Zero-modes on orbifolds: Magnetized orbifold models by modular transformation

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We study T^2/Z_N orbifold models with magnetic fluxes. We propose a systematic way to analyze the number of zero-modes and their wave functions by use of modular transformation. Our results are consistent with the previous results, and our approach is more direct and analytical than the previous ones. The index theorem implies that the zero-mode number of the Dirac operator on T^2 is equal to the index M, which corresponds to the magnetic flux in a certain unit. Our results show that the zero-mode number of the Dirac operator on T^2/Z_N is equal to $\lfloor M/N \rfloor + 1$ except one case on the T^2/Z_3 orbifold.

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I. INTRODUCTION

Superstring theory is a promising candidate for unified theory including gravity and leads to six-dimensional space in addition to our four-dimensional (4D) spacetime. Thus, extra dimensional models are well-motivated. Indeed, many studies have been carried out. It is a key point how to derive 4D chiral theory starting from extra dimensional theories, because the standard model is a chiral theory. For example, the toroidal compactification is one of the simplest compactifications, but leads to nonchiral theory. Then, the simple toroidal compactification is not realistic. However, the torus compactification with magnetic fluxes can lead to 4D chiral theory from extra dimensional theories as well as superstring theories [1–4]. In addition, the magnitude of magnetic flux determines the number of zero-modes, which would correspond to the generation number. Also zero-mode profiles are quasilocalized and can lead to suppressed couplings depending on their localized points. Hence, the torus compactification with magnetic fluxes is quite interesting. Indeed, several studies have been done, e.g. on computation of Yukawa couplings [5], higher order couplings [6], non-Abelian flavor symmetries [7,8], massive modes and their phenomenological effects [8–11], etc.[1]

The orbifold models with magnetic fluxes are also interesting. Orbifolding can project out the adjoint matter fields corresponding to open string moduli, which remain massless in the toroidal compactification with magnetic fluxes. The number of zero-modes and their profiles in orbifold models are different from those in toroidal models [14]. Thus, orbifold models with magnetic fluxes have rich structures in model building. Indeed, Z_2 orbifold models have been studied on several aspects, e.g. model building [15–19], realization of quark and lepton masses and their mixing angles and CP phase [20–23]. In addition, it is possible to introduce some degree of freedom on orbifold fixed points, e.g. localized modes and localized operators. That makes phenomenological aspects richer [24–27].

Other Z_N orbifold models with $N = 3$, 4, 6 have been also studied. Zero-mode wave functions were studied by numerical studies [28] and the corresponding states were studied by operator analysis in quantum mechanism [29]. By use of those results, model building and fermion mass matrices were also studied [30–32]. However, the numerical study is not analytical, and results from both approaches were rather complicated. A simpler approach would be useful for further applications.

Here, we study Z_N orbifold models with magnetic fluxes. In particular we study the number of zero-modes and their wave functions directly by using modular transformation. The modular transformation is a geometrical transformation of the lattice which is used to construct T^2. Zero-mode wave functions can be written in terms of theta functions, which have a characteristic behavior under modular transformation. When we fix a value of complex structure properly, certain modular transformation behaves as Z_N twists with $N = 3$, 4, 6. Using such behavior, we can obtain zero-mode wave functions on Z_N orbifolds. For generic values of magnetic flux, we compute the number of zero-modes with each Z_N eigenvalue on T^2/Z_N. We show that the number of Z_N invariant zero-modes is almost universal on different T^2/Z_N orbifolds, and it is equal to $\lfloor M/N \rfloor + 1$ for magnetic flux M in a certain unit except one case in the T^2/Z_3 orbifold, where $\lfloor r \rfloor$ denotes the maximum integer n satisfying $n \leq r$. Alternatively, the number of Z_3 invariant zero-modes is written by $2\lfloor M/(2N) \rfloor + 1$.

This paper is organized as follows. In Sec. II, we review wave functions on the two-dimension torus T^2 with magnetic fluxes as well as the T^2/Z_2 orbifold. In Sec. III, we study the T^2/Z_4 orbifold. In Sec. IV we study the T^2/Z_3 orbifold as well as T^2/Z_6 orbifold. In Sec. V, we give a comment on our universal result on the number of Z_N invariant zero-modes. Section VI is the Conclusion. In Appendix A, we show computations on the normalization

[1] See also [12,13].

factor of zero-mode wave function and inner product of two types of wave functions. Such computations are useful for Secs. III and IV. In Appendix B, we show the computation on products of the Z_3 matrix. In Appendix C, we show explicitly zero-mode wave functions on the T^2/Z_4 orbifold.

II. TORUS MODEL WITH MAGNETIC FLUX

Our starting point is the gauge theory with $2n$ extra dimensions, which are chosen as $(T^2)^n$. Our theory includes the spinor field λ, and its Lagrangian is written by

$$\mathcal{L} = -\frac{1}{4g^2}\mathrm{Tr}F^{MN}F_{FM} - \frac{i}{2g^2}\bar{\lambda}^M D_M \lambda, \tag{1}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$. Here, we set the kinetic term of λ as one in super Yang-Mills theory, because we are motivated from such a theory. For simplicity, we concentrate on U(1) gauge theory with $n = 1$ and spinor field with charge q. Similarly, we can extend our analysis to a non-Abelian gauge theory with $n \geq 1$.

