Curved Witten-Dijkgraaf-Verlinde-Verlinde equation and $\mathcal{N} = 4$ mechanics

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We propose a generalization of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation from $\mathbb{R}^n$ to an arbitrary Riemannian manifold. Its form is obtained by extending the relation of the WDVV equation with $\mathcal{N} = 4$ supersymmetric n-dimensional mechanics from flat to curved space. The resulting "curved WDVV equation" is written in terms of a third-rank Codazzi tensor. For every flat-space WDVV solution subject to a simple constraint, we provide a curved-space solution on any isotropic space, in terms of the rotationally invariant conformal factor of the metric.

I. WDVV EQUATION IN CURVED SPACE

A celebrated mathematical structure common to several areas in geometry and mathematical physics is the (generalized) Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation [1,2]

$$F_{kip}^{(0)} \delta^{pq} F_{qjm}^{(0)} - F_{kjp}^{(0)} \delta^{pq} F_{qim}^{(0)} = 0 \quad \text{with} \quad F_{ijk}^{(0)} = \partial_i \partial_j \partial_k F^{(0)} \tag{1.1}$$

for a real-valued function $F^{(0)}(x)$ of n real variables $x = (x^i) = (x^1, x^2, \ldots, x^n)$ which can be taken as coordinates of Euclidean space $\mathbb{R}^n$. There is considerable mathematics and physics literature on solutions to this equation. In this short paper, we propose a generalization of this equation to any Riemannian manifold $\mathcal{M}$ given by a metric,

$$ds^2 = g_{ik}(x) dx^i dx^k. \tag{1.2}$$

In order to motivate the form of the purported "curved WDVV equation," let us reformulate the flat-space version by defining n $n \times n$ matrices $\hat{F}_i^{(0)}$ and a matrix-valued one-form $\hat{F}^{(0)}$ via

$$(\hat{F}_i^{(0)})^\ell{}_p = \delta^{\ell k} F_{kip}^{(0)} \quad \text{and} \quad \hat{F}^{(0)} = dx^i \hat{F}_i^{(0)}, \tag{1.3}$$

respectively [3]. In these terms, the WDVV equation (1.1) is equivalent to the flatness or nilpotency condition

$$(d \pm \hat{F}^{(0)})^2 = 0 \Leftrightarrow \hat{F} \wedge \hat{F} = 0 \Leftrightarrow [\hat{F}_i^{(0)}, \hat{F}_j^{(0)}] = 0, \tag{1.4}$$

where both signs are allowed.

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This form suggests a natural geometric generalization to curved space, by simply covariantizing with respect to diffeomorphisms,

$$(d + \hat{\Gamma} \pm \hat{F})^2 = 0 \Leftrightarrow (d + \hat{\Gamma})\hat{F} = 0 \quad \text{and} \quad \hat{F} \wedge \hat{F} + \hat{R} = 0$$
$$\Leftrightarrow \nabla_{[i}\hat{F}_{j]} = 0 \quad \text{and} \quad [\hat{F}_i, \hat{F}_j] + \hat{R}_{ij} = 0, \tag{1.5}$$

where

$$(d + \hat{\Gamma})^\ell{}_p = dx^i (\nabla_i)^\ell{}_p = dx^i (\delta^\ell{}_p \partial_i + \Gamma^\ell{}_{pi})$$
$$\text{and} \quad (\hat{R})^\ell{}_m = dx^i \wedge dx^j (\hat{R}_{ij})^\ell{}_m = dx^i \wedge dx^j R^\ell{}_{mij} \tag{1.6}$$

contain the Levi-Cività connection $\hat{\Gamma}$, the general coordinate-covariant derivative ∇_i, and the Riemann curvature tensor $\hat{R}_{ij}$. In components with all indices down, Eq. (1.5) reads

