi_{ij}^{\top(2)}$ does not appear in (28).

The (ij) component of the 2nd-order perturbed Einstein equation is

$$G_{ij}^{(2)} \equiv R_{ij}^{(2)} - \frac{1}{2}\delta_{ij}a^2R^{(2)} - \frac{1}{4}a^2\gamma_{ij}^{(2)}R^{(0)} - \frac{1}{2}a^2\gamma_{ij}^{(1)}R^{(1)} = 0. \tag{32}$$

To compare with the equation (4.30) in Ref. [51], one can combine Eq. (32) into the following

$$2G_{ij}^{(2)} - \delta_{ij}G_{kl}^{(2)}\delta^{kl} = 0. \tag{33}$$

Plugging Eqs. (B7) and (B8) and the 1st-order solutions (14) and (15) into above, and using Eq. (12) gives the 2nd-order evolution equation [51]:

$$\begin{aligned}
& - \left(\phi^{(2)''} + \frac{4}{\tau} \phi^{(2)'} \right) \delta_{ij} + \phi_{,ij}^{(2)} + \frac{1}{2} \left(D_{ij} \chi^{\parallel(2)''} + \frac{4}{\tau} D_{ij} \chi^{\parallel(2)'} \right) + \frac{1}{2} \left(\chi_{ij}^{\perp(2)''} + \frac{4}{\tau} \chi_{ij}^{\perp(2)'} \right) \\
& + \frac{1}{2} \left(\chi_{ij}^{\top(2)''} + \frac{4}{\tau} \chi_{ij}^{\top(2)'} - \nabla^2 \chi_{ij}^{\top(2)} \right) - \frac{1}{4} D_{kl} \chi^{\parallel(2),kl} \delta_{ij} + \frac{2}{3} \nabla^2 \chi_{,ij}^{\parallel(2)} - \frac{1}{2} \nabla^2 D_{ij} \chi^{\parallel(2)} \\
& = S_{Sij} + S_{s(t)ij} + S_{Tij},
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
S_{Sij} \equiv & -\frac{100}{9} \varphi \varphi_{,ij} + \frac{25}{9} \varphi^k \varphi_{,k} \delta_{ij} - \frac{50}{3} \varphi_{,i} \varphi_{,j} + \frac{24\tau^2}{9} \varphi_{,i}^k \varphi_{,kj} + \frac{11\tau^2}{18} (\nabla^2 \varphi)^2 \delta_{ij} - \frac{22\tau^2}{9} \varphi_{,ij} \nabla^2 \varphi - \frac{11\tau^2}{18} \varphi^{,kl} \varphi_{,kl} \delta_{ij} - \frac{5\tau^2}{9} \varphi_{,k} \varphi_{,ij}^k \\
& + \frac{\tau^4}{18} \varphi_{,ij}^k \nabla^2 \varphi_{,k} - \frac{\tau^4}{72} \nabla^2 \varphi^m \nabla^2 \varphi_{,m} \delta_{ij} - \frac{\tau^4}{18} \varphi_{,j}^{kl} \varphi_{,ikl} + \frac{\tau^4}{72} \varphi_{,m}^{kl} \varphi_{,kl}^m \delta_{ij} + \frac{81}{2\tau^8} X^2 \delta_{ij} - \frac{81}{2\tau^8} \nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl} \delta_{ij} - \frac{162}{\tau^8} X \nabla^{-2} X_{,ij} \\
& + \frac{324}{\tau^8} \nabla^{-2} X_{,i}^k \nabla^{-2} X_{,kj} - \frac{9}{2\tau^6} X^k X_{,k} \delta_{ij} + \frac{9}{2\tau^6} \nabla^{-2} X^{,klm} \nabla^{-2} X_{,klm} \delta_{ij} + \frac{18}{\tau^6} X^k \nabla^{-2} X_{,kij} - \frac{18}{\tau^6} \nabla^{-2} X_{,i}^{kl} \nabla^{-2} X_{,klj} + \frac{7}{\tau^3} X \nabla^2 \varphi \delta_{ij} \\
& - \frac{7}{\tau^3} \varphi^{,kl} \nabla^{-2} X_{,kl} \delta_{ij} - \frac{14}{\tau^3} \varphi_{,ij} X - \frac{14}{\tau^3} \nabla^2 \varphi \nabla^{-2} X_{,ij} - \frac{10}{\tau^3} \varphi^k \nabla^{-2} X_{,kij} + \frac{8}{\tau^3} \varphi_{,i}^k \nabla^{-2} X_{,kj} + \frac{8}{\tau^3} \varphi_{,j}^k \nabla^{-2} X_{,ki} - \frac{1}{2\tau} X_{,k} \nabla^2 \varphi^k \delta_{ij} \\
& + \frac{1}{2\tau} \varphi_{,klm} \nabla^{-2} X^{,klm} \delta_{ij} + \frac{1}{\tau} X^k \varphi_{,kij} + \frac{1}{\tau} \nabla^{-2} X_{,kij} \nabla^2 \varphi^k - \frac{1}{\tau} \varphi_{,klj} \nabla^{-2} X_{,i}^{kl} - \frac{1}{\tau} \varphi_{,kli} \nabla^{-2} X_{,j}^{kl},
\end{aligned} \tag{35}$$

$$\begin{aligned}
S_{s(t)ij} \equiv & -\frac{\tau^2}{6} \varphi^{,kl} \chi_{kl}^{\top(1)'} \delta_{ij} - \frac{2\tau}{3} \varphi_{,i}^k \chi_{kj}^{\top(1)'} - \frac{2\tau}{3} \varphi_{,j}^k \chi_{ki}^{\top(1)'} + \frac{\tau}{3} \chi_{ij}^{\top(1)'} \nabla^2 \varphi - \frac{\tau}{2} \varphi^{,kl} \chi_{kl}^{\top(1)'} \delta_{ij} + \frac{10}{3} \chi_{ij}^{\top(1)} \nabla^2 \varphi + \frac{5}{3} \varphi_{,kl} \chi^{\top(1)kl} \delta_{ij} \\
& - \frac{10}{3} \varphi_{,j}^k \chi_{ki}^{\top(1)} - \frac{10}{3} \varphi_{,i}^k \chi_{kj}^{\top(1)} + \frac{10}{3} \varphi \nabla^2 \chi_{ij}^{\top(1)} + \frac{\tau^2}{3} \varphi^{,kl} \chi_{ij,kl}^{\top(1)} + \frac{\tau^2}{3} \varphi^{,kl} \chi_{kl,ij}^{\top(1)} - \frac{\tau^2}{3} \varphi^{,kl} \chi_{li,jk}^{\top(1)} - \frac{\tau^2}{3} \varphi^{,kl} \chi_{lj,ik}^{\top(1)} + 5\varphi^k \chi_{ij,k}^{\top(1)} \\
& - \frac{5}{3} \varphi^k \chi_{ki,j}^{\top(1)} - \frac{5}{3} \varphi^k \chi_{kj,i}^{\top(1)} + \frac{\tau^2}{6} \chi_{ij,k}^{\top(1)} \nabla^2 \varphi^k - \frac{\tau^2}{6} \chi_{ki,j}^{\top(1)} \nabla^2 \varphi^k - \frac{\tau^2}{6} \chi_{kj,i}^{\top(1)} \nabla^2 \varphi^k + \frac{\tau^2}{6} \varphi_{,i}^{kl} \chi_{kl,j}^{\top(1)} + \frac{\tau^2}{6} \varphi_{,j}^{kl} \chi_{kl,i}^{\top(1)} \\
& - \frac{\tau^2}{12} \varphi^{,klm} \chi_{km,l}^{\top(1)} \delta_{ij} - \frac{33}{2\tau^4} \chi_{kl}^{\top(1)'} \nabla^{-2} X^{,kl} \delta_{ij} - \frac{3}{2\tau^3} \chi_{kl,m}^{\top(1)} \nabla^{-2} X^{,klm} \delta_{ij} - \frac{3}{\tau^3} \chi_{kl}^{\top(1)''} \nabla^{-2} X^{,kl} \delta_{ij} - \frac{9}{\tau^4} X \chi_{ij}^{\top(1)'} \\
& + \frac{18}{\tau^4} \chi_{ki}^{\top(1)'} \nabla^{-2} X_{,j}^k + \frac{18}{\tau^4} \chi_{kj}^{\top(1)'} \nabla^{-2} X_{,i}^k - \frac{3}{\tau^3} X^k \chi_{kj,i}^{\top(1)} - \frac{3}{\tau^3} X^k \chi_{ki,j}^{\top(1)} + \frac{3}{\tau^3} X^k \chi_{ij,k}^{\top(1)} - \frac{6}{\tau^3} \chi_{lj,ik}^{\top(1)} \nabla^{-2} X^{,kl} \\
& - \frac{6}{\tau^3} \chi_{li,jk}^{\top(1)} \nabla^{-2} X^{,kl} + \frac{6}{\tau^3} \chi_{ij,kl}^{\top(1)} \nabla^{-2} X^{,kl} + \frac{6}{\tau^3} \chi_{kl,ij}^{\top(1)} \nabla^{-2} X^{,kl} + \frac{3}{\tau^3} \chi_{kl,i}^{\top(1)} \nabla^{-2} X_{,j}^{kl} + \frac{3}{\tau^3} \chi_{kl,j}^{\top(1)} \nabla^{-2} X_{,i}^{kl},
\end{aligned} \tag{36}$$

$$\begin{aligned}
S_{Tij} \equiv & \chi_i^{\top(1)k} \chi_{kj}^{\top(1)'} - \frac{1}{8} \chi^{\top(1)kl} \chi_{kl}^{\top(1)'} \delta_{ij} + \chi^{\top(1)kl} \chi_{li,jk}^{\top(1)} + \chi^{\top(1)kl} \chi_{lj,ik}^{\top(1)} - \chi^{\top(1)kl} \chi_{kl,ij}^{\top(1)} - \chi^{\top(1)kl} \chi_{ij,kl}^{\top(1)} + \chi_{li}^{\top(1),k} \chi_{kj}^{\top(1),l} \\
& - \chi_{i,l}^{\top(1),k} \chi_{jk}^{\top(1),l} - \frac{1}{2} \chi_{,i}^{\top(1)kl} \chi_{kl,j}^{\top(1)} + \frac{3}{8} \chi^{\top(1)klm} \chi_{kl,m}^{\top(1)} \delta_{ij} - \frac{1}{4} \chi_{ml,k}^{\top(1)} \chi^{\top(1)mk,l} \delta_{ij} + \frac{1}{2} \chi^{\top(1)kl} \chi_{kl}^{\top(1)''} \delta_{ij} + \frac{2}{\tau} \chi^{\top(1)kl} \chi_{kl}^{\top(1)'} \delta_{ij},
\end{aligned} \tag{37}$$

are the couplings and play a role of source of evolution. Note that the 2nd-order perturbations, scalar $\phi^{(2)}$, $\chi^{\parallel(2)}$, vector $\chi_{ij}^{\perp(2)}$, and tensor $\chi_{ij}^{\top(1)}$, all appear in the evolution equation (34).

There are three types of couplings: scalar-scalar, scalar-tensor, and tensor-tensor in (23), (28), and (34). In order to deal with these equations separately, we split the 2nd-order perturbations into three parts according to the type of couplings

$$\phi^{(2)} \equiv \phi_S^{(2)} + \phi_{s(t)}^{(2)} + \phi_T^{(2)}, \tag{38}$$

$$D_{ij} \chi^{\parallel(2)} \equiv D_{ij} \chi_S^{\parallel(2)} + D_{ij} \chi_{s(t)}^{\parallel(2)} + D_{ij} \chi_T^{\parallel(2)}, \tag{39}$$

$$\chi_{ij}^{\perp(2)} = \chi_{Sij}^{\perp(2)} + \chi_{s(t)ij}^{\perp(2)} + \chi_{Tij}^{\perp(2)}, \tag{40}$$

$$\chi_{ij}^{\top(2)} = \chi_{Sij}^{\top(2)} + \chi_{s(t)ij}^{\top(2)} + \chi_{Tij}^{\top(2)}, \tag{41}$$

and the equations (23), (28), and (34) into three sets as follows. The first set involves only the scalar-scalar coupling:

$$\frac{2}{\tau}\phi_S^{(2)'} - \frac{1}{3}\nabla^2\phi_S^{(2)} + \frac{6}{\tau^2}\phi_S^{(2)} - \frac{1}{12}D^{ij}\chi_{S,ij}^{\parallel(2)} = E_S, \quad (42)$$

$$2\phi_{S,j}^{(2)'} + \frac{1}{2}D_{ij}\chi_S^{\parallel(2)'} + \frac{1}{2}\chi_{Sij}^{\perp(2)'} = M_{Sj}, \quad (43)$$

$$\begin{aligned} & - \left(\phi_S^{(2)''} + \frac{4}{\tau}\phi_S^{(2)'} \right) \delta_{ij} + \phi_{S,ij}^{(2)} + \frac{1}{2} \left(D_{ij}\chi_S^{\parallel(2)''} + \frac{4}{\tau}D_{ij}\chi_S^{\parallel(2)'} \right) \\ & + \frac{1}{2} \left(\chi_{Sij}^{\perp(2)''} + \frac{4}{\tau}\chi_{Sij}^{\perp(2)'} \right) + \frac{1}{2} \left(\chi_{Sij}^{\top(2)''} + \frac{4}{\tau}\chi_{Sij}^{\top(2)'} - \nabla^2\chi_{Sij}^{\top(2)} \right) \\ & - \frac{1}{4}D_{kl}\chi_S^{\parallel(2),kl}\delta_{ij} + \frac{2}{3}\nabla^2\chi_{S,ij}^{\parallel(2)} - \frac{1}{2}\nabla^2D_{ij}\chi_S^{\parallel(2)} = S_{Sij}. \end{aligned} \quad (44)$$

Observe that M_{Sj} on the rhs of (43) has a nonvanishing curl, $\epsilon^{ikj}\partial_k M_{Sj} \neq 0$ with ϵ^{ikj} as the Levi-Civita symbol, commanding the introduction of the 2nd-order vector $\chi_{Sij}^{\perp(2)}$ on the lhs.

The second set involves the scalar-tensor coupling:

$$\frac{2}{\tau}\phi_{s(t)}^{(2)'} - \frac{1}{3}\nabla^2\phi_{s(t)}^{(2)} + \frac{6}{\tau^2}\phi_{s(t)}^{(2)} - \frac{1}{12}D^{ij}\chi_{s(t),ij}^{\parallel(2)} = E_{s(t)}, \quad (45)$$

$$2\phi_{s(t),j}^{(2)'} + \frac{1}{2}D_{ij}\chi_{s(t)}^{\parallel(2)'} + \frac{1}{2}\chi_{s(t)ij}^{\perp(2)'} = M_{s(t)j}, \quad (46)$$

$$\begin{aligned} & - \left(\phi_{s(t)}^{(2)''} + \frac{4}{\tau}\phi_{s(t)}^{(2)'} \right) \delta_{ij} + \phi_{s(t),ij}^{(2)} + \frac{1}{2} \left(D_{ij}\chi_{s(t)}^{\parallel(2)''} + \frac{4}{\tau}D_{ij}\chi_{s(t)}^{\parallel(2)'} \right) + \frac{1}{2} \left(\chi_{s(t)ij}^{\perp(2)''} + \frac{4}{\tau}\chi_{s(t)ij}^{\perp(2)'} \right) + \frac{1}{2} \left(\chi_{s(t)ij}^{\top(2)''} + \frac{4}{\tau}\chi_{s(t)ij}^{\top(2)'} - \nabla^2\chi_{s(t)ij}^{\top(2)} \right) \\ & - \frac{1}{4}D_{kl}\chi_{s(t)}^{\parallel(2),kl}\delta_{ij} + \frac{2}{3}\nabla^2\chi_{s(t),ij}^{\parallel(2)} - \frac{1}{2}\nabla^2D_{ij}\chi_{s(t)}^{\parallel(2)} = S_{s(t)ij}. \end{aligned} \quad (47)$$

The third set involves the tensor-tensor coupling:

$$\frac{2}{\tau}\phi_T^{(2)'} - \frac{1}{3}\nabla^2\phi_T^{(2)} + \frac{6}{\tau^2}\phi_T^{(2)} - \frac{1}{12}D^{ij}\chi_{T,ij}^{\parallel(2)} = E_T, \quad (48)$$

$$2\phi_{T,j}^{(2)'} + \frac{1}{2}D_{ij}\chi_T^{\parallel(2)'} + \frac{1}{2}\chi_{Tij}^{\perp(2)'} = M_{Tj}, \quad (49)$$

$$\begin{aligned} & - \left(\phi_T^{(2)''} + \frac{4}{\tau}\phi_T^{(2)'} \right) \delta_{ij} + \phi_{T,ij}^{(2)} + \frac{1}{2} \left(D_{ij}\chi_T^{\parallel(2)''} + \frac{4}{\tau}D_{ij}\chi_T^{\parallel(2)'} \right) + \frac{1}{2} \left(\chi_{Tij}^{\perp(2)''} + \frac{4}{\tau}\chi_{Tij}^{\perp(2)'} \right) + \frac{1}{2} \left(\chi_{Tij}^{\top(2)''} + \frac{4}{\tau}\chi_{Tij}^{\top(2)'} - \nabla^2\chi_{Tij}^{\top(2)} \right) \\ & - \frac{1}{4}D_{kl}\chi_T^{\parallel(2),kl}\delta_{ij} + \frac{2}{3}\nabla^2\chi_{T,ij}^{\parallel(2)} - \frac{1}{2}\nabla^2D_{ij}\chi_T^{\parallel(2)} = S_{Tij}. \end{aligned} \quad (50)$$

The lhs of these equations involve only the 2nd-order metric perturbations with a similar structure to those of the 1st-order equations, but the difference is the couplings on the rhs.

During the matter era, the 1st-order tensor has a lower amplitude than the 1st-order scalar. As the 1st-order solutions (14), (15), and (19) show, the scalar grows as $\phi^{(1)}, \chi^{\parallel(1)} \propto a(\tau)$, whereas the short wavelengths modes of tensor decrease as $h_k(\tau) \propto 1/a(\tau)$, and those of long wavelength remain constant $h_k(\tau) \propto \text{const}$ [19,28]. Given

the upper bound of tensor-scalar ratio $r < 0.1$ [37,38], the amplitude of scalar is growing dominant over that of tensor. This leads to an estimation that, among the three types of couplings, the scalar-scalar is greater in magnitude than the scalar-tensor and tensor-tensor for the matter stage. In the following, we shall focus on the set of equations of scalar-scalar coupling. As for the scalar-tensor and tensor-tensor couplings, they are more involved and will be reported in a subsequent paper separately in the future.

IV. 2ND-ORDER PERTURBATIONS WITH THE SCALAR-SCALAR COUPLING SOURCE

A. Scalar perturbation $\phi_S^{(2)}$

We shall solve the set of equations (42), (43), and (44), which have a structure similar to that of the 1st-order equations [(A2), (A6), and (A9) in Appendix A] except for the coupling terms on the rhs. Applying $\partial^j \int_{\tau_0}^{\tau} d\tau'$ on Eq. (43) gives

$$\begin{aligned} \phi_S^{(2)'} + \frac{3}{\tau} \phi_S^{(2)} &= \frac{\tau}{2} E_S - \frac{5\tau^3}{108} (\nabla^2 \varphi \nabla^2 \varphi + 2\varphi^i \nabla^2 \varphi_{,i} + \varphi^{ki} \varphi_{,ki}) + \frac{\tau^5}{432} (\nabla^2 \varphi^k \nabla^2 \varphi_{,k} - \varphi^{ijk} \varphi_{,ijk}) - \frac{\tau}{12} F \\ &+ \frac{3}{4\tau^5} (X^{,k} X_{,k} - \nabla^{-2} X^{,klm} \nabla^{-2} X_{,klm}) - \frac{5}{6\tau^2} (2\varphi_{,k} X^{,k} + X \nabla^2 \varphi + \varphi^{kl} \nabla^{-2} X_{,kl}) \\ &+ \frac{1}{12} (X_{,k} \nabla^2 \varphi^{,k} - \varphi_{,klm} \nabla^{-2} X^{,klm}), \end{aligned} \quad (52)$$

where

$$\begin{aligned} F &\equiv -2\nabla^2 \phi_{S0}^{(2)} - \frac{1}{3} \nabla^2 \nabla^2 \chi_{S0}^{(2)} - \frac{5\tau_0^2}{9} (\nabla^2 \varphi \nabla^2 \varphi + 2\varphi^i \nabla^2 \varphi_{,i} + \varphi^{ki} \varphi_{,ki}) + \frac{\tau_0^4}{36} (\nabla^2 \varphi^k \nabla^2 \varphi_{,k} - \varphi^{ijk} \varphi_{,ijk}) \\ &+ \frac{9}{\tau_0^6} (X^{,k} X_{,k} - \nabla^{-2} X^{,klm} \nabla^{-2} X_{,klm}) - \frac{10}{\tau_0^3} (2\varphi_{,k} X^{,k} + X \nabla^2 \varphi + \varphi^{kl} \nabla^{-2} X_{,kl}) + \frac{1}{\tau_0} (X_{,k} \nabla^2 \varphi^{,k} - \varphi_{,klm} \nabla^{-2} X^{,klm}), \end{aligned} \quad (53)$$

depending on the initial values $\phi_{S0}^{(2)}, \chi_{S0}^{(2)}$ at τ_0 . The solution of Eq. (52) is

$$\begin{aligned} \phi_S^{(2)} &= \frac{1}{7} \tau^4 \left(\frac{1}{36} \nabla^2 \varphi \nabla^2 \varphi - \frac{5}{54} \varphi_{,ki} \varphi^{ki} \right) + \tau^2 \left(\frac{10}{27} \varphi \nabla^2 \varphi + \frac{5}{18} \varphi_{,i} \varphi^i \right) - \frac{\tau^2}{60} F - \frac{3}{4\tau^6} (5\nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl} - X^2) + \frac{Z(\mathbf{x})}{\tau^3} \\ &- \frac{1}{3\tau} X \nabla^2 \varphi + \left(\phi_{S0}^{(2)} - \frac{1}{3} \delta_{S0}^{(2)} + \frac{6}{\tau_0^6} \nabla^{-2} X^{,kl} \nabla^{-2} X_{,kl} + \frac{3}{\tau_0^6} X^2 + \frac{20}{3\tau_0^3} \varphi X + \frac{2}{3\tau_0} \varphi_{,ij} \nabla^{-2} X^{,ij} + \frac{1}{3\tau_0} X \nabla^2 \varphi \right. \\ &\left. + \frac{10\tau_0^2}{27} \varphi \nabla^2 \varphi + \frac{\tau_0^4}{54} \varphi_{,ij} \varphi^{ij} + \frac{\tau_0^4}{108} (\nabla^2 \varphi)^2 \right), \end{aligned} \quad (54)$$

where

$$\begin{aligned} Z &= \frac{\tau_0^3}{3} \delta_{S0}^{(2)} + \frac{\tau_0^5}{60} F - \tau_0^5 \left(\frac{20}{27} \varphi \nabla^2 \varphi + \frac{5}{18} \varphi_{,i} \varphi^i \right) - \frac{\tau_0^7}{7} \left(\frac{5}{54} \nabla^2 \varphi \nabla^2 \varphi + \frac{1}{27} \varphi_{,ki} \varphi^{ki} \right) \\ &+ \frac{3}{4\tau_0^3} (-3\nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl} - 5X^2) - \frac{20}{3} \varphi X - \frac{2\tau_0^2}{3} \varphi_{,ij} \nabla^{-2} X^{,ij}, \end{aligned} \quad (55)$$

in which $\phi_{S0}^{(2)}, \chi_{S0}^{(2)}, \delta_{S0}^{(2)}$ need to be fixed by joining conditions with the radiation dominated stage. In particular, the 2nd-order density contrast $\delta_{S0}^{(2)}$ is generally nonvanishing and has been inherited from the previous expansion stages. In the above all the X terms are contributed by the 1st-order decaying modes.

Note that the solution (54) can be also derived in another way, using the trace part of the evolution equation (44) and the energy constraint (42); i.e., the trace of Eq. (44) plus three times of Eq. (42) is the Raychaudhuri equation of $\phi_S^{(2)}$, and the solution is the same as (54).

$$\begin{aligned} 2\nabla^2 \phi_S^{(2)} + \frac{1}{2} D_{ij} \chi_S^{(2),ij} \\ = \int_{\tau_0}^{\tau} d\tau' M_{Sj}^j + 2\nabla^2 \phi_{S0}^{(2)} + \frac{1}{2} D_{ij} \chi_{S0}^{(2),ij}, \end{aligned} \quad (51)$$

with $\chi_{S0}^{(2)}$ being the value at τ_0 . A combination [Eq. (42) + $\frac{1}{6} \partial^j \int_{\tau_0}^{\tau} d\tau'$ Eq. (43)], yields the first-order differential equation of $\phi_S^{(2)}$ as follows:

The expression (54) extends (4.31) of Ref. [51] to general initial conditions at τ_0 . For a realistic cosmological model, the initial metric perturbations at the radiation-matter equality are important and determine the spectra of CMB anisotropies and polarization [16].

B. Scalar perturbation $D_{ij} \chi_S^{(2)}$

Substituting M_{Sj} of (29) and $\phi_S^{(2)}$ of (54) into Eq. (51), one obtains the scalar

$$\begin{aligned}
\chi_S^{\parallel(2)} = & \frac{\tau^4}{84} \nabla^{-2} \left[\frac{20}{3} \varphi^{,kl} \varphi_{,kl} - 2\nabla^2 \varphi \nabla^2 \varphi + \nabla^{-2} (7\nabla^2 \varphi^{,k} \nabla^2 \varphi_{,k} - 7\varphi^{,klm} \varphi_{,klm}) \right] - \frac{5\tau^2}{18} [4\varphi\varphi + 4\nabla^{-2}(\varphi_{,k}\varphi^{,k}) \\
& + \nabla^{-2}\nabla^{-2}(6\nabla^2\varphi\nabla^2\varphi - 6\varphi_{,kl}\varphi^{,kl})] + \nabla^{-2}\nabla^{-2}A + \frac{\tau^2}{10} \nabla^{-2}F + \frac{9}{2\tau^6} [\nabla^{-2}(5\nabla^{-2}X_{,kl}\nabla^{-2}X^{,kl} - X^2) \\
& + \nabla^{-2}\nabla^{-2}(6X^{,k}X_{,k} - 6\nabla^{-2}X^{,klm}\nabla^{-2}X_{,klm})] + \frac{6}{\tau^3} [-\nabla^{-2}(Z + 5\varphi X) + \nabla^{-2}\nabla^{-2}(5\varphi\nabla^2X - 5\varphi^{,kl}\nabla^{-2}X_{,kl})] \\
& + \frac{1}{\tau} [\nabla^{-2}(2X\nabla^2\varphi) + 3\nabla^{-2}\nabla^{-2}(X_{,k}\nabla^2\varphi^k - \varphi_{,klm}\nabla^{-2}X^{,klm})], \tag{56}
\end{aligned}$$

where

$$\begin{aligned}
A \equiv & \nabla^2\nabla^2\chi_{S0}^{\parallel(2)} + 2\nabla^2\delta_{S0}^{(2)} + \frac{5\tau_0^2}{3} \left[\nabla^2\varphi\nabla^2\varphi - \varphi_{,kl}\varphi^{,kl} + \nabla^2 \left(\varphi_{,k}\varphi^{,k} - \frac{4}{3}\varphi\nabla^2\varphi \right) \right] \\
& - \frac{\tau_0^4}{12} \left[\nabla^2\varphi_{,k}\nabla^2\varphi^{,k} - \varphi_{,klm}\varphi^{,klm} + \nabla^2 \left(\frac{4}{3}\varphi_{,kl}\varphi^{,kl} + \frac{2}{3}\nabla^2\varphi\nabla^2\varphi \right) \right] + \frac{9}{\tau_0^6} [3\nabla^{-2}X^{,klm}\nabla^{-2}X_{,klm} - 3X^{,k}X_{,k} \\
& - \nabla^2(4\nabla^{-2}X^{,kl}\nabla^{-2}X_{,kl} + 2X^2)] + \frac{10}{\tau_0^3} [3\varphi^{,kl}\nabla^{-2}X_{,kl} - 3\varphi\nabla^2X - \nabla^2(\varphi X)] + \frac{1}{\tau_0} [3\varphi_{,klm}\nabla^{-2}X^{,klm} - 3X_{,k}\nabla^2\varphi^{,k} \\
& - 2\nabla^2(2\varphi_{,ij}\nabla^{-2}X^{,ij} + X\nabla^2\varphi)], \tag{57}
\end{aligned}$$

which depends on the initial values at τ_0 . Thus, the scalar metric perturbation $D_{ij}\chi_S^{\parallel(2)}$ is obtained,

$$\begin{aligned}
D_{ij}\chi_S^{\parallel(2)} = & \frac{\tau^4}{84} D_{ij} \nabla^{-2} \left[\frac{20}{3} \varphi^{,kl} \varphi_{,kl} - 2\nabla^2 \varphi \nabla^2 \varphi + \nabla^{-2} (7\nabla^2 \varphi^{,k} \nabla^2 \varphi_{,k} - 7\varphi^{,klm} \varphi_{,klm}) \right] \\
& - \frac{5\tau^2}{18} D_{ij} [4\varphi\varphi + 4\nabla^{-2}(\varphi_{,k}\varphi^{,k}) + \nabla^{-2}\nabla^{-2}(6\nabla^2\varphi\nabla^2\varphi - 6\varphi_{,kl}\varphi^{,kl})] \\
& + D_{ij} \nabla^{-2}\nabla^{-2}A + \frac{\tau^2}{10} D_{ij} \nabla^{-2}F + \frac{9}{2\tau^6} D_{ij} [\nabla^{-2}(5\nabla^{-2}X_{,kl}\nabla^{-2}X^{,kl} - X^2) \\
& + \nabla^{-2}\nabla^{-2}(6X^{,k}X_{,k} - 6\nabla^{-2}X^{,klm}\nabla^{-2}X_{,klm})] + \frac{6}{\tau^3} D_{ij} [-\nabla^{-2}(Z + 5\varphi X) + \nabla^{-2}\nabla^{-2}(5\varphi\nabla^2X - 5\varphi^{,kl}\nabla^{-2}X_{,kl})] \\
& + \frac{1}{\tau} D_{ij} [\nabla^{-2}(2X\nabla^2\varphi) + 3\nabla^{-2}\nabla^{-2}(X_{,k}\nabla^2\varphi^k - \varphi_{,klm}\nabla^{-2}X^{,klm})], \tag{58}
\end{aligned}$$

which holds for general initial conditions at τ_0 . The solution (58) can be also obtained by the traceless part of the evolution equation (44) together with the momentum constraint (43).