We decompose

$$\lambda(x^\mu, y^m) = \sum_n \eta_n(x^\mu) \otimes \psi_n(y^m), \tag{2}$$

where x^μ denotes coordinates of four-dimensional spacetime, while y^m with $m = 1, 2$ denotes coordinates on T^2. $\psi_n(y^m)$ are eigenfunctions of a Dirac operator on T^2. In what follows, we concentrate on the zero-modes, $\psi_0(y)$, which correspond to massless modes in 4D effective field theory, and we denote them by $\psi(y)$.

A. Magnetized torus models

Here, we give a review on the T^2 model with a magnetic flux, in particular zero-mode wave functions [5]. We use the complex coordinate $z = y^1 + \tau y^2$ instead of the real coordinates, y^1 and y^2, where τ is a complex, and the metric is given as $ds^2 = g_{\alpha\beta} dz^\alpha d\bar{z}^\beta$,

$$g_{\alpha\beta} = \begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = (2\pi R)^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{3}$$

To realize the T^2, we identify $z \sim z + 1$ and $z \sim z + \tau$.

We consider the U(1) magnetic flux F on T^2,

$$F = i\frac{\pi M}{\mathrm{Im}\tau}(dz \wedge d\bar{z}). \tag{4}$$

Such a magnetic flux can be obtained from the following vector potential:

$$A(z) = \frac{\pi M}{\mathrm{Im}\tau}\mathrm{Im}(\bar{z}dz). \tag{5}$$

It satisfies the boundary conditions,

$$A(z+1) = A(z) + d\phi_1, \qquad A(z+\tau) = A(z) + d\phi_2, \tag{6}$$

where

$$\phi_1 = \frac{\pi M}{\mathrm{Im}\tau}\mathrm{Im}z, \qquad \phi_2 = \frac{\pi M}{\mathrm{Im}\tau}\mathrm{Im}\bar{\tau}z. \tag{7}$$

Now, let us study the spinor field with U(1) charge q on T^2,

$$\psi(z, \bar{z}) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{8}$$

We use the gamma matrices,

$$\Gamma^z = (2\pi R)^{-1}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad \Gamma^{\bar{z}} = (2\pi R)^{-1}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \tag{9}$$

Then, the Dirac operator on ψ is written by

$$i\not{D} = i\Gamma^z\nabla_z + i\Gamma^{\bar{z}}\nabla_{\bar{z}} = \frac{i}{\pi R}\begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}, \tag{10}$$

where

$$D^\dagger \equiv \partial - q\frac{\pi M}{2\mathrm{Im}\tau}\bar{z}, \qquad D \equiv \bar{\partial} + q\frac{\pi M}{2\mathrm{Im}\tau}z. \tag{11}$$

Thus, the zero mode equations of spinor are written by

$$D\psi_+ = 0, \qquad D^\dagger\psi_- = 0. \tag{12}$$

Also, they must satisfy the following boundary condition:

$$\psi_\pm(z+1) = e^{iq\phi_1(z)}\psi_\pm(z) = \exp\left\{i\frac{\pi qM}{\mathrm{Im}\tau}\mathrm{Im}z\right\}\psi_\pm(z), \tag{13}$$

$$\psi_\pm(z+\tau) = e^{iq\phi_2(z)}\psi_\pm(z) = \exp\left\{i\frac{\pi qM}{\mathrm{Im}\tau}\mathrm{Im}\bar{\tau}z\right\}\psi_\pm(z), \tag{14}$$

because of Eq. (6). The magnetic flux should be quantized and qM must be an integer.

If $qM > 0$, ψ_- has no zero-mode, but ψ_+ has qM zero-modes and their wave functions are written as

$$\psi_+^{j,M}(z) = \mathcal{N}e^{i\pi qMz\frac{\mathrm{Im}z}{\mathrm{Im}\tau}} \cdot \vartheta\begin{bmatrix} \frac{j}{qM} \\ 0 \end{bmatrix}(qMz, qM\tau), \tag{15}$$

with $j = 0, 1, \ldots, (qM-1)$, where ϑ denotes the Jacobi theta function,

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau) = \sum_{l\in\mathbf{Z}} e^{\pi i(a+l)^2\tau} e^{2\pi i(a+l)(\nu+b)}. \qquad (16)$$

Here, $\mathcal{N}$ denotes the normalization factor given by

$$\mathcal{N} = \left(\frac{2\mathrm{Im}\tau|qM|}{\mathcal{A}^2}\right)^{1/4}, \qquad (17)$$

with $\mathcal{A} = 4\pi^2 R^2 \mathrm{Im}\tau$. See Appendix A for computation of $\mathcal{N}$.

If $qM < 0$, ψ_+ has no zero-mode, but ψ_- has $|qM|$ zero-modes. Their wave functions are the same as the above except replacing qM by $|qM|$. Thus, introducing magnetic flux leads to a chiral theory.

For simplicity, we normalize the charge $q = 1$. We can discuss other charges $q \neq 1$ by replacing $M \to qM$ in the following analysis. Hereafter, we also set $M > 0$. Thus, in what follows, we consider the zero-mode wave functions,

$$\psi^{j,M}(z,\tau) = \mathcal{N}\cdot e^{i\pi M z \mathrm{Im}z/\mathrm{Im}\tau}\cdot\vartheta\begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix}(Mz, M\tau). \qquad (18)$$

Here, we write τ explicitly in $\psi^{j,M}(z,\tau)$ because τ dependence is important in the following analysis. We can use another basis of zero-mode solutions,

$$\chi^{j,M}(\tau,z) = \frac{\mathcal{N}}{\sqrt{M}}\cdot e^{i\pi M z \mathrm{Im}z/\mathrm{Im}\tau}\cdot\vartheta\begin{bmatrix} 0 \\ \frac{j}{M} \end{bmatrix}(z, \tau/M). \qquad (19)$$

These are related with each other as

$$\chi^{j,M} = \frac{1}{\sqrt{M}}\sum_k e^{2\pi i\frac{jk}{M}}\psi^{k,M}, \qquad (20)$$

$$\psi^{j,M} = \frac{1}{\sqrt{M}}\sum_k e^{-2\pi i\frac{jk}{M}}\chi^{k,M}. \qquad (21)$$

See Appendix B for these relations.