$$\nabla_i F_{kjp} - \nabla_j F_{kip} = 0 \tag{1.7a}$$

$$F_{kip} g^{pq} F_{qjm} - F_{kjp} g^{pq} F_{qim} = -R_{kmij}, \tag{1.7b}$$

where, as usual and with $g^{\ell k} g_{km} = \delta^\ell{}_m$,

$$\Gamma^\ell{}_{pi} = \frac{1}{2} g^{\ell k} (\partial_p g_{ik} + \partial_i g_{pk} - \partial_k g_{pi}),$$
$$R^\ell{}_{pij} = \partial_i \Gamma^\ell{}_{pj} - \partial_j \Gamma^\ell{}_{pi} + \Gamma^\ell{}_{mi} \Gamma^m{}_{pj} - \Gamma^\ell{}_{mj} \Gamma^m{}_{pi},$$
$$\nabla_i F_{jkp} = \partial_i F_{jkp} - \Gamma^m{}_{ij} F_{mkp} - \Gamma^m{}_{ik} F_{jmp} - \Gamma^m{}_{ip} F_{jkm}. \tag{1.8}$$

In general, F_{kip} is no longer the third flat derivative of some function (prepotential) F, but we keep the total symmetry in all three indices,

$$F_{kip} = F_{ikp} = F_{pik}. \tag{1.9}$$

Then, Eq. (1.7a) qualifies (F_{kip}) as a so-called third-rank Codazzi tensor [4], while (1.7b) is our curved-space proposal for the WDVV equation. Note that it is no longer homogeneous in F but depends on the Riemann tensor of the manifold. With a more natural index position, it takes the form

$$F^k_{\ ip}F^p_{\ jm} - F^k_{\ jp}F^p_{\ im} = -R^k_{\ mij}. \qquad (1.10)$$

II. $\mathcal{N} = 4$ SUPERSYMMETRIC MULTIDIMENSIONAL MECHANICS

Our proposal (1.10) can in fact be "derived" from the relation of the WDVV equation with $\mathcal{N} = 4$ supersymmetric classical (and also quantum) mechanics for n non-relativistic identical particles on a real line. The phase space of this system is given by n commuting coordinates x^i and momenta p_k as well as $4n$ anticommuting variables

$$\psi^{ia} \quad \text{and} \quad \bar{\psi}^k_{\ b} = (\psi^{kb})^\dagger \quad \text{with} \quad a, b = 1, 2 \qquad (2.1)$$

and canonical Poisson brackets between x and p as well as between ψ and $\bar{\psi}$. The starting point then is an ansatz for the $\mathcal{N} = 4$ supercharges,

$$Q^a = p_i\psi^{ia} + iF^{(0)}_{ijk}(x)\psi^{ib}\psi^j_{\ b}\bar{\psi}^{ka}$$
$$\text{and} \quad \bar{Q}_a = p_i\bar{\psi}^i_{\ a} + iF^{(0)}_{ijk}(x)\bar{\psi}^i_{\ b}\bar{\psi}^{jb}\psi^k_{\ a}, \qquad (2.2)$$

with real structure functions $F^{(0)}_{ink}$ symmetric in the first two indices. Imposing the supersymmetry algebra

$$\{Q^a, Q^b\} = 0 = \{\bar{Q}_a, \bar{Q}_b\}$$
$$\text{and} \quad \{Q^a, \bar{Q}_b\} = \frac{i}{2}\delta^a_{\ b}H \qquad (2.3)$$

defines the Hamiltonian

$$H = \delta^{ij}p_ip_j - 2(\partial_iF^{(0)}_{kjp} + \partial_jF^{(0)}_{kip})\psi^{ia}\bar{\psi}^j_{\ a}\psi^{kb}\bar{\psi}^p_{\ b} \qquad (2.4)$$

and constrains the functions $F^{(0)}_{kip}(x)$ to be symmetric in all three indices and to obey the flat WDVV equations (1.1), as was first demonstrated by Wyllard [5] and put into the WDVV context by Bellucci *et al.* [6]. In addition, a further condition,

$$\partial_iF^{(0)}_{kjp} - \partial_jF^{(0)}_{kip} = 0, \qquad (2.5)$$

is resolved by the existence of the prepotential $F^{(0)}$ in (1.1). Hence, the four-fermion coefficient in brackets in the Hamiltonian (2.4) equals $2\partial_i\partial_j\partial_k\partial_pF^{(0)}$.