C. Vector perturbation $\chi_{Sij}^{\perp(2)}$

The time integration of the momentum constraint (43) from τ_0 to τ yields

$$2\phi_{S,j}^{(2)} + \frac{1}{3}\nabla^2\chi_{S,j}^{\parallel(2)} + \frac{1}{2}\chi_{Sij}^{\perp(2),i} = \int_{\tau_0}^{\tau} d\tau' M_{Sj} + 2\phi_{S0,j}^{(2)} + \frac{1}{3}\nabla^2\chi_{S0,j}^{\parallel(2)} + \frac{1}{2}\chi_{S0ij}^{\perp(2),i}, \tag{59}$$

where $\chi_{S0ij}^{\perp(2)}$ is the initial value of $\chi_{Sij}^{\perp(2)}$ at τ_0 . Plugging M_{Sj} of (29), $\phi_S^{(2)}$ of (54), and $\chi_S^{\parallel(2)}$ of (56) into (59), one gets

$$\begin{aligned}
\chi_{Sij}^{\perp(2),j} = & \frac{10\tau^2}{9} [-\varphi_{,i}\nabla^2\varphi + \partial_i\nabla^{-2}(\varphi^k\nabla^2\varphi_{,k} + \nabla^2\varphi\nabla^2\varphi)] + \frac{\tau^4}{18} [\varphi_{,ki}\nabla^2\varphi^k + \partial_i\nabla^{-2}(-\varphi^{kl}\nabla^2\varphi_{,kl} - \nabla^2\varphi_{,k}\nabla^2\varphi^k)] \\
& + G_i + \frac{18}{\tau^6} [X^{,k}\nabla^{-2}X_{,ik} + \partial_i\nabla^{-2}(-X^{,kl}\nabla^{-2}X_{,kl} - X^{,k}X_{,k})] + \frac{20}{\tau^3} [-\varphi_{,i}X - \varphi^k\nabla^{-2}X_{,ik} \\
& + \partial_i\nabla^{-2}(X\nabla^2\varphi + 2\varphi_{,k}X^{,k} + \varphi^{kl}\nabla^{-2}X_{,kl})] + \frac{2}{\tau} [3\nabla^2\varphi^k\nabla^{-2}X_{,ik} - 2X^{,k}\varphi_{,ik} + 5\varphi^{kl}\nabla^{-2}X_{,kli} \\
& + \partial_i\nabla^{-2}(-3\nabla^2\varphi^{kl}\nabla^{-2}X_{,kl} - X_{,k}\nabla^2\varphi^k - 3\varphi^{kl}X_{,kl} - 5\varphi^{klm}\nabla^{-2}X_{,klm})], \tag{60}
\end{aligned}$$

where

$$\begin{aligned}
G_i \equiv & \chi_{S0ij}^{\perp(2),j} + \frac{10\tau_0^2}{9} [\varphi_{,i}\nabla^2\varphi + \partial_i\nabla^{-2}(-\varphi^k\nabla^2\varphi_{,k} - \nabla^2\varphi\nabla^2\varphi)] + \frac{\tau_0^4}{18} [-\varphi_{,ki}\nabla^2\varphi^k + \partial_i\nabla^{-2}(\varphi^{kl}\nabla^2\varphi_{,kl} + \nabla^2\varphi_{,k}\nabla^2\varphi^k)] \\
& + \frac{18}{\tau_0^6} [-X^{,k}\nabla^{-2}X_{,ik} + \partial_i\nabla^{-2}(X^{,kl}\nabla^{-2}X_{,kl} + X^{,k}X_{,k})] + \frac{20}{\tau_0^3} [\varphi_{,i}X + \varphi^k\nabla^{-2}X_{,ik} + \partial_i\nabla^{-2}(-X\nabla^2\varphi - 2\varphi_{,k}X^{,k} - \varphi^{kl}\nabla^{-2}X_{,kl})] \\
& + \frac{2}{\tau_0} [-3\nabla^2\varphi^k\nabla^{-2}X_{,ik} + 2X^{,k}\varphi_{,ik} - 5\varphi^{kl}\nabla^{-2}X_{,kli} + \partial_i\nabla^{-2}(3\nabla^2\varphi^{kl}\nabla^{-2}X_{,kl} + X_{,k}\nabla^2\varphi^k + 3\varphi^{kl}X_{,kl} + 5\varphi^{klm}\nabla^{-2}X_{,klm})], \tag{61}
\end{aligned}$$

depending on the initial values at τ_0 . To get $\chi_{Sij}^{\perp(2)}$ from Eq. (60), one need to remove ∂^j . By writing $\chi_{Sij}^{\perp(2)} = A_{Si,j} + A_{Sj,i}$ in terms of a 3-vector A_{Si} as Eqs. (12), (60) becomes an equation of A_{Si} , whose solution is

$$\begin{aligned}
A_{Si} = & \frac{10\tau^2}{9} \nabla^{-2} [-\varphi_{,i}\nabla^2\varphi + \partial_i\nabla^{-2}(\varphi^k\nabla^2\varphi_{,k} + \nabla^2\varphi\nabla^2\varphi)] + \frac{\tau^4}{18} \nabla^{-2} [\varphi_{,ki}\nabla^2\varphi^k + \partial_i\nabla^{-2}(-\varphi^{kl}\nabla^2\varphi_{,kl} - \nabla^2\varphi_{,k}\nabla^2\varphi^k)] \\
& + \nabla^{-2}G_i + \frac{18}{\tau^6} \nabla^{-2} [X^{,k}\nabla^{-2}X_{,ik} + \partial_i\nabla^{-2}(-X^{,kl}\nabla^{-2}X_{,kl} - X^{,k}X_{,k})] + \frac{20}{\tau^3} \nabla^{-2} [-\varphi_{,i}X - \varphi^k\nabla^{-2}X_{,ik} \\
& + \partial_i\nabla^{-2}(X\nabla^2\varphi + 2\varphi_{,k}X^{,k} + \varphi^{kl}\nabla^{-2}X_{,kl})] + \frac{2}{\tau} \nabla^{-2} [3\nabla^2\varphi^k\nabla^{-2}X_{,ik} - 2X^{,k}\varphi_{,ik} + 5\varphi^{kl}\nabla^{-2}X_{,kli} \\
& + \partial_i\nabla^{-2}(-3\nabla^2\varphi^{kl}\nabla^{-2}X_{,kl} - X_{,k}\nabla^2\varphi^k - 3\varphi^{kl}X_{,kl} - 5\varphi^{klm}\nabla^{-2}X_{,klm})]. \tag{62}
\end{aligned}$$

Thus, the vector perturbation is obtained

$$\begin{aligned}
\chi_{Sij}^{\perp(2)} = & \frac{10\tau^2}{9} \nabla^{-2} [-\partial_i(\varphi_{,j}\nabla^2\varphi) - \partial_j(\varphi_{,i}\nabla^2\varphi) + \partial_i\partial_j\nabla^{-2}(2\varphi^k\nabla^2\varphi_{,k} + 2\nabla^2\varphi\nabla^2\varphi)] + \frac{\tau^4}{18} \nabla^{-2} [\partial_i(\varphi_{,kj}\nabla^2\varphi^k) \\
& + \partial_j(\varphi_{,ki}\nabla^2\varphi^k) + \partial_i\partial_j\nabla^{-2}(-2\varphi^{kl}\nabla^2\varphi_{,kl} - 2\nabla^2\varphi_{,k}\nabla^2\varphi^k)] + \nabla^{-2}(G_{i,j} + G_{j,i}) \\
& + \frac{18}{\tau^6} \nabla^{-2} [\partial_i(X^{,k}\nabla^{-2}X_{,jk}) + \partial_j(X^{,k}\nabla^{-2}X_{,ik}) + \partial_i\partial_j\nabla^{-2}(-2X^{,kl}\nabla^{-2}X_{,kl} - 2X^{,k}X_{,k})] \\
& + \frac{20}{\tau^3} \nabla^{-2} [\partial_i(-\varphi_{,j}X - \varphi^k\nabla^{-2}X_{,jk}) + \partial_j(-\varphi_{,i}X - \varphi^k\nabla^{-2}X_{,ik}) + \partial_i\partial_j\nabla^{-2}(2X\nabla^2\varphi + 4\varphi_{,k}X^{,k} + 2\varphi^{kl}\nabla^{-2}X_{,kl})] \\
& + \frac{2}{\tau} \nabla^{-2} [\partial_i(3\nabla^2\varphi^k\nabla^{-2}X_{,jk} - 2X^{,k}\varphi_{,jk} + 5\varphi^{kl}\nabla^{-2}X_{,klij}) + \partial_j(3\nabla^2\varphi^k\nabla^{-2}X_{,ik} - 2X^{,k}\varphi_{,ik} + 5\varphi^{kl}\nabla^{-2}X_{,klij}) \\
& + \partial_i\partial_j\nabla^{-2}(-6\nabla^2\varphi^{kl}\nabla^{-2}X_{,kl} - 2X_{,k}\nabla^2\varphi^k - 6\varphi^{kl}X_{,kl} - 10\varphi^{klm}\nabla^{-2}X_{,klm})]. \tag{63}
\end{aligned}$$

As mentioned earlier, $\chi_{Sij}^{\perp(2)}$ has two polarizations, their solutions as above contain two unknown functions in $\chi_{S0ij}^{\perp(2),j}(\mathbf{x})$ with $\partial^i\chi_{S0ij}^{\perp(2),j} = 0$ through G_i as the initial values. This is consistent since their initial first-order time derivatives are fixed via the momentum constraint. The solution (63) can be also derived from the curl portion of the momentum constraint (43) itself without explicitly using the solutions $\phi_S^{(2)}$ of (54), and $\chi_S^{\parallel(2)}$ of (56). The result (63) tells us that the 2nd-order vector perturbation is generated by the coupling of 1st-order scalar perturbations, even though the 1st-order vector perturbation is absent.

D. Tensor perturbation $\chi_{Sij}^{\top(2)}$

Finally consider the traceless part of the evolution equation (44):

$$\chi_{Sij}^{\top(2)''} + \frac{4}{\tau} \chi_{Sij}^{\top(2)'} - \nabla^2 \chi_{Sij}^{\top(2)} = 2\bar{S}_{Sij} - \left(2D_{ij}\phi_S^{(2)} + \frac{1}{3}\nabla^2 D_{ij}\chi_S^{\parallel(2)} \right) - \left(D_{ij}\chi_S^{\parallel(2)''} + \frac{4}{\tau} D_{ij}\chi_S^{\parallel(2)'} \right) - \left(\chi_{Sij}^{\perp(2)''} + \frac{4}{\tau} \chi_{Sij}^{\perp(2)'} \right), \quad (64)$$

where $\bar{S}_{Sij} \equiv S_{Sij} - \frac{1}{3}\delta_{ij}S_{Sk}^k$, explicitly given by

$$\begin{aligned} \bar{S}_{Sij} = & \tau^4 \left(\frac{1}{18} \varphi_{,ij}^k \nabla^2 \varphi_{,k} - \frac{1}{18} \varphi_{,ikl} \varphi_{,j}^{kl} \right) + \tau^2 \left(\frac{8}{3} \varphi_{,i}^k \varphi_{,kj} - \frac{22}{9} \varphi_{,ij} \nabla^2 \varphi - \frac{5}{9} \varphi_{,k} \varphi_{,ij}^k \right) - \frac{100}{9} \varphi \varphi_{,ij} - \frac{50}{3} \varphi_{,i} \varphi_{,j} \\ & + \delta_{ij} \left[\tau^4 \left(\frac{1}{54} \varphi_{,klm} \varphi^{klm} - \frac{1}{54} \nabla^2 \varphi_{,k} \nabla^2 \varphi^{,k} \right) + \tau^2 \left(-\frac{8}{9} \varphi^{,kl} \varphi_{,kl} + \frac{22}{27} \nabla^2 \varphi \nabla^2 \varphi + \frac{5}{27} \varphi_{,k} \nabla^2 \varphi^{,k} \right) + \frac{100}{27} \varphi \nabla^2 \varphi \right. \\ & + \frac{50}{9} \varphi_{,k} \varphi^k + \frac{1}{\tau^8} (54X^2 - 108\nabla^{-2}X_{,kl} \nabla^{-2}X^{,kl}) + \frac{1}{\tau^6} (6\nabla^{-2}X^{,klm} \nabla^{-2}X_{,klm} - 6X^{,k}X_{,k}) \\ & + \left. \frac{1}{\tau^3} \left(\frac{28}{3} X \nabla^2 \varphi - \frac{16}{3} \varphi^{,kl} \nabla^{-2}X_{,kl} + \frac{10}{3} \varphi^{,k} X_{,k} \right) + \frac{2}{3\tau} (\varphi_{,klm} \nabla^{-2}X^{,klm} - X_{,k} \nabla^2 \varphi^{,k}) \right] \\ & + \frac{1}{\tau^8} (324\nabla^{-2}X_{,i}^k \nabla^{-2}X_{,kj} - 162X \nabla^{-2}X_{,ij}) + \frac{1}{\tau^6} (18X^{,k} \nabla^{-2}X_{,kij} - 18\nabla^{-2}X_{,i}^{,kl} \nabla^{-2}X_{,klj}) \\ & + \frac{1}{\tau^3} (-14\varphi_{,ij} X - 14\nabla^2 \varphi \nabla^{-2}X_{,ij} - 10\varphi^{,k} \nabla^{-2}X_{,kij} + 8\varphi_{,i}^k \nabla^{-2}X_{,kj} + 8\varphi_{,j}^k \nabla^{-2}X_{,ki}) \\ & + \frac{1}{\tau} (X^{,k} \varphi_{,kij} + \nabla^{-2}X_{,kij} \nabla^2 \varphi^k - \varphi_{,klj} \nabla^{-2}X_{,i}^{,kl} - \varphi_{,kli} \nabla^{-2}X_{,j}^{,kl}). \end{aligned} \quad (65)$$

One can substitute the known $\phi_S^{(2)}$, $D_{ij}\chi_S^{\parallel(2)}$, $\chi_{Sij}^{\perp(2)}$ into Eq. (64), and solve for $\chi_{Sij}^{\top(2)}$. But the following calculation is simpler and will yield the same result. Applying consecutively $\partial^i \partial^j$, $\nabla^{-2} \nabla^{-2}$, and D_{ij} on (64) leads to

$$-\left(2D_{ij}\phi_S^{(2)} + \frac{1}{3}\nabla^2 D_{ij}\chi_S^{\parallel(2)} \right) - \left(D_{ij}\chi_S^{\parallel(2)''} + \frac{4}{\tau} D_{ij}\chi_S^{\parallel(2)'} \right) = -3D_{ij} \nabla^{-2} \nabla^{-2} \bar{S}_{Skl}^{,kl}. \quad (66)$$

Substituting Eq. (66) into the rhs of Eq. (64) gives

$$\chi_{Sij}^{\top(2)''} + \frac{4}{\tau} \chi_{Sij}^{\top(2)'} - \nabla^2 \chi_{Sij}^{\top(2)} = 2\bar{S}_{Sij} - 3D_{ij} \nabla^{-2} \nabla^{-2} \bar{S}_{Skl}^{,kl} - \left(\chi_{Sij}^{\perp(2)''} + \frac{4}{\tau} \chi_{Sij}^{\perp(2)'} \right). \quad (67)$$

Applying ∂^j to (67) and together with Eq. (12) leads to an equation of A_{Si} as the following

$$0 = 2\bar{S}_{Sij}^j - 2\nabla^{-2} \bar{S}_{Skl,i}^{,kl} - \nabla^2 \left(A_{Si}'' + \frac{4}{\tau} A_{Si}' \right). \quad (68)$$

With the help of Eq. (68), one has

$$\begin{aligned} -\left(\chi_{Sij}^{\perp(2)''} + \frac{4}{\tau} \chi_{Sij}^{\perp(2)'} \right) &= -\partial_j \left(A_{Si}'' + \frac{4}{\tau} A_{Si}' \right) - \partial_i \left(A_{Sj}'' + \frac{4}{\tau} A_{Sj}' \right) \\ &= -2\nabla^{-2} \bar{S}_{Ski,j}^{,k} - 2\nabla^{-2} \bar{S}_{Skj,i}^{,k} + 4\nabla^{-2} \nabla^{-2} \bar{S}_{Skl,i}^{,kl}. \end{aligned} \quad (69)$$

Substituting (69) into the rhs of Eq. (67) yields the equation of 2nd-order tensor perturbation

$$\chi_{Sij}^{\top(2)''} + \frac{4}{\tau} \chi_{Sij}^{\top(2)'} - \nabla^2 \chi_{Sij}^{\top(2)} = 2\bar{S}_{Sij} + \nabla^{-2} \nabla^{-2} \bar{S}_{Skl,i}^{,kl} + \delta_{ij} \nabla^{-2} \bar{S}_{Skl}^{,kl} - 2\nabla^{-2} \bar{S}_{Ski,j}^{,k} - 2\nabla^{-2} \bar{S}_{Skj,i}^{,k}. \quad (70)$$

This is a second-order, hyperbolic differential wave equation with the source constructed from \bar{S}_{Sij} . Substituting (65) into the rhs of (70) and regrouping by powers of τ , one obtains the equation

$$\mathcal{X}_{Sij}^{\top(2)''} + \frac{4}{\tau} \mathcal{X}_{Sij}^{\top(2)'} - \nabla^2 \mathcal{X}_{Sij}^{\top(2)} = B_{1ij} + \tau^2 B_{2ij} + \tau^4 B_{3ij} + \frac{B_{4ij}}{\tau^8} + \frac{B_{5ij}}{\tau^6} + \frac{B_{6ij}}{\tau^3} + \frac{B_{7ij}}{\tau}, \quad (71)$$

where

$$B_{1ij} = \delta_{ij} \nabla^{-2} \left(\frac{50}{9} \varphi_{,kl} \varphi^{,kl} - \frac{50}{9} \nabla^2 \varphi \nabla^2 \varphi \right) + \nabla^{-2} \left(\frac{200}{9} \varphi_{,ij} \nabla^2 \varphi - \frac{200}{9} \varphi_{,ki} \varphi_{,j}^k \right) + \nabla^{-2} \nabla^{-2} \partial_i \partial_j \left(\frac{50}{9} \varphi^{,kl} \varphi_{,kl} - \frac{50}{9} \nabla^2 \varphi \nabla^2 \varphi \right), \quad (72)$$

$$\begin{aligned} B_{2ij} = & \delta_{ij} \left[\frac{11}{9} \nabla^2 \varphi \nabla^2 \varphi - \frac{11}{9} \varphi^{,kl} \varphi_{,kl} + \nabla^{-2} \left(\frac{7}{9} \nabla^2 \varphi_{,k} \nabla^2 \varphi^k - \frac{7}{9} \varphi_{,klm} \varphi^{,klm} \right) \right] + \frac{44}{9} \varphi_{,i}^k \varphi_{,kj} - \frac{44}{9} \varphi_{,ij} \nabla^2 \varphi \\ & + \nabla^{-2} \left(\frac{28}{9} \varphi_{,kli} \varphi_{,j}^{kl} - \frac{28}{9} \varphi_{,kij} \nabla^2 \varphi^k \right) + \nabla^{-2} \nabla^{-2} \partial_i \partial_j \left(\frac{7}{9} \nabla^2 \varphi_{,k} \nabla^2 \varphi^k - \frac{7}{9} \varphi^{,klm} \varphi_{,klm} \right) \\ & + \nabla^{-2} \partial_i \partial_j \left(\frac{11}{9} \nabla^2 \varphi \nabla^2 \varphi - \frac{11}{9} \varphi_{,kl} \varphi^{,kl} \right), \end{aligned} \quad (73)$$

$$B_{3ij} = \delta_{ij} \left(\frac{1}{36} \varphi_{,klm} \varphi^{,klm} - \frac{1}{36} \nabla^2 \varphi_{,k} \nabla^2 \varphi^k \right) + \frac{1}{9} \varphi_{,ij}^k \nabla^2 \varphi_{,k} - \frac{1}{9} \varphi_{,ikl} \varphi_{,j}^{kl} + \nabla^{-2} \partial_i \partial_j \left(\frac{1}{36} \varphi_{,klm} \varphi^{,klm} - \frac{1}{36} \nabla^2 \varphi_{,k} \nabla^2 \varphi^k \right), \quad (74)$$

$$\begin{aligned} B_{4ij} = & \delta_{ij} [81X^2 - 162\nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl} + \nabla^{-2} (162X^{,kl} \nabla^{-2} X_{,kl} + 162X^{,k} X_{,k})] - 324X \nabla^{-2} X_{,ij} + 648\nabla^{-2} X_{,i}^k \nabla^{-2} X_{,kj} \\ & - \nabla^{-2} \partial_i (324X^{,k} \nabla^{-2} X_{,kj}) - \nabla^{-2} \partial_j (324X^{,k} \nabla^{-2} X_{,ki}) + \nabla^{-2} \partial_i \partial_j (81X^2 - 162\nabla^{-2} X^{,kl} \nabla^{-2} X_{,kl}) \\ & + \nabla^{-2} \nabla^{-2} \partial_i \partial_j (162X^{,kl} \nabla^{-2} X_{,kl} + 162X^{,k} X_{,k}), \end{aligned} \quad (75)$$

$$\begin{aligned} B_{5ij} = & \delta_{ij} [-9X^{,k} X_{,k} + 9\nabla^{-2} X^{,klm} \nabla^{-2} X_{,klm}] + 36X^{,k} \nabla^{-2} X_{,kij} - 36\nabla^{-2} X_{,i}^{kl} \nabla^{-2} X_{,klj} \\ & + \nabla^{-2} \partial_i \partial_j (-9X^{,k} X_{,k} + 9\nabla^{-2} X^{,klm} \nabla^{-2} X_{,klm}), \end{aligned} \quad (76)$$

$$\begin{aligned} B_{6ij} = & \delta_{ij} [14X \nabla^2 \varphi - 14\varphi^{,kl} \nabla^{-2} X_{,kl} + \nabla^{-2} (2\varphi^{,klm} \nabla^{-2} X_{,klm} - 2X_{,k} \nabla^2 \varphi^k)] - 28\varphi_{,ij} X - 28\nabla^2 \varphi \nabla^{-2} X_{,ij} - 20\varphi^{,k} \nabla^{-2} X_{,kij} \\ & + 16\varphi_{,i}^k \nabla^{-2} X_{,kj} + 16\varphi_{,j}^k \nabla^{-2} X_{,ki} + \nabla^{-2} \partial_i (-8\varphi_{,j}^k X_{,k} + 12\nabla^2 \varphi^k \nabla^{-2} X_{,kj} + 20\varphi^{,kl} \nabla^{-2} X_{,klj}) \\ & + \nabla^{-2} \partial_j (-8\varphi_{,i}^k X_{,k} + 12\nabla^2 \varphi^k \nabla^{-2} X_{,ki} + 20\varphi^{,kl} \nabla^{-2} X_{,kli}) + \nabla^{-2} \partial_i \partial_j (14X \nabla^2 \varphi - 14\varphi^{,kl} \nabla^{-2} X_{,kl} + 20\varphi^{,k} X_{,k}) \\ & + \nabla^{-2} \nabla^{-2} \partial_i \partial_j (2\varphi^{,klm} \nabla^{-2} X_{,klm} - 2X_{,k} \nabla^2 \varphi^k), \end{aligned} \quad (77)$$

$$\begin{aligned} B_{7ij} = & \delta_{ij} [-X_{,k} \nabla^2 \varphi^k + \varphi_{,klm} \nabla^{-2} X^{,klm}] + 2X^{,k} \varphi_{,kij} + 2\nabla^{-2} X_{,kij} \nabla^2 \varphi^k - 2\varphi_{,klj} \nabla^{-2} X_{,i}^{kl} - 2\varphi_{,kli} \nabla^{-2} X_{,j}^{kl} \\ & + \nabla^{-2} \partial_i \partial_j (-X^{,k} \nabla^2 \varphi_{,k} + \varphi_{,klm} \nabla^{-2} X^{,klm}). \end{aligned} \quad (78)$$

The solution of Eq. (71) is

$$\mathcal{X}_{Sij}^{\top(2)} = \frac{1}{(2\pi)^{3/2}} \int d^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \left[\bar{y}_{ij}(\mathbf{k}, \tau) + \sum_{s=+, \times} \bar{\epsilon}_{ij}^s(k) \bar{h}_k^s(\tau) \right], \quad (79)$$

where

$$\begin{aligned} \bar{y}_{ij} = & \frac{\bar{B}_{1ij}}{k^2} - \frac{10\bar{B}_{2ij}}{k^4} + \frac{280\bar{B}_{3ij}}{k^6} + \tau^2 \frac{\bar{B}_{2ij}}{k^2} - \tau^2 \frac{28\bar{B}_{3ij}}{k^4} + \tau^4 \frac{\bar{B}_{3ij}}{k^2} + \left(\frac{k \cos(k\tau) \bar{B}_{5ij}}{8\tau^3} - \frac{k^3 \cos(k\tau) \bar{B}_{4ij}}{144\tau^3} \right. \\ & + \frac{k^2 \sin(k\tau) \bar{B}_{5ij}}{8\tau^2} - \frac{k^4 \sin(k\tau) \bar{B}_{4ij}}{144\tau^2} \left. \right) \left(\int_0^{k\tau} \frac{\sin s}{s} ds \right) + \left(\frac{k \sin(k\tau) \bar{B}_{5ij}}{8\tau^3} - \frac{k^3 \sin(k\tau) \bar{B}_{4ij}}{144\tau^3} - \frac{k^2 \cos(k\tau) \bar{B}_{5ij}}{8\tau^2} \right. \\ & + \frac{k^4 \cos(k\tau) \bar{B}_{4ij}}{144\tau^2} \left. \right) \left(\int_{k\tau}^{+\infty} \frac{\cos s}{s} ds \right) + \frac{\bar{B}_{4ij}}{18\tau^6} + \frac{\bar{B}_{5ij}}{4\tau^4} - \frac{k^2 \bar{B}_{4ij}}{72\tau^4} + \frac{2\bar{B}_{7ij}}{k^4 \tau^3} + \frac{\bar{B}_{6ij}}{k^2 \tau^3} + \frac{\bar{B}_{7ij}}{k^2 \tau}, \end{aligned} \quad (80)$$

$\bar{B}_{1ij}, \bar{B}_{2ij}, \bar{B}_{3ij}, \bar{B}_{4ij}, \bar{B}_{5ij}, \bar{B}_{6ij}, \bar{B}_{7ij}$ are the Fourier components of $B_{1ij}, B_{2ij}, B_{3ij}, B_{4ij}, B_{5ij}, B_{6ij}, B_{7ij}$ respectively, and the term $\sum_s \epsilon_{ij}^s h_k$ in (79) is a 2nd-order homogeneous solution similar to (18) and (19), whose coefficients d_1^s, d_2^s are to be determined by the initial condition of 2nd-order RGW. Using the relation

$$\frac{1}{(2\pi)^{3/2}} \int d^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{k^2} \bar{B}_{1ij} = -\nabla^{-2} B_{1ij}, \quad (81)$$

and the like, the solution (79) is written as

$$\begin{aligned} \chi_{Sij}^{\top(2)} = & -\nabla^{-2} B_{1ij} - 10\nabla^{-2}\nabla^{-2} B_{2ij} - 280\nabla^{-2}\nabla^{-2}\nabla^{-2} B_{3ij} - \tau^2\nabla^{-2} B_{2ij} - 28\tau^2\nabla^{-2}\nabla^{-2} B_{3ij} - \tau^4\nabla^{-2} B_{3ij} + \frac{B_{4ij}}{18\tau^6} + \frac{B_{5ij}}{4\tau^4} \\ & + \frac{\nabla^2 B_{4ij}}{72\tau^4} + \frac{2\nabla^{-2}\nabla^{-2} B_{7ij}}{\tau^3} - \frac{\nabla^{-2} B_{6ij}}{\tau^3} - \frac{\nabla^{-2} B_{7ij}}{\tau} + \frac{1}{(2\pi)^{3/2}} \int d^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \left[\left(\frac{k \cos(k\tau) \bar{B}_{5ij}}{8\tau^3} - \frac{k^3 \cos(k\tau) \bar{B}_{4ij}}{144\tau^3} \right. \right. \\ & + \frac{k^2 \sin(k\tau) \bar{B}_{5ij}}{8\tau^2} - \frac{k^4 \sin(k\tau) \bar{B}_{4ij}}{144\tau^2} \left. \right) \left(\int_0^{k\tau} \frac{\sin s}{s} ds \right) + \left(\frac{k \sin(k\tau) \bar{B}_{5ij}}{8\tau^3} - \frac{k^3 \sin(k\tau) \bar{B}_{4ij}}{144\tau^3} - \frac{k^2 \cos(k\tau) \bar{B}_{5ij}}{8\tau^2} \right. \\ & \left. \left. + \frac{k^4 \cos(k\tau) \bar{B}_{4ij}}{144\tau^2} \right) \left(\int_{k\tau}^{+\infty} \frac{\cos s}{s} ds \right) + \sum_{s=+, \times} \epsilon_{ij}^s(k) h_k^s(\tau) \right]. \quad (82) \end{aligned}$$