Using these wave functions, we can compute 3-point coupling [5],

$$\int d^2z\psi^{j_1,M_1}(z)\psi^{j_2,M_2}(z)\psi^{j_2,M_2}(z), \qquad (22)$$

as well as n-point couplings [6],

$$\int d^2z\psi^{j_1,M_1}(z)\psi^{j_2,M_2}(z)\cdots\psi^{j_n,M_n}(z). \qquad (23)$$

B. T^2/Z_2 orbifold

In [14], the zero-mode wave functions on the T^2/Z_2 orbifold were studied. On the T^2/Z_2 orbifold, we identify $z \sim -z$. Under the Z_2 twist, the zero-mode wave functions satisfy the following simple relation:

$$\psi^{j,M}(-z) = \psi^{M-j,M}(z). \qquad (24)$$

Note that $\psi^{0,M}(z) = \psi^{M,M}(z)$. The other basis, $\chi^{j,M}(z)$, also satisfies the same relation. Thus, the Z_2 even and odd wave functions $\Theta^{j,M}_{\pm1}(z)$ can be written by

$$\Theta^{j,M}_{\pm1}(z) = \frac{1}{\sqrt{2}}(\psi^{j,M}(z) \pm \psi^{M-j,M}(z)). \qquad (25)$$

The numbers of even and odd modes are shown in Table I.

By using Z_2 eigenfunctions, $\Theta^{j,M}_{\pm1}(z)$, we can compute 3-point couplings and higher order couplings similar to Eqs. (22) and (23). Then, we obtain phenomenological interesting results e.g., the realization of quark and lepton mass hierarchies and their mixing angles [20–23].

We also give a comment on Scherk-Schwarz phases and discrete Wilson lines. These degrees of freedom are equivalent to each other [28]. Hence, we restrict ourselves to Scherk-Schwarz phases. With Scherk-Schwarz phases (β_1, β_τ), the boundary conditions (13) and (14) change as

$$\psi(z+1) = e^{i\phi_1(z)+2\pi i\beta_1}\psi(z), \qquad (26)$$

$$\psi(z+\tau) = e^{i\phi_2(z)+2\pi i\beta_\tau}\psi(z), \qquad (27)$$

for $q = 1$. On the orbifold, discrete values of Scherk-Schwarz phases are possible [28]. (See also [33].) On the T^2/Z_2 orbifold, there are four possible Scherk-Schwarz phases,

$$(\beta_1,\beta_\tau) = (0,0), \quad (0,1/2), \quad (1/2,0), \quad (1/2,1/2). \qquad (28)$$

For such boundary conditions, the zero-mode wave functions are obtained as [28]

$$\psi^{j+\beta_1,\beta_\tau,M}(z) = \mathcal{N}\cdot e^{i\pi M z\mathrm{Im}z/\mathrm{Im}\tau}\cdot\vartheta\begin{bmatrix} \frac{j+\beta_1}{M} \\ -\beta_\tau \end{bmatrix}(Mz, M\tau). \qquad (29)$$

Under the Z_2 twist, these wave functions behave as

$$\begin{aligned}\psi^{j+\beta_1,\beta_\tau,M}(-z) &= \psi^{M-j-\beta_1,-\beta_\tau,M}(z)\\ &= e^{-4\pi i\frac{(j+\beta_1)\beta_\tau}{M}}\psi^{M-j-\beta_1,\beta_\tau,M}(z).\end{aligned} \qquad (30)$$

Using this behavior, we can construct Z_2 eigenstates similar to Eq. (25).

TABLE I. The numbers of Z_2 even and odd zero-modes.

M	$2n$	$2n+1$
Z_2 even	$n+1$	$n+1$
Z_2 odd	$n-1$	n

III. T^2/Z_4 ORBIFOLD

Here, we study T^2/Z_4 orbifold models.

A. Modular transformation

We denote the basis vectors of the lattice Λ by (α_1, α_2) to construct $T^2 = R^2/\Lambda$, i.e., $\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\tau$ in the complex basis. The same lattice can be described by another basis, (α'_1, α'_2), and these lattice bases are related with each other as,

$$\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}, \tag{31}$$

where a, b, c, d are integer with satisfying $ad - bd = 1$. That is $SL(2,Z)$ transformation. The lattice basis (α_1, α_2) spans exactly the same lattice as the basis $(-\alpha_1, -\alpha_2)$. Thus, the modular transformation is $SL(2,Z)/Z_2$.

Under the above transformation (31), the modular parameter τ transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{32}$$

This transformation includes two important generators, S and T,

$$S\colon\ \tau \to -\frac{1}{\tau}, \tag{33}$$

$$T\colon\ \tau \to \tau + 1. \tag{34}$$

Here, we study S because it is relevant to the Z_4 twist. S transforms the lattice basis as

$$(\alpha_1, \alpha_2) \to (-\alpha_2, \alpha_1). \tag{35}$$

This is nothing but the Z_4 twist, for $\tau = i$. More precisely we can refer to this as the inverse of the Z_4 twist, i.e., the $-\pi/2$ rotation.

