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Exact solution of the Schwarzian theory

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The explicit evaluation of the partition function in the Schwarzian theory is presented.

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The Schwarzian theory [1] is the basic element of various physical models including the SYK model and the two-dimensional dilaton gravity (see, e.g., [2–5], and references therein). The action of the theory is

$$I = -\frac{1}{g^2}\int_0^{2\pi}\left[\mathcal{S}_\phi(\tau) + \frac{1}{2}(\phi')^2(\tau)\right]d\tau. \tag{1}$$

Here,

$$\mathcal{S}_\phi(t) \equiv \left(\frac{\phi''(t)}{\phi'(t)}\right)' - \frac{1}{2}\left(\frac{\phi''(t)}{\phi'(t)}\right)^2 \tag{2}$$

is the Schwarzian derivative, $\phi \in \mathrm{Diff}^3([0,2\pi])$, and $\phi'(0) = \phi'(2\pi)$.

It is convenient to rewrite the action in the form

$$I = -\frac{1}{\sigma^2}\int_0^1 [\mathcal{S}_\varphi(t) + 2\pi^2\dot\varphi^2(t)]dt, \tag{3}$$

where

$$\sigma = \sqrt{2\pi}g, \quad t = \frac{1}{2\pi}\tau, \quad \varphi(t) = \frac{1}{2\pi}\phi(\tau),$$
$$\varphi \in \mathrm{Diff}^3([0,1]), \quad \dot\varphi(0) = \dot\varphi(1).$$

The functional integral for the partition function

$$Z(g) = \int_{\dot\varphi(0)=\dot\varphi(1)} \exp\{-I\}d\varphi \tag{4}$$

diverges [1]. However, as we will see later on [Eq. (17)], the integral

$$Z_\alpha(g) = \int_{\mathrm{Diff}^1([0,1])} \exp\{-I\}\exp\left\{\frac{-2[\pi^2-\alpha^2]}{\sigma^2}\int_0^1 \dot\varphi^2(t)dt\right\}d\varphi$$
$$= \int_{\dot\varphi(0)=\dot\varphi(1)} \exp\left\{\frac{1}{\sigma^2}\int_0^1 [\mathcal{S}_\varphi(t) + 2\alpha^2\dot\varphi^2(t)]dt\right\}d\varphi \tag{5}$$

converges for $0 \le \alpha < \pi$. Therefore, let us evaluate the integral (5) first.

The measure

$$\mu_\sigma(X) = \int_X \exp\left\{\frac{1}{\sigma^2}\int_0^1 \mathcal{S}_\varphi(t)dt\right\}d\varphi \tag{6}$$

is quasi-invariant, and the Radon-Nikodym derivative of the measure is [6,7]

$$\frac{d\mu_\sigma^f}{d\mu_\sigma}(\varphi) = \frac{1}{\sqrt{\dot f(0)\dot f(1)}} \times \exp\left\{\frac{1}{\sigma^2}\left[\frac{\ddot f(0)}{\dot f(0)}\dot\varphi(0) - \frac{\ddot f(1)}{\dot f(1)}\dot\varphi(1)\right] + \frac{1}{\sigma^2}\int_0^1 \mathcal{S}_f(\varphi(t))\dot\varphi^2 dt\right\}, \tag{7}$$

where

$$\mu_\sigma^f(X) = \mu_\sigma(f\circ X).$$

Here, we have used the well-known property of the Schwarzian derivative:

$$\mathcal{S}_{f\circ\varphi}(t) = \mathcal{S}_f(\varphi(t))\dot\varphi^2(t) + \mathcal{S}_\varphi(t), \quad (f\circ\varphi)(t) = f(\varphi(t)).$$

Take the function f to be

$$f(t) = \frac{1}{2}\left[\frac{1}{\tan\frac{\alpha}{2}}\tan\left(\alpha\left(t-\frac{1}{2}\right)\right) + 1\right]. \tag{8}$$

In this case,

$$\mathcal{S}_f(t) = 2\alpha^2, \qquad \dot f(0) = \dot f(1) = \frac{\alpha}{\sin\alpha},$$
$$-\frac{\ddot f(0)}{\dot f(0)} = \frac{\ddot f(1)}{\dot f(1)} = 2\alpha\tan\frac{\alpha}{2}. \tag{9}$$

Now we have the following functional integrals equality:

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$$\frac{\alpha}{\sin\alpha}\int\limits_{\dot\varphi(0)=\dot\varphi(1)} F(\varphi)\mu_\sigma(d\varphi) = \int\limits_{\dot\varphi(0)=\dot\varphi(1)} F(f(\varphi))\exp\left\{-\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\dot\varphi(0)\right\} \times\exp\left\{\frac{1}{\sigma^2}\int\limits_0^1[\mathcal{S}_\varphi(t)+2\alpha^2\dot\varphi^2(t)]dt\right\}d\varphi. \quad (10)$$

The next step is the choice of the function F. Let it be

$$F(f(\varphi)) = \exp\left\{\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\dot\varphi(0)\right\}. \quad (11)$$

To find $F(\varphi)$ from the previous equation, note that for $u(t)=f(\varphi(t))$,

$$\dot\varphi(0)=\frac{1}{\dot f(0)}\dot u(0).$$

Then

$$F(u)=\exp\left\{\frac{8\sin^2\frac{\alpha}{2}}{\sigma^2}\dot u(0)\right\}. \quad (12)$$

Thus for the regularized partition function we have

$$Z_\alpha(g)=\frac{\alpha}{\sin\alpha}\int\limits_{\dot\varphi(0)=\dot\varphi(1)}\exp\left\{\frac{8\sin^2\frac{\alpha}{2}}{\sigma^2}\dot\varphi(0)\right\}\mu_\sigma(d\varphi). \quad (13)$$

Under the substitution

$$\varphi(t)=\frac{\int_0^t\exp\{\xi(\eta)\}d\eta}{\int_0^1\exp\{\xi(\eta)\}d\eta}, \quad (14)$$

the measure $\mu_\sigma(d\varphi)$ turns into the Wiener measure [6,7]

$$w_\sigma(d\xi)=\exp\left\{-\frac{1}{2\sigma^2}\int\limits_0^1\dot\xi^2(t)dt\right\}d\xi. \quad (15)$$

In this case,

$$\xi(t)=\ln\dot\varphi(t)-\ln\dot\varphi(0), \qquad \xi\in C([0,1]), \quad (16)$$

and $\xi(0)=\xi(1)=0$.