Plugging (75)–(78) into Eq. (82), we obtain the solution of 2nd-order tensor perturbation:

$$\begin{aligned} \chi_{Sij}^{\top(2)} = & + \left[\frac{20}{3} \delta_{ij} \nabla^{-2} \nabla^{-2} (\varphi_{,kl} \varphi^{,kl} - \nabla^2 \varphi \nabla^2 \varphi) + \frac{80}{3} \nabla^{-2} \nabla^{-2} (\varphi_{,ij} \nabla^2 \varphi - \varphi_{,i}^k \varphi_{,kj}) + \frac{20}{3} \nabla^{-2} \nabla^{-2} \nabla^{-2} \partial_i \partial_j (\varphi_{,kl} \varphi^{,kl} - \nabla^2 \varphi \nabla^2 \varphi) \right] \\ & + \tau^2 \left[\frac{11}{9} \delta_{ij} \nabla^{-2} (\varphi^{,kl} \varphi_{,kl} - \nabla^2 \varphi \nabla^2 \varphi) + \frac{44}{9} \nabla^{-2} (\varphi_{,ij} \nabla^2 \varphi - \varphi_{,i}^k \varphi_{,kj}) + \frac{11}{9} \nabla^{-2} \nabla^{-2} \partial_i \partial_j (\varphi_{,kl} \varphi^{,kl} - \nabla^2 \varphi \nabla^2 \varphi) \right] \\ & + \tau^4 \left[\delta_{ij} \nabla^{-2} \left(-\frac{1}{36} \varphi_{,klm} \varphi^{,klm} + \frac{1}{36} \nabla^2 \varphi_{,k} \nabla^2 \varphi^{,k} \right) + \frac{1}{9} \nabla^{-2} (\varphi_{,ikl} \varphi_{,j}^{,kl} - \varphi_{,ij}^k \nabla^2 \varphi_{,k}) \right. \\ & + \frac{1}{36} \nabla^{-2} \nabla^{-2} \partial_i \partial_j (\nabla^2 \varphi_{,k} \nabla^2 \varphi^{,k} - \varphi_{,klm} \varphi^{,klm}) \left. \right] + \frac{9}{2\tau^6} [\delta_{ij} (X^2 - 2\nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl} + \nabla^{-2} (2X^{,kl} \nabla^{-2} X_{,kl} + 2X^{,k} X_{,k})) \\ & - 4X \nabla^{-2} X_{,ij} + 8\nabla^{-2} X_{,i}^k \nabla^{-2} X_{,kj} - \nabla^{-2} \partial_i (4X^{,k} \nabla^{-2} X_{,kj}) - \nabla^{-2} \partial_j (4X^{,k} \nabla^{-2} X_{,ki}) \\ & + \nabla^{-2} \partial_i \partial_j (X^2 - 2\nabla^{-2} X^{,kl} \nabla^{-2} X_{,kl}) + \nabla^{-2} \nabla^{-2} \partial_i \partial_j (2X^{,kl} \nabla^{-2} X_{,kl} + 2X^{,k} X_{,k})] \\ & + \frac{9}{8\tau^4} [\delta_{ij} \nabla^2 (X^2 - \nabla^{-2} X^{,kl} \nabla^{-2} X_{,kl}) + \partial_i \partial_j (X^2 - \nabla^{-2} X^{,kl} \nabla^{-2} X_{,kl}) + \nabla^2 (4\nabla^{-2} X_{,i}^k \nabla^{-2} X_{,kj} - 4X \nabla^{-2} X_{,ij})] \\ & + \frac{7}{\tau^3} [\delta_{ij} \nabla^{-2} (2\varphi^{,kl} \nabla^{-2} X_{,kl} - 2X \nabla^2 \varphi) + \nabla^{-2} \nabla^{-2} \partial_i \partial_j (-2X \nabla^2 \varphi + 2\varphi^{,kl} \nabla^{-2} X_{,kl}) \\ & + \nabla^{-2} (4\varphi_{,ij} X + 4\nabla^2 \varphi \nabla^{-2} X_{,ij} - 4\varphi_{,i}^k \nabla^{-2} X_{,kj} - 4\varphi_{,j}^k \nabla^{-2} X_{,ki})] \\ & + \frac{1}{\tau} [\delta_{ij} \nabla^{-2} (X_{,k} \nabla^2 \varphi^k - \varphi_{,klm} \nabla^{-2} X^{,klm}) + \nabla^{-2} \nabla^{-2} \partial_i \partial_j (X^{,k} \nabla^2 \varphi_{,k} - \varphi_{,klm} \nabla^{-2} X^{,klm}) \\ & + \nabla^{-2} (-2X^{,k} \varphi_{,kij} - 2\nabla^{-2} X_{,kij} \nabla^2 \varphi^k + 2\varphi_{,klj} \nabla^{-2} X_{,i}^{kl} + 2\varphi_{,kli} \nabla^{-2} X_{,j}^{kl})] \\ & + \frac{1}{(2\pi)^{3/2}} \int d^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \left[\left(\frac{k \cos(k\tau) \bar{B}_{5ij}}{8\tau^3} - \frac{k^3 \cos(k\tau) \bar{B}_{4ij}}{144\tau^3} + \frac{k^2 \sin(k\tau) \bar{B}_{5ij}}{8\tau^2} \right. \right. \\ & - \frac{k^4 \sin(k\tau) \bar{B}_{4ij}}{144\tau^2} \left. \right) \left(\int_0^{k\tau} \frac{\sin s}{s} ds \right) + \left(\frac{k \sin(k\tau) \bar{B}_{5ij}}{8\tau^3} - \frac{k^3 \sin(k\tau) \bar{B}_{4ij}}{144\tau^3} \right. \\ & \left. \left. - \frac{k^2 \cos(k\tau) \bar{B}_{5ij}}{8\tau^2} + \frac{k^4 \cos(k\tau) \bar{B}_{4ij}}{144\tau^2} \right) \left(\int_{k\tau}^{+\infty} \frac{\cos s}{s} ds \right) + \sum_{s=+, \times} \epsilon_{ij}^s(k) h_k^s(\tau) \right]. \quad (83) \end{aligned}$$

Let us examine the traceless metric perturbation given by Eq. (4.33) in Ref. [51] as the following:

$$\chi_{Sij}^{(2)} = \pi_{Sij} + \frac{5\tau^2}{9} \left(-6\varphi_{,i}\varphi_{,j} - 4\varphi\varphi_{,ij} + 2\delta_{ij}\varphi_{,k}\varphi^{,k} + \frac{4}{3}\delta_{ij}\varphi\nabla^2\varphi \right) + \frac{\tau^4}{126} \left(19\varphi_{,i}^k\varphi_{,jk} - 12\varphi_{,ij}\nabla^2\varphi + 4(\nabla^2\varphi)^2\delta_{ij} - \frac{19}{3}\delta_{ij}\varphi^{kl}\varphi_{,kl} \right), \quad (84)$$

due to the scalar-scalar coupling, where π_{Sij} is the tensor given by (4.37) in Ref. [51]. (84) contains the growing modes only, which correspond to the case $X = 0$. We notice that the last two terms on the rhs of (84) still contain a tensor portion beside π_{Sij} . Now we can decompose Eq. (84) into the scalar, vector, and tensor: $\chi_{Sij}^{(2)} = D_{ij}\chi_S^{\parallel(2)} + \chi_{Sij}^{\perp(2)} + \chi_{Sij}^{\top(2)}$. By calculation, we find that the scalar $D_{ij}\chi_S^{\parallel(2)}$ is just the expression of Eq. (58) (with $X = 0$, $\delta_{S0}^{(2)}, \phi_{S0}^{(2)}, \chi_{S0}^{\parallel(2)} = 0$ at $\tau_0 = 0$), the vector $\chi_{Sij}^{\perp(2)}$ is just the expression of Eq. (63) (with $X = 0$, $\chi_{S0ij}^{\perp(2)} = 0$ at $\tau_0 = 0$), and the tensor is

$$\begin{aligned} \chi_{Sij}^{\top(2)} = & \pi_{Sij} + \frac{5\tau^2}{9} [\delta_{ij}\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) - 2\varphi_{,i}\varphi_{,j} + \partial_i\nabla^{-2}(2\varphi_{,j}\nabla^2\varphi) \\ & + \partial_j\nabla^{-2}(2\varphi_{,i}\nabla^2\varphi) - \partial_i\partial_j\nabla^{-2}\nabla^{-2}(\nabla^2\varphi\nabla^2\varphi - \varphi_{,kl}\varphi^{,kl})] \\ & + \frac{\tau^4}{252} [\delta_{ij}(6\nabla^2\varphi\nabla^2\varphi - 6\varphi^{,kl}\varphi_{,kl} + \nabla^{-2}(7\nabla^2\varphi_{,k}\nabla^2\varphi^{,k} - 7\varphi_{,klm}\varphi^{,klm})) \\ & + 38\varphi_{,i}^k\varphi_{,jk} - 24\varphi_{,ij}\nabla^2\varphi - \partial_i\nabla^{-2}(14\varphi_{,jk}\nabla^2\varphi^{,k}) - \partial_j\nabla^{-2}(14\varphi_{,ik}\nabla^2\varphi^{,k}) \\ & + \partial_i\partial_j\nabla^{-2}(6\nabla^2\varphi\nabla^2\varphi - 6\varphi^{,kl}\varphi_{,kl}) \\ & + \partial_i\partial_j\nabla^{-2}\nabla^{-2}(-7\varphi_{,klm}\varphi^{,klm} + 7\nabla^2\varphi_{,k}\nabla^2\varphi^{,k})]. \end{aligned} \quad (85)$$

The expression π_{Sij} in Ref. [51] can be written as the following:

$$\begin{aligned} \pi_{Sij} = & \frac{20}{3}\delta_{ij}\nabla^{-2}\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) + \frac{20}{3}\partial_i\partial_j\nabla^{-2}\nabla^{-2}\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) \\ & + \frac{80}{3}\nabla^{-2}\nabla^{-2}(\varphi_{,ij}\nabla^2\varphi - \varphi_{,ik}\varphi_{,j}^k) + \tau^2 \left[\frac{2}{3}\delta_{ij}\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) \right. \\ & \left. + \frac{2}{3}\partial_i\partial_j\nabla^{-2}\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) + \frac{8}{3}\nabla^{-2}(\varphi_{,ij}\nabla^2\varphi - \varphi_{,ik}\varphi_{,j}^k) \right] \\ & + \tau^4 \left[\frac{1}{42}\delta_{ij}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) + \frac{1}{42}\partial_i\partial_j\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) \right. \\ & \left. + \frac{2}{21}(\varphi_{,ij}\nabla^2\varphi - \varphi_{,ik}\varphi_{,j}^k) \right] + \frac{1}{(2\pi)^{3/2}} \int d^3k e^{ik\cdot x} \left[\sum_{s=+, \times} \tilde{\epsilon}_{ij}^s(k) \tilde{h}_k^s(\tau) \right]. \end{aligned} \quad (86)$$

(Notice that the term containing $j_1(k\tau)/k\tau$ in (4.38) of Ref. [51] can be absorbed in the homogenous solution.) Substituting (86) into (85) recovers our solution (83) of the case of $X = 0$. Thus, we have proven that the solution (83) is the full expression of tensor, whereas π_{Sij} is only a portion of tensor.

We can also derive the 2nd-order density contrast in terms of gravitational potential. Substituting the 1st order of (17) and (20) and the 2nd orders of (9) and (54) into Eq. (A29) in Appendix A yields

$$\begin{aligned} \delta_S^{(2)} = & \frac{\tau^4}{126}(5\nabla^2\varphi\nabla^2\varphi + 2\varphi_{,ki}\varphi^{,ki}) + \frac{\tau^2}{18}(40\varphi\nabla^2\varphi + 15\varphi_{,k}\varphi^{,k}) - \frac{\tau^2}{20}F \\ & + \frac{9}{4\tau^6}(3\nabla^{-2}X_{,kl}\nabla^{-2}X^{,kl} + 5X^2) + \frac{3}{\tau^3}Z + \frac{20}{\tau^3}\varphi X + \frac{2}{\tau}\varphi_{,kl}\nabla^{-2}X^{,kl}. \end{aligned} \quad (87)$$

This extends (4.39) of Ref. [51] to general initial conditions $\delta_{S0}^{(2)}, \phi_{S0}^{(2)}, \chi_{S0}^{\parallel(2)}$ through Z and F .

So far, the solutions of the 2nd-order scalar, vector, and tensor metric perturbations, as well as the 2nd-order density contrast, have been obtained. However, in synchronous coordinates, there are still residual gauge transformations, and, correspondingly, the solutions will contain 2nd-order residual gauge modes. In regard to applications, one must find out these 2nd-order gauge modes. We shall address this issue in the next section.

V. THE 2ND-ORDER RESIDUAL GAUGE TRANSFORMATIONS IN SYNCHRONOUS COORDINATES

The general 2nd-order gauge transformations of metric perturbations are generated by a 1st-order vector field $\xi^{(1)}$ and a 2nd-order vector field $\xi^{(2)}$, which are specified by Eqs. (C1)–(C18) in Appendix C.

Consider the special case from synchronous-to-synchronous for the dust model with $a(\tau) \propto \tau^2$ in this section. The 1st-order vector field $\xi^{(1)\mu}$ is listed in (C17), (C26), (C27), and (C28), and the 1st-order gauge transformation of metric perturbations is listed in (C25), (C29), and (C30) in Appendix C. We shall give the corresponding 2nd-order ones for the case of scalar-scalar coupling. From the general formulas (C43), (C48), and (C49) for the case $a \propto \tau^2$, we get the 2nd-order vector field $\xi^{(2)\mu}$ in the presence of $\xi^{(1)\mu}$ as the following

$$\alpha^{(2)} = \frac{A^{(2)}(\mathbf{x})}{\tau^2}, \quad (88)$$

$$\begin{aligned} \beta^{(2)} = \nabla^{-2} & \left[-\frac{20}{3\tau} \varphi^{,k} A_{,k}^{(1)} - \frac{20}{3\tau} \varphi \nabla^2 A^{(1)} + \frac{2\tau}{3} A^{(1),k} \nabla^2 \varphi_{,k} + \frac{2\tau}{3} \varphi_{,kl} A^{(1),kl} - \frac{2}{\tau} A^{(1),k} \nabla^2 C_{,k}^{\parallel(1)} - \frac{2}{\tau} C_{,kl}^{\parallel(1)} A^{(1),kl} - \frac{3}{\tau^4} A_{,k}^{(1)} X^{,k} \right. \\ & \left. - \frac{3}{\tau^4} A^{(1),kl} \nabla^{-2} X_{,kl} \right] - \frac{1}{2\tau^4} A^{(1)} A^{(1)} + \frac{1}{2\tau^2} A_{,k}^{(1)} A^{(1),k} - \frac{1}{\tau} A^{(2)} + C^{\parallel(2)}, \end{aligned} \quad (89)$$

$$\begin{aligned} d_i^{(2)} = \partial_i \nabla^{-2} & \left[\frac{3}{\tau^4} A_{,k}^{(1)} X^{,k} + \frac{3}{\tau^4} A^{(1),kl} \nabla^{-2} X_{,kl} + \frac{20}{3\tau} \varphi^{,k} A_{,k}^{(1)} + \frac{20}{3\tau} \varphi \nabla^2 A^{(1)} - \frac{2\tau}{3} A^{(1),k} \nabla^2 \varphi_{,k} - \frac{2\tau}{3} \varphi_{,kl} A^{(1),kl} + \frac{2}{\tau} A^{(1),k} \nabla^2 C_{,k}^{\parallel(1)} \right. \\ & \left. + \frac{2}{\tau} C_{,kl}^{\parallel(1)} A^{(1),kl} \right] - \frac{3}{\tau^4} A^{(1),k} \nabla^{-2} X_{,ki} - \frac{2}{\tau} \left[\frac{10}{3} \varphi A_{,i}^{(1)} + C_{,ik}^{\parallel(1)} A^{(1),k} \right] + \frac{2\tau}{3} \varphi_{,ik} A^{(1),k} + C_i^{\perp(2)}, \end{aligned} \quad (90)$$

where the solutions (14) and (15) of scalar perturbations have been used, and the $\chi_{ij}^{\top(1)}$ -dependent parts have been dropped as they belong to the scalar-tensor coupling. In the above, $A^{(1)}(\mathbf{x})$ is an arbitrary function, and $C_i^{(1)}(\mathbf{x})$ is an arbitrary 3-vector which can be written into two parts: $C_i^{(1)} = C_i^{\parallel(1)} + C_i^{\perp(1)}$, with $C_i^{\parallel(1)}$ being the longitudinal part and $C_i^{\perp(1)}$ the transverse part. See Eqs. (C21) and (C22) in Appendix C. Similar for $A^{(2)}(\mathbf{x})$ and $C_i^{(2)}(\mathbf{x})$. The expressions of components of $\xi^{(2)\mu}$ in (89) and (90) are rather lengthy as they involve the complicated functions of the 1st-order metric perturbations and components of $\xi^{(1)\mu}$. This complication of $\xi^{(2)\mu}$ is caused by the requirement $\bar{g}_{0i}^{(2)} = 0$ in the presence of $\xi^{(1)\mu}$. The general formulas (C50), (C56), (C57), and (C58), follow the required residual gauge transformations of 2nd-order metric perturbations in the Einstein-de Sitter model:

$$\begin{aligned} \bar{\phi}_S^{(2)} = \phi_S^{(2)} - \frac{2}{\tau^6} & [XA^{(1)} + A^{(1)}A^{(1)}] + \frac{1}{\tau^4} \left[A_{,k}^{(1)} X^{,k} + 3A^{(1),kl} \nabla^{-2} X_{,kl} + 2A^{(1)} \nabla^2 A^{(1)} + \frac{5}{3} A_{,k}^{(1)} A^{(1),k} \right] \\ & + \frac{1}{\tau^3} \left[-2X_{,k} C^{\parallel(1),k} - 4C^{\parallel(1),kl} \nabla^{-2} X_{,kl} - \frac{40}{3} \varphi A^{(1)} - \frac{8}{3} A^{(1)} \nabla^2 C^{\parallel(1)} - 2A_{,k}^{(1)} C^{\parallel(1),k} \right] - \frac{1}{3\tau^2} A_{,kl}^{(1)} A^{(1),kl} \\ & + \frac{1}{\tau} \left[-\frac{2}{3} A^{(1)} \nabla^2 \varphi + \frac{10}{9} \varphi_{,k} A^{(1),k} + \frac{1}{3} C^{\parallel(1),k} \nabla^2 A_{,k}^{(1)} - \frac{1}{3} A^{(1),k} \nabla^2 C_{,k}^{\parallel(1)} + \frac{2}{3} A_{,kl}^{(1)} C^{\parallel(1),kl} \right] \\ & + \left[-\frac{20}{9} \varphi \nabla^2 C^{\parallel(1)} - \frac{10}{3} \varphi_{,k} C^{\parallel(1),k} - \frac{1}{3} C^{\parallel(1),k} \nabla^2 C_{,k}^{\parallel(1)} - \frac{2}{3} C_{,kl}^{\parallel(1)} C^{\parallel(1),kl} \right] \\ & + \tau \left[\frac{1}{3} A^{(1),k} \nabla^2 \varphi_{,k} + \frac{4}{9} \varphi_{,kl} A^{(1),kl} \right] + \tau^2 \left[-\frac{1}{9} \nabla^2 \varphi_{,k} C^{\parallel(1),k} - \frac{2}{9} \varphi_{,kl} C^{\parallel(1),kl} \right] \\ & + \frac{2A^{(2)}}{\tau^3} - \frac{1}{3\tau} \nabla^2 A^{(2)} + \frac{1}{3} \nabla^2 C^{\parallel(2)}, \end{aligned} \quad (91)$$

$$\begin{aligned}
D_{ij}\bar{\chi}_S^{\parallel(2)} = & D_{ij}\chi_S^{\parallel(2)} + \frac{1}{\tau^6}D_{ij}[12\nabla^{-2}(A^{(1)}X) + 18\nabla^{-2}\nabla^{-2}(A^{(1),kl}\nabla^{-2}X_{,kl} - X\nabla^2A^{(1)})] \\
& + \frac{1}{\tau^4}D_{ij}[-6\nabla^{-2}(A^{(1)}X_{,k} + 3A^{(1),kl}\nabla^{-2}X_{,kl}) + 18\nabla^{-2}\nabla^{-2}(A^{(1),klm}\nabla^{-2}X_{,klm} - X^k\nabla^2A^{(1)}) + A^{(1)}A^{(1)} \\
& - 14\nabla^{-2}(A^{(1)}\nabla^2A^{\parallel(1)}) + 21\nabla^{-2}\nabla^{-2}(-A^{(1),kl}A^{\parallel(1)} + \nabla^2A^{(1)}\nabla^2A^{\parallel(1)})] + \frac{1}{\tau^3}D_{ij}[12C^{\parallel(1),k}\nabla^{-2}X_{,k} \\
& - 12\nabla^{-2}(\nabla^2C^{\parallel(1),k}\nabla^{-2}X_{,k}) + 18\nabla^{-2}\nabla^{-2}(X^k\nabla^2C^{\parallel(1)} - C^{\parallel(1),klm}\nabla^{-2}X_{,klm}) + 16\nabla^{-2}(A^{(1)}\nabla^2C^{\parallel(1)}) \\
& + 24\nabla^{-2}\nabla^{-2}(A^{(1),kl}C^{\parallel(1)} - \nabla^2A^{(1)}\nabla^2C^{\parallel(1)})] + \frac{1}{\tau^2}D_{ij}[A_{,k}^{(1)}A^{(1),k} - 2\nabla^{-2}(A^{(1),k}\nabla^2A_{,k}^{(1)}) + 3\nabla^{-2}\nabla^{-2}(\nabla^2A^{(1),k}\nabla^2A_{,k}^{(1)} \\
& - A^{(1),klm}A_{,klm}^{(1)})] + \frac{1}{\tau}D_{ij}\left[\frac{20}{3}\varphi A^{(1)} - 2A^{(1),k}C^{\parallel(1)} + \frac{4}{3}\nabla^{-2}(-5\varphi\nabla^2A^{(1)} - 4A^{(1)}\nabla^2\varphi + 3A^{(1),k}\nabla^2C^{\parallel(1)})\right. \\
& \left. + 6\nabla^{-2}\nabla^{-2}(-3\varphi^{,kl}A_{,kl}^{(1)} + 3\nabla^2\varphi\nabla^2A^{(1)} + A^{(1),klm}C^{\parallel(1)} - \nabla^2A^{(1),k}\nabla^2C^{\parallel(1)})\right] + D_{ij}\left[2C^{\parallel(1),k}C^{\parallel(1)}\right. \\
& \left. - \nabla^{-2}\left(-\frac{40}{3}\varphi\nabla^2C^{\parallel(1)} + 2C^{\parallel(1),k}\nabla^2C^{\parallel(1),k}\right) + \nabla^{-2}\nabla^{-2}(-20\nabla^2\varphi\nabla^2C^{\parallel(1)} + 20\varphi^{,kl}C^{\parallel(1)} + 3\nabla^2C^{\parallel(1)}\nabla^2C^{\parallel(1),k}\right. \\
& \left. - 3C^{\parallel(1),klm}C^{\parallel(1)}\right] + \tau D_{ij}\left[-\frac{4}{3}A^{(1),k}\varphi_{,k} + \nabla^{-2}\left(\frac{4}{3}\varphi_{,k}\nabla^2A^{(1),k} - \frac{2}{3}A^{(1)}\nabla^2\varphi^k\right) + \nabla^{-2}\nabla^{-2}(-\nabla^2A_{,k}^{(1)}\nabla^2\varphi^k\right. \\
& \left. + A^{(1),klm}\varphi_{,klm})\right] + \tau^2 D_{ij}\left[\frac{2}{3}C^{\parallel(1),k}\varphi_{,k} - \nabla^{-2}\left(\frac{2}{3}\varphi^k\nabla^2C^{\parallel(1)}\right) + \nabla^{-2}\nabla^{-2}(-\varphi^{,klm}C^{\parallel(1)} + \nabla^2\varphi^k\nabla^2C^{\parallel(1)})\right] \\
& + \frac{2}{\tau}D_{ij}A^{(2)} - 2D_{ij}C^{\parallel(2)}, \tag{92}
\end{aligned}$$

$$\begin{aligned}
\bar{\chi}_{Sij}^{\perp(2)} = & \chi_{Sij}^{\perp(2)} + \frac{12}{\tau^6}[\partial_i\nabla^{-2}(A^{(1),k}\nabla^{-2}X_{,kj} - A_j^{(1)}X) + \partial_i\partial_j\nabla^{-2}\nabla^{-2}(X\nabla^2A^{(1)} - A^{(1),kl}\nabla^{-2}X_{,kl})] + \frac{1}{\tau^4}[-\partial_i(9A^{(1),k}\nabla^{-2}X_{,kj}) \\
& + \partial_i\nabla^{-2}(12A^{(1),k}X_{,kj} - 12A_j^{(1),kl}\nabla^{-2}X_{,kl}) + \partial_i\partial_j\nabla^{-2}(9X_{,k}A^{(1),k} - 3A^{(1),kl}\nabla^{-2}X_{,kl}) + \partial_i\partial_j\nabla^{-2}\nabla^{-2}(-12A^{(1),k}\nabla^2X_{,k} \\
& + 24\nabla^{-2}X_{,kl}\nabla^2A^{(1),kl} + 36A^{(1),klm}\nabla^{-2}X_{,klm}) + 14\partial_i\nabla^{-2}(A_j^{(1)}\nabla^2A^{(1)}) - 7\partial_i\partial_j\nabla^{-2}(A^{(1),k}A_{,k}^{(1)}) \\
& + 14\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),kl}A_{,kl}^{(1)} - \nabla^2A^{(1)}\nabla^2A^{(1)})] + \frac{1}{\tau^3}[\partial_i\nabla^{-2}(12C^{\parallel(1),kl}\nabla^{-2}X_{,klj} + 12\nabla^{-2}X_{,kj}\nabla^2C^{\parallel(1),k}) \\
& + \partial_i\partial_j\nabla^{-2}(12C^{\parallel(1),kl}\nabla^{-2}X_{,kl} - 12X_{,k}C^{\parallel(1),k}) + \partial_i\partial_j\nabla^{-2}\nabla^{-2}(12C^{\parallel(1),k}\nabla^2X_{,k} - 24\nabla^{-2}X_{,kl}\nabla^2C^{\parallel(1),kl} \\
& - 36C^{\parallel(1),klm}\nabla^{-2}X_{,klm}) + 16\partial_i\nabla^{-2}(A^{(1),k}C^{\parallel(1)} - A_j^{(1)}\nabla^2C^{\parallel(1)}) - 16\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),kl}C^{\parallel(1)} - \nabla^2A^{(1)}\nabla^2C^{\parallel(1)})] \\
& + \frac{1}{\tau^2}[2\partial_i\nabla^{-2}(A_{,jk}^{(1)}\nabla^2A^{(1),k}) - \partial_i\partial_j\nabla^{-2}(A^{(1),kl}A_{,kl}^{(1)}) + 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),klm}A_{,klm}^{(1)} - \nabla^2A_{,k}^{(1)}\nabla^2A^{(1),k})] \\
& + \frac{1}{\tau}\left[\frac{4}{3}\partial_i\nabla^{-2}(-5\varphi^k_{,jk}A_{,jk}^{(1)} - 4A^{(1),k}\varphi_{,jk} + 5\varphi_{,j}\nabla^2A^{(1)} + 4A_j^{(1)}\nabla^2\varphi + 3A^{(1),kl}C^{\parallel(1)} - 3A_{,jk}^{(1)}\nabla^2C^{\parallel(1),k})\right. \\
& \left. + 4\partial_i\partial_j\nabla^{-2}\nabla^{-2}(-3\nabla^2\varphi\nabla^2A^{(1)} + 3A^{(1),kl}\varphi_{,kl} + \nabla^2A_{,k}^{(1)}\nabla^2C^{\parallel(1),k} - A^{(1),klm}C^{\parallel(1)})\right] \\
& + \left[2\partial_j\nabla^{-2}\left(\frac{20}{3}\varphi^k C^{\parallel(1)} - \frac{20}{3}\varphi_{,i}\nabla^2C^{\parallel(1)} - C^{\parallel(1),kl}C^{\parallel(1)} + C^{\parallel(1)}_{,ik}\nabla^2C^{\parallel(1),k}\right) + 2\partial_i\partial_j\nabla^{-2}\nabla^{-2}\left(-\frac{20}{3}\varphi^{,kl}C^{\parallel(1)}\right.\right. \\
& \left. + \frac{20}{3}\nabla^2\varphi\nabla^2C^{\parallel(1)} + C^{\parallel(1),klm}C^{\parallel(1)} - \nabla^2C^{\parallel(1),k}\nabla^2C^{\parallel(1)}\right)] + \tau\left[\partial_j\nabla^{-2}\left(\frac{4}{3}A^{(1),kl}\varphi_{,ikl} - \frac{2}{3}A^{(1),ikl}\varphi_{,kl}\right.\right. \\
& \left. + \frac{2}{3}A_{,i}^{(1),k}\nabla^2\varphi_{,k} - \frac{4}{3}\varphi^k_{,i}\nabla^2A_{,k}^{(1)}\right) + \partial_i\partial_j\nabla^{-2}\nabla^{-2}\left(-\frac{2}{3}A^{(1),klm}\varphi_{,klm} + \frac{2}{3}\nabla^2A^{(1),k}\nabla^2\varphi_{,k}\right)] \\
& + \tau^2\left[\partial_j\nabla^{-2}\left(-\frac{2}{3}\varphi^{,kl}C^{\parallel(1)} + \frac{2}{3}\varphi_{,ik}\nabla^2C^{\parallel(1),k}\right) + \frac{2}{3}\partial_i\partial_j\nabla^{-2}\nabla^{-2}(\varphi^{,klm}C^{\parallel(1)} - \nabla^2\varphi^k\nabla^2C^{\parallel(1)})\right] \\
& - C_{ij}^{\perp(2)} + (i \leftrightarrow j), \tag{93}
\end{aligned}$$