B. T^2/Z_4 orbifold model

Here, we study the transformation behavior of zero-mode wave functions under S. Let us start with $\chi^{j,M}(z,\tau)$. Then, we examine its S transformation. That is, we replace $\tau \to -1/\tau$, $z \to z/\tau$ in $\chi^{j,M}(z,\tau)$. It is found that

$$\chi^{j,M}(z/\tau, -1/\tau) = \psi^{j,M}(z,\tau). \tag{36}$$

To show this transformation, we have used the following relation:

$$\vartheta\begin{bmatrix} 0 \\ a \end{bmatrix}\left(\frac{\nu}{\kappa}, -\frac{1}{\kappa}\right) = (-i\kappa)^{1/2} e^{i\pi\nu^2/\kappa} \cdot \vartheta\begin{bmatrix} a \\ 0 \end{bmatrix}(\nu, \kappa). \tag{37}$$

That is, the ϑ function in $\chi^{j,M}(z,\tau)$ transforms

$$\vartheta\begin{bmatrix} 0 \\ \frac{j}{M} \end{bmatrix}\left(z, \frac{\tau}{M}\right) \to (-iM\tau)^{1/2} e^{i\pi M \frac{z^2}{\tau}} \cdot \vartheta\begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix}(Mz, M\tau). \tag{38}$$

In addition, we combine the S transformation of the phase $e^{i\pi M z \frac{\mathrm{Im} z}{\mathrm{Im}\tau}}$ with the phase factor $e^{i\pi M \frac{z^2}{\tau}}$ in the above equation (38) to find

$$\exp\left\{\pi i M \frac{z}{\tau} \frac{\mathrm{Im} z/\tau}{\mathrm{Im}(-1/\tau)} + \pi i M \frac{z^2}{\tau}\right\} = \exp\left\{\pi i M \frac{z \cdot \mathrm{Im} z}{\mathrm{Im}\tau}\right\}. \tag{39}$$

Also the normalization factor transforms under S,

$$\mathcal{N} \to \left(\frac{1}{|\tau|^2}\right)^{1/4} \mathcal{N}. \tag{40}$$

Using these results, we can derive the transformation (36) [5].[2]

On the other hand, we replace

$$\tau \to -1/\tau, \qquad z \to \tau z, \tag{41}$$

in $\psi^{j,M}(z,\tau)$. Similarly, we find that

$$\psi^{j,M}\left(\tau z, -\frac{1}{\tau}\right) = \chi^{j,M}(z,\tau). \tag{42}$$

We require that the torus is invariant under the S transformation, i.e.

$$\tau = -\frac{1}{\tau}. \tag{43}$$

Its solution is $\tau = \pm i$. Here, we set $\tau = i$. Then, the above transformation (41) is nothing but the Z_4 twist, $z \to \tau z = iz$. Thus, under such Z_4 twist, wave functions transform,

$$\begin{aligned} \psi^{j,M}(z, \tau = i) &\to \psi^{j,M}(iz, -1/\tau = i) \\ &= \chi^{j,M}(z, \tau = i) \\ &= C^j_{k,M} \psi^{k,M}(z, \tau = i). \end{aligned} \tag{44}$$

In the last equality, we have used the relation (20), and the coefficients $C^j_{k,M}$ are written by

[2]Such a transformation behavior is important in modular symmetry of 4D low-energy effective field theory [34].

$$C^j_{k,M} = \frac{1}{\sqrt{M}} e^{2\pi i \frac{jk}{M}}. \tag{45}$$

The matrix $C^j_{k,M}$ satisfies

$$\sum_k C^j_{k,M} C^k_{l,M} = \frac{1}{M} \sum_k e^{2\pi i (j+l)k/M} = \delta_{(j+l),nM}, \tag{46}$$

where n is integer. That is, we find that the Z_4 transformation,

$$\psi^{j,M}(z,i) \to \chi^{j,M}(z,i) \to \psi^{M-j,M}(z,i) \to \chi^{M-j,M}(z,i) \to \psi^{j,M}(z,i). \tag{47}$$

This transformation property is consistent with the Z_2 transformation (24). That is, we can write

$$\psi^{j,M}(z,i) \to \chi^{j,M}(z,i) \to \psi^{j,M}(-z,i) \to \chi^{j,M}(-z,i) \to \psi^{j,M}(z,i), \tag{48}$$

and the operation of the Z_4 twist 2 times is just the Z_2 twist.

Now, we can write the zero-mode wave functions with Z_4 eigenvalues $\gamma = \pm 1, \pm i$ as

$$\frac{1}{2}(\psi^{j,M}(z,i) + \gamma^{-1}\chi^{j,M}(z,i) + \gamma^{-2}\psi^{M-j,M}(z,i) + \gamma^{-3}\chi^{M-j,M}(z,i)), \tag{49}$$

i.e.,

$$\frac{1}{2}\left(\psi^{j,M}(z,i) + \gamma^{-1}\sum_k C^j_{k,M}\psi^{k,M}(z,i) + \gamma^{-2}\psi^{M-j,M}(z,i) + \gamma^{-3}\sum_k C^{M-j}_{k,M}\psi^{k,M}(z,i)\right). \tag{50}$$

Obviously, we can construct the Z_4 eigenstates as those of the matrix $C^j_{k,M}$. As an illustrating example, we study the model with $M = 3$, where the matrix $C^j_{k,M}$ is obtained as

$$C^j_{k,M} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho & \rho^2 \\ 1 & \rho^2 & \rho \end{pmatrix}, \tag{51}$$

with $\rho = 2\pi i/3$.[3] This matrix has eigenvalues, $\gamma = 1, -1, i$, and eigenvectors in the basis $\sum_j a_j \psi^{j,3}$,

[3]This matrix is the same as the matrix representation of S in heterotic string theory on the Z_3 orbifold [35].