Now $Z_\alpha(g)$ is written as

$$Z_\alpha(g)=\frac{\alpha}{\sin\alpha}\times\int\limits_{\xi(0)=\xi(1)=0}\exp\left\{\frac{8\sin^2\frac{\alpha}{2}}{\sigma^2}\frac{1}{\int_0^1\exp\{\xi(\eta)\}d\eta}\right\}w_\sigma(d\xi). \quad (17)$$

The singularity at $\alpha=\pi$ is canceled out in the ratio

$$\frac{Z_\alpha(g)}{Z_\alpha(1)},$$

and we can remove the regularization there

$$\frac{Z(g)}{Z(1)}=\lim_{\alpha\to\pi-0}\frac{Z_\alpha(g)}{Z_\alpha(1)}. \quad (18)$$

To evaluate the integral

$$\int\limits_{\xi(0)=\xi(1)=0}\exp\left\{\frac{8}{\sigma^2}\frac{1}{\int_0^1\exp\{\xi(\eta)\}d\eta}\right\}w_\sigma(d\xi), \quad (19)$$

we use the following equation:

$$\int\limits_{\xi(0)=\xi(1)=0}\exp\left\{\frac{-2\beta^2}{\sigma^2(\beta+1)}\frac{1}{\int_0^1\exp\{\xi(\eta)\}d\eta}\right\}w_\sigma(d\xi) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{4(\ln(\beta+1))^2}{2\sigma^2}\right\}. \quad (20)$$

The proof of the more general formula will be given in the forthcoming paper (also, see [7]). For the integral (19)

$$(\beta+1)=-1.$$

Thus the final result is

$$\frac{Z(g)}{Z(1)}=\frac{\exp\{-\pi\}}{g}\exp\left\{\frac{\pi}{g^2}\right\}. \quad (21)$$

It is interesting to compare the one-loop results for $Z(g)$ in [1] with the Eq. (21). Note that the power of the constant g in the denominator is determined by the number of gauge fixing conditions. The one-loop result for the orbit $\mathrm{Diff}(S^1)/U(1)$ [Eq. (3.45) in [1]] has the same form as our exact result (21).

Unlike its compact subgroup $U(1)$, the group $SL(2,\mathbf{R})$ is noncompact. Therefore, integrating over the quotient space $\mathrm{Diff}^1([0,1])/SL(2,\mathbf{R})$ we get the finite result for the partition function in the Schwarzian theory.

We define the Schwarzian partition function as a limit

$$Z_{\mathrm{Schw}}(g) = \lim_{\alpha\to\pi-0} \frac{Z_\alpha(g)}{V_\alpha(g)}. \tag{22}$$

Here, $Z_\alpha(g)$ is given by the Eq. (5) and Eq. (17), and $V_\alpha(g)$ is the regularized volume of the group $SL(2,\mathbf{R})$

$$V_\alpha(g) = \int\limits_{SL(2,\mathbf{R})} \exp\left\{\frac{-2[\pi^2-\alpha^2]}{\sigma^2}\int\limits_0^1 \dot{\varphi}^2(t)dt\right\} d\mu_H. \tag{23}$$

Note that the functional measure in the Eq. (5) and the Haar measure $d\mu_H$ on the group $SL(2,\mathbf{R})$ in the Eq. (23) are regularized in the same manner.

To perform the integration over the group $SL(2,\mathbf{R})$ in the Eq. (23) we choose the representation [8]

$$\varphi_z(t) = -\frac{i}{2\pi}\ln\frac{e^{i2\pi t}+z}{\bar{z}e^{i2\pi t}+1}, \qquad z=\rho e^{i\theta}, \qquad \rho<1. \tag{24}$$

In this case, the Haar measure is [8]

$$\mu_H(dz) = \frac{4\rho d\rho d\theta}{(1-\rho^2)^2}. \tag{25}$$

The integral

$$\int_0^1 \dot{\varphi}_z^2(t)dt = \int_0^1 \frac{(1-|z|^2)^2 dt}{(e^{i2\pi t}+z)^2(e^{-i2\pi t}+\bar{z})^2} = \frac{1+\rho^2}{1-\rho^2} \tag{26}$$

does not depend on θ. And the regularized volume of the group has the form

$$V_\alpha(g) = \int\limits_0^1 \exp\left\{-\frac{[\pi^2-\alpha^2]}{\pi g^2}\frac{(1+\rho^2)}{(1-\rho^2)}\right\}\frac{8\pi\rho d\rho}{(1-\rho^2)^2} = \exp\left\{-\frac{[\pi^2-\alpha^2]}{\pi g^2}\right\}\frac{2\pi^2 g^2}{[\pi^2-\alpha^2]}. \tag{27}$$

Thus we can evaluate the Schwarzian partition function

$$Z_{Schw}(g) = \frac{1}{2\pi g^3}\exp\left\{\frac{\pi}{g^2}\right\}. \tag{28}$$

Note that the one-loop result in [1,2] has the same form as the exact partition function (28) obtained by the direct functional integration.

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