$$\begin{aligned}
\tilde{\chi}_{Sij}^{\top(2)} = & \chi_{Sij}^{\top(2)} + \frac{1}{\tau^6} [12A^{(1)}\nabla^{-2}X_{,ij} + \delta_{ij}\nabla^{-2}(6A^{(1),kl}\nabla^{-2}X_{,kl} - 6X\nabla^2A^{(1)}) + 12\partial_i\nabla^{-2}(A_j^{(1)}X \\
& - A^{(1),k}\nabla^{-2}X_{,kj}) + 12\partial_j\nabla^{-2}(A_i^{(1)}X - A^{(1),k}\nabla^{-2}X_{,ki}) - \partial_i\partial_j\nabla^{-2}(12A^{(1)}X) \\
& + 6\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),kl}\nabla^{-2}X_{,kl} - X\nabla^2A^{(1)})] + \frac{1}{\tau^4} [8A_{,ij}^{(1)}X + 12A^{(1),k}\nabla^{-2}X_{,kij} \\
& + \delta_{ij}\nabla^{-2}(6A^{(1),klm}\nabla^{-2}X_{,klm} - 6X_{,k}\nabla^2A^{(1),k}) + 8\nabla^{-2}(X_{,i}^kA_{,kj}^{(1)} + X_{,j}^kA_{,ki}^{(1)} - X_{,ij}\nabla^2A^{(1)} \\
& - A_{,ij}^{(1)}\nabla^2X) + 4\partial_i\nabla^{-2}(3A_{,j}^{(1),kl}\nabla^{-2}X_{,kl} - X_{,kj}A^{(1),k} - 2X\nabla^2A_{,j}^{(1)}) + 4\partial_j\nabla^{-2}(3A_{,i}^{(1),kl}\nabla^{-2}X_{,kl} \\
& - X_{,ki}A^{(1),k} - 2X\nabla^2A_{,i}^{(1)}) + 4\partial_i\partial_j\nabla^{-2}(2X\nabla^2A^{(1)} - X_{,k}A^{(1),k} - 6A^{(1),kl}\nabla^{-2}X_{,kl}) \\
& + 6\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),klm}\nabla^{-2}X_{,klm} - X_{,k}\nabla^2A^{(1),k}) - 7\nabla^{-2}(-4A_{,i}^{(1),k}A_{,jk}^{(1)} + 4A_{,ij}^{(1)}\nabla^2A^{(1)} \\
& + A^{(1),kl}A_{,kl}^{(1)}\delta_{ij} - \nabla^2A^{(1)}\nabla^2A^{(1)}\delta_{ij}) - 7\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),kl}A_{,kl}^{(1)} - \nabla^2A^{(1)}\nabla^2A^{(1)})] \\
& + \frac{1}{\tau^3} [-8C_{,ij}^{\parallel(1)}X - 12C^{\parallel(1),k}\nabla^{-2}X_{,kij} + \delta_{ij}\nabla^{-2}(6X_{,k}\nabla^2C^{\parallel(1),k} - 6C^{\parallel(1),klm}\nabla^{-2}X_{,klm}) \\
& + 8\nabla^{-2}(X_{,ij}\nabla^2C^{\parallel(1)} + C_{,ij}^{\parallel(1)}\nabla^2X - X_{,i}^kC_{,kj}^{\parallel(1)} - X_{,j}^kC_{,ki}^{\parallel(1)}) + 4\partial_i\nabla^{-2}(X_{,kj}C^{\parallel(1),k} + 2X\nabla^2C_{,j}^{\parallel(1)} \\
& - 3C_{,j}^{\parallel(1),kl}\nabla^{-2}X_{,kl}) + 4\partial_j\nabla^{-2}(X_{,ki}C^{\parallel(1),k} + 2X\nabla^2C_{,i}^{\parallel(1)} - 3C_{,i}^{\parallel(1),kl}\nabla^{-2}X_{,kl}) \\
& + 4\partial_i\partial_j\nabla^{-2}(X_{,k}C^{\parallel(1),k} - 2X\nabla^2C^{\parallel(1)} + 6C^{\parallel(1),kl}\nabla^{-2}X_{,kl}) + 6\partial_i\partial_j\nabla^{-2}\nabla^{-2}(X_{,k}\nabla^2C^{\parallel(1),k} \\
& - C^{\parallel(1),klm}\nabla^{-2}X_{,klm}) + 8\nabla^{-2}(-2A_{,j}^{(1),k}C_{,ik}^{\parallel(1)} - 2A_{,i}^{(1),k}C_{,jk}^{\parallel(1)} + 2C_{,ij}^{\parallel(1)}\nabla^2A^{(1)} + 2A_{,ij}^{(1)}\nabla^2C^{\parallel(1)} \\
& + A^{(1),kl}C_{,kl}^{\parallel(1)}\delta_{ij} - \nabla^2A^{(1)}\nabla^2C^{\parallel(1)}\delta_{ij}) + 8\partial_i\partial_j\nabla^{-2}\nabla^{-2}(A^{(1),kl}C_{,kl}^{\parallel(1)} - \nabla^2A^{(1)}\nabla^2C^{\parallel(1)})] \\
& + \frac{1}{\tau^2} \left[\nabla^{-2}(4A_{,i}^{(1),kl}A_{,jkl}^{(1)} - 4A_{,ijk}^{(1)}\nabla^2A^{(1),k} - A^{(1),klm}A_{,klm}^{(1)}\delta_{ij} + \nabla^2A_{,k}^{(1)}\nabla^2A^{(1),k}\delta_{ij}) \right. \\
& \left. + \partial_i\partial_j\nabla^{-2}\nabla^{-2}(-A^{(1),klm}A_{,klm}^{(1)} + \nabla^2A_{,k}^{(1)}\nabla^2A^{(1),k}) \right] + \frac{1}{\tau} \nabla^{-2} [2(6\varphi_{,j}^kA_{,ik}^{(1)} + 6\varphi_{,i}^kA_{,jk}^{(1)} \\
& - 6A_{,ij}^{(1)}\nabla^2\varphi - 6\varphi_{,ij}\nabla^2A^{(1)} - 3A^{(1),kl}\varphi_{,kl}\delta_{ij} + 3\nabla^2A^{(1)}\nabla^2\varphi\delta_{ij} - 2A_{,j}^{(1),kl}C_{,ikl}^{\parallel(1)} - 2A_{,i}^{(1),kl}C_{,jkl}^{\parallel(1)} \\
& + 2C_{,ijk}^{\parallel(1)}\nabla^2A^{(1),k} + 2A_{,ijk}^{(1)}\nabla^2C^{\parallel(1),k} + A^{(1),klm}C_{,klm}^{\parallel(1)}\delta_{ij} - \nabla^2A_{,k}^{(1)}\nabla^2C^{\parallel(1),k}\delta_{ij}) \\
& + 2\partial_i\partial_j\nabla^{-2}(-3A^{(1),kl}\varphi_{,kl} + 3\nabla^2A^{(1)}\nabla^2\varphi + A^{(1),klm}C_{,klm}^{\parallel(1)} - \nabla^2A_{,k}^{(1)}\nabla^2C^{\parallel(1),k}) \\
& + \left[\nabla^{-2} \left(-\frac{40}{3}\varphi_{,j}^kC_{,ik}^{\parallel(1)} - \frac{40}{3}\varphi_{,i}^kC_{,jk}^{\parallel(1)} + \frac{40}{3}C_{,ij}^{\parallel(1)}\nabla^2\varphi + \frac{40}{3}\varphi_{,ij}\nabla^2C^{\parallel(1)} + \frac{20}{3}\varphi^{kl}C_{,kl}^{\parallel(1)}\delta_{ij} \right. \right. \\
& \left. \left. - \frac{20}{3}\nabla^2\varphi\nabla^2C^{\parallel(1)}\delta_{ij} + 4C_{,j}^{\parallel(1),kl}C_{,ikl}^{\parallel(1)} - 4C_{,ijk}^{\parallel(1)}\nabla^2C^{\parallel(1),k} - C^{\parallel(1),klm}C_{,klm}^{\parallel(1)}\delta_{ij} \right. \right. \\
& \left. \left. + \nabla^2C^{\parallel(1),k}\nabla^2C_{,k}^{\parallel(1)}\delta_{ij} \right) + \partial_i\partial_j\nabla^{-2}\nabla^{-2} \left(\frac{20}{3}\varphi^{kl}C_{,kl}^{\parallel(1)} - \frac{20}{3}\nabla^2\varphi\nabla^2C^{\parallel(1)} + \nabla^2C^{\parallel(1),k}\nabla^2C_{,k}^{\parallel(1)} \right. \right. \\
& \left. \left. - C^{\parallel(1),klm}C_{,klm}^{\parallel(1)} \right) \right] + \tau \left[\nabla^{-2} \left(-\frac{2}{3}A_{,j}^{(1),kl}\varphi_{,ikl} - \frac{2}{3}A_{,i}^{(1),kl}\varphi_{,jkl} + \frac{2}{3}A_{,ijk}^{(1)}\nabla^2\varphi^k + \frac{2}{3}\varphi_{,ijk}\nabla^2A^{(1),k} \right. \right. \\
& \left. \left. + \frac{1}{3}A^{(1),klm}\varphi_{,klm}\delta_{ij} - \frac{1}{3}\nabla^2A^{(1),k}\nabla^2\varphi_{,k}\delta_{ij} \right) + \partial_i\partial_j\nabla^{-2}\nabla^{-2} \left(-\frac{1}{3}\nabla^2A^{(1),k}\nabla^2\varphi_{,k} + \frac{1}{3}A^{(1),klm}\varphi_{,klm} \right) \right] \\
& + \tau^2 \left[\nabla^{-2} \left(\frac{2}{3}\varphi_{,j}^klC_{,ikl}^{\parallel(1)} + \frac{2}{3}\varphi_{,i}^klC_{,jkl}^{\parallel(1)} - \frac{2}{3}C_{,ijk}^{\parallel(1)}\nabla^2\varphi^k - \frac{2}{3}\varphi_{,ijk}\nabla^2C^{\parallel(1),k} - \frac{1}{3}\varphi^{klm}C_{,klm}^{\parallel(1)}\delta_{ij} \right. \right. \\
& \left. \left. + \frac{1}{3}\nabla^2\varphi^k\nabla^2C_{,k}^{\parallel(1)}\delta_{ij} \right) + \frac{1}{3}\partial_i\partial_j\nabla^{-2}\nabla^{-2}(\nabla^2\varphi^k\nabla^2C_{,k}^{\parallel(1)} - \varphi^{klm}C_{,klm}^{\parallel(1)}) \right]. \tag{94}
\end{aligned}$$

The transformation formulas (91)–(94) are lengthy because of those terms brought about by $\xi^{(1)\mu}$. Eq. (94) tells us that the transformation of 2nd-order tensor involves only $\xi^{(1)}$, independent of the 2nd-order vector field $\xi^{(2)}$.

The synchronous-to-synchronous transformation of 2nd-order density perturbation is derived by applying (C5) to the 2nd order $T_{00}^{(2)}$,

$$\bar{T}_{00}^{(2)} = T_{00}^{(2)} - 2\mathcal{L}_{\xi^{(1)}}T_{00}^{(1)} + \mathcal{L}_{\xi^{(1)}}(\mathcal{L}_{\xi^{(1)}}T_{00}^{(0)}) - \mathcal{L}_{\xi^{(2)}}T_{00}^{(0)}. \quad (95)$$

Up to the 2nd order, one calculates $T_{00}^{(2)} = \rho^{(2)}a^2$, $\bar{U}_0^{(2)} = -\frac{A^{(1),i}A_i^{(1)}}{\tau^2}$, $\bar{T}_{00}^{(2)} = a^2\bar{\rho}^{(2)} + 2\rho^{(0)}A^{(1),i}A_{,i}^{(1)}$. Plugging these into (95) leads to the result

$$\begin{aligned} \bar{\rho}_S^{(2)} &= \rho_S^{(2)} - 2\rho_{,0}^{(1)}\frac{A^{(1)}}{\tau^2} - 2\rho_{,k}^{(1)}\left(-\frac{A^{(1),k}}{\tau} + C^{\parallel(1),k}\right) \\ &\quad + 54\rho^{(0)}\frac{A^{(1)}A^{(1)}}{\tau^6} - 6\rho^{(0)}\frac{A_{,k}^{(1)}}{\tau^3}\left(-\frac{A^{(1),k}}{\tau} + C^{\parallel(1),k}\right) \\ &\quad + 6\rho^{(0)}\frac{A^{(2)}}{\tau^3}. \end{aligned} \quad (96)$$

This result can be also derived by applying (C10) to $\rho^{(2)}$ as a scalar function,

$$\bar{\rho}^{(2)} = \rho^{(2)} - 2\mathcal{L}_{\xi^{(1)}}\rho^{(1)} + \mathcal{L}_{\xi^{(1)}}(\mathcal{L}_{\xi^{(1)}}\rho^{(0)}) - \mathcal{L}_{\xi^{(2)}}\rho^{(0)}. \quad (97)$$

(96) is also written in terms of the density contrast

$$\begin{aligned} \bar{\delta}_S^{(2)} &= \delta_S^{(2)} + \left(\frac{12}{\tau}\delta^{(1)} - 2\delta_{,0}^{(1)}\right)\frac{A^{(1)}}{\tau^2} - 2\delta_{,k}^{(1)}\left(-\frac{A^{(1),k}}{\tau} + C^{\parallel(1),k}\right) \\ &\quad + 54\frac{A^{(1)}A^{(1)}}{\tau^6} - 6\frac{A_{,k}^{(1)}}{\tau^3}\left(-\frac{A^{(1),k}}{\tau} + C^{\parallel(1),k}\right) + 6\frac{A^{(2)}}{\tau^3}. \end{aligned} \quad (98)$$

So far in this paper, the perturbations of 4-velocity are taken to be zero, $U^{(1)\mu} = U^{(2)\mu} = 0$, for the dust model. When one also requires in the new synchronous coordinate \bar{x}^μ the transformed 3-velocity is zero $\bar{U}^{(1)i} = 0$, [42], one gets an extra constraint on the transformation vector field:

$$A^{(1)}(\mathbf{x})_{,i} = 0, \quad (99)$$

i.e, $A^{(1)} = \text{const.}$ [See Eq. (C35) in Appendix C.]

When one further requires the transformed 2nd-order perturbed velocity $\bar{U}^{(2)i} = 0$ by using (C14), this leads to $\mathcal{L}_{\xi^{(2)}}U^{(0)i} = 0$, which gives another constraint:

$$A^{(2)}(\mathbf{x})_{,i} = 0, \quad (100)$$

i.e, $A^{(2)} = \text{const.}$ Under the condition (99) and (100), the components of $\xi^{(2)\mu}$ in (88), (89), and (90) are much simplified as the following:

$$\alpha^{(2)} = \frac{A^{(2)}}{\tau^2}, \quad (101)$$

$$\beta^{(2)} = -\frac{1}{2\tau^4}A^{(1)}A^{(1)} - \frac{1}{\tau}A^{(2)} + C^{\parallel(2)}(\mathbf{x}), \quad (102)$$

$$d_i^{(2)} = C_i^{\perp(2)}(\mathbf{x}), \quad (103)$$

and (91), (92), (93), and (94) substantially reduce to

$$\begin{aligned} \bar{\phi}_S^{(2)} &= \phi_S^{(2)} - \frac{2}{\tau^6}[XA^{(1)} + A^{(1)}A^{(1)}] + \frac{1}{\tau^3}\left[-2X_{,k}C^{\parallel(1),k} - 4C^{\parallel(1),kl}\nabla^{-2}X_{,kl} - \frac{40}{3}\phi A^{(1)} - \frac{8}{3}A^{(1)}\nabla^2C^{\parallel(1)}\right] + \frac{1}{\tau}\left[-\frac{2}{3}A^{(1)}\nabla^2\phi\right] \\ &\quad + \left[-\frac{20}{9}\phi\nabla^2C^{\parallel(1)} - \frac{10}{3}\phi_{,k}C^{\parallel(1),k} - \frac{1}{3}C^{\parallel(1),k}\nabla^2C_{,k}^{\parallel(1)} - \frac{2}{3}C_{,kl}^{\parallel(1)}C^{\parallel(1),kl}\right] + \tau^2\left[-\frac{1}{9}\nabla^2\phi_{,k}C^{\parallel(1),k} - \frac{2}{9}\phi_{,kl}C^{\parallel(1),kl}\right] \\ &\quad + 2\frac{A^{(2)}}{\tau^3} + \frac{1}{3}\nabla^2C^{\parallel(2)}(\mathbf{x}), \end{aligned} \quad (104)$$

$$\begin{aligned} D_{ij}\bar{\chi}_S^{\parallel(2)} &= D_{ij}\chi_S^{\parallel(2)} + \frac{1}{\tau^6}D_{ij}[12\nabla^{-2}(A^{(1)}X)] + \frac{1}{\tau^3}D_{ij}[12C^{\parallel(1),k}\nabla^{-2}X_{,k} - 12\nabla^{-2}(\nabla^2C^{\parallel(1),k}\nabla^{-2}X_{,k}) \\ &\quad + 18\nabla^{-2}\nabla^{-2}(X^{,k}\nabla^2C_{,k}^{\parallel(1)} - C^{\parallel(1),klm}\nabla^{-2}X_{,klm}) + 16A^{(1)}C^{\parallel(1)}] + \frac{1}{\tau}D_{ij}\left[\frac{20}{3}\phi A^{(1)} + \frac{4}{3}\nabla^{-2}(-4A^{(1)}\nabla^2\phi)\right] \\ &\quad + D_{ij}\left[2C^{\parallel(1),k}C_{,k}^{\parallel(1)} - \nabla^{-2}\left(-\frac{40}{3}\phi\nabla^2C^{\parallel(1)} + 2C_{,k}^{\parallel(1)}\nabla^2C^{\parallel(1),k}\right)\right. \\ &\quad \left.+ \nabla^{-2}\nabla^{-2}(-20\nabla^2\phi\nabla^2C^{\parallel(1)} + 20\phi^{,kl}C_{,kl}^{\parallel(1)} + 3\nabla^2C_{,k}^{\parallel(1)}\nabla^2C^{\parallel(1),k} - 3C^{\parallel(1),klm}C_{,klm}^{\parallel(1)})\right] \\ &\quad + \tau^2D_{ij}\left[\frac{2}{3}C^{\parallel(1),k}\phi_{,k} - \nabla^{-2}\left(\frac{2}{3}\phi^{,k}\nabla^2C_{,k}^{\parallel(1)}\right) + \nabla^{-2}\nabla^{-2}(-\phi^{,klm}C_{,klm}^{\parallel(1)} + \nabla^2\phi^{,k}\nabla^2C_{,k}^{\parallel(1)})\right] - 2D_{ij}C^{\parallel(2)}, \end{aligned} \quad (105)$$

$$\begin{aligned}
\bar{\chi}_{Sij}^{\perp(2)} = & \chi_{Sij}^{\perp(2)} + \frac{1}{\tau^3} [\partial_i \nabla^{-2} (12C^{\parallel(1),kl} \nabla^{-2} X_{klj} + 12\nabla^{-2} X_{kj} \nabla^2 C^{\parallel(1),k}) + \partial_i \partial_j \nabla^{-2} (12C^{\parallel(1),kl} \nabla^{-2} X_{kl} - 12X_{,k} C^{\parallel(1),k}) \\
& + \partial_i \partial_j \nabla^{-2} \nabla^{-2} (12C^{\parallel(1),k} \nabla^2 X_{,k} - 24\nabla^{-2} X_{,kl} \nabla^2 C^{\parallel(1),kl} - 36C^{\parallel(1),klm} \nabla^{-2} X_{,klm})] \\
& + \left[2\partial_j \nabla^{-2} \left(\frac{20}{3} \varphi^{,k} C_{,ik}^{\parallel(1)} - \frac{20}{3} \varphi_{,i} \nabla^2 C^{\parallel(1)} - C^{\parallel(1),kl} C_{,ikl}^{\parallel(1)} + C_{,ik}^{\parallel(1)} \nabla^2 C^{\parallel(1),k} \right) \right. \\
& + 2\partial_i \partial_j \nabla^{-2} \nabla^{-2} \left(-\frac{20}{3} \varphi^{,kl} C_{,kl}^{\parallel(1)} + \frac{20}{3} \nabla^2 \varphi \nabla^2 C^{\parallel(1)} + C^{\parallel(1),klm} C_{,klm}^{\parallel(1)} - \nabla^2 C^{\parallel(1),k} \nabla^2 C_{,k}^{\parallel(1)} \right) \left. \right] \\
& + \frac{2}{3} \tau^2 [\partial_j \nabla^{-2} (\varphi_{,ik} \nabla^2 C^{\parallel(1),k} - \varphi^{,kl} C_{,ikl}^{\parallel(1)}) + \partial_i \partial_j \nabla^{-2} \nabla^{-2} (\varphi^{,klm} C_{,klm}^{\parallel(1)} - \nabla^2 \varphi^{,k} \nabla^2 C_{,k}^{\parallel(1)})] - C_{ij}^{\perp(2)} + (i \leftrightarrow j), \quad (106)
\end{aligned}$$

$$\begin{aligned}
\bar{\chi}_{Sij}^{\top(2)} = & \chi_{Sij}^{\top(2)} + \frac{1}{\tau^6} [12A^{(1)} \nabla^{-2} X_{,ij} - \partial_i \partial_j \nabla^{-2} (12A^{(1)} X)] + \frac{1}{\tau^3} [-8C_{,ij}^{\parallel(1)} X - 12C^{\parallel(1),k} \nabla^{-2} X_{,kij} \\
& + \delta_{ij} \nabla^{-2} (6X_{,k} \nabla^2 C^{\parallel(1),k} - 6C^{\parallel(1),klm} \nabla^{-2} X_{,klm}) + 8\nabla^{-2} (X_{,ij} \nabla^2 C^{\parallel(1)} + C_{,ij}^{\parallel(1)} \nabla^2 X - X_{,i}^k C_{,kj}^{\parallel(1)} - X_{,j}^k C_{,ki}^{\parallel(1)}) \\
& + 4\partial_i \nabla^{-2} (X_{,kj} C^{\parallel(1),k} + 2X \nabla^2 C_{,j}^{\parallel(1)} - 3C_{,j}^{\parallel(1),kl} \nabla^{-2} X_{,kl}) + 4\partial_j \nabla^{-2} (X_{,ki} C^{\parallel(1),k} + 2X \nabla^2 C_{,i}^{\parallel(1)} - 3C_{,i}^{\parallel(1),kl} \nabla^{-2} X_{,kl}) \\
& + 4\partial_i \partial_j \nabla^{-2} (X_{,k} C^{\parallel(1),k} - 2X \nabla^2 C^{\parallel(1)} + 6C^{\parallel(1),kl} \nabla^{-2} X_{,kl}) + 6\partial_i \partial_j \nabla^{-2} \nabla^{-2} (X_{,k} \nabla^2 C^{\parallel(1),k} - C^{\parallel(1),klm} \nabla^{-2} X_{,klm})] \\
& + \left[\nabla^{-2} \left(-\frac{40}{3} \varphi_{,j}^k C_{,ik}^{\parallel(1)} - \frac{40}{3} \varphi_{,i}^k C_{,jk}^{\parallel(1)} + \frac{40}{3} C_{,ij}^{\parallel(1)} \nabla^2 \varphi + \frac{40}{3} \varphi_{,ij} \nabla^2 C^{\parallel(1)} + \frac{20}{3} \varphi^{,kl} C_{,kl}^{\parallel(1)} \delta_{ij} - \frac{20}{3} \nabla^2 \varphi \nabla^2 C^{\parallel(1)} \delta_{ij} \right. \right. \\
& + 4C_{,j}^{\parallel(1),kl} C_{,ikl}^{\parallel(1)} - 4C_{,ijk}^{\parallel(1)} \nabla^2 C^{\parallel(1),k} - C^{\parallel(1),klm} C_{,klm}^{\parallel(1)} \delta_{ij} + \nabla^2 C^{\parallel(1),k} \nabla^2 C_{,k}^{\parallel(1)} \delta_{ij} \left. \left. \right) \right. \\
& + \partial_i \partial_j \nabla^{-2} \nabla^{-2} \left(-\frac{20}{3} \nabla^2 \varphi \nabla^2 C^{\parallel(1)} + \frac{20}{3} \varphi^{,kl} C_{,kl}^{\parallel(1)} + \nabla^2 C^{\parallel(1),k} \nabla^2 C_{,k}^{\parallel(1)} - C^{\parallel(1),klm} C_{,klm}^{\parallel(1)} \right) \left. \right] \\
& + \tau^2 \left[\nabla^{-2} \left(\frac{2}{3} \varphi_{,j}^{,kl} C_{,ikl}^{\parallel(1)} + \frac{2}{3} \varphi_{,i}^{,kl} C_{,jkl}^{\parallel(1)} - \frac{2}{3} C_{,ijk}^{\parallel(1)} \nabla^2 \varphi^{,k} - \frac{2}{3} \varphi_{,ijk} \nabla^2 C^{\parallel(1),k} - \frac{1}{3} \varphi^{,klm} C_{,klm}^{\parallel(1)} \delta_{ij} + \frac{1}{3} \nabla^2 \varphi^{,k} \nabla^2 C_{,k}^{\parallel(1)} \delta_{ij} \right) \right. \\
& \left. + \frac{1}{3} \partial_i \partial_j \nabla^{-2} \nabla^{-2} (-\varphi^{,klm} C_{,klm}^{\parallel(1)} + \nabla^2 \varphi^{,k} \nabla^2 C_{,k}^{\parallel(1)}) \right], \quad (107)
\end{aligned}$$

and Eqs. (96) and (98) reduce to

$$\begin{aligned}
\bar{\rho}_S^{(2)} = & \rho_S^{(2)} - 2\rho^{(1)} \frac{A^{(1)}}{\tau^2} - 2\rho_{,k}^{(1)} C^{\parallel(1),k} \\
& + 54 \frac{A^{(1)} A^{(1)}}{\tau^6} \rho^{(0)} + 6 \frac{A^{(2)}}{\tau^3} \rho^{(0)}. \quad (108)
\end{aligned}$$

$$\begin{aligned}
\bar{\delta}_S^{(2)} = & \delta_S^{(2)} + \left(\frac{12}{\tau} \delta^{(1)} - 2\delta_{,0}^{(1)} \right) \frac{A^{(1)}}{\tau^2} - 2\delta_{,k}^{(1)} C^{\parallel(1),k} \\
& + 54 \frac{A^{(1)} A^{(1)}}{\tau^6} + 6 \frac{A^{(2)}}{\tau^3}. \quad (109)
\end{aligned}$$

The above results of synchronous-to-synchronous transformations are general ones, in the sense that two vector fields $\xi^{(1)}$ and $\xi^{(2)}$ are involved simultaneously. However, in applications, certain distinctions should be made in regard to transformations due to $\xi^{(1)}$ and $\xi^{(2)}$. If one sets $\xi^{(2)} = 0$ [46,57], only $\xi^{(1)}$ remains, which ensures $\bar{g}_{00}^{(1)} = 0$ and

$\bar{g}_{0i}^{(1)} = 0$, one has no freedom to make $\bar{g}_{00}^{(2)} = 0$ and $\bar{g}_{0i}^{(2)} = 0$ anymore, because $\xi^{(1)}$ has been already fixed, as specified by (C19) and (C21). Thus, effective 2nd-order transformations from synchronous to synchronous can not be made when $\xi^{(2)} = 0$. On the other hand, if the 1st-order solutions are held fixed and only the 2nd-order metric perturbations are transformed [58], one simply sets $\xi^{(1)} = 0$ but $\xi^{(2)} \neq 0$, so that (88), (89), and (90) reduce to

$$\alpha^{(2)}(\tau, \mathbf{x}) = \frac{A^{(2)}(\mathbf{x})}{\tau^2}, \quad (110)$$

$$\beta_{,i}^{(2)}(\tau, \mathbf{x}) = -\frac{A^{(2)}(\mathbf{x})_{,i}}{\tau} + C^{\parallel(2)}(\mathbf{x})_{,i}, \quad (111)$$

$$d_i^{(2)} = 0, \quad (112)$$

and (91), (92), (93), (94), and (109) reduce to

$$\bar{\phi}_S^{(2)} = \phi_S^{(2)} + \frac{2}{\tau^3} A^{(2)} - \frac{1}{3\tau} \nabla^2 A^{(2)} + \frac{1}{3} \nabla^2 C^{\parallel(2)}, \quad (113)$$

$$D_{ij} \bar{\chi}_S^{\parallel(2)} = D_{ij} \chi_S^{\parallel(2)} + \frac{2}{\tau} D_{ij} A^{(2)} - 2D_{ij} C^{\parallel(2)}, \quad (114)$$

$$\bar{\chi}_{Sij}^{\perp(2)} = \chi_{Sij}^{\perp(2)} - [C_{i,j}^{\perp(2)} + C_{j,i}^{\perp(2)}], \quad (115)$$

$$\bar{\chi}_{Sij}^{\top(2)} = \chi_{Sij}^{\top(2)}, \quad (116)$$

$$\bar{\delta}_S^{(2)} = \delta_S^{(2)} + 6 \frac{A^{(2)}}{\tau^3}. \quad (117)$$

Thus, it is seen that only $\xi^{(2)\mu}$ is effective in carrying out 2nd-order transformations that we consider, because $\xi^{(1)\mu}$ has been used in obtaining the 1st-order perturbations. This is also consistent with the fact that one can have 4 degrees of freedom at each order of transformation. The gauge transformations (113)–(117) have a similar structure to the 1st-order gauge transformations in (C25), (C29), (C30), and (C34) in Appendix C. In particular, the gauge mode of 2nd-order vector is time-independent, and the 2nd-order tensor contains no gauge mode. Furthermore, under the condition $\bar{U}^{(2)i} = 0$ (i.e., $A_i^{(2)} = 0$), (110), (112), (115), (116), and (117) remain unchanged, whereas (111), (113), and (114) reduce to

$$\beta_{,i}^{(2)}(\mathbf{x}) = C^{\parallel(2)}(\mathbf{x})_{,i}, \quad (118)$$

$$\bar{\phi}_S^{(2)} = \phi_S^{(2)} + 2 \frac{A^{(2)}}{\tau^3} + \frac{1}{3} \nabla^2 C^{\parallel(2)}(\mathbf{x}), \quad (119)$$

$$D_{ij} \bar{\chi}_S^{\parallel(2)} = D_{ij} \chi_S^{\parallel(2)0} - 2D_{ij} C^{\parallel(2)}(\mathbf{x}). \quad (120)$$

Though having residual gauge freedom, synchronous gauges have advantages in regard to interpretation of cosmological observations in terms of calculational results. The RW spacetime as given by Eq. (2) in synchronous coordinate can be written as $ds^2 = -dt^2 + a^2(t) \gamma_{ij} dx^i dx^j$, where $dt \equiv a(\tau) d\tau$ is the cosmic time, and is equal to the proper time $ds/c = \sqrt{-g_{00}} d\tau^2$ measured by a comoving observer ($dx^i = 0$). For galaxies without peculiar velocity (or, the typical peculiar velocity $v \sim 10^2$ km/s $\sim 10^{-3}$ c is neglected as an approximation), an observer on a galaxy is a comoving observer. This holds for a perturbed RW spacetime as $g_{00}^{(1)} = g_{00}^{(2)} = 0$ by definition of synchronous coordinate. Therefore, the time τ occurring in perturbed cosmological quantities is directly related to the proper time t used by the observer on our Galaxy.

On the other hand, in the Poisson coordinates (conformal-Newtonian coordinates) (see Appendix D) for the perturbed RW spacetime, the proper time of a comoving observer is $ds/c = \sqrt{(1 + 2\psi) dt_P^2}$, where dt_P is

the time in Poisson coordinates and $\psi = \psi(t_P, \mathbf{x})$ is the lapse function, which is also the gravitational potential at the position of observer. Thus, the proper time measured by a comoving observer on our Galaxy is not equal to the time dt_P appearing in the perturbed cosmological quantities in Poisson gauges. In principle, the observer would need to know the potential $\psi(t_P, \mathbf{x})$ at the Galaxy, in order to relate his proper time to the Poisson coordinate time t_P .

VI. CONCLUSION

We have conducted a comprehensive study of the 2nd-order perturbed Einstein equation for the Einstein-de Sitter model in synchronous coordinates. There are three types of couplings of 1st-order metric perturbations: scalar-scalar, scalar-tensor, and tensor-tensor, which serve as the source for 2nd-order metric perturbations. We have decomposed the 2nd-order perturbed Einstein equation into three sets according to the couplings. For the scalar-scalar coupling in this paper, we have obtained the solutions of the 2nd-order scalar, vector and tensor perturbations for general initial conditions. In particular, the 2nd-order vector perturbations in Eq. (63) are produced by the coupling of 1st-order scalar perturbations, even though there is no 1st-order vector metric perturbation. Besides, the complete expression of 2nd-order tensor metric perturbation is given in Eqs. (83) and (85), which includes some extra terms as well as the homogeneous solution, correcting that in literature.

Moreover, we have also performed a detailed study of general synchronous-to-synchronous 2nd-order gauge transformations, which are generated by both a 1st-order vector field $\xi^{(1)\mu}$ and an independent 2nd-order vector field $\xi^{(2)\mu}$ as well. While $\xi^{(1)\mu}$ is actually fixed for the usual 1st-order gauge transformation, $\xi^{(2)\mu}$ is effective for carrying out the required 2nd-order transformations. As a main result, the residual gauge modes of 2nd-order metric perturbations and density contrast have been found explicitly, listed in (C50), (C56), (C57), and (C58) for a general RW spacetime, and in (91), (92), (93), (94), and (98) for the Einstein-de Sitter model. We have also found that, when one further requires that the transformed 3-velocity perturbations are also zero, $\bar{U}^{(1)i} = \bar{U}^{(2)i} = 0$, the vector fields $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$ are consequently constrained, and the residual gauge modes of perturbations are substantially reduced, which are listed in (104), (105), (106), (107), and (109). Furthermore, if we fix the 1st-order solutions of perturbations ($\xi^{(1)\mu} = 0$) and transform only the 2nd-order perturbations by $\xi^{(2)\mu}$, all the things become very simple and the resulting formulas are listed in (110)–(120), which have exactly the same structure as the 1st-order transformations. In particular, the 2nd-order vector contains a time-independent, residual gauge mode, and the 2nd-order tensor contains no residual gauge mode.