TABLE II. The number of zero-modes in the Z_4 orbifold model.

M	1	2	3	4	5	6	7	8	9	10	11	12
Z_4 eigenvalue: $+1$	1	1	1	2	2	2	2	3	3	3	3	4
Z_4 eigenvalue: -1	0	1	1	1	1	2	2	2	2	3	3	3
Z_4 eigenvalue: $+i$	0	0	1	1	1	1	2	2	2	2	3	3
Z_4 eigenvalue: $-i$	0	0	0	0	1	1	1	1	2	2	2	2

TABLE III. Generic results on the numbers of Z_4 zero-modes.

M	$4n$	$4n+1$	$4n+2$	$4n+3$
Z_4 eigenvalue: $+1$	$n+1$	$n+1$	$n+1$	$n+1$
Z_4 eigenvalue: -1	n	n	$n+1$	$n+1$
Z_4 eigenvalue: $+i$	n	n	n	$n+1$
Z_4 eigenvalue: $-i$	$n-1$	n	n	n

$$\begin{aligned} (a_0, a_1, a_2) &= (1+\sqrt{3}, 1, 1) \quad \text{for } \gamma = 1, \\ &(1-\sqrt{3}, 1, 1) \quad \text{for } \gamma = -1, \\ &(0, 1, -1) \quad \text{for } \gamma = i, \end{aligned} \tag{52}$$

up to normalization factors.

Similarly, we can obtain Z_4 eigenvalues and eigenstates by using explicit matrices, $C^j_{k,M}$ for each value of M, in particular small values of M. Table II shows the numbers of Z_4 zero-modes for small values of M. This result is consistent with the previous results [28,29] up to the definition of the Z_4 twist.[4] The corresponding Z_4 eigenstates are shown in Appendix C.

We give a comment on Z_4 eigenstates. The Z_2 even states, $(\psi^{j,M} + \psi^{M-j,M})$, correspond to the Z_4 eigenstates with eigenvalues $\gamma = \pm 1$, while Z_2 odd states, $(\psi^{j,M} - \psi^{M-j,M})$, correspond to the Z_4 states with eigenvalues $\gamma = \pm i$. Explicit results on eigenstates for small number of M are shown in Appendix C. For $M =$ even, Z_4 eigenvectors are relatively simple, while for $M =$ odd Z_4 eigenvectors are complicated.

From the above explicit results, we can expect generic results on the numbers of zero-modes, which are shown in Table III. Indeed, we can prove this result. First, we compute $\mathrm{tr}C^j_{k,M}$,

$$\mathrm{tr}C = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} e^{2\pi i \frac{k^2}{M}}. \tag{53}$$

In our computation, the following Landsberg-Schaar relation:

[4]The number of zero-modes with Z_4 eigenvalue $\gamma = i$ is exchanged for the number of zero-modes with eigenvalue $\gamma = -i$ when we replace the definition of Z_4 twist by its inverse.

$$\frac{1}{\sqrt{p}}\sum_{n=0}^{p-1}e^{\frac{2\pi i n^2 q}{p}}=\frac{e^{\frac{\pi i}{4}}}{\sqrt{2q}}\sum_{n=0}^{2q-1}e^{-\frac{\pi i n^2 p}{2q}},\qquad p,q\in\mathbb{N},\tag{54}$$

is very useful. We take $p=M$, $q=1$ in the Landsberg-Schaar relation to compute tr$C$,

$$\mathrm{tr}C=e^{\frac{\pi i}{4}}\frac{1}{\sqrt{2}}\sum_{k=0}^{1}e^{-\frac{\pi i k^2 M}{2}}=e^{\frac{\pi i}{4}}\frac{1}{\sqrt{2}}(1+e^{-\pi i\frac{M}{2}}).\tag{55}$$

Then, we find that

$$\mathrm{tr}C=\begin{cases}1+i & \text{for } M=4n\\ 1 & \text{for } M=4n+1\\ 0 & \text{for } M=4n+2\\ i & \text{for } M=4n+3\end{cases}.\tag{56}$$

For example, recall that when $M=4n$, there are $(2n+1)$ Z_2 even zero-modes and $(2n-1)$ Z_2 odd zero-modes. That is, the sum of the numbers of Z_4 zero-modes with eigenvalues $\gamma=\pm1$ is equal to $(2n+1)$, while the sum of the numbers of Z_4 zero-modes with eigenvalues $\gamma=\pm i$ is equal to $(2n-1)$. Combination of these with Eq. (56) leads to the result for $M=4n$ in Table III. Similarly, we can derive the numbers of Z_4 zero-modes with other values of M as shown in Table III.

IV. T^2/Z_3 ORBIFOLD

In this section, we study the zero-modes on T^2/Z_3 and T^2/Z_6 orbifolds.