In this context, the curved-space generalization deforms the canonical Poisson brackets to

$$\{x^i, p_j\} = \delta^i_j, \quad \{\psi^{ia}, \bar{\psi}^j_{\ b}\} = \frac{i}{2}\delta^a_{\ b}g^{ij}, \quad \{p_i, \psi^{aj}\} = \Gamma^j_{ik}\psi^{ka},$$
$$\{p_i, \bar{\psi}^j_{\ a}\} = \Gamma^j_{ik}\bar{\psi}^k_{\ a}, \quad \{p_i, p_j\} = -2iR_{ijkm}\psi^{ka}\bar{\psi}^m_{\ a}, \qquad (2.6)$$

which unambiguously follows from the first two brackets by employing the Jacobi identities. We now use exactly the same ansatz (2.2) for the supercharges, except that we drop the (0) superscript on the structure functions,

$$Q^a = p_i\psi^{ia} + iF_{ijk}(x)\psi^{ib}\psi^j_{\ b}\bar{\psi}^{ka}$$
$$\text{and} \quad \bar{Q}_a = p_i\bar{\psi}^i_{\ a} + iF_{ijk}(x)\bar{\psi}^i_{\ b}\bar{\psi}^{jb}\psi^k_{\ a}. \qquad (2.7)$$

Demanding again the supersymmetry algebra (2.3) once more forces the F_{kip} to be totally symmetric and, using the deformed Poisson brackets (2.6), leads precisely to our curved equations (1.7). The corresponding Hamiltonian acquires the form

$$H = g^{ij}p_ip_j - 2(\nabla_iF_{kjp} + \nabla_jF_{kip} + 2R_{kpij})\psi^{ia}\bar{\psi}^j_{\ a}\psi^{kb}\bar{\psi}^p_{\ b}. \qquad (2.8)$$

Remark: If our manifold $\mathcal{M}$ is of constant curvature, i.e. maximally symmetric, then a third-rank Codazzi tensor is determined by a single prepotential [4],

$$F_{ijk} = \frac{1}{3}(\nabla_i\nabla_j\nabla_kF + \nabla_j\nabla_i\nabla_kF + \nabla_k\nabla_i\nabla_jF)$$
$$+ \frac{4R}{3n(n-1)}(g_{jk}\nabla_iF + g_{ik}\nabla_jF + g_{ij}\nabla_kF)$$
$$= \nabla_i\nabla_j\nabla_kF + \frac{R}{n(n-1)}(2g_{jk}\partial_iF + g_{ik}\partial_jF + g_{ij}\partial_kF)$$

with $R = R^\ell_{\ j\ell m}g^{jm} = \text{const.}$ \qquad (2.9)

Having solved (1.7a) in terms of $F(x)$, the curved WDVV equation (1.7b) yields a rather complicated equation for this prepotential, which we have not investigated further. Instead, we have found large classes of solutions on isotropic (not necessarily homogeneous) spaces, which we shall display in the remainder of this paper.

III. PARTICULAR SOLUTIONS OF CURVED WDVV EQUATION

A. Potential metric

Motivated by the results of Refs. [7,8] on $\mathcal{N} = 4$ n-dimensional supersymmetric mechanics, we consider metrics given by a potential function,

$$g_{ij} = \partial_i\partial_jF(x). \qquad (3.1)$$

For a manifold admitting such a metric, one finds that

$$\Gamma_{ijk} = \frac{1}{2}\partial_i\partial_j\partial_k F$$

$$\text{and} \quad R_{ijkm} = g^{pq}(\Gamma_{imp}\Gamma_{jkq} - \Gamma_{ikp}\Gamma_{jmq}). \quad (3.2)$$

It is then rather easy to see that the choice

$$F_{ijk} = \pm\Gamma_{ijk} \quad (3.3)$$

solves both curved WDVV equations (1.7) for any choice of the potential F.