There are several related issues that can be explored in future research. Firstly, we can extend the above method to the case of the scalar-tensor and tensor-tensor types of couplings of 1st-order metric perturbations as the source for the 2nd-order metric perturbations. This will be more involved and will be presented in a subsequent paper. Next, one can extend the work to other stages of expansion, such as the radiation and inflationary stages, and the calculations would be more involved. For these earlier stages, the scalar-tensor and tensor-tensor couplings will be more significant than for the matter stage. Besides, based on our results of the 2nd-order perturbations with scalar-scalar coupling, a detailed computation of the evolution during the matter stage in a realistic Big-Bang cosmology will be a whole new project in the future.

ACKNOWLEDGMENTS

Yang Zhang is supported by NSFC Grant No. 11275187, NSFC No. 11421303, No. 11675165, No. 11633001, SRFDP, and CAS, the Strategic Priority Research Program “The Emergence of Cosmological Structures” of the Chinese Academy of Sciences, Grant No. XDB09000000.

APPENDIX A: THE 1ST ORDER PERTURBATIONS

The formulas of the Ricci tensor are $R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\lambda\alpha}^\alpha \Gamma_{\mu\nu}^\lambda - \Gamma_{\lambda\nu}^\alpha \Gamma_{\alpha\mu}^\lambda$ and $\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\rho} (g_{\rho\gamma,\beta} + g_{\beta\rho,\gamma} - g_{\beta\gamma,\rho})$. In the 0th order, $R_{00}^{(0)} = -\frac{3a''}{a} + 3(\frac{a'}{a})^2$, $R_{ij}^{(0)} = \delta_{ij}[\frac{a''}{a} + (\frac{a'}{a})^2]$, $R^{(0)} = \frac{6}{a^2} \frac{a''}{a}$. The 1st-order Ricci tensor $R_{00}^{(1)} = 3\phi^{(1)''} + 3\frac{a'}{a}\phi^{(1)'}$, $R_{0i}^{(1)} = 2\phi_{,i}^{(1)'} + \frac{1}{2}D_{ij}\chi^{\parallel(1)'j}$,

$$R_{ij}^{(1)} = \phi_{,ij}^{(1)} - \left(\phi^{(1)''} + 5\frac{a'}{a}\phi^{(1)'} + 2\left(\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right)\phi^{(1)} - \nabla^2\phi^{(1)} \right) \delta_{ij} + \frac{1}{2}D_{ij}\chi^{\parallel(1)''} + \frac{a'}{a}D_{ij}\chi^{\parallel(1)'} \\ + \left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)D_{ij}\chi^{\parallel(1)} - \frac{1}{2}\nabla^2D_{ij}\chi^{\parallel(1)} + D_{ik}\chi_{,jk}^{\parallel(1)} + \frac{1}{2}\chi_{ij}^{\top(1)''} + \frac{a'}{a}\chi_{ij}^{\top(1)'} + \left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\chi_{ij}^{\top(1)} - \frac{1}{2}\nabla^2\chi_{ij}^{\top(1)},$$

and $R^{(1)} = \frac{1}{a^2}(-6\phi^{(1)''} - 18\frac{a'}{a}\phi^{(1)'} + 4\nabla^2\phi^{(1)} + D_{ij}\chi^{\parallel(1),ij})$.

The $(0i)$ component of the 1st-order perturbed Einstein equation is

$$G_{0i}^{(1)} \equiv R_{0i}^{(1)} = 0, \quad (\text{A1})$$

which leads the 1st-order momentum constraint

$$\phi_{,i}^{(1)'} + \frac{1}{6}\nabla^2\chi_{,i}^{\parallel(1)'} = 0. \quad (\text{A2})$$

This relates the two scalar modes and can be written as

$$\phi^{(1)'} + \frac{1}{6}\nabla^2\chi^{\parallel(1)'} = 0, \quad (\text{A3})$$

after dropping a space-independent constant. Integration of (A3) gives

$$\phi^{(1)} + \frac{1}{6}\nabla^2\chi^{\parallel(1)} = \phi_0^{(1)} + \frac{1}{6}\nabla^2\chi_0^{\parallel(1)}, \quad (\text{A4})$$

where $\phi_0^{(1)}$ and $\chi_0^{\parallel(1)}$ denote the initial values of the two scalar modes, respectively.

The (00) component of 1st-order perturbed Einstein equation is

$$G_{00}^{(1)} \equiv R_{00}^{(1)} - \frac{1}{2}g_{00}^{(0)}R^{(1)} = 8\pi G\rho^{(0)}a^2\delta^{(1)}, \quad (\text{A5})$$

which leads to the 1st-order energy constraint

$$-6\frac{a'}{a}\phi^{(1)'} + 2\left(\nabla^2\phi^{(1)} + \frac{1}{6}\nabla^2\nabla^2\chi^{\parallel(1)}\right) = 8\pi G\rho^{(0)}a^2\delta^{(1)}. \quad (\text{A6})$$

Using (A3), (A27) into the above gives

$$\nabla^2\left(\frac{2}{\tau}\chi^{\parallel(1)'} + \frac{6}{\tau^2}(\chi^{\parallel(1)} - \chi_0^{\parallel(1)}) + 2\phi_0^{(1)} + \frac{1}{3}\nabla^2\chi_0^{\parallel(1)}\right) = \frac{12}{\tau^2}\delta_0^{(1)}. \quad (\text{A7})$$

The (ij) component of 1st-order perturbed Einstein equation is:

$$G_{ij}^{(1)} \equiv R_{ij}^{(1)} - \frac{1}{2}\delta_{ij}a^2R^{(1)} - \frac{1}{2}a^2\gamma_{ij}^{(1)}R^{(0)} = 0, \quad (\text{A8})$$

which gives the evolution equation

$$2\phi^{(1)''}\delta_{ij} + \frac{8}{\tau}\phi^{(1)'}\delta_{ij} + \phi_{,ij}^{(1)} - \nabla^2\phi^{(1)}\delta_{ij} + \frac{1}{2}D_{ij}\chi^{\parallel(1)''} \\ + \frac{2}{\tau}D_{ij}\chi^{\parallel(1)'} + \frac{1}{6}\nabla^2D_{ij}\chi^{\parallel(1)} - \frac{1}{9}\delta_{ij}\nabla^2\nabla^2\chi^{\parallel(1)} \\ + \frac{1}{2}\chi_{ij}^{\top(1)''} + \frac{2}{\tau}\chi_{ij}^{\top(1)'} - \frac{1}{2}\nabla^2\chi_{ij}^{\top(1)} = 0, \quad (\text{A9})$$

involving all the metric perturbations. The traceless part of Eq. (A9) is

$$\begin{aligned} & \left(D_{ij}\chi^{\parallel(1)''} + \frac{4}{\tau}D_{ij}\chi^{\parallel(1)'} + \frac{1}{3}\nabla^2 D_{ij}\chi^{\parallel(1)} \right) \\ & + \left(\chi_{ij}^{\top(1)''} + \frac{4}{\tau}\chi_{ij}^{\top(1)'} - \nabla^2\chi_{ij}^{\top(1)} \right) \\ & + 2\left(\partial_i\partial_j\phi^{(1)} - \frac{1}{3}\delta_{ij}\nabla^2\phi^{(1)} \right) = 0. \end{aligned} \quad (\text{A10})$$

The transverse part of (A10) directly gives the hyperbolic, partial differential equation of 1st-order tensor

$$\chi_{ij}^{\top(1)''} + \frac{4}{\tau}\chi_{ij}^{\top(1)'} - \nabla^2\chi_{ij}^{\top(1)} = 0, \quad (\text{A11})$$

which describes gravitational wave propagating at the speed of light. Eqs. (A11), (A2), and Eq. (A7) show that, at 1st order, the tensor and the scalar are independent. The solution of (A11) is given by (18) and (19).

Taking the trace of (A9) gives

$$6\phi^{(1)''} + \frac{24}{\tau}\phi^{(1)'} - 2\left(\nabla^2\phi^{(1)} + \frac{1}{6}\nabla^2\nabla^2\chi^{\parallel(1)} \right) = 0, \quad (\text{A12})$$

which, by the momentum constraint (A4), is also written as the equation of $\phi^{(1)}$:

$$\phi^{(1)''} + \frac{4}{\tau}\phi^{(1)'} = \frac{1}{3}\left(\nabla^2\phi_0^{(1)} + \frac{1}{6}\nabla^2\nabla^2\chi_0^{\parallel(1)} \right). \quad (\text{A13})$$

By (A3), it can be also written as the equation of $\chi^{\parallel(1)}$:

$$\chi^{\parallel(1)''} + \frac{4}{\tau}\chi^{\parallel(1)'} = -2\left(\phi_0^{(1)} + \frac{1}{6}\nabla^2\chi_0^{\parallel(1)} \right). \quad (\text{A14})$$

Note that the evolution equations (A13) and (A14) are not hyperbolic differential equations; thus, the scalar perturbations $\phi^{(1)}$ and $\chi^{\parallel(1)}$ just follow where the density perturbation is distributed and do not propagate at speed of light, in contrast to the tensor $\chi_{ij}^{\top(1)}$. Combining (A27) and (A7) yields the equation of the 1st-order density contrast

$$\delta^{(1)''} + \frac{2}{\tau}\delta^{(1)'} - \frac{6}{\tau^2}\delta^{(1)} = 0. \quad (\text{A15})$$

Note that Eq. (A15) has no sound speed term because the pressure is zero for the dust model. The general solution of (A15) consists of a growing mode $\propto \tau^2$ and a decaying mode τ^{-3} ,

$$\delta^{(1)} = \delta_{0g}^{(1)}\frac{\tau^2}{\tau_0^2} + \frac{3X}{\tau^3}, \quad (\text{A16})$$

where $X = X(\mathbf{x})$ represents the decaying mode of the density contrast. $\delta_0^{(1)} \equiv \delta_{0g}^{(1)} + \frac{3X}{\tau_0^3}$ will denote the initial value of $\delta^{(1)}$. Combining (A28) and (A15) leads to

$$\chi^{\parallel(1)''} + \frac{2}{\tau}\chi^{\parallel(1)'} - \frac{6}{\tau^2}\chi^{\parallel(1)} = 0, \quad (\text{A17})$$

which is the same equation as (A15). Its solution is taken as

$$\nabla^2\chi^{\parallel(1)} = -2\frac{\delta_{0g}^{(1)}}{\tau_0^2}\tau^2 - \frac{6X}{\tau^3}. \quad (\text{A18})$$

$\phi^{(1)}$ satisfies a similar equation to (A17) with the inhomogeneous terms $-\frac{6}{\tau^2}(\phi_0^{(1)} + \frac{1}{6}\nabla^2\chi_0^{\parallel(1)})$. For convenience, introduce the gravitational potential φ defined by

$$\nabla^2\varphi(\mathbf{x}) = \frac{6}{\tau_0^2}\delta_{0g}^{(1)}(\mathbf{x}). \quad (\text{A19})$$

Then one obtains the solution

$$\phi^{(1)} = \frac{5}{3}\varphi + \frac{\tau^2}{18}\nabla^2\varphi + \frac{X}{\tau^3} \quad (\text{A20})$$

$$D_{ij}\chi^{\parallel(1)} = -\frac{\tau^2}{3}\left(\varphi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\varphi \right) - \frac{6\nabla^{-2}D_{ij}X}{\tau^3}, \quad (\text{A21})$$

by (A4) and (A14). We shall keep the decaying terms $\propto X/\tau^3$ in (A16), (A20), and (A21). The scalar perturbations $\phi^{(1)}$ and $D_{ij}\chi^{\parallel(1)}$ are independent dynamic fields, but the two fields and their first time derivatives are related through the energy and momentum constraints. This leads to the fact that the growing mode of $D_{ij}\chi^{\parallel(1)}$ is related to that of $\phi^{(1)}$ and, respectively, so is the decaying mode. Thus, there are only two unknown functions φ and X in the solutions (A21) and (A20) which will be determined by the initial condition at τ_0 . We remark that the solutions (A21) and (A20) are consistent with each other in a fixed gauge. If the time-independent term $\frac{5}{3}\varphi$ in Eq. (A21) was discarded as a gauge term, $D_{ij}\chi^{\parallel(1)}$ of (A20) would acquire a term $10D_{ij}(\nabla^{-2}\varphi)$ simultaneously, according to the gauge transformation (C38) and (C39) of Appendix C.

1. The energy density contrast

For the matter source of gravity, the equation of dust is determined by the conservation of energy-momentum tensor, by which the perturbations of density will be related to the metric perturbations as the following. Define the density contrast

$$\delta = \frac{\rho - \rho^{(0)}}{\rho^{(0)}}, \quad (\text{A22})$$

where $\rho = \rho(\mathbf{x}, \tau)$ is the mass density, $\rho^{(0)}$ is its mean density. From the energy conservation $T^{0\nu}_{;\nu} = 0$, one obtains the solution δ in terms of metric perturbations

$$\delta(\tau, \mathbf{x}) = (1 + \delta_0(\mathbf{x})) \left[\frac{\gamma(\tau, \mathbf{x})}{\gamma_0(\mathbf{x})} \right]^{-\frac{1}{2}} - 1, \quad (\text{A23})$$

where δ_0 and γ_0 are the initial values, and

$$\begin{aligned} \gamma &\equiv \det(\gamma_{ij}) \\ &= 1 + \gamma_i^{(1)i} + \frac{1}{2}\gamma_i^{(2)i} + \frac{1}{2}\gamma_i^{(1)i}\gamma_j^{(1)j} - \frac{1}{2}\gamma^{(1)ij}\gamma_{ij}^{(1)}. \end{aligned} \quad (\text{A24})$$

Expanding (A23) up to 2nd order gives

$$\begin{aligned} \delta &= -\frac{1}{2}\gamma_i^{(1)i} + \frac{1}{2}\gamma_{0i}^{(1)i} + \delta_0^{(1)} + \frac{1}{2}\delta_0^{(2)} + \frac{1}{4}\gamma_{0i}^{(2)i} - \frac{1}{4}\gamma_i^{(2)i} \\ &\quad + \frac{1}{8}(\gamma_i^{(1)i})^2 + \frac{1}{8}(\gamma_{0i}^{(1)i})^2 - \frac{1}{4}\gamma_i^{(1)i}\gamma_{0j}^{(1)j} + \frac{1}{4}\gamma^{(1)ij}\gamma_{ij}^{(1)} \\ &\quad - \frac{1}{4}\gamma_0^{(1)ij}\gamma_{0ij}^{(1)} - \frac{1}{2}\gamma_i^{(1)i}\delta_0^{(1)} + \frac{1}{2}\gamma_{0i}^{(1)i}\delta_0^{(1)}, \end{aligned} \quad (\text{A25})$$

where $\delta_0 \equiv \delta_0^{(1)} + \frac{1}{2}\delta_0^{(2)}$. From (A25) one reads off the 1st-order density contrast

$$\delta^{(1)} = \delta_0^{(1)} + \frac{1}{2}(6\phi^{(1)} - 6\phi_0^{(1)}), \quad (\text{A26})$$

which, by Eq. (A4), can be also written as

$$\delta^{(1)} = \delta_0^{(1)} - \frac{1}{2}\nabla^2(\chi^{\parallel(1)} - \chi_0^{\parallel(1)}). \quad (\text{A27})$$

One can use the residual gauge freedom to take $\nabla^2\chi_0^{\parallel(1)} = -2\delta_0^{(1)}$ [see Eq. (C31) in Appendix C], so that (A27) reduces to

$$\delta^{(1)} = -\frac{1}{2}\nabla^2\chi^{\parallel(1)}. \quad (\text{A28})$$

From (A25) one reads off the 2nd-order density contrast

$$\begin{aligned} \delta^{(2)} &= \delta_0^{(2)} + \frac{1}{2}\gamma_{0i}^{(2)i} - \frac{1}{2}\gamma_i^{(2)i} + \frac{1}{4}(\gamma_i^{(1)i})^2 + \frac{1}{4}(\gamma_{0i}^{(1)i})^2 \\ &\quad - \frac{1}{2}\gamma_i^{(1)i}\gamma_{0j}^{(1)j} + \frac{1}{2}\gamma^{(1)ij}\gamma_{ij}^{(1)} - \frac{1}{2}\gamma_0^{(1)ij}\gamma_{0ij}^{(1)} \\ &\quad - \gamma_i^{(1)i}\delta_0^{(1)} + \gamma_{0i}^{(1)i}\delta_0^{(1)}, \end{aligned} \quad (\text{A29})$$

which depends on 2nd-order perturbations only through $\gamma_i^{(2)i} = -6\phi^{(2)}$, independent of the 2nd-order tensor. But $\delta^{(2)}$ depends on the 1st-order tensor via the term $\frac{1}{4}\gamma^{(1)ij}\gamma_{ij}^{(1)} = \frac{1}{4}(2\chi_{ij}^{\top(1)}D^{ij}\chi^{\parallel(1)} + \chi_{ij}^{\top(1)}\chi^{\top(1)ij})$.

APPENDIX B: EXPRESSIONS OF 2ND ORDER PERTURBED RICCI TENSORS

By Eqs. (2) and (3), we calculate the 2nd-order Ricci and Einstein tensors. The 2nd-order Ricci tensors are

$$\begin{aligned} R_{00}^{(2)} &= \frac{3a'}{2a}\phi^{(2)'} + \frac{3}{2}\phi^{(2)''} + 6\frac{a'}{a}\phi^{(1)}\phi^{(1)'} + 6\phi^{(1)}\phi^{(1)''} + 3\phi^{(1)'}\phi^{(1)'} + \frac{1}{2}D^{ij}\chi^{\parallel(1)}D_{ij}\chi^{\parallel(1)''} + \frac{1}{4}D^{ij}\chi^{\parallel(1)'}D_{ij}\chi^{\parallel(1)'} \\ &\quad + \frac{a'}{2a}D^{ij}\chi^{\parallel(1)}D_{ij}\chi^{\parallel(1)'} + \frac{1}{2}\chi^{\top(1)ij}\chi_{ij}^{\top(1)''} + \frac{1}{4}\chi^{\top(1)ij}\chi_{ij}^{\top(1)'} + \frac{a'}{2a}\chi^{\top(1)ij}\chi_{ij}^{\top(1)'} + \frac{1}{2}\chi^{\top(1)ij}D_{ij}\chi^{\parallel(1)''} + \frac{1}{2}\chi^{\top(1)ij}D_{ij}\chi^{\parallel(1)'} \\ &\quad + \frac{a'}{2a}\chi^{\top(1)ij}D_{ij}\chi^{\parallel(1)'} + \frac{1}{2}\chi_{ij}^{\top(1)''}D^{ij}\chi^{\parallel(1)} + \frac{a'}{2a}\chi_{ij}^{\top(1)'}D^{ij}\chi^{\parallel(1)}, \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} R_{0i}^{(2)} &= \phi_{,i}^{(2)'} + \frac{1}{4}D_{ij}\chi^{\parallel(2)'} + \frac{1}{4}\chi_{ij}^{\perp(2)'} + 4\phi^{(1)'}\phi_{,i}^{(1)} + 4\phi^{(1)}\phi_{,i}^{(1)'} + \phi^{(1)}D_{ij}\chi^{\parallel(1)'} + \phi^{(1)'}D_{ij}\chi^{\parallel(1)'} - \frac{1}{2}\phi^{(1),j}D_{ij}\chi^{\parallel(1)'} \\ &\quad + \phi^{(1)'}D_{ij}\chi^{\parallel(1)} - \frac{1}{2}\phi^{(1),j}\chi_{ij}^{\top(1)'} + \phi^{(1)'}\chi_{ij}^{\top(1)} - \frac{1}{2}D_{jk}^k\chi^{\parallel(1),j}D_{ik}\chi^{\parallel(1)'} - \frac{1}{2}D_{jk}^k\chi^{\parallel(1)}D_{ik}\chi^{\parallel(1),j} + \frac{1}{2}D^{jk}\chi_{,i}^{\parallel(1)'}D_{jk}\chi^{\parallel(1)} \\ &\quad + \frac{1}{4}D^{jk}\chi^{\parallel(1)'}D_{jk}\chi_{,i}^{\parallel(1)} - \frac{1}{2}\chi_j^{\top(1)k}D_{ik}\chi^{\parallel(1),j} + \frac{1}{2}\chi^{\top(1)jk}D_{jk}\chi_{,i}^{\parallel(1)'} + \frac{1}{4}\chi_{,i}^{\top(1)jk}D_{jk}\chi^{\parallel(1)'} - \frac{1}{2}\chi_{ik}^{\top(1),j}D_{jk}\chi^{\parallel(1)} \\ &\quad + \frac{1}{2}\chi_{jk,i}^{\top(1)'}D^{jk}\chi^{\parallel(1)} + \frac{1}{4}\chi_{jk}^{\top(1)'}D^{jk}\chi_{,i}^{\parallel(1)} - \frac{1}{2}\chi_{ik}^{\top(1)'}D_{jk}\chi^{\parallel(1),j} - \frac{1}{2}\chi_j^{\top(1)k}\chi_{ik}^{\top(1),j} + \frac{1}{2}\chi^{\top(1)jk}\chi_{jk,i}^{\top(1)'} + \frac{1}{4}\chi_{,i}^{\top(1)jk}\chi_{jk}^{\top(1)'} \end{aligned} \quad (\text{B2})$$

$$\begin{aligned}
R_{ij}^{(2)} = & \delta_{ij} \left[-\frac{5a'}{2a} \phi^{(2)'} - \left(\frac{a'}{a}\right)^2 \phi^{(2)} - \frac{a''}{a} \phi^{(2)} - \frac{1}{2} \phi^{(2)''} + \frac{1}{2} \nabla^2 \phi^{(2)} + (\phi^{(1)'})^2 + \phi^{(1),k} \phi_{,k}^{(1)} + 2\phi^{(1)} \nabla^2 \phi^{(1)} - \phi_{,k}^{(1)} D^{lk} \chi_{,l}^{(1)} \right. \\
& - \phi_{,kl}^{(1)} D^{kl} \chi^{(1)} - \phi_{,kl}^{(1)} \chi^{\top(1)kl} - \frac{a'}{2a} D^{kl} \chi^{(1)} D_{kl} \chi^{(1)'} - \frac{a'}{2a} \chi_{kl}^{\top(1)'} D^{kl} \chi^{(1)} - \frac{a'}{2a} \chi^{\top(1)kl} D_{kl} \chi^{(1)'} - \frac{a'}{2a} \chi^{\top(1)kl} \chi_{kl}^{\top(1)'} \left. \right] \\
& + \frac{1}{2} \phi_{,ij}^{(2)} + \frac{1}{4} D_{ij} \chi^{(2)''} + \left[\left(\frac{a'}{a}\right)^2 + \frac{a''}{a} \right] D_{ij} \chi^{(2)} + \frac{a'}{2a} D_{ij} \chi^{(2)'} + \frac{1}{4} D_{jk}^k \chi_{,ki}^{(2)} + \frac{1}{4} D_{ik}^k \chi_{,kj}^{(2)} - \frac{1}{4} \nabla^2 D_{ij} \chi^{(2)} + \frac{1}{4} \chi_{ij}^{\perp(2)''} \\
& + \left[\left(\frac{a'}{a}\right)^2 + \frac{a''}{a} \right] \chi_{ij}^{\perp(2)} + \frac{a'}{2a} \chi_{ij}^{\perp(2)'} + \frac{1}{4} \chi_{kj,i}^{\perp(2),k} + \frac{1}{4} \chi_{ki,j}^{\perp(2),k} - \frac{1}{4} \nabla^2 \chi_{ij}^{\perp(2)} + \frac{1}{4} \chi_{ij}^{\top(2)''} + \left[\left(\frac{a'}{a}\right)^2 + \frac{a''}{a} \right] \chi_{ij}^{\top(2)} + \frac{a'}{2a} \chi_{ij}^{\top(2)'} \\
& - \frac{1}{4} \nabla^2 \chi_{ij}^{\top(2)} + 3\phi_{,i}^{(1)} \phi_{,j}^{(1)} + 2\phi^{(1)} \phi_{,ij}^{(1)} - \frac{3a'}{a} \phi^{(1)'} D_{ij} \chi^{(1)} + \frac{1}{2} \phi^{(1)'} D_{ij} \chi^{(1)'} + \frac{1}{2} \phi_{,k}^{(1)} D_{jk}^k \chi_{,i}^{(1)} + \frac{1}{2} \phi_{,k}^{(1)} D_{ik}^k \chi_{,j}^{(1)} \\
& - \frac{3}{2} \phi_{,k}^{(1)} D_{ij} \chi^{(1),k} + \phi^{(1)} D_{jk}^k \chi_{,ik}^{(1)} + \phi^{(1)} D_{ik}^k \chi_{,jk}^{(1)} - \phi^{(1)} \nabla^2 D_{ij} \chi^{(1)} + \phi_{,i}^{(1)} D_{jk}^k \chi_{,k}^{(1)} + \phi_{,j}^{(1)} D_{ik}^k \chi_{,k}^{(1)} + \phi_{,ki}^{(1)} D_{jk}^k \chi_{,l}^{(1)} \\
& + \phi_{,kj}^{(1)} D_{ik}^k \chi_{,l}^{(1)} - \frac{3a'}{a} \phi^{(1)'} \chi_{ij}^{\top(1)} + \frac{1}{2} \phi^{(1)'} \chi_{ij}^{\top(1)'} + \frac{1}{2} \phi_{,k}^{(1)} \chi_{j,i}^{\top(1)k} + \frac{1}{2} \phi_{,k}^{(1)} \chi_{i,j}^{\top(1)k} - \frac{3}{2} \phi_{,k}^{(1)} \chi_{ij}^{\top(1),k} - \phi^{(1)} \nabla^2 \chi_{ij}^{\top(1)} \\
& + \phi_{,ki}^{(1)} \chi_{j}^{\top(1)k} + \phi_{,kj}^{(1)} \chi_{i}^{\top(1)k} - \frac{1}{2} D_{ik}^k \chi_{,l}^{(1)'} D_{kj} \chi_{,l}^{(1)'} - \frac{1}{2} D^{kl} \chi_{,l}^{(1)} D_{kj} \chi_{,i}^{(1)} - \frac{1}{2} D^{kl} \chi_{,l}^{(1)} D_{ki} \chi_{,j}^{(1)} + \frac{1}{2} D^{lk} \chi_{,l}^{(1)} D_{ij} \chi_{,k}^{(1)} \\
& - \frac{1}{2} D^{kl} \chi_{,l}^{(1)} D_{li} \chi_{,ik}^{(1)} - \frac{1}{2} D^{kl} \chi_{,l}^{(1)} D_{li} \chi_{,jk}^{(1)} + \frac{1}{2} D^{kl} \chi_{,l}^{(1)} D_{ij} \chi_{,kl}^{(1)} + \frac{1}{2} D^{kl} \chi_{,l}^{(1)} D_{kl} \chi_{,ij}^{(1)} + \frac{1}{4} D^{kl} \chi_{,i}^{(1)} D_{kl} \chi_{,j}^{(1)} \\
& - \frac{1}{2} D_{ik}^k \chi_{,l}^{(1)} D_{jk}^k \chi_{,l}^{(1)} + \frac{1}{2} D_{ik}^k \chi_{,l}^{(1)} D_{kj} \chi_{,l}^{(1),l} - \frac{1}{2} \chi_{kj}^{\top(1)'} D_{ik}^k \chi_{,l}^{(1)'} - \frac{1}{2} \chi_{i}^{\top(1)'} D_{kj} \chi_{,l}^{(1)'} - \frac{1}{2} \chi_{kj,i}^{\top(1)} D^{kl} \chi_{,l}^{(1)} - \frac{1}{2} \chi_{ki,j}^{\top(1)} D^{kl} \chi_{,l}^{(1)} \\
& + \frac{1}{2} \chi_{ij,k}^{\top(1)} D^{lk} \chi_{,l}^{(1)} - \frac{1}{2} \chi_{lj,ik}^{\top(1)} D^{kl} \chi_{,l}^{(1)} - \frac{1}{2} \chi^{\top(1)kl} D_{lj} \chi_{,ik}^{(1)} - \frac{1}{2} \chi_{li,jk}^{\top(1)} D^{kl} \chi_{,l}^{(1)} - \frac{1}{2} \chi^{\top(1)kl} D_{li} \chi_{,jk}^{(1)} + \frac{1}{2} \chi_{ij,kl}^{\top(1)} D^{kl} \chi_{,l}^{(1)} \\
& + \frac{1}{2} \chi^{\top(1)kl} D_{ij} \chi_{,kl}^{(1)} + \frac{1}{2} \chi_{kl,ij}^{\top(1)} D^{kl} \chi_{,l}^{(1)} + \frac{1}{2} \chi^{\top(1)kl} D_{kl} \chi_{,ij}^{(1)} + \frac{1}{4} \chi_{kl,j}^{\top(1)} D^{kl} \chi_{,i}^{(1)} + \frac{1}{4} \chi_{,i}^{\top(1)kl} D_{kl} \chi_{,j}^{(1)} - \frac{1}{2} \chi_{j,l}^{\top(1)k} D_{ik}^k \chi_{,k}^{(1)} \\
& - \frac{1}{2} \chi_{i,k}^{\top(1)l} D_{jk}^k \chi_{,l}^{(1)} + \frac{1}{2} \chi_{kj}^{\top(1),l} D_{ik}^k \chi_{,l}^{(1)} + \frac{1}{2} \chi_{i,l}^{\top(1)k} D_{kj} \chi_{,l}^{(1),l} - \frac{1}{2} \chi_{i}^{\top(1)'} \chi_{kj}^{\top(1)'} - \frac{1}{2} \chi^{\top(1)kl} \chi_{lj,ik}^{\top(1)} - \frac{1}{2} \chi^{\top(1)kl} \chi_{li,jk}^{\top(1)} \\
& + \frac{1}{2} \chi^{\top(1)kl} \chi^{\top(1)}_{ij,kl} + \frac{1}{2} \chi^{\top(1)kl} \chi_{kl,ij}^{\top(1)} + \frac{1}{4} \chi_{,i}^{\top(1)kl} \chi_{kl,j}^{\top(1)} - \frac{1}{2} \chi_{i,k}^{\top(1)l} \chi_{j,l}^{\top(1)k} + \frac{1}{2} \chi_{i,l}^{\top(1)k} \chi_{kj}^{\top(1),l}, \tag{B3}
\end{aligned}$$

and the 2nd-order Ricci scalar is

$$\begin{aligned}
R^{(2)} = & \frac{1}{a^2} \left[2\nabla^2 \phi^{(2)} - 9\frac{a'}{a} \phi^{(2)'} - 3\phi^{(2)''} + \frac{1}{2} D^{kl} \chi_{,kl}^{(2)} - 12\phi^{(1)} \phi^{(1)''} - 36\frac{a'}{a} \phi^{(1)'} \phi^{(1)} + 6\phi_{,k}^{(1)} \phi^{(1),k} + 16\phi^{(1)} \nabla^2 \phi^{(1)} \right. \\
& + 4\phi^{(1)} D^{kl} \chi_{,kl}^{(1)} - 2\phi_{,kl}^{(1)} D^{kl} \chi^{(1)} - 2\phi_{,kl}^{(1)} \chi^{\top(1)kl} - D^{kl} \chi^{(1)} D_{kl} \chi^{(1)''} - \frac{3}{4} D^{kl} \chi^{(1)'} D_{kl} \chi^{(1)'} - 3\frac{a'}{a} D^{kl} \chi^{(1)} D_{kl} \chi^{(1)'} \\
& - 2D_{ml} \chi_{,k}^{(1),l} D^{km} \chi^{(1)} + D^{km} \chi^{(1)} \nabla^2 D_{km} \chi^{(1)} - D^{km} \chi_{,k}^{(1)} D_{ml} \chi^{(1),l} + \frac{3}{4} D^{km} \chi_{,l}^{(1)} D_{km} \chi^{(1),l} - \frac{1}{2} D^{km} \chi_{,l}^{(1)} D_{kl} \chi_{,m}^{(1)} \\
& - D^{kl} \chi^{(1)} \chi_{kl}^{\top(1)''} - \chi^{\top(1)kl} D_{kl} \chi^{(1)''} - \frac{3}{2} \chi_{kl}^{\top(1)'} D^{kl} \chi_{,l}^{(1)'} - 3\frac{a'}{a} \chi_{kl}^{\top(1)'} D^{kl} \chi_{,l}^{(1)} - 3\frac{a'}{a} \chi^{\top(1)kl} D_{kl} \chi_{,l}^{(1)'} - 2\chi^{\top(1)km} D_{ml} \chi_{,k}^{(1),l} \\
& + \chi^{\top(1)km} \nabla^2 D_{km} \chi^{(1)} + D^{km} \chi^{(1)} \nabla^2 \chi_{km}^{\top(1)} + \frac{3}{2} \chi_{km}^{\top(1),l} D^{km} \chi_{,l}^{(1)} - \chi_{k,m}^{\top(1)l} D^{km} \chi_{,l}^{(1)} - \chi^{\top(1)kl} \chi_{kl}^{\top(1)''} - \frac{3}{4} \chi^{\top(1)kl} \chi_{kl}^{\top(1)'} \\
& \left. - 3\frac{a'}{a} \chi^{\top(1)kl} \chi_{kl}^{\top(1)'} + \chi^{\top(1)km} \nabla^2 \chi_{km}^{\top(1)} + \frac{3}{4} \chi_{,l}^{\top(1)km} \chi_{km}^{\top(1),l} - \frac{1}{2} \chi_{,l}^{\top(1)km} \chi_{k,m}^{\top(1)l} \right]. \tag{B4}
\end{aligned}$$