A. T^2/Z_3 orbifold

Here, we study the Z_3 orbifold models. Our strategy is the same as one in the previous section. That is, we examine the modular transformation corresponding to the Z_3 twist. A good candidate for the Z_3 twist is ST transformation, because it satisfies $(ST)^3=1$ on τ. Under ST, the modular parameter τ transforms as

$$\tau\to-\frac{1}{\tau+1}.\tag{57}$$

When $\tau=e^{\pm2\pi i/3}$, the modular parameter is invariant under ST, i.e.

$$\tau=-\frac{1}{\tau+1}.\tag{58}$$

For such a transformation, the Z_3 twist (its inverse) can be defined by

$$z\to\tau z,\tag{59}$$

when $\tau=e^{2\pi i/3}$ ($\tau=e^{-2\pi i/3}$). Alternatively, we can define the Z_3 twist by

$$z\to\frac{-z}{\tau+1},\tag{60}$$

because of the relation (58). In what follows, we study the transformation of wave functions under Eqs. (57) and (60). We restrict ourselves to the models with $M=$ even, because the following transformation behavior is valid only for $M=$ even.

We find that

$$\begin{aligned}\chi^{j,M}(-z/(\tau+1),-1/(\tau+1))&=e^{\pi i\frac{j^2}{M}}\cdot\psi^{j,M}(-z,\tau)\\&=e^{\pi i\frac{j^2}{M}}\cdot\psi^{M-j,M}(z,\tau).\end{aligned}\tag{61}$$

Here, we have used the relation (37) and the following relation:

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(\nu,\tau+1)=e^{-i\pi a(a-1)}\cdot\theta\begin{bmatrix}a\\b+a-\frac{1}{2}\end{bmatrix}(\nu,\tau).\tag{62}$$

Since $\psi^{M-j,M}=C_{jk}\chi^{k,M}$, the transformation in the χ basis is written by

$$\chi^{j,M}(z,\tau)\to D^j_{k,M}\chi^{k,M}(z,\tau),\qquad D^j_{k,M}=e^{\pi i\frac{j^2}{M}}C^j_{k,M}.\tag{63}$$

When we examine the inverse transformation,

$$\tau\to-\frac{1}{\tau}-1,\qquad z\to\frac{1}{\tau}z,\tag{64}$$

on the wave function $\psi^{j,M}(z,\tau)$, we find that

$$\psi^{j,M}\left(\frac{z}{\tau},-1/\tau-1\right)=e^{-\pi i\frac{j^2}{M}}\cdot\chi^{M-j}(z,\tau).\tag{65}$$

Thus, it is found that under the above inverse transformation, the wave function $\psi^{j,M}$ transforms as

$$\psi^{j,M}\to(D^{-1})^j_{k,M}\psi^{j,M},\tag{66}$$

where D^{-1} is the inverse matrix of $D^j_{k,M}$.

For example, for $M=2$, we obtain

$$D^j_{k,M=2}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\i&-i\end{pmatrix}.\tag{67}$$

However, we find that

$$(D^j_{k,M=2})^3=\frac{1}{\sqrt{2}}\begin{pmatrix}1+i&0\\0&1+i\end{pmatrix}.\tag{68}$$

This matrix does not realize exactly the Z_3 twist.

Indeed, for generic even number M, we can find

$$D^j_{k,M} D^k_{\ell,M} D^\ell_{m,M} = \frac{1}{\sqrt{2}}(1+i)\delta_{j,m} = e^{\pi i/4}\delta_{j,m}. \tag{69}$$

See Appendix B. Thus, the matrix $D^j_{k,M}$ on $\chi^{j,M}$ does not represent the Z_3 twist exactly.

Here, we allow the constant phase for all modes under the above transformation, e.g.[5]

$$\chi^{j,M}(z,\tau) \to \tilde{D}^j_{k,M}\chi^{k,M}(z,\tau), \qquad \tilde{D}^j_{k,M} = e^{-\frac{\pi i}{12}} D^j_{k,M}. \tag{70}$$

Then, we can realize the Z_3 twist,

$$\tilde{D}^j_{k,M}\tilde{D}^k_{\ell,M}\tilde{D}^\ell_{m,M} = \delta_{j,m}. \tag{71}$$

Here, we employ this matrix $\tilde{D}$ as the Z_3 twist.

For example, for $M=2$, we use the following matrix for the Z_3 twist on $\chi^{j,M}$:

$$\tilde{D}^j_{k,M=2} = \frac{e^{-\frac{\pi i}{12}}}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{72}$$

TABLE IV. The number of zero-modes in the Z_3 orbifold model.

M	2	4	6	8	10	12
Z_3 eigenvalue: 1	1	1	3	3	3	5
Z_3 eigenvalue: $e^{2\pi i/3}$	0	2	2	2	4	4
Z_3 eigenvalue: $e^{-2\pi i/3}$	1	1	1	3	3	3

Its eigenvalues are obtained as $\gamma = 1, e^{-2\pi i/3}$, and eigenvectors are given as

$$(1, \sqrt{2}\gamma e^{\frac{\pi i}{12}} - 1), \tag{73}$$

in the $(\chi^{0,M}, \chi^{1,M})$ basis, up to a normalization factor.