B. Isotropic

Suppose that the manifold $\mathcal{M}$ is isotropic, i.e. it has $\frac{1}{2}n(n-1)$ Killing vectors and admits an $SO(n)$-invariant metric,

$$ds^2 = g_{ik}(r)dx^i dx^k \quad \text{for } r^2 = \delta_{ij}x^i x^j. \quad (3.4)$$

Such a space is a cone over S^{n-1} and is conformally flat, so we can find coordinates in which the metric takes the form

$$ds^2 = dr^2 + h(r)^2 d\Omega_{n-1}^2 \quad \text{or} \quad ds^2 = f(r)^{-2}\delta_{ik}dx^i dx^k,$$
$$(3.5)$$

where $d\Omega_{n-1}^2$ is the metric on the round $(n-1)$-sphere.

Let us make the following ansatz, inspired by Ref. [9],

$$F_{ijk} = a(r)x^i x^j x^k + b(r)(\delta_{ij}x^k + \delta_{jk}x^i + \delta_{ki}x^j)$$
$$+ cf(r)^{-2}F^{(0)}_{ijk}(x), \quad (3.6)$$

including an arbitrary flat-space solution $F^{(0)}$ (the f^{-2} prefactor arises from pulling down the first index of F). The two functions $a(r)$ and $b(r)$ are to be determined depending on the constant c. One may check that the linear equation (1.7a) is satisfied if

$$r(rf' - f)a + 4f'b + fb' = 0 \quad \text{and} \quad x^i F^{(0)}_{ijk} = \delta_{jk}, \quad (3.7)$$

where we fixed the scale of $F^{(0)}$. Here, the prime means differentiation with respect to r.

The curved WDVV equation (1.7b) further imposes the quadratic conditions

$$f^2 b(r^2 a + b) + ca = -\frac{1}{rf^3}\left(\frac{f'}{r}\right)'$$

$$\text{and} \quad r^2 f^2 b^2 + 2cb = \frac{r^2 f'}{f^4}\left(\frac{f}{r^2}\right)'. \quad (3.8)$$

Interestingly, these equations already imply the condition (3.7), for any value of c, meaning that (1.7a) follows from (1.7b) for our ansatz (3.6). The equations (3.8) may be easily solved as

$$a = \frac{2cf\sqrt{c^2f^2 - 2rff' + r^2(f')^2} \pm (2c^2f^2 - 3rff' + r^2(f')^2 + r^2ff'')}{r^4f^3\sqrt{c^2f^2 - 2rff' + r^2(f')^2}},$$

$$b = -\frac{cf \pm \sqrt{c^2f^2 - 2rff' + r^2(f')^2}}{r^2f^3}. \quad (3.9)$$

For $c = 1$, it simplifies to

$$a = \frac{2f(\frac{f}{r})' \mp r(f(\frac{f}{r})')'}{r^4f^3(\frac{f}{r})'} \quad \text{and} \quad b = -\frac{f \mp r^2(\frac{f}{r})'}{r^2f^3}. \quad (3.10)$$

Thus, for isotropic metrics, any flat-space WDVV solution $F^{(0)}$ may be lifted to a solution of the curved WDVV equation via (3.6).

IV. CONCLUSION

We have employed the relation between n-dimensional $\mathcal{N} = 4$ supersymmetric mechanics and the WDVV equation to generalize the latter to curved spaces, i.e. to arbitrary Riemannian manifolds. In this curved WDVV equation, the third derivative of the prepotential, $F^{(0)}_{ijk} = \partial_i\partial_j\partial_k F^{(0)}$, is replaced by the third-rank Codazzi tensor F_{ijk}, while the WDVV equation itself acquires a nontrivial right-hand side given by the Riemann curvature tensor. We have found

solutions of the curved WDVV equation for metrics with a potential and on arbitrary isotropic spaces. The latter solution is built on an arbitrary solution $F^{(0)}$ of the *flat* WDVV equation subject to $x^i F^{(0)}_{ijk} \sim \delta_{jk}$. Thus, any such flat solution can be lifted to a curved solution on an isotropic space.

Here, we have worked with a restricted form of $\mathcal{N} = 4$ supersymmetric mechanics, with vanishing potential W. So, the obvious reverse application to multidimensional supersymmetric mechanics will extend the supercharges by the potential terms (as it was done in Refs. [5,9,10] for the flat case) and to find admissible potentials. This task will be considered elsewhere.

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