The 2nd-order perturbed Einstein tensors are

$$\begin{aligned}
G_{00}^{(2)} &\equiv R_{00}^{(2)} - \frac{1}{2}g_{00}^{(0)}R^{(2)} \\
&= \nabla^2\phi^{(2)} - \frac{3a'}{a}\phi^{(2)'} + \frac{1}{4}D^{kl}\chi_{,kl}^{(2)} - 12\frac{a'}{a}\phi^{(1)'}\phi^{(1)} + 3\phi^{(1)'}\phi^{(1)'} + 3\phi_{,k}^{(1)}\phi^{(1),k} + 8\phi^{(1)}\nabla^2\phi^{(1)} + 2\phi^{(1)}D^{kl}\chi_{,kl}^{(1)} \\
&\quad - \phi_{,kl}^{(1)}D^{kl}\chi^{(1)} - \frac{1}{8}D^{kl}\chi^{(1)'}D_{kl}\chi^{(1)'} - \frac{a'}{a}D^{kl}\chi^{(1)}D_{kl}\chi^{(1)'} - D_{ml}\chi_{,k}^{(1),l}D^{km}\chi^{(1)} + \frac{1}{2}D^{km}\chi^{(1)}\nabla^2D_{km}\chi^{(1)} \\
&\quad - \frac{1}{2}D^{km}\chi_{,k}^{(1)}D_{ml}\chi^{(1),l} + \frac{3}{8}D^{km}\chi_{,l}^{(1)}D_{km}\chi^{(1),l} - \frac{1}{4}D^{km}\chi_{,l}^{(1)}D_{k,l}^{(1)} - \phi_{,kl}^{(1)}\chi^{\top(1)kl} - \frac{1}{4}\chi_{kl}^{\top(1)'}D^{kl}\chi^{(1)'} \\
&\quad - \frac{a'}{a}\chi^{\top(1)kl}D_{kl}\chi^{(1)'} - \frac{a'}{a}\chi_{kl}^{\top(1)'}D^{kl}\chi^{(1)} - \chi^{\top(1)km}D_{ml}\chi_{,k}^{(1),l} + \frac{1}{2}\chi^{\top(1)km}\nabla^2D_{km}\chi^{(1)} + \frac{1}{2}D^{km}\chi^{(1)}\nabla^2\chi_{km}^{\top(1)} \\
&\quad + \frac{3}{4}\chi_{km}^{\top(1),l}D^{km}\chi_{,l}^{(1)} - \frac{1}{2}\chi_{k,m}^{\top(1)l}D^{km}\chi_{,l}^{(1)} - \frac{1}{8}\chi^{\top(1)kl}\chi_{kl}^{\top(1)'} - \frac{a'}{a}\chi^{\top(1)kl}\chi_{kl}^{\top(1)'} + \frac{1}{2}\chi^{\top(1)km}\nabla^2\chi_{km}^{\top(1)} + \frac{3}{8}\chi_{,l}^{\top(1)km}\chi_{km}^{\top(1),l} \\
&\quad - \frac{1}{4}\chi_{,l}^{\top(1)km}\chi_{k,m}^{\top(1)l}, \tag{B5}
\end{aligned}$$

$$G_{0i}^{(2)} \equiv R_{0i}^{(2)}, \tag{B6}$$

$$\begin{aligned}
G_{ij}^{(2)} &= \delta_{ij} \left[-\frac{1}{2}\nabla^2\phi^{(2)} + \phi^{(2)''} + 2\frac{a'}{a}\phi^{(2)'} - \frac{1}{4}D^{kl}\chi_{,kl}^{(2)} - 2\phi^{(1),k}\phi_{,k}^{(1)} - 2\phi^{(1)}\nabla^2\phi^{(1)} + (\phi^{(1)'})^2 - \phi_{,k}^{(1)}D^{lk}\chi_{,l}^{(1)} - \phi^{(1)}D^{kl}\chi_{,kl}^{(1)} \right. \\
&\quad + \frac{a'}{a}D^{kl}\chi^{(1)'}D_{kl}\chi^{(1)'} + D_{ml}\chi_{,k}^{(1),l}D^{km}\chi^{(1)} - \frac{1}{2}D^{km}\chi^{(1)}\nabla^2D_{km}\chi^{(1)} - \frac{3}{8}D^{km}\chi^{(1),l}D_{km}\chi_{,l}^{(1)} + \frac{1}{4}D^{km}\chi^{(1),l}D_{kl}\chi_{,m}^{(1)} \\
&\quad + \frac{1}{2}D^{kl}\chi_{,l}^{(1)}D_{k,m}^{(1)} + \frac{3}{8}D^{kl}\chi^{(1)'}D_{kl}\chi^{(1)'} + \frac{1}{2}D^{kl}\chi^{(1)}D_{kl}\chi^{(1)'} + \frac{a'}{a}\chi^{\top(1)kl}D_{kl}\chi^{(1)'} + \frac{a'}{a}D^{kl}\chi^{(1)}\chi_{kl}^{\top(1)'} \\
&\quad + \chi^{\top(1)km}D_{ml}\chi_{,k}^{(1),l} - \frac{1}{2}\chi^{\top(1)km}\nabla^2D_{km}\chi^{(1)} - \frac{1}{2}D^{km}\chi^{(1)}\nabla^2\chi_{km}^{\top(1)} - \frac{3}{4}D^{km}\chi^{(1),l}\chi_{km,l}^{\top(1)} + \frac{1}{2}D^{km}\chi^{(1),l}\chi_{kl,m}^{\top(1)} \\
&\quad + \frac{3}{4}\chi^{\top(1)kl}D_{kl}\chi^{(1)'} + \frac{1}{2}\chi_{kl}^{\top(1)'}D^{kl}\chi^{(1)} + \frac{1}{2}\chi^{\top(1)kl}D_{kl}\chi^{(1)''} + \frac{a'}{a}\chi^{\top(1)kl}\chi_{kl}^{\top(1)'} - \frac{1}{2}\chi^{\top(1)km}\nabla^2\chi_{km}^{\top(1)} - \frac{3}{8}\chi^{\top(1)km,l}\chi_{km,l}^{\top(1)} \\
&\quad \left. + \frac{1}{4}\chi^{\top(1)km,l}\chi_{kl,m}^{\top(1)} + \frac{3}{8}\chi^{\top(1)kl}\chi_{kl}^{\top(1)'} + \frac{1}{2}\chi^{\top(1)kl}\chi_{kl}^{\top(1)''} \right] + \frac{1}{2}\phi_{,ij}^{(2)} + \frac{a'}{2a}D_{ij}\chi^{(2)'} + \frac{1}{4}D_{j,k}^k\chi_{,ki}^{(2)} + \frac{1}{4}D_{i,kj}^k\chi_{,kj}^{(2)} - \frac{1}{4}\nabla^2D_{ij}\chi^{(2)} \\
&\quad + \frac{1}{4}D_{ij}\chi^{(2)''} + \frac{a'}{2a}\chi_{ij}^{\perp(2)'} + \frac{1}{4}\chi_{j,ki}^{\perp(2)k} + \frac{1}{4}\chi_{i,kj}^{\perp(2)k} - \frac{1}{4}\nabla^2\chi_{ij}^{\perp(2)} + \frac{1}{4}\chi_{ij}^{\perp(2)''} + \frac{a'}{2a}\chi_{ij}^{\top(2)'} - \frac{1}{4}\nabla^2\chi_{ij}^{\top(2)} + \frac{1}{4}\chi_{ij}^{\top(2)''} + 3\phi_{,i}^{(1)}\phi_{,j}^{(1)} \\
&\quad + 2\phi^{(1)}\phi_{,ij}^{(1)} + \frac{1}{2}\phi^{(1)'}D_{ij}\chi^{(1)'} + \frac{1}{2}\phi_{,k}^{(1)}D_{j,i}^k\chi_{,i}^{(1)} + \frac{1}{2}\phi_{,k}^{(1)}D_{i,j}^k\chi_{,j}^{(1)} - \frac{3}{2}\phi_{,k}^{(1)}D_{ij}\chi^{(1),k} + \phi^{(1)}D_{j,i}^k\chi_{,ik}^{(1)} + \phi^{(1)}D_{i,j}^k\chi_{,jk}^{(1)} \\
&\quad - \phi^{(1)}\nabla^2D_{ij}\chi^{(1)} - 2D_{ij}\chi^{(1)}\nabla^2\phi^{(1)} + \phi_{,i}^{(1)}D_{j,k}^k\chi_{,i}^{(1)} + \phi_{,j}^{(1)}D_{i,k}^k\chi_{,j}^{(1)} + \phi_{,ki}^{(1)}D_{j,i}^k\chi_{,i}^{(1)} + \phi_{,kj}^{(1)}D_{i,j}^k\chi_{,j}^{(1)} + 3\phi^{(1)''}D_{ij}\chi^{(1)} \\
&\quad + 6\frac{a'}{a}\phi^{(1)'}D_{ij}\chi^{(1)} + \frac{1}{2}\phi^{(1)'}\chi_{ij}^{\top(1)'} + \frac{1}{2}\phi_{,k}^{(1)}\chi_{j,i}^{\top(1)k} + \frac{1}{2}\phi_{,k}^{(1)}\chi_{i,j}^{\top(1)k} - \frac{3}{2}\phi_{,k}^{(1)}\chi_{ij}^{\top(1),k} - \phi^{(1)}\nabla^2\chi_{ij}^{\top(1)} - 2\chi_{ij}^{\top(1)}\nabla^2\phi^{(1)} \\
&\quad + \phi_{,ki}^{(1)}\chi_j^{\top(1)k} + \phi_{,kj}^{(1)}\chi_i^{\top(1)k} + 3\phi^{(1)''}\chi_{ij}^{\top(1)} + 6\frac{a'}{a}\phi^{(1)'}\chi_{ij}^{\top(1)} - \frac{1}{2}D_{i,j}^k\chi_{,i}^{(1)'}D_{k,j}\chi_{,i}^{(1)'} - \frac{1}{2}D^{kl}\chi_{,l}^{(1)}D_{kj}\chi_{,i}^{(1)} \\
&\quad - \frac{1}{2}D^{kl}\chi_{,l}^{(1)}D_{ki}\chi_{,j}^{(1)} + \frac{1}{2}D^{lk}\chi_{,l}^{(1)}D_{ij}\chi_{,k}^{(1)} - \frac{1}{2}D^{kl}\chi^{(1)}D_{ij}\chi_{,ik}^{(1)} - \frac{1}{2}D^{kl}\chi^{(1)}D_{li}\chi_{,jk}^{(1)} + \frac{1}{2}D^{kl}\chi^{(1)}D_{ij}\chi_{,kl}^{(1)} \\
&\quad + \frac{1}{2}D^{kl}\chi^{(1)}D_{ki}\chi_{,ij}^{(1)} + \frac{1}{4}D^{kl}\chi_{,i}^{(1)}D_{kl}\chi_{,j}^{(1)} - \frac{1}{2}D_{ki}\chi^{(1),kl}D_{ij}\chi^{(1)} - \frac{1}{2}D_{i,k}^k\chi_{,l}^{(1)}D_{j,i}^k\chi_{,l}^{(1)} + \frac{1}{2}D_{i,l}^k\chi_{,l}^{(1)}D_{k,j}\chi_{,i}^{(1),l} \\
&\quad - \frac{1}{2}\chi_{kj}^{\top(1)'}D_{i,j}^k\chi_{,i}^{(1)'} - \frac{1}{2}\chi_i^{\top(1)k}D_{k,j}\chi_{,i}^{(1)'} - \frac{1}{2}\chi_{k,j,i}^{\top(1)}D^{kl}\chi_{,l}^{(1)} - \frac{1}{2}\chi_{k,i,j}^{\top(1)}D^{kl}\chi_{,l}^{(1)} + \frac{1}{2}\chi_{i,j,k}^{\top(1)}D^{lk}\chi_{,l}^{(1)} - \frac{1}{2}\chi_{l,j,ik}^{\top(1)}D^{kl}\chi_{,l}^{(1)} \\
&\quad - \frac{1}{2}\chi^{\top(1)kl}D_{lj}\chi_{,ik}^{(1)} - \frac{1}{2}\chi_{li,jk}^{\top(1)}D^{kl}\chi_{,l}^{(1)} - \frac{1}{2}\chi^{\top(1)kl}D_{li}\chi_{,jk}^{(1)} + \frac{1}{2}\chi_{ij,kl}^{\top(1)}D^{kl}\chi_{,l}^{(1)} + \frac{1}{2}\chi^{\top(1)kl}D_{ij}\chi_{,kl}^{(1)} + \frac{1}{2}\chi_{kl,ij}^{\top(1)}D^{kl}\chi_{,l}^{(1)} \\
&\quad + \frac{1}{2}\chi^{\top(1)kl}D_{kl}\chi_{,ij}^{(1)} + \frac{1}{4}\chi_{kl,j}^{\top(1)}D^{kl}\chi_{,i}^{(1)} + \frac{1}{4}\chi_{,i}^{\top(1)kl}D_{kl}\chi_{,j}^{(1)} - \frac{1}{2}\chi_{ij}^{\top(1)}D_{kl}\chi^{(1),kl} - \frac{1}{2}\chi_{j,l}^{\top(1)k}D_{i,j}^k\chi_{,k}^{(1)} - \frac{1}{2}\chi_{i,k}^{\top(1)l}D_{j,i}^k\chi_{,l}^{(1)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \chi_{kj}^{\top(1),l} D_{i,l}^k \chi_{i,l}^{\top(1)} + \frac{1}{2} \chi_{i,l}^{\top(1),k} D_{kj} \chi_{i,l}^{\top(1)} - \frac{1}{2} \chi_i^{\top(1)'k} \chi_{kj}^{\top(1)'} - \frac{1}{2} \chi^{\top(1)kl} \chi_{lj,ik}^{\top(1)} - \frac{1}{2} \chi^{\top(1)kl} \chi_{li,jk}^{\top(1)} + \frac{1}{2} \chi^{\top(1)kl} \chi_{ij,kl}^{\top(1)} \\
& + \frac{1}{2} \chi^{\top(1)kl} \chi_{kl,ij}^{\top(1)} + \frac{1}{4} \chi_{i,l}^{\top(1)kl} \chi_{kl,j}^{\top(1)} - \frac{1}{2} \chi_{i,k}^{\top(1)l} \chi_{j,l}^{\top(1)k} + \frac{1}{2} \chi_{i,l}^{\top(1)k} \chi_{kj}^{\top(1),l}, \tag{B7}
\end{aligned}$$

and its trace is given by

$$\begin{aligned}
\delta^{kl} G_{kl}^{(2)} &= -\nabla^2 \phi^{(2)} + 3\phi^{(2)''} + 6\frac{a'}{a} \phi^{(2)'} - \frac{1}{4} D^{kl} \chi_{,kl}^{\top(2)} - 3\phi^{(1),k} \phi_{,k}^{(1)} - 4\phi^{(1)} \nabla^2 \phi^{(1)} + 3(\phi^{(1)'})^2 - \phi^{(1)} D^{kl} \chi_{,kl}^{\top(1)} \\
&+ 2\phi_{,kl}^{(1)} D^{kl} \chi^{\top(1)} + 2\phi_{,kl}^{(1)} \chi^{\top(1)kl} + 3\frac{a'}{a} D^{kl} \chi^{\top(1)} D_{kl} \chi^{\top(1)'} + 2D_{ml} \chi_{,k}^{\top(1),l} D^{km} \chi^{\top(1)} - D^{km} \chi^{\top(1)} \nabla^2 D_{km} \chi^{\top(1)} \\
&- \frac{3}{8} D^{km} \chi_{,l}^{\top(1)} D_{km} \chi^{\top(1),l} + \frac{1}{2} D^{kl} \chi_{,l}^{\top(1)} D_k^m \chi_{,m}^{\top(1)} + \frac{5}{8} D^{kl} \chi^{\top(1)'} D_{kl} \chi^{\top(1)'} + \frac{3}{2} D^{kl} \chi^{\top(1)} D_{kl} \chi^{\top(1)''} + \frac{1}{4} D^{km} \chi^{\top(1),l} D_{lm} \chi_{,k}^{\top(1)} \\
&+ 3\frac{a'}{a} \chi_{kl}^{\top(1)'} D^{kl} \chi^{\top(1)} + 3\frac{a'}{a} \chi^{\top(1)kl} D_{kl} \chi^{\top(1)'} + 2\chi^{\top(1)km} D_{ml} \chi_{,k}^{\top(1),l} - \chi^{\top(1)km} \nabla^2 D_{km} \chi^{\top(1)} - D^{km} \chi^{\top(1)} \nabla^2 \chi_{km}^{\top(1)} \\
&- \frac{3}{4} \chi_{,l}^{\top(1)km} D_{km} \chi^{\top(1),l} + \frac{5}{4} \chi^{\top(1)kl} D_{kl} \chi^{\top(1)'} + \frac{3}{2} \chi_{kl}^{\top(1)''} D^{kl} \chi^{\top(1)} + \frac{3}{2} \chi^{\top(1)kl} D_{kl} \chi^{\top(1)''} + \frac{1}{2} \chi_{lm,k}^{\top(1)} D^{km} \chi^{\top(1),l} \\
&+ 3\frac{a'}{a} \chi^{\top(1)kl} \chi_{kl}^{\top(1)'} - \chi^{\top(1)km} \nabla^2 \chi_{km}^{\top(1)} - \frac{3}{8} \chi_{,l}^{\top(1)km} \chi_{km}^{\top(1),l} + \frac{5}{8} \chi^{\top(1)'kl} \chi_{kl}^{\top(1)'} + \frac{3}{2} \chi^{\top(1)kl} \chi_{kl}^{\top(1)'} \\
&+ \frac{1}{4} \chi^{\top(1)km,l} \chi_{lm,k}^{\top(1)}. \tag{B8}
\end{aligned}$$

APPENDIX C: GAUGE TRANSFORMATIONS FROM SYNCHRONOUS TO SYNCHRONOUS

First we give the formulas for the gauge transformation between two general coordinates. Consider a general coordinate transformation [51,59]

$$x^\mu \rightarrow \bar{x}^\mu = x^\mu + \xi^{(1)\mu} + \frac{1}{2} \xi_{,\alpha}^{(1)\mu} \xi^{(1)\alpha} + \frac{1}{2} \xi^{(2)\mu}, \tag{C1}$$

where $\xi^{(1)\mu}$ is a 1st-order vector field, and $\xi^{(2)\mu}$ is an independent vector field whose magnitude is of 2nd order. The corresponding transformations of metric [42,59]

$$g_{\mu\nu}(x) = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu} \bar{g}_{\alpha\beta}(\bar{x}). \tag{C2}$$

The metric is written as $g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} + \frac{1}{2} g_{\mu\nu}^{(2)}$ to the 2nd order, and similar for $\bar{g}_{\mu\nu}$. Eq. (C2) leads to the following transformations to each order:

$$g_{\mu\nu}^{(0)}(x) = \bar{g}_{\mu\nu}^{(0)}(x), \tag{C3}$$

$$\bar{g}_{\mu\nu}^{(1)}(x) = g_{\mu\nu}^{(1)}(x) - \mathcal{L}_{\xi^{(1)}} g_{\mu\nu}^{(0)}(x), \tag{C4}$$

$$\begin{aligned}
\bar{g}_{\mu\nu}^{(2)}(x) &= g_{\mu\nu}^{(2)}(x) - 2\mathcal{L}_{\xi^{(1)}} g_{\mu\nu}^{(1)}(x) + \mathcal{L}_{\xi^{(1)}} (\mathcal{L}_{\xi^{(1)}} g_{\mu\nu}^{(0)}(x)) \\
&- \mathcal{L}_{\xi^{(2)}} g_{\mu\nu}^{(0)}(x), \tag{C5}
\end{aligned}$$

where the Lie derivative along $\xi^{(1)\mu}$ is defined as

$$\mathcal{L}_{\xi^{(1)}} g_{\mu\nu}^{(0)} \equiv g_{\mu\nu,\alpha}^{(0)} \xi^{(1)\alpha} + g_{\mu\alpha}^{(0)} \xi_{,\nu}^{(1)\alpha} + g_{\nu\alpha}^{(0)} \xi_{,\mu}^{(1)\alpha}, \tag{C6}$$

and others are similarly defined. It is checked that, under the transformation (C1) and (C2), the spacetime line element remains invariant,

$$ds^2 = \bar{g}_{\mu\nu}(\bar{x}) d\bar{x}^\mu d\bar{x}^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{C7}$$

The formulas (C3), (C4), and (C5) also apply to the energy-momentum tensor $T_{\mu\nu} = \rho U^\mu U^\nu$. Similarly, under the coordinate transformation (C1), a scalar function transforms as $f(x) = \bar{f}(\bar{x})$. By writing $f(x) = f^{(0)}(x) + f^{(1)}(x) + \frac{1}{2} f^{(2)}(x)$, one has

$$\bar{f}^{(0)}(x) = f^{(0)}(x), \tag{C8}$$

$$\bar{f}^{(1)}(x) = f^{(1)}(x) - \mathcal{L}_{\xi^{(1)}} f^{(0)}(x), \tag{C9}$$

$$\begin{aligned}
\bar{f}^{(2)}(x) &= f^{(2)}(x) - 2\mathcal{L}_{\xi^{(1)}} f^{(1)}(x) + \mathcal{L}_{\xi^{(1)}} (\mathcal{L}_{\xi^{(1)}} f^{(0)}(x)) \\
&- \mathcal{L}_{\xi^{(2)}} f^{(0)}(x), \tag{C10}
\end{aligned}$$

where

$$\mathcal{L}_{\xi} f \equiv f_{,\alpha} \xi^\alpha. \tag{C11}$$

Under (C1), a 4-vector Z^μ transforms as $\bar{Z}^\mu(\bar{x}) = (\partial \bar{x}^\mu / \partial x^\alpha) Z^\alpha(x)$. Writing $Z^\mu(x) = Z^{(0)\mu}(x) + Z^{(1)\mu}(x) + \frac{1}{2} Z^{(2)\mu}(x)$, one has

$$\bar{Z}^{(0)\mu}(x) = Z^{(0)\mu}(x), \quad (\text{C12})$$

$$\bar{Z}^{(1)\mu}(x) = Z^{(1)\mu}(x) - \mathcal{L}_{\xi^{(1)}} Z^{(0)\mu}(x), \quad (\text{C13})$$

$$\begin{aligned} \bar{Z}^{(2)\mu}(x) &= Z^{(2)\mu}(x) - 2\mathcal{L}_{\xi^{(1)}} Z^{(1)\mu}(x) + \mathcal{L}_{\xi^{(1)}}(\mathcal{L}_{\xi^{(1)}} Z^{(0)\mu}(x)) \\ &\quad - \mathcal{L}_{\xi^{(2)}} Z^{(0)\mu}(x), \end{aligned} \quad (\text{C14})$$

where

$$\mathcal{L}_{\xi} Z^{\mu} \equiv Z^{\mu}_{,\alpha} \xi^{\alpha} - \xi^{\mu} Z^{\alpha}, \quad (\text{C15})$$

$$\mathcal{L}_{\xi} Z_{\mu} \equiv Z_{\mu,\alpha} \xi^{\alpha} + \xi_{\mu,\alpha} Z^{\alpha}. \quad (\text{C16})$$

The components of the vector fields $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$ can be denoted by the parameters

$$\xi^{(1)0} = \alpha^{(1)}, \quad \xi^{(1)i} = \partial^i \beta^{(1)} + d^{(1)i}, \quad (\text{C17})$$

$$\xi^{(2)0} = \alpha^{(2)}, \quad \xi^{(2)i} = \partial^i \beta^{(2)} + d^{(2)i}, \quad (\text{C18})$$

with the constraints $\partial_i d^{(1)i} = 0$ and $\partial_i d^{(2)i} = 0$.

We first apply the above formulas to the 1st-order gauge transformations between synchronous coordinates. By (C4), requiring $\bar{g}_{00}^{(1)}(x) = g_{00}^{(1)}(x) = 0$ leads to $\mathcal{L}_{\xi^{(1)}} g_{00}^{(0)} = 0$, and its solution is

$$\xi^{(1)0}(\tau, \mathbf{x}) = \frac{A^{(1)}(\mathbf{x})}{a(\tau)}. \quad (\text{C19})$$

Requiring $\bar{g}_{0i}^{(1)}(x) = g_{0i}^{(1)}(x) = 0$ leads to $\mathcal{L}_{\xi^{(1)}} g_{0i}^{(0)} = 0$, ie,

$$-\xi_{,i}^{(1)0} + \xi_{i,0}^{(1)} = 0, \quad (\text{C20})$$

and its solution is

$$\xi_i^{(1)}(\tau, \mathbf{x}) = A^{(1)}(\mathbf{x})_{,i} \int^{\tau} \frac{d\tau'}{a(\tau')} + C_i^{(1)}(\mathbf{x}), \quad (\text{C21})$$

where $A^{(1)}(\mathbf{x})$ is an arbitrary function, and $C_i^{(1)}(\mathbf{x})$ is an arbitrary 3-vector. $A^{(1)}(\mathbf{x})$ and $C_i^{(1)}(\mathbf{x})$ together represent 4 degrees of freedom of 1st-order gauge transformation under consideration. One can write $C_i^{(1)}(\mathbf{x})$ as

$$C_i^{(1)} = C_i^{\parallel(1)} + C_i^{\perp(1)}, \quad (\text{C22})$$

where the longitudinal part $C_i^{\parallel(1)} = C_{,i}^{\parallel(1)}$ with $C^{\parallel(1)}$ being an arbitrary function, and $C_i^{\perp(1)}$ is a transverse part so that $C_i^{\perp(1),i} = 0$. One can set $C_i^{\perp(1)} = 0$, since the 1st-order vector metric perturbation is zero for an irrotational dust in this paper.

Now we explain why a 1st-order vector metric perturbation $\chi_{ij}^{\perp(1)}$ is a residual gauge mode and can be set at 0. Assume that it exists as the following:

$$\chi_{ij}^{\perp(1)} = v_{i,j}^{(1)} + v_{j,i}^{(1)},$$

where the vector satisfies $\partial^i v_i^{(1)} = 0$. It is easy to give the momentum constraint: $\frac{1}{4} \chi_{ij}^{\perp(1),j} = 0$, which leads to $\frac{1}{4} \nabla^2 v_i^{(1)} = 0$ and the solution $v_i^{(1)} = v_i(\mathbf{x}) + e_i(\tau)$, where $\partial^i v_i(\mathbf{x}) = 0$ and $e_i(\tau)$ is an arbitrary vector depending on τ only. By a residual gauge transformation using $\xi_i^{(1)}$ of (C21) with the parameter $C_i^{\perp(1)}$ only, $\chi_{ij}^{\perp(1)}$ is changed to $\bar{\chi}_{ij}^{\perp(1)} = (v_{i,j} + v_{j,i}) - (C_{i,j}^{\perp(1)} + C_{j,i}^{\perp(1)})$. Setting $C_i^{\perp(1)} = v_i$, one gets $\bar{\chi}_{ij}^{\perp(1)} = 0$. Hence, the 1st-order vector is a gauge mode and can be eliminated.