Similarly, we study the model with $M=4$. The eigenvalues of the matrix $\tilde{D}^j_{k,M}$ are

$$(1, e^{2\pi i/3}, e^{2\pi i/3}, e^{-2\pi i/3}), \tag{74}$$

and their eigenvectors are obtained in the basis $a_i\chi_i$,

$$\begin{aligned}
&[0,-1,0,1],\\
&[-i(-1+\sqrt{2}), (1+i)-\sqrt{2}, 1, 0],\\
&\left[\frac{(-6+6i)+3i\sqrt{2}-(2+2i)\sqrt{3}+(1-2i)\sqrt{6}}{3i\sqrt{2}+(2-2i)\sqrt{3}}, 1, \frac{\sqrt{2}(3i+(1+2i)\sqrt{3})}{3i\sqrt{2}+(2-2i)\sqrt{3}+\sqrt{6}}, 1\right],\\
&\left[\frac{6i+3(-1)^{1/4}+(1+3i)\sqrt{3/2}+2\sqrt{3}}{3(-1)^{1/4}-(1-i)\sqrt{3/2}+2i\sqrt{3}}, 1, \frac{\sqrt{2}(-3i+(1+2i)\sqrt{3}}{-3i\sqrt{2}+(2-2i)\sqrt{3}+\sqrt{6}}, 1\right],
\end{aligned} \tag{75}$$

up to normalization factors.

Similarly, we can analyze the eigenvalues and eigenvectors for other M. Table IV shows the numbers of Z_3 zero-modes with each eigenvalue for small values of M. This result is consistent with the previous results [28,29]. We can derive eigenvectors, but their explicit forms are, in general, very complicated.

We can analyze the number of Z_3 zero-modes for generic even number M. First we compute the trace of the inverse of $\tilde{D}$,

$$(\tilde{D}^{-1})^j_k = e^{\pi i\frac{1}{12}} \cdot e^{-\pi i\frac{j^2}{M}} \cdot C^\dagger_{jk}. \tag{76}$$

That is, its trace is written by

$$\mathrm{tr}(\tilde{D}^{-1}) = e^{\pi i\frac{1}{12}}\sum_{k=0}^{M-1} e^{-\pi i\frac{k^2}{M}} \cdot e^{-2\pi i\frac{k^2}{M}} = e^{\pi i\frac{1}{12}}\sum_{k=0}^{M-1} e^{-3\pi i\frac{k^2}{M}}. \tag{77}$$

Here, we use the Landsberg-Schaar relation (54) with $p=3$ and $2q=M$. Then, we find

$$\mathrm{tr}(\tilde{D}^{-1}) = \frac{ie^{-\pi i\frac{2}{3}}}{\sqrt{3}}\left(1+2e^{\pi i\frac{M}{3}}\right). \tag{78}$$

Explicitly, we obtain the following results:

$$\mathrm{tr}\tilde{D}^{-1} = \begin{cases} 1+\omega & \text{for } M=6n+2 \\ \omega^2 & \text{for } M=6n+4\,, \\ 2+\omega^2 & \text{for } M=6n \end{cases} \tag{79}$$

[5]We have other two values for candidates of the constant phase, and totally there are three possibilities. Different constant phases lead to a change of degeneracy factors for each Z_3 eigenvalues. Such possibilities of constant phases may correspond to the possibility of the introduction of Scherk-Schwarz phases. Similarly, we have the degree of freedom to define the Z_4 twist by $e^{\pi i n/2}C^j_{k,M}$ with $n=0, 1, 2, 3$.

TABLE V. Generic results on Z_3 zero-modes.

M	$6n$	$6n+2$	$6n+4$
Z_3 eigenvalue: 1	$2n+1$	$2n+1$	$2n+1$
Z_3 eigenvalue: $e^{2\pi i/3}$	$2n$	$2n$	$2n+2$
Z_3 eigenvalue: $e^{2\pi i/3}$	$2n-1$	$2n+1$	$2n+1$

where $\omega = e^{2\pi i/3}$. Then, the trace of its inverse can be obtained by replacing $\omega \to \omega^2$,

$$\mathrm{tr}\tilde{D} = \begin{cases} 1+\omega^2 & \text{for } M = 6n+2 \\ \omega & \text{for } M = 6n+4\,. \\ 2+\omega & \text{for } M = 6n \end{cases} \tag{80}$$

From this result, we can derive the number of Z_3 eigenstates as shown in Table V. Note that $1+\omega+\omega^2 = 0$.

B. Z_6 orbifold

Obviously, the Z_6 twist can realized by the product of the Z_2 and Z_3 twists. Also, recall that the Z_2 twist on $\psi^{j,M}(z)$ and $\chi^{j,M}(z)$ is realized by

$$\begin{aligned} \psi^{j,M}(z) &\to \psi^{j,M}(-z) = \psi^{M-j,M}(z), \\ \chi^{j,M}(z) &\to \chi^{j,M}(-z) = \chi^{M-j,M}(z). \end{aligned} \tag{81}$$

Here, we restrict ourselves to the models with $M =$ even. From the analysis on the T^2/Z_3 orbifold, the Z_6 twist can be realized by

$$F^j_{k,M} = e^{\frac{\pi i}{12}} e^{-\pi i \frac{j^2}{M}} C^j_{k,M}. \tag{82}$$

Again, using the Landsberg-Schaar relation (54), we compute the trace of $F^j_{k,M}$ matrix,

$$\mathrm{tr}F = e^{\frac{\pi i}{12}} \sum_k e^{\pi i \frac{k^2}{M}} = e^{\frac{\pi i}{12}} e^{\frac{\pi i}{4}}. \tag{83}$$

The possible eigenvalues of F-matrix are $\gamma = \rho^k$ with $k = 0, 1, \ldots, 5$ and $\rho = e^{\pi i/3}$. Here, we denote the number of zero-modes with eigenvalues γ by N_γ. Since $(F^j_{k,M})^3$ corresponds to the Z_2 twist, the zero-mode numbers, N_γ must satisfy

$$N_1 + N_{\rho^2} + N_{\rho^4} = n+1, \qquad N_\rho + N_{\rho^3} + M_{\rho^5} = n-1, \tag{84}$$

for $M = 2n$. Similarly, $(F^j_{k,M})^2$ corresponds to the Z_3 twist, the zero-mode numbers must satisfy

$$N_1 + N_{\rho^3} = 2n+1, \tag{85}$$

TABLE VI. Generic results on Z_6 zero-modes.