Under the transformation of (C19) and (C21), the metric perturbation changes to

$$\bar{\phi}^{(1)} = \phi^{(1)} + \frac{a'}{a} \frac{A^{(1)}}{a} + \frac{1}{3} \nabla^2 A^{(1)} \int^{\tau} \frac{d\tau'}{a(\tau')} + \frac{1}{3} \nabla^2 C^{\parallel(1)}, \quad (\text{C23})$$

$$\bar{\chi}^{\parallel(1)} = \chi^{\parallel(1)} - 2A^{(1)} \int^{\tau} \frac{d\tau'}{a(\tau')} - 2C^{\parallel(1)}, \quad (\text{C24})$$

$$\bar{\chi}_{ij}^{\top(1)} = \chi_{ij}^{\top(1)}. \quad (\text{C25})$$

The results (C19), (C21), (C23), (C24), and (C25) are valid for a general scale factor $a(\tau)$. For the MD era with $a(\tau) \propto \tau^2$, (C19) and (C21) reduce to

$$\alpha^{(1)}(\tau, \mathbf{x}) = \frac{A^{(1)}(\mathbf{x})}{\tau^2}, \quad (\text{C26})$$

$$\beta_{,i}^{(1)}(\tau, \mathbf{x}) = -\frac{A^{(1)}(\mathbf{x})_{,i}}{\tau} + C^{\parallel(1)}(\mathbf{x})_{,i}, \quad (\text{C27})$$

$$d_i^{(1)} = 0, \quad (\text{C28})$$

and (C23) and (C24) become

$$\bar{\phi}^{(1)} = \phi^{(1)} + 2 \frac{A^{(1)}(\mathbf{x})}{\tau^3} - \frac{\nabla^2 A^{(1)}(\mathbf{x})}{3\tau} + \frac{1}{3} \nabla^2 C^{\parallel(1)}(\mathbf{x}), \quad (\text{C29})$$

$$\bar{D}_{ij} \chi^{\parallel(1)} = D_{ij} \chi^{\parallel(1)} + \frac{2D_{ij} A^{(1)}(\mathbf{x})}{\tau} - 2D_{ij} C^{\parallel(1)}(\mathbf{x}). \quad (\text{C30})$$

From (A28) one sees that for a given initial density contrast $\delta_0^{(1)}(\mathbf{x})$, one can always choose $C^{\parallel(1)}(\mathbf{x})$ to satisfy

$$\nabla^2 \bar{\chi}_0^{\parallel(1)}(\mathbf{x}) = -2\delta_0^{(1)}(\mathbf{x}). \quad (\text{C31})$$

The synchronous-to-synchronous transformation of 1st-order density perturbation can be also derived from the following:

$$\bar{T}_{00}^{(1)}(x) = T_{00}^{(1)}(x) - \mathcal{L}_{\xi^{(1)}} T_{00}^{(0)}(x), \quad (\text{C32})$$

as an application of (C4). Up to the 1st order, one has $T_{00}^{(1)} = \rho^{(1)}a^2$, $\bar{U}_0^{(1)} = 0$, $\bar{T}_{00}^{(1)} = \bar{\rho}^{(1)}a^2$, and $\mathcal{L}_{\xi^{(1)}} T_{00}^{(0)} = -6a^2\rho^{(0)}\frac{A^{(1)}}{\tau^3}$, leading to the result

$$\bar{\rho}^{(1)} = \rho^{(1)} + 6\rho^{(0)}\frac{A^{(1)}}{\tau^3}. \quad (\text{C33})$$

This result also follows from $\bar{\rho}^{(1)} = \rho^{(1)} - \mathcal{L}_{\xi^{(1)}}\rho^{(0)}$ as an application of (C9). In terms of the 1st-order density contrast, (C33) gives

$$\bar{\delta}^{(1)} = \delta^{(1)} + 6\frac{A^{(1)}}{\tau^3}. \quad (\text{C34})$$

Thus, the residual gauge mode of $\delta^{(1)}$ is τ^{-3} .

In this paper on the dust model, the 4-velocity $U^{(0)\mu} = (a^{-1}, 0, 0, 0)$, and its 1st-order perturbation $U^{(1)\mu} = 0$ has been taken. When one further requires the transformed $\bar{U}^{(1)i} = 0$ [42], the formula (C13) leads to

$$A^{(1)}(\mathbf{x})_{,i} = 0, \quad (\text{C35})$$

i.e., $A^{(1)} = \text{const}$, so that (C26), (C27), (C29), and (C30) reduce to

$$\alpha^{(1)}(\tau) = \frac{A^{(1)}}{\tau^2}, \quad (\text{C36})$$

$$\beta_{,i}^{(1)}(\mathbf{x}) = C_{,i}^{\parallel(1)}(\mathbf{x}), \quad (\text{C37})$$

$$\bar{\phi}^{(1)} = \phi^{(1)} + 2\frac{A^{(1)}}{\tau^3} + \frac{1}{3}\nabla^2 C^{\parallel(1)}(\mathbf{x}), \quad (\text{C38})$$

$$\bar{D}_{ij}\chi^{\parallel(1)} = D_{ij}\chi^{\parallel(1)} - 2D_{ij}C^{\parallel(1)}(\mathbf{x}). \quad (\text{C39})$$

If starting with a velocity $U^{(1)\mu} \neq 0$ in a given synchronous coordinate, we can make a coordinate transform to render the velocity vanish at $\bar{U}^{(1)\mu} = 0$; however, at the same time, the new coordinate is no longer synchronous in general. This is explained as follows: by (C13), the vector $U^{(1)\mu}$ transforms as

$$\bar{U}^{(1)0} = U^{(1)0} - [U_{,0}^{(0)0}\xi^{(1)0} - \xi_{,0}^{(1)0}U^{(0)0}],$$

$$\bar{U}^{(1)i} = U^{(1)i} + \xi_{,0}^{(1)i}U^{(0)0},$$

under a general $\xi^{(1)\nu}$. Requiring $\bar{U}^{(1)0} = 0$, $\bar{U}^{(1)i} = 0$ in the new coordinate leads to

$$U^{(1)0} - [U_{,0}^{(0)0}\xi^{(1)0} - \xi_{,0}^{(1)0}U^{(0)0}] = 0,$$

$$U^{(1)i} + \xi_{,0}^{(1)i}U^{(0)0} = 0.$$

The solution of the transformation vector is

$$\begin{aligned} \xi^{(1)0} &= \frac{H(\mathbf{x})}{a(\tau)} - \frac{1}{a(\tau)} \int^\tau a^2(\tau')U^{(1)0}d\tau', \\ \xi^{(1)i} &= - \int^\tau a(\tau')U^{(1)i}d\tau' + E^i(\mathbf{x}), \end{aligned} \quad (\text{C40})$$

where $H(\mathbf{x})$ is an arbitrary function and $E^i(\mathbf{x})$ is an arbitrary vector. For this solution to satisfy the synchronous-to-synchronous conditions (C19) and (C21), the given velocity must be of the following special form:

$$U^{(1)0} = 0, \quad U^{(1)i} \propto \frac{1}{a^2}. \quad (\text{C41})$$

In fact, for a pressureless dust with the energy-momentum tensor $T^{\mu\nu} = \rho U^\mu U^\nu$ in a synchronous coordinate, one can write $U^\mu = U^{(0)\mu} + U^{(1)\mu}$, and show that $U^{(1)0} = 0$ as a result of $U^\mu U_\mu = -1$, that $(a^2 U^{(1)i})' = 0$ as a result of the momentum conservation $T^{i\mu}{}_{;\mu} = 0$ at the 1st order; that is, (C41) is satisfied by a pressureless dust. Thus, one can set the 1st-order velocity $\bar{U}^{(1)\mu} = 0$ by a 1st-order gauge transformation. Similar calculation analysis can be also performed for the 2nd order, leading to a similar result; i.e., the 2nd-order velocity $\bar{U}^{(2)\mu}$ can be brought zero by a 2nd-order gauge transformation. Therefore, for a pressureless dust in a synchronous coordinate, one can take the 4-velocity $U^\mu = (a^{-1}, 0, 0, 0)$.

Now we determine the 2nd-order vector $\xi^{(2)}$ of synchronous-to-synchronous residual gauge transformations. By the requirement $\bar{g}_{00}^{(2)}(x) = g_{00}^{(2)}(x) = 0$, the formula (C5) gives

$$0 = 0 - 2\mathcal{L}_{\xi^{(1)}}g_{00}^{(1)} + \mathcal{L}_{\xi^{(1)}}(\mathcal{L}_{\xi^{(1)}}g_{00}^{(0)}) - \mathcal{L}_{\xi^{(2)}}g_{00}^{(0)}. \quad (\text{C42})$$

As is checked, $\mathcal{L}_{\xi^{(1)}}g_{00}^{(1)} = 0$ and $\mathcal{L}_{\xi^{(1)}}\mathcal{L}_{\xi^{(1)}}g_{00}^{(0)} = 0$ for the given $\xi^{(1)}$, so that the above reduces to $\mathcal{L}_{\xi^{(2)}}g_{00}^{(0)} = 0$, which yields a solution

$$\alpha^{(2)} = \frac{A^{(2)}(\mathbf{x})}{a(\tau)}, \quad (\text{C43})$$

with $A^{(2)}(\mathbf{x})$ being an arbitrary function. By the requirement $\bar{g}_{0i}^{(2)}(x) = g_{0i}^{(2)}(x) = 0$, the formula (C5) gives

$$0 = 0 - 2\mathcal{L}_{\xi^{(1)}}g_{0i}^{(1)} + \mathcal{L}_{\xi^{(1)}}(\mathcal{L}_{\xi^{(1)}}g_{0i}^{(0)}) - \mathcal{L}_{\xi^{(2)}}g_{0i}^{(0)}, \quad (\text{C44})$$

which leads to the equation

$$\frac{d\xi^{(2)i}}{d\tau} = 2(2\phi^{(1)}\delta_{ik} - \chi_{ik}^{(1)})\frac{A^{(1),k}}{a} + 2a'\frac{A_{,i}^{(1)}A^{(1)}}{a^3} + 2\frac{A^{(1),k}}{a}A_{,ik}^{(1)}\int^\tau \frac{d\tau'}{a(\tau')} + 2\frac{A^{(1),k}}{a}C_{,ik}^{||(1)} + \frac{A_{,i}^{(2)}}{a}. \quad (\text{C45})$$

Its solution is

$$\begin{aligned} \xi_i^{(2)}(\tau, \mathbf{x}) &= 4A^{(1)}(\mathbf{x})_{,i}\int^\tau \frac{\phi^{(1)}(\tau', \mathbf{x})}{a(\tau')}d\tau' - 2A^{(1)}(\mathbf{x})^k\int^\tau \frac{\chi_{ki}^{(1)}(\tau', \mathbf{x})}{a(\tau')}d\tau' - \frac{1}{a^2}A^{(1)}(\mathbf{x})A^{(1)}(\mathbf{x})_{,i} \\ &\quad + 2A^{(1)}(\mathbf{x})^kA^{(1)}(\mathbf{x})_{,ki}\int^\tau \frac{d\tau'}{a(\tau')}\int^{\tau'} \frac{d\tau''}{a(\tau'')} + 2A^{(1)}(\mathbf{x})^kC^{||(1)}(\mathbf{x})_{,ki}\int^\tau \frac{d\tau'}{a(\tau')} \\ &\quad + A^{(2)}(\mathbf{x})_{,i}\int^\tau \frac{d\tau'}{a(\tau')} + C_i^{(2)}(\mathbf{x}), \end{aligned} \quad (\text{C46})$$

where $C_i^{(2)}$ is an integration constant 3-vector and can be decomposed as

$$C_i^{(2)} = C_{,i}^{||(2)} + C_i^{\perp(2)}. \quad (\text{C47})$$

$A^{(2)}(\mathbf{x})$ and $C_i^{(2)}(\mathbf{x})$ together represent 4 degrees of freedom of 2nd-order gauge transformation under consideration. From Eq. (C46), one has

$$\begin{aligned} \beta^{(2)} &= \nabla^{-2}\left\{[\nabla^2A^{(1)}(\mathbf{x})]\int^\tau \frac{4\phi^{(1)}(\tau', \mathbf{x})}{a(\tau')}d\tau' + A^{(1)}(\mathbf{x})_{,k}\int^\tau \frac{4\phi^{(1)}(\tau', \mathbf{x})^k}{a(\tau')}d\tau' - A^{(1)}(\mathbf{x})^{,ki}\int^\tau \frac{2\chi_{ki}^{(1)}(\tau', \mathbf{x})}{a(\tau')}d\tau' \right. \\ &\quad \left. - A^{(1)}(\mathbf{x})^k\int^\tau \frac{2\chi_{ki}^{(1)}(\tau', \mathbf{x})^i}{a(\tau')}d\tau' + 2A^{(1)}(\mathbf{x})^{,ki}C^{||(1)}(\mathbf{x})_{,ki}\int^\tau \frac{d\tau'}{a(\tau')} + 2A^{(1)}(\mathbf{x})^k\nabla^2C^{||(1)}(\mathbf{x})_{,k}\int^\tau \frac{d\tau'}{a(\tau')}\right\} \\ &\quad - \frac{1}{2a^2(\tau)}A^{(1)}(\mathbf{x})A^{(1)}(\mathbf{x}) + A^{(1)}(\mathbf{x})^kA^{(1)}(\mathbf{x})_{,k}\int^\tau \frac{d\tau'}{a(\tau')}\int^{\tau'} \frac{d\tau''}{a(\tau'')} + A^{(2)}(\mathbf{x})\int^\tau \frac{d\tau'}{a(\tau')} + C^{||(2)}(\mathbf{x}), \end{aligned} \quad (\text{C48})$$

$$\begin{aligned} d_i^{(2)} &= \xi_i^{(2)} - \beta_{,i}^{(2)} \\ &= \partial_i\nabla^{-2}\left\{-[\nabla^2A^{(1)}(\mathbf{x})]\int^\tau \frac{4\phi^{(1)}(\tau', \mathbf{x})}{a(\tau')}d\tau' - A^{(1)}(\mathbf{x})_{,k}\int^\tau \frac{4\phi^{(1)}(\tau', \mathbf{x})^k}{a(\tau')}d\tau' + 2A^{(1)}(\mathbf{x})^{,kl}\int^\tau \frac{\chi_{kl}^{(1)}(\tau', \mathbf{x})}{a(\tau')}d\tau' \right. \\ &\quad \left. + 2A^{(1)}(\mathbf{x})^k\int^\tau \frac{\chi_{kl}^{(1)}(\tau', \mathbf{x})^l}{a(\tau')}d\tau' - 2A^{(1)}(\mathbf{x})^{,kl}C^{||(1)}(\mathbf{x})_{,kl}\int^\tau \frac{d\tau'}{a(\tau')} - 2A^{(1)}(\mathbf{x})^k\nabla^2C^{||(1)}(\mathbf{x})_{,k}\int^\tau \frac{d\tau'}{a(\tau')}\right\} \\ &\quad + 4A^{(1)}(\mathbf{x})_{,i}\int^\tau \frac{\phi^{(1)}(\tau', \mathbf{x})}{a(\tau')}d\tau' - 2A^{(1)}(\mathbf{x})^k\int^\tau \frac{\chi_{ki}^{(1)}(\tau', \mathbf{x})}{a(\tau')}d\tau' + 2A^{(1)}(\mathbf{x})^kC^{||(1)}(\mathbf{x})_{,ki}\int^\tau \frac{d\tau'}{a(\tau')} + C_i^{\perp(2)}(\mathbf{x}). \end{aligned} \quad (\text{C49})$$

The results (C43), (C48), and (C49) are valid for a general $a(\tau)$.

We now determine the transformation of 2nd-order metric perturbations. Applying the formula (C5) to the (ij) components yields

$$\begin{aligned} \bar{\phi}^{(2)} &= \phi^{(2)} - \left[\left(4\frac{a'}{a}\phi^{(1)} + 2\phi^{(1)'}\right) + \left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\alpha^{(1)} + \frac{a'}{a}\alpha^{(1)'}\right]\alpha^{(1)} - \frac{1}{3}\left(4\phi^{(1)} + \alpha^{(1)}\partial_0 + \beta^{(1),k}\partial_k + 4\frac{a'}{a}\alpha^{(1)}\right)\nabla^2\beta^{(1)} \\ &\quad - \left(2\phi_{,k}^{(1)} + \frac{a'}{a}\alpha_{,k}^{(1)}\right)\beta^{(1),k} - \frac{2}{3}\left(-\chi_{kl}^{(1)} + \beta_{,kl}^{(1)}\right)\beta^{(1),kl} + \frac{a'}{a}\alpha^{(2)} + \frac{1}{3}\nabla^2\beta^{(2)}, \end{aligned} \quad (\text{C50})$$

$$\bar{\chi}_{ij}^{(2)} = \chi_{ij}^{(2)} + W_{ij}, \quad (\text{C51})$$

where

$$\begin{aligned}
W_{ij} \equiv & -\frac{2}{a}\chi_{ij}^{(1)'}A^{(1)} - 4\frac{a'}{a^2}\chi_{ij}^{(1)}A^{(1)} - 2\chi_{ij,k}^{(1)}A^{(1),k} \int^\tau \frac{d\tau'}{a(\tau')} - 2\chi_{ij,k}^{(1)}C^{\parallel(1),k} + 8\phi^{(1)}D_{ij}A^{(1)} \int^\tau \frac{d\tau'}{a(\tau')} + 8\phi^{(1)}D_{ij}C^{\parallel(1)} \\
& + \frac{2}{a^2}A^{(1)}D_{ij}A^{(1)} - 2A^{(1),k}D_{ij}A_{,k}^{(1)} \left[\int^\tau \frac{d\tau'}{a(\tau')} \right]^2 - 2A^{(1),k}D_{ij}C_{,k}^{\parallel(1)} \int^\tau \frac{d\tau'}{a(\tau')} \\
& - 2C^{\parallel(1),k}D_{ij}A_{,k}^{(1)} \int^\tau \frac{d\tau'}{a(\tau')} - 2C^{\parallel(1),k}D_{ij}C_{,k}^{\parallel(1)} + \frac{8a'}{a^2}A^{(1)}D_{ij}A^{(1)} \int^\tau \frac{d\tau'}{a(\tau')} + \frac{8a'}{a^2}A^{(1)}D_{ij}C^{\parallel(1)} \\
& - 4\left(\chi_{k(i}^{(1)}A_{,j)}^{(1),k} - \frac{1}{3}\chi_{kl}^{(1)}A^{(1),kl}\delta_{ij}\right) \int^\tau \frac{d\tau'}{a(\tau')} - 4\left(\chi_{k(i}^{(1)}C_{,j)}^{\parallel(1),k} - \frac{1}{3}\chi_{kl}^{(1)}C^{\parallel(1),kl}\delta_{ij}\right) \\
& + 2D_{ij}\left(A_{,k}^{(1)}A^{(1),k} \left[\int^\tau \frac{d\tau'}{a(\tau')} \right]^2 + 2A_{,k}^{(1)}C^{\parallel(1),k} \int^\tau \frac{d\tau'}{a(\tau')} + C_{,k}^{\parallel(1)}C^{\parallel(1),k}\right) - 2(D_{ij}\beta^{(2)} + d_{(i,j)}^{(2)}). \tag{C52}
\end{aligned}$$

Eq. (C51) is to be decomposed into scalar, vector, tensor: $\bar{\chi}_{ij}^{(2)} = \bar{\chi}_{ij}^{\top(2)} + \bar{\chi}_{ij}^{\perp(2)} + D_{ij}\bar{\chi}^{\parallel(2)}$. By calculations, we get

$$\bar{\chi}^{\parallel(2)} = \chi^{\parallel(2)} + \frac{3}{2}\nabla^{-2}\nabla^{-2}W_{kl}^{kl}, \tag{C53}$$

$$\bar{\chi}_{ij}^{\perp(2)} = \chi_{ij}^{\perp(2)} + \nabla^{-2}(W_{ki,j}^k + W_{kj,i}^k) - 2\nabla^{-2}\nabla^{-2}W_{kl,ij}^{kl}, \tag{C54}$$

$$\bar{\chi}_{ij}^{\top(2)} = \chi_{ij}^{\top(2)} + W_{ij} - \nabla^{-2}(W_{ki,j}^k + W_{kj,i}^k) + \frac{1}{2}\nabla^{-2}\nabla^{-2}W_{kl,ij}^{kl} + \frac{\delta_{ij}}{2}\nabla^{-2}W_{kl}^{kl}. \tag{C55}$$

Substituting (C52) into Eqs. (C53), (C54), and Eq. (C55), we obtain the residual gauge transformations of the scalar, vector and tensor as the following:

$$\begin{aligned}
\bar{\chi}^{\parallel(2)} = & \chi^{\parallel(2)} + \frac{1}{a^2}[A^{(1)}A^{(1)} + 2\nabla^{-2}(A^{(1)}\nabla^2A^{(1)}) + 3\nabla^{-2}\nabla^{-2}(A^{(1),kl}A_{,kl}^{(1)} - \nabla^2A^{(1)}\nabla^2A^{(1)})] \\
& + \frac{4a'}{a^2}\left[\int^\tau \frac{d\tau'}{a(\tau')}\right][2\nabla^{-2}(A^{(1)}\nabla^2A^{(1)}) + 3\nabla^{-2}\nabla^{-2}(A^{(1),kl}A_{,kl}^{(1)} - \nabla^2A^{(1)}\nabla^2A^{(1)})] \\
& + \frac{a'}{a^2}[8\nabla^{-2}(A^{(1)}\nabla^2C^{\parallel(1)}) + 6\nabla^{-2}\nabla^{-2}(-\chi_{kl}^{(1),kl}A^{(1)} - \chi_{kl}^{(1)}A^{(1),kl} - 2\chi_{kl}^{(1),k}A^{(1),l} \\
& + 2A^{(1),kl}C_{,kl}^{\parallel(1)} - 2\nabla^2A^{(1)}\nabla^2C^{\parallel(1)})] - \frac{3}{a}\nabla^{-2}\nabla^{-2}[\chi_{kl}^{(1),kl}A^{(1)} + \chi_{kl}^{(1)'}A^{(1),kl} + 2\chi_{kl}^{(1),k}A^{(1),l}] \\
& - 2A^{(1),k}A_{,k}^{(1)} \int^\tau d\tau' \frac{1}{a(\tau')} \int^{\tau'} \frac{d\tau''}{a(\tau'')} \\
& + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right]^2 [2A_{,k}^{(1)}A^{(1),k} - 2\nabla^{-2}(A^{(1),k}\nabla^2A_{,k}^{(1)}) + 3\nabla^{-2}\nabla^{-2}(\nabla^2A^{(1),k}\nabla^2A_{,k}^{(1)} - A^{(1),klm}A_{,klm}^{(1)})] \\
& + \left[\int^\tau \frac{d\tau'}{a(\tau')}\right][2A^{(1),k}C_{,k}^{\parallel(1)} + 2\nabla^{-2}(4\phi^{(1)}\nabla^2A^{(1)} + \chi_{kl}^{(1)}A^{(1),kl} - 2A^{(1),k}\nabla^2C_{,k}^{\parallel(1)}) \\
& + 3\nabla^{-2}\nabla^{-2}(-\chi_{kl}^{(1),klm}A_{,m}^{(1)} - 3\chi_{kl,m}^{(1)}A^{(1),klm} - 4\chi_{kl,m}^{(1),k}A^{(1),lm} - 2\chi_{kl}^{(1),k}\nabla^2A^{(1),l} \\
& - 2\chi_{kl}^{(1)}\nabla^2A^{(1),kl} + 4\phi^{(1),kl}A_{,kl}^{(1)} - 4\nabla^2\phi^{(1)}\nabla^2A^{(1)} + 2\nabla^2A^{(1),k}\nabla^2C_{,k}^{\parallel(1)} - 2A^{(1),klm}C_{,klm}^{\parallel(1)})]
\end{aligned}$$

$$\begin{aligned}
 & + [2C_{,k}^{||(1)} C^{||(1),k} + \nabla^{-2}(8\phi^{(1)} \nabla^2 C^{||(1)} + 2\chi_{kl}^{(1)} C^{||(1),kl} - 2C^{||(1),k} \nabla^2 C_{,k}^{||(1)}) \\
 & + 3\nabla^{-2} \nabla^{-2}(4\phi^{(1),kl} C_{,kl}^{||(1)} - 4\nabla^2 \phi^{(1)} \nabla^2 C^{||(1)} - \chi_{kl}^{(1),klm} C_{,m}^{||(1)} - 3\chi_{kl,m}^{(1)} C^{||(1),klm} \\
 & - 4\chi_{kl,m}^{(1),k} C^{||(1),lm} - 2\chi_{kl}^{(1),k} \nabla^2 C^{||(1),l} - 2\chi_{kl}^{(1)} \nabla^2 C^{||(1),kl} + \nabla^2 C^{||(1),k} \nabla^2 C_{,k}^{||(1)} - C^{||(1),klm} C_{,klm}^{||(1)})] \\
 & - 4\nabla^{-2} \left[\nabla^2 A^{(1)} \int^\tau \frac{2\phi^{(1)}(\tau', \mathbf{x})}{a(\tau')} d\tau' + A_{,k}^{(1)} \int^\tau \frac{2\phi^{(1)}(\tau', \mathbf{x})^k}{a(\tau')} d\tau' - A^{(1),kl} \int^\tau \frac{\chi_{kl}^{(1)}(\tau', \mathbf{x})}{a(\tau')} d\tau' \right. \\
 & \left. - A^{(1),k} \int^\tau \frac{\chi_{km}^{(1)}(\tau', \mathbf{x})^m}{a(\tau')} d\tau' \right] - 2A^{(2)} \int^\tau \frac{d\tau'}{a(\tau')} - 2C^{||(2)}, \tag{C56}
 \end{aligned}$$

$$\begin{aligned}
 \bar{\chi}_{ij}^{\perp(2)} = & \chi_{ij}^{\perp(2)} + \frac{1}{a^2} [-2\partial_i \nabla^{-2}(A_j^{(1)} \nabla^2 A^{(1)}) + \partial_i \partial_j \nabla^{-2}(A^{(1),k} A_{,k}^{(1)}) - 2\partial_i \partial_j \nabla^{-2} \nabla^{-2}(A^{(1),kl} A_{,kl}^{(1)} - \nabla^2 A^{(1)} \nabla^2 A^{(1)})] \\
 & + \frac{4a'}{a^2} \left[\int^\tau \frac{d\tau'}{a(\tau')} \right] [-2\partial_i \nabla^{-2}(A_j^{(1)} \nabla^2 A^{(1)}) + \partial_i \partial_j \nabla^{-2}(A^{(1),k} A_{,k}^{(1)}) - 2\partial_i \partial_j \nabla^{-2} \nabla^{-2}(A^{(1),kl} A_{,kl}^{(1)} - \nabla^2 A^{(1)} \nabla^2 A^{(1)})] \\
 & + \frac{2a'}{a^2} [2\partial_i \nabla^{-2}(2A^{(1),k} C_{,kj}^{||(1)} - 2A_j^{(1)} \nabla^2 C^{||(1)} - \chi_{kj}^{(1),k} A^{(1)} - \chi_{kj}^{(1)} A^{(1),k}) \\
 & + 2\partial_i \partial_j \nabla^{-2} \nabla^{-2}(2\nabla^2 A^{(1)} \nabla^2 C^{||(1)} - 2A^{(1),kl} C_{,kl}^{||(1)} + \chi_{kl}^{(1),kl} A^{(1)} + \chi_{kl}^{(1)} A^{(1),kl} + 2\chi_{kl}^{(1),k} A^{(1),l})] \\
 & - \frac{1}{a} [2\partial_i \nabla^{-2}(\chi_{kj}^{(1),k} A^{(1)} + \chi_{kj}^{(1)'} A^{(1),k}) - 2\partial_i \partial_j \nabla^{-2} \nabla^{-2}(\chi_{kl}^{(1),kl} A^{(1)} + \chi_{kl}^{(1)'} A^{(1),kl} + 2\chi_{kl}^{(1),k} A^{(1),l})] \\
 & + \left[\int^\tau \frac{d\tau'}{a(\tau')} \right]^2 [2\partial_i \nabla^{-2}(A_{,kj}^{(1)} \nabla^2 A^{(1),k}) - \partial_i \partial_j \nabla^{-2}(A^{(1),km} A_{,km}^{(1)}) + 2\partial_i \partial_j \nabla^{-2} \nabla^{-2}(A^{(1),klm} A_{,klm}^{(1)} - \nabla^2 A^{(1),k} \nabla^2 A_{,k}^{(1)})] \\
 & - \left[\int^\tau \frac{d\tau'}{a(\tau')} \right] [2\partial_i \nabla^{-2}(4\phi_j^{(1)} \nabla^2 A^{(1)} - 4\phi^{(1),k} A_{,kj}^{(1)} + \chi_{kj,m}^{(1),k} A^{(1),m} + 2\chi_{kj,m}^{(1)} A^{(1),km} + \chi_{kl}^{(1),l} A_j^{(1),k} + \chi_{kl}^{(1)} A_j^{(1),kl} \\
 & + \chi_{kj}^{(1)} \nabla^2 A^{(1),k} + 2A^{(1),km} C_{,kmj}^{||(1)} - 2A_{,kj}^{(1)} \nabla^2 C^{||(1),k}) - 2\partial_i \partial_j \nabla^{-2} \nabla^{-2}(4\nabla^2 \phi^{(1)} \nabla^2 A^{(1)} - 4\phi^{(1),kl} A_{,kl}^{(1)} + \chi_{kl,m}^{(1),k} A^{(1),m} \\
 & + 3\chi_{kl,m}^{(1)} A^{(1),klm} + 4\chi_{kl,m}^{(1),k} A^{(1),lm} + 2\chi_{kl}^{(1),k} \nabla^2 A^{(1),l} + 2\chi_{kl}^{(1)} \nabla^2 A^{(1),kl} + 2A^{(1),klm} C_{,klm}^{||(1)} - 2\nabla^2 A^{(1),k} \nabla^2 C_{,k}^{||(1)})] \\
 & + [-2\partial_i \nabla^{-2}(4\phi_j^{(1)} \nabla^2 C^{||(1)} - 4\phi^{(1),k} C_{,kj}^{||(1)} + \chi_{kj,m}^{(1),k} C^{||(1),m} + 2\chi_{kj,m}^{(1)} C^{||(1),km} + \chi_{kl}^{(1),l} C_j^{||(1),k} + \chi_{kl}^{(1)} C_j^{||(1),kl} \\
 & + \chi_{kj}^{(1)} \nabla^2 C^{||(1),k} + C^{||(1),km} C_{,kmj}^{||(1)} - C_{,kj}^{||(1)} \nabla^2 C^{||(1),k}) + 2\partial_i \partial_j \nabla^{-2} \nabla^{-2}(4\nabla^2 \phi^{(1)} \nabla^2 C^{||(1)} - 4\phi^{(1),kl} C_{,kl}^{||(1)} \\
 & + \chi_{kl,m}^{(1),kl} C^{||(1),m} + 4\chi_{kl,m}^{(1),k} C^{||(1),lm} + 3\chi_{kl,m}^{(1)} C^{||(1),klm} + 2\chi_{kl}^{(1),k} \nabla^2 C^{||(1),l} + 2\chi_{kl}^{(1)} \nabla^2 C^{||(1),kl} + C^{||(1),klm} C_{,klm}^{||(1)} \\
 & - \nabla^2 C^{||(1),k} \nabla^2 C_{,k}^{||(1)})] + \partial_i \left[A^{(1),k} \int^\tau \frac{2\chi_{kj}^{(1)}(\tau', \mathbf{x})}{a(\tau')} d\tau' - A_j^{(1)} \int^\tau \frac{4\phi^{(1)}(\tau', \mathbf{x})}{a(\tau')} d\tau' \right] \\
 & + \partial_i \partial_j \nabla^{-2} \left[\nabla^2 A^{(1)} \int^\tau \frac{4\phi^{(1)}(\tau', \mathbf{x})}{a(\tau')} d\tau' + A_{,k}^{(1)} \int^\tau \frac{4\phi^{(1)}(\tau', \mathbf{x})^k}{a(\tau')} d\tau' - A^{(1),kl} \int^\tau \frac{2\chi_{kl}^{(1)}(\tau', \mathbf{x})}{a(\tau')} d\tau' \right. \\
 & \left. - A^{(1),k} \int^\tau \frac{2\chi_{km}^{(1)}(\tau', \mathbf{x})^m}{a(\tau')} d\tau' \right] - C_{i,j}^{\perp(2)} + (i \leftrightarrow j), \tag{C57}
 \end{aligned}$$

showing that transformation of $\chi_{ij}^{\perp(2)}$ depends on $\xi^{(2)}$ only through $C_{(i,j)}^{\perp(2)}$,

$$\begin{aligned}
\bar{\chi}_{ij}^{\top(2)} = & \chi_{ij}^{\top(2)} + \left[\frac{1}{a^2} \right] [\delta_{ij} \nabla^{-2} (A^{(1),kl} A_{,kl}^{(1)} - \nabla^2 A^{(1)} \nabla^2 A^{(1)}) + 4 \nabla^{-2} (A_{,ij}^{(1)} \nabla^2 A^{(1)} - A_{,i}^{(1),k} A_{,kj}^{(1)}) \\
& + \partial_i \partial_j \nabla^{-2} \nabla^{-2} (A^{(1),kl} A_{,kl}^{(1)} - \nabla^2 A^{(1)} \nabla^2 A^{(1)})] + \frac{4a'}{a^2} \left[\int^\tau \frac{d\tau'}{a(\tau')} \right] [\delta_{ij} \nabla^{-2} (A^{(1),kl} A_{,kl}^{(1)} \\
& - \nabla^2 A^{(1)} \nabla^2 A^{(1)}) + 4 \nabla^{-2} (A_{,ij}^{(1)} \nabla^2 A^{(1)} - A_{,i}^{(1),k} A_{,kj}^{(1)}) + \partial_i \partial_j \nabla^{-2} \nabla^{-2} (A^{(1),kl} A_{,kl}^{(1)} \\
& - \nabla^2 A^{(1)} \nabla^2 A^{(1)})] + \left[\frac{2a'}{a^2} \right] [-\delta_{ij} \nabla^{-2} (\chi_{kl}^{(1),kl} A^{(1)} + \chi_{kl}^{(1)} A^{(1),kl} + 2\chi_{kl}^{(1),k} A^{(1),l} \\
& + 2\nabla^2 A^{(1)} \nabla^2 C^{\parallel(1)} - 2A^{(1),kl} C_{,kl}^{\parallel(1)}) - 2\chi_{ij}^{(1)} A^{(1)} + 2\partial_i \nabla^{-2} (\chi_{kj}^{(1),k} A^{(1)} + \chi_{kj}^{(1)} A^{(1),k}) \\
& + 2\partial_j \nabla^{-2} (\chi_{ki}^{(1),k} A^{(1)} + \chi_{ki}^{(1)} A^{(1),k}) + 4\nabla^{-2} (A_{,ij}^{(1)} \nabla^2 C^{\parallel(1)} + C_{,ij}^{\parallel(1)} \nabla^2 A^{(1)} - A_{,i}^{(1),k} C_{,kj}^{\parallel(1)} \\
& - A_{,j}^{(1),k} C_{,ki}^{\parallel(1)}) - \partial_i \partial_j \nabla^{-2} \nabla^{-2} (\chi_{kl}^{(1),kl} A^{(1)} + \chi_{kl}^{(1)} A^{(1),kl} + 2\chi_{kl}^{(1),k} A^{(1),l} + 2A^{(1),kl} C_{,kl}^{\parallel(1)} \\
& - 2\nabla^2 A^{(1)} \nabla^2 C^{\parallel(1)})] - \left[\frac{1}{a} \right] [\delta_{ij} \nabla^{-2} (\chi_{kl}^{(1)'} A^{(1)} + \chi_{kl}^{(1)'} A^{(1),kl} + 2\chi_{kl}^{(1)'} A^{(1),l}) \\
& + 2\chi_{ij}^{(1)'} A^{(1)} - 2\partial_i \nabla^{-2} (\chi_{kj}^{(1)'} A^{(1)} + \chi_{kj}^{(1)'} A^{(1),k}) - 2\partial_j \nabla^{-2} (\chi_{ki}^{(1)'} A^{(1)} + \chi_{ki}^{(1)'} A^{(1),k}) \\
& + \partial_i \partial_j \nabla^{-2} \nabla^{-2} (\chi_{kl}^{(1)'} A^{(1)} + \chi_{kl}^{(1)'} A^{(1),kl} + 2\chi_{kl}^{(1)'} A^{(1),l})] - \left[\int^\tau \frac{d\tau'}{a(\tau')} \right]^2 [\delta_{ij} \nabla^{-2} (A^{(1),klm} A_{,klm}^{(1)} \\
& - \nabla^2 A^{(1),k} \nabla^2 A_{,k}^{(1)}) + 2\nabla^{-2} (-2A_{,i}^{(1),kl} A_{,klj}^{(1)} + 2A_{,kij}^{(1)} \nabla^2 A^{(1),k}) + \partial_i \partial_j \nabla^{-2} \nabla^{-2} (A^{(1),klm} A_{,klm}^{(1)} \\
& - \nabla^2 A^{(1),k} \nabla^2 A_{,k}^{(1)})] + \left[\int^\tau \frac{d\tau'}{a(\tau')} \right] [\delta_{ij} \nabla^{-2} (4\phi^{(1),kl} A_{,kl}^{(1)} - 4\nabla^2 \phi^{(1)} \nabla^2 A^{(1)} - \chi_{kl,m}^{(1)} A^{(1),m} \\
& - 4\chi_{kl,m}^{(1),k} A^{(1),lm} + \chi_{kl,m}^{(1)} A^{(1),klm} - 2\chi_{kl}^{(1),k} \nabla^2 A^{(1),l} + 2A^{(1),kl} \nabla^2 \chi_{kl}^{(1)} - 2A^{(1),klm} C_{,klm}^{\parallel(1)} \\
& + 2\nabla^2 A^{(1),k} \nabla^2 C_{,k}^{\parallel(1)}) - 2\chi_{ij,k}^{(1)} A^{(1),k} - 2\chi_{ki}^{(1)} A_{,j}^{(1),k} - 2\chi_{kj}^{(1)} A_{,i}^{(1),k} - 4\nabla^{-2} (2\phi_{,i}^{(1),k} A_{,kj}^{(1)} + 2\phi_{,j}^{(1),k} A_{,ki}^{(1)} \\
& - 2\phi_{,ij}^{(1)} \nabla^2 A^{(1)} - 2A_{,ij}^{(1)} \nabla^2 \phi^{(1)} - A_{,i}^{(1),kl} C_{,klj}^{\parallel(1)} - A_{,j}^{(1),kl} C_{,kli}^{\parallel(1)} + A_{,kij}^{(1)} \nabla^2 C^{\parallel(1),k} + C_{,kij}^{\parallel(1)} \nabla^2 A^{(1),k}) \\
& + 2\partial_i \nabla^{-2} (\chi_{kj,l}^{(1),k} A^{(1),l} + 2\chi_{kj,l}^{(1)} A^{(1),kl} + \chi_{kl}^{(1),l} A_{,j}^{(1),k} + \chi_{kl}^{(1)} A_{,j}^{(1),kl} + \chi_{kj}^{(1)} \nabla^2 A^{(1),k}) \\
& + 2\partial_j \nabla^{-2} (\chi_{ki,l}^{(1),k} A^{(1),l} + 2\chi_{ki,l}^{(1)} A^{(1),kl} + \chi_{kl}^{(1),l} A_{,i}^{(1),k} + \chi_{kl}^{(1)} A_{,i}^{(1),kl} + \chi_{ki}^{(1)} \nabla^2 A^{(1),k}) \\
& + \partial_i \partial_j \nabla^{-2} \nabla^{-2} (4\phi^{(1),kl} A_{,kl}^{(1)} - 4\nabla^2 \phi^{(1)} \nabla^2 A^{(1)} - \chi_{kl,m}^{(1)} A^{(1),m} - 4\chi_{kl,m}^{(1),k} A^{(1),lm} - 7\chi_{kl,m}^{(1)} A^{(1),klm} \\
& - 2\chi_{kl}^{(1),k} \nabla^2 A^{(1),l} - 4\chi_{kl}^{(1)} \nabla^2 A^{(1),kl} - 2A^{(1),kl} \nabla^2 \chi_{kl}^{(1)} - 2A^{(1),klm} C_{,klm}^{\parallel(1)} + 2\nabla^2 A^{(1),k} \nabla^2 C_{,k}^{\parallel(1)})] \\
& + [\delta_{ij} \nabla^{-2} (4\phi^{(1),kl} C_{,kl}^{\parallel(1)} - 4\nabla^2 \phi^{(1)} \nabla^2 C^{\parallel(1)} - \chi_{kl,m}^{(1)} C^{\parallel(1),m} - 4\chi_{kl,m}^{(1),k} C^{\parallel(1),lm} + \chi_{kl,m}^{(1)} C^{\parallel(1),klm} \\
& - 2\chi_{kl}^{(1),k} \nabla^2 C^{\parallel(1),l} + 2C^{\parallel(1),kl} \nabla^2 \chi_{kl}^{(1)} + \nabla^2 C^{\parallel(1),k} \nabla^2 C_{,k}^{\parallel(1)} - C^{\parallel(1),klm} C_{,klm}^{\parallel(1)} - 2\chi_{ij,k}^{(1)} C^{\parallel(1),k} \\
& - 2\chi_{ki}^{(1)} C_{,j}^{\parallel(1),k} - 2\chi_{kj}^{(1)} C_{,i}^{\parallel(1),k} - 4\nabla^{-2} (2\phi_{,i}^{(1),k} C_{,kj}^{\parallel(1)} + 2\phi_{,j}^{(1),k} C_{,ki}^{\parallel(1)} - 2\phi_{,ij}^{(1)} \nabla^2 C^{\parallel(1)} - 2C_{,ij}^{\parallel(1)} \nabla^2 \phi^{(1)} \\
& - C_{,i}^{\parallel(1),kl} C_{,klj}^{\parallel(1)} + C_{,kij}^{\parallel(1)} \nabla^2 C^{\parallel(1),k}) + 2\partial_i \nabla^{-2} (\chi_{kj,l}^{(1),k} C^{\parallel(1),l} + 2\chi_{kj,l}^{(1)} C^{\parallel(1),kl} + \chi_{kl}^{(1),l} C_{,j}^{\parallel(1),k} \\
& + \chi_{kl}^{(1)} C_{,j}^{\parallel(1),kl} + \chi_{kj}^{(1)} \nabla^2 C^{\parallel(1),k}) + 2\partial_j \nabla^{-2} (\chi_{ki,l}^{(1),k} C^{\parallel(1),l} + 2\chi_{ki,l}^{(1)} C^{\parallel(1),kl} + \chi_{kl}^{(1),l} C_{,i}^{\parallel(1),k} \\
& + \chi_{kl}^{(1)} C_{,i}^{\parallel(1),kl} + \chi_{ki}^{(1)} \nabla^2 C^{\parallel(1),k}) - \partial_i \partial_j \nabla^{-2} \nabla^{-2} (4\nabla^2 \phi^{(1)} \nabla^2 C^{\parallel(1)} - 4\phi^{(1),kl} C_{,kl}^{\parallel(1)} + \chi_{kl,m}^{(1),kl} C^{\parallel(1),m} \\
& + 4\chi_{kl,m}^{(1),k} C^{\parallel(1),lm} + 7\chi_{kl,m}^{(1)} C^{\parallel(1),klm} + 2\chi_{kl}^{(1),k} \nabla^2 C^{\parallel(1),l} + 4\chi_{kl}^{(1)} \nabla^2 C^{\parallel(1),kl} + 2C^{\parallel(1),kl} \nabla^2 \chi_{kl}^{(1)} \\
& + C^{\parallel(1),klm} C_{,klm}^{\parallel(1)} - \nabla^2 C^{\parallel(1),k} \nabla^2 C_{,k}^{\parallel(1)})]. \tag{C58}
\end{aligned}$$

Eq. (C58) tells that transformation of $\chi_{ij}^{\top(2)}$ involves $\xi^{(1)}$ only, not $\xi^{(2)}$. The above formulas are valid for a general scale factor $a(\tau)$. Note that, in (C56), (C57), and (C58), $\chi_{ij}^{(1)}$ contains the tensor $\chi_{ij}^{\top(1)}$, which belongs to the scalar-tensor coupling and will not be considered in this paper.

APPENDIX D: GAUGE TRANSFORMATIONS FROM SYNCHRONOUS TO POISSON

The perturbed metric in the Poisson gauge up to 2nd order is generally written as [10,51]

$$g_{00} = -a^2[1 + 2\psi_P^{(1)} + \psi_P^{(2)}], \quad (\text{D1})$$

$$g_{0i} = a^2 \left[w_{Pi}^{(1)} + \frac{1}{2} w_{Pi}^{(2)} \right], \quad (\text{D2})$$

$$g_{ij} = a^2 \left[\delta_{ij} - 2 \left(\phi_P^{(1)} + \frac{1}{2} \phi_P^{(2)} \right) \delta_{ij} + \chi_{Pij}^{\top(1)} + \frac{1}{2} \chi_{Pij}^{\top(2)} \right], \quad (\text{D3})$$

where the vector (shift) is transverse

$$\partial^i w_{Pi}^{(A)} = 0, \quad A = 1, 2, \quad (\text{D4})$$

and the tensor is

$$\chi_{Pi}^{\top(A)i} = 0, \quad \partial^i \chi_{Pij}^{\top(A)} = 0. \quad (\text{D5})$$

Consider transformations of the metric perturbations from a synchronous to a Poisson coordinate. Given the 1st-order solutions $\phi^{(1)}$, $D_{ij}\chi^{\parallel(1)}$, $\chi_{ij}^{\top(1)}$ in synchronous gauge without vector mode, one gets, by the formula (C4), the 1st-order perturbation in Poisson gauge as the following [51]

$$\psi_P^{(1)} = -\alpha^{(1)'} - \frac{a'}{a} \alpha^{(1)}, \quad (\text{D6})$$

$$w_{Pi}^{(1)} = \alpha_{,i}^{(1)} - \beta_{,i}^{(1)'} - d_i^{(1)'}, \quad (\text{D7})$$

$$\phi_P^{(1)} = \phi^{(1)} + \frac{1}{3} \nabla^2 \beta^{(1)} + \frac{a'}{a} \alpha^{(1)}, \quad (\text{D8})$$

$$\chi_{Pij}^{\top(1)} = D_{ij}\chi^{\parallel(1)} + \chi_{ij}^{\top(1)} - 2D_{ij}\beta^{(1)} - d_{i,j}^{(1)} - d_{j,i}^{(1)}. \quad (\text{D9})$$

By the conditions (D4) and (D5) and the solution $D_{ij}\chi^{\parallel(1)}$ of (15), one gets the 1st-order vector field of transformation

$$\begin{aligned} \alpha^{(1)} &= -\frac{\tau}{3} \varphi + \frac{9}{\tau^4} \nabla^{-2} X, \\ \beta^{(1)} &= -\frac{\tau^2}{6} \varphi - \frac{3}{\tau^3} \nabla^{-2} X, \quad d_i^{(1)} = 0. \end{aligned} \quad (\text{D10})$$

Substituting (D10) into (D6)–(D9), one obtains the 1st-order perturbations in Poisson gauge [51]

$$\begin{aligned} \psi_P^{(1)} &= \phi_P^{(1)} = \varphi + \frac{18}{\tau^5} \nabla^{-2} X, \quad w_{Pi}^{(1)} = 0, \\ \chi_{Pij}^{\top(1)} &= \chi_{ij}^{\top(1)}, \end{aligned} \quad (\text{D11})$$

which tells that the two scalars are equal and contain decaying modes, the vector is absent, and the tensor is equal to that in synchronous gauge.

Next is the 2nd-order transformation. By the formula (C5), keeping only the scalar-scalar coupling, using the solutions $\phi^{(1)}$ of (14), $\beta^{(1)}$ of (D10), $\chi_S^{\parallel(2)}$ of (63), $\chi_{Sij}^{\perp(2)}$ of (63), after lengthy calculations, one obtains

$$\begin{aligned} \alpha^{(2)} &= \frac{\tau}{10} \nabla^{-2} F - \frac{2\tau}{9} [5\varphi\varphi - 6\nabla^{-2}(\varphi_{,k}\varphi^{,k}) + 9\nabla^{-2}\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi)] - \frac{\tau^3}{21} [\nabla^{-2}(\nabla^2\varphi\nabla^2\varphi - \varphi^{,kl}\varphi_{,kl})] \\ &+ \frac{162}{\tau^9} [-3\nabla^{-2}X\nabla^{-2}X + \nabla^{-2}(5\nabla^{-2}X_{,k}\nabla^{-2}X^{,k}) + \nabla^{-2}\nabla^{-2}(6\nabla^2X\nabla^{-2}X + 6X_{,k}\nabla^{-2}X^{,k})] \\ &+ \frac{3}{2\tau^7} [2\nabla^{-2}X_{,k}\nabla^{-2}X^{,k} + \nabla^{-2}(9X^2 - 9\nabla^{-2}X^{,kl}\nabla^{-2}X_{,kl})] \\ &+ \frac{3}{\tau^4} [3\nabla^{-2}Z + \nabla^{-2}\nabla^{-2}(5\nabla^{-2}X\nabla^2\nabla^2\varphi - 5\varphi\nabla^2X - 6X\nabla^2\varphi + 6\varphi_{,k}X^{,k} - 4\nabla^2\varphi^{,k}\nabla^{-2}X_{,k} + 8\varphi^{,kl}\nabla^{-2}X_{,kl})] \\ &+ \frac{1}{2\tau^2} [5\varphi^{,k}\nabla^{-2}X_{,k} + \nabla^{-2}(-2X\nabla^2\varphi - 8\varphi^{,kl}\nabla^{-2}X_{,kl} - 10\nabla^2\varphi^{,k}\nabla^{-2}X_{,k})], \end{aligned} \quad (\text{D12})$$

$$\begin{aligned}
\beta^{(2)} = & \frac{1}{2} \nabla^{-2} \nabla^{-2} A - \frac{\tau^2}{6} \left[-\frac{3}{10} \nabla^{-2} F + 7\varphi\varphi - 4\nabla^{-2}(\varphi_{,k}\varphi^{,k}) + 6\nabla^{-2}\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) \right] \\
& - \frac{\tau^4}{504} [7\varphi^{,k}\varphi_{,k} + 6\nabla^{-2}(\nabla^2\varphi\nabla^2\varphi - \varphi^{,kl}\varphi_{,kl})] \\
& + \frac{81}{4\tau^8} [\nabla^{-2}X\nabla^{-2}X + \nabla^{-2}(-5\nabla^{-2}X_{,k}\nabla^{-2}X^{,k}) + \nabla^{-2}\nabla^{-2}(-6\nabla^2X\nabla^{-2}X - 6X_{,k}\nabla^{-2}X^{,k})] \\
& + \frac{9}{4\tau^6} [\nabla^{-2}(-4X_{,k}\nabla^{-2}X^{,k} - X^2) + \nabla^{-2}\nabla^{-2}(-6X_{,kl}\nabla^{-2}X^{,kl} - 6\nabla^{-2}X^{,klm}\nabla^{-2}X_{,klm})] \\
& + \frac{1}{\tau^3} [\nabla^{-2}(3\nabla^{-2}X\nabla^2\varphi - 3Z) + \nabla^{-2}\nabla^{-2}(-17\varphi\nabla^2X - 11X\nabla^2\varphi - 64\varphi_{,k}X^{,k} - 36\varphi^{,kl}\nabla^{-2}X_{,kl})] \\
& + \frac{1}{2\tau} [-\varphi^{,k}\nabla^{-2}X_{,k} + \nabla^{-2}(2X\nabla^2\varphi - 2\varphi_{,kl}\nabla^{-2}X^{,kl})], \tag{D13}
\end{aligned}$$

$$\begin{aligned}
d_i^{(2)} = & \nabla^{-2}G_i - \frac{4\tau^2}{3} \nabla^{-2}[-\varphi_{,i}\nabla^2\varphi + \varphi^{,k}\varphi_{,ki} + \partial_i\nabla^{-2}(\nabla^2\varphi\nabla^2\varphi - \varphi_{,kl}\varphi^{,kl})] \\
& + \frac{162}{\tau^8} \nabla^{-2}[-X_{,i}\nabla^{-2}X + \partial_i\nabla^{-2}(\nabla^2X\nabla^{-2}X + X_{,k}\nabla^{-2}X^{,k})] \\
& + \frac{2}{\tau^3} \nabla^{-2}[7\varphi_{,i}X + 15\varphi_{,ki}\nabla^{-2}X^{,k} - 15\varphi^{,k}\nabla^{-2}X_{,ik} - 3\nabla^{-2}X_{,i}\nabla^2\varphi + \partial_i\nabla^{-2}(-4X\nabla^2\varphi + 8X_{,k}\varphi^{,k} - 12\nabla^2\varphi_{,k}\nabla^{-2}X^{,k})] \\
& + \frac{1}{\tau} \nabla^{-2}[5\nabla^2\varphi^{,k}\nabla^{-2}X_{,ik} - 5\varphi_{,ik}X^{,k} + 5\varphi^{,kl}\nabla^{-2}X_{,kli} - 5\varphi_{,kli}\nabla^{-2}X^{,kl}], \tag{D14}
\end{aligned}$$

where F , Z , A , G_i are given by (53), (55), (57), and (61), all depending on the initial values at τ_0 , and the 2nd metric perturbations in Poisson gauge are

$$\begin{aligned}
\psi_P^{(2)} = & -\frac{3}{10} \nabla^{-2} F + \left[\frac{16}{3} \varphi\varphi - 4\nabla^{-2}(\varphi_{,k}\varphi^{,k}) + 6\nabla^{-2}\nabla^{-2}(\varphi_{,kl}\varphi^{,kl} - \nabla^2\varphi\nabla^2\varphi) \right] + \tau^2 \left[\frac{1}{6} \varphi^{,k}\varphi_{,k} + \frac{5}{21} \nabla^{-2}(\nabla^2\varphi\nabla^2\varphi - \varphi^{,kl}\varphi_{,kl}) \right] \\
& + \frac{162}{\tau^{10}} [-12\nabla^{-2}X\nabla^{-2}X + \nabla^{-2}(35\nabla^{-2}X_{,k}\nabla^{-2}X^{,k}) + \nabla^{-2}\nabla^{-2}(42\nabla^2X\nabla^{-2}X + 42X_{,k}\nabla^{-2}X^{,k})] \\
& + \frac{3}{2\tau^8} [46\nabla^{-2}X_{,k}\nabla^{-2}X^{,k} + \nabla^{-2}(45X^2 - 45\nabla^{-2}X^{,kl}\nabla^{-2}X_{,kl})] \\
& + \frac{6}{\tau^5} [3\nabla^{-2}Z + 7\varphi\nabla^{-2}X + \nabla^{-2}\nabla^{-2}(5\nabla^{-2}X\nabla^2\nabla^2\varphi - 5\varphi\nabla^2X - 6X\nabla^2\varphi + 6\varphi_{,k}X^{,k} - 4\nabla^2\varphi^{,k}\nabla^{-2}X_{,k} + 8\varphi^{,kl}\nabla^{-2}X_{,kl})] \\
& + \frac{6}{\tau^3} \varphi^{,k}\nabla^{-2}X_{,k}, \tag{D15}
\end{aligned}$$

$$\begin{aligned}
\phi_P^{(2)} = & \left[\frac{3}{5} \phi_{S0}^{(2)} + \frac{1}{10} \nabla^2 \chi_{S0}^{(2)} + \frac{\tau_0^2}{6} \varphi_{,k}\varphi^{,k} + \frac{\tau_0^2}{6} \nabla^{-2}(\nabla^2\varphi\nabla^2\varphi - \varphi^{,kl}\varphi_{,kl}) - \frac{\tau_0^4}{120} \nabla^{-2}(\nabla^2\varphi^{,k}\nabla^2\varphi_{,k} - \varphi^{,klm}\varphi_{,klm}) \right] \\
& + \left[\frac{4}{3} \varphi\varphi + \frac{8}{3} \nabla^{-2}(\varphi_{,k}\varphi^{,k}) + 4\nabla^{-2}\nabla^{-2}(\nabla^2\varphi\nabla^2\varphi - \varphi_{,kl}\varphi^{,kl}) \right] + \tau^2 \left[\frac{1}{6} \varphi_{,k}\varphi^{,k} + \frac{5}{21} \nabla^{-2}(\nabla^2\varphi\nabla^2\varphi - \varphi_{,kl}\varphi^{,kl}) \right] \\
& + \frac{324}{\tau^{10}} [-\nabla^{-2}(5X\nabla^{-2}X) + \nabla^{-2}\nabla^{-2}(6\nabla^2X\nabla^{-2}X + 6X_{,k}\nabla^{-2}X^{,k})] + \frac{15}{2\tau^8} [8\nabla^{-2}X_{,k}\nabla^{-2}X^{,k} + \nabla^{-2}(9X^2 - 9\nabla^{-2}X^{,kl}\nabla^{-2}X_{,kl})] \\
& + \frac{6}{\tau^5} [3\nabla^{-2}Z - 17\varphi\nabla^{-2}X + \nabla^{-2}\nabla^{-2}(5\nabla^{-2}X\nabla^2\nabla^2\varphi - 5\varphi\nabla^2X - 6X\nabla^2\varphi + 6\varphi_{,k}X^{,k} - 4\nabla^2\varphi^{,k}\nabla^{-2}X_{,k} + 8\varphi^{,kl}\nabla^{-2}X_{,kl})] \\
& + \frac{6}{\tau^3} \varphi^{,k}\nabla^{-2}X_{,k} + \left[\frac{27}{10\tau_0^6} \nabla^{-2}(\nabla^{-2}X^{,klm}\nabla^{-2}X_{,klm} - X^{,k}X_{,k}) + \frac{3}{\tau_0^3} \nabla^{-2}(X\nabla^2\varphi + \varphi^{,kl}\nabla^{-2}X_{,kl} + 2X^{,k}\varphi_{,k}) \right. \\
& \left. + \frac{3}{10\tau_0} \nabla^{-2}(\varphi_{,klm}\nabla^{-2}X^{,klm} - X_{,k}\nabla^2\varphi^{,k}) \right], \tag{D16}
\end{aligned}$$

$$w_{P_i}^{(2)} = \frac{8}{3} \tau \nabla^{-2} [-\varphi_{,i} \nabla^2 \varphi + \varphi^{,k} \varphi_{,ki} + \partial_i \nabla^{-2} (\nabla^2 \varphi \nabla^2 \varphi - \varphi_{,kl} \varphi^{,kl})] + \frac{1296}{\tau^9} \nabla^{-2} [-X_{,i} \nabla^{-2} X + \partial_i \nabla^{-2} (\nabla^2 X \nabla^{-2} X + X_{,k} \nabla^{-2} X^{,k})] + \frac{6}{\tau^4} \nabla^{-2} [-8\varphi_{,i} X + 12 \nabla^{-2} X_{,i} \nabla^2 \varphi + \partial_i \nabla^{-2} (-4X \nabla^2 \varphi + 8X_{,k} \varphi^{,k} - 12 \nabla^2 \varphi_{,k} \nabla^{-2} X^{,k})], \quad (\text{D17})$$

$$\begin{aligned} \chi_{P_{ij}}^{\top(2)} = & \frac{20}{3} \delta_{ij} \nabla^{-2} \nabla^{-2} (\varphi_{,kl} \varphi^{,kl} - \nabla^2 \varphi \nabla^2 \varphi) + \frac{80}{3} \nabla^{-2} \nabla^{-2} (\varphi_{,ij} \nabla^2 \varphi - \varphi_{,i}^{,k} \varphi_{,kj}) + \frac{20}{3} \nabla^{-2} \nabla^{-2} \nabla^{-2} \partial_i \partial_j (\varphi_{,kl} \varphi^{,kl} - \nabla^2 \varphi \nabla^2 \varphi) \\ & + \frac{1}{\tau^8} [\nabla^{-2} (81X^2 - 81 \nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl}) \delta_{ij} - 162 \nabla^{-2} X_{,ij} \nabla^{-2} X + \nabla^{-2} \partial_i (162X_{,j} \nabla^{-2} X) + \nabla^{-2} \partial_j (162X_{,i} \nabla^{-2} X) \\ & + \nabla^{-2} \partial_i \partial_j (81 \nabla^{-2} X_{,k} \nabla^{-2} X^{,k} - 81X \nabla^{-2} X) + \nabla^{-2} \nabla^{-2} \partial_i \partial_j (81 \nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl} - 81 \nabla^2 X \nabla^{-2} X)] \\ & + \frac{9}{2\tau^6} [(X^2 - \nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl}) \delta_{ij} - 4X \nabla^{-2} X_{,ij} + 4 \nabla^{-2} X_{,i}^{,k} \nabla^{-2} X_{,kj} + \nabla^{-2} \partial_i \partial_j (X^2 - \nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl})] \\ & + \frac{9}{8\tau^4} [\delta_{ij} \nabla^2 (X^2 - \nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl}) + \partial_i \partial_j (X^2 - \nabla^{-2} X_{,kl} \nabla^{-2} X^{,kl}) + \nabla^2 (4 \nabla^{-2} X_{,i}^{,k} \nabla^{-2} X_{,kj} - 4X \nabla^{-2} X_{,ij})] \\ & + \frac{1}{(2\pi)^{3/2}} \int d^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \left[\left(\frac{k \cos(k\tau) \bar{B}_{5ij}}{8\tau^3} - \frac{k^3 \cos(k\tau) \bar{B}_{4ij}}{144\tau^3} + \frac{k^2 \sin(k\tau) \bar{B}_{5ij}}{8\tau^2} - \frac{k^4 \sin(k\tau) \bar{B}_{4ij}}{144\tau^2} \right) \left(\int_0^{k\tau} \frac{\sin s}{s} ds \right) \right. \\ & \left. + \left(\frac{k \sin(k\tau) \bar{B}_{5ij}}{8\tau^3} - \frac{k^3 \sin(k\tau) \bar{B}_{4ij}}{144\tau^3} - \frac{k^2 \cos(k\tau) \bar{B}_{5ij}}{8\tau^2} + \frac{k^4 \cos(k\tau) \bar{B}_{4ij}}{144\tau^2} \right) \left(\int_{k\tau}^{+\infty} \frac{\cos s}{s} ds \right) + \sum_{s=+,x} \epsilon_{ij}^s(k) h_k^s(\tau) \right]. \quad (\text{D18}) \end{aligned}$$

The results in (D15)–(D18) contain decaying modes, and extend (6.8) of Ref. [51] to general initial conditions at time τ_0 . The results tell that the scalars $\propto \tau^0, \tau^2, \tau^{-10}, \tau^{-8}, \tau^{-5}, \tau^{-3}$, the vector $\propto \tau^1, \tau^{-9}, \tau^{-4}$, and the tensor $\propto \tau^{-8}, \tau^{-6}, \tau^{-4}, \tau^0$ in Poisson gauge.

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