M	$6n$	$6n+2$	$6n+4$
eigenvalue: 1	$n+1$	$n+1$	$n+1$
eigenvalue: $e^{\pi i/3}$	n	n	$n+1$
eigenvalue: $e^{2\pi i/3}$	n	$n+1$	$n+1$
eigenvalue: $e^{3\pi i/3}$	n	n	n
eigenvalue: $e^{4\pi i/3}$	n	n	$n+1$
eigenvalue: $e^{5\pi i/3}$	$n-1$	n	n

for $M = 6n, 6n+2, 6n+4$,

$$N_\rho + N_{\rho^4} = \begin{cases} 2n & \text{for } M = 6n, 6n+2 \\ 2n+2 & \text{for } M = 6n+4 \end{cases}, \tag{86}$$

and

$$N_{\rho^2} + N_{\rho^5} = \begin{cases} 2n-1 & \text{for } M = 6n \\ 2n+1 & \text{for } M = 6n+2, 6n+4 \end{cases}. \tag{87}$$

Combining these relations with the trace (83), we find the number of eigenstates, which is shown in Table VI.

In principle, we can derive zero-mode wave functions with eigenvalues γ, but its explicit form is complicated.

V. ZERO-MODES ON ORBIFOLDS

We have studied the zero-modes on several orbifolds, T^2/Z_N with $N = 2, 3, 4, 6$. Now, let us compare our results between different T^2/Z_N orbifolds. We examine the Z_N invariant zero-modes. It is found that the number of Z_N invariant zero-modes is written by

$$I_{M,N} = \lfloor M/N \rfloor + 1, \tag{88}$$

on T^2/Z_N orbifold with magnetic flux M except the Z_3 orbifold with $M = 6n+4$. Here, $\lfloor r \rfloor$ denotes the maximum integer n, which satisfies $n \le r$. Alternatively, the number of Z_3 invariant zero-modes is written by

$$I^{(3)}_{M,N} = 2\lfloor M/(2N) \rfloor + 1. \tag{89}$$

Our results are quite universal for different T^2/Z_N orbifolds.

The index theorem tells that the number of zero-modes of the Dirac operators on T^2 with flux M is equal to M. The above number $I_{M,N}$ as well as $I^{(3)}_{M,N}$ would correspond to such an index on the T^2/Z_N orbifolds.

It would be useful to rewrite the numbers of zero-modes with other eigenvalues by using the symbol $\lfloor r \rfloor$. These are shown in Table VII. Note that the number of zero-modes with Z_N eigenvalue γ is exchanged for one with Z_N eigenvalue γ^{-1} when we replace the definition of Z_N twist by its inverse.

TABLE VII. Generic results on Z_N zero-modes.

Eigenvalues (γ)	Number of zero-modes
$Z_{N=2n}$ invariant	$\lfloor M/N \rfloor + 1$
Z_3 invariant	$2\lfloor M/(2N) \rfloor + 1$
Z_2 ($\gamma = -1$)	$\lfloor (M-1)/2 \rfloor$
Z_4 ($\gamma = -1$)	$\lfloor (M-2)/4 \rfloor + 1$
Z_4 ($\gamma = i$)	$\lfloor (M-3)/4 \rfloor + 1$
Z_4 ($\gamma = -i$)	$\lfloor (M-1)/4 \rfloor$
Z_3 ($\gamma = e^{2\pi i/3}$)	$2\lfloor (M-4)/6 \rfloor + 2$
Z_3 ($\gamma = e^{-2\pi i/3}$)	$2\lfloor (M-2)/6 \rfloor + 1$
Z_6 ($\gamma = e^{\pi i/3}$)	$\lfloor (M-4)/6 \rfloor + 1$
Z_6 ($\gamma = e^{2\pi i/3}$)	$\lfloor (M-2)/6 \rfloor + 1$
Z_6 ($\gamma = e^{3\pi i/3}$)	$\lfloor M/6 \rfloor$
Z_6 ($\gamma = e^{4\pi i/3}$)	$\lfloor (M-4)/6 \rfloor + 1$
Z_6 ($\gamma = e^{5\pi i/3}$)	$\lfloor (M-2)/6 \rfloor$

It seems that the T^2/Z_3 orbifold has the zero-mode structure different from the other orbifolds. The numbers of zero-modes on T^2/Z_N with $N =$ even have the structure with the period N for M. That is, the number of zero-modes increases by one when we replace M by $M+N$. On the other hand, the number of zero-modes on T^2/Z_3 has the structure with the period 6, and the number of zero-modes increases by 2 when replace M by $M+6$. Such a structure of T^2/Z_3 is similar to one of T^2/Z_6 and seems to be originated from the T^2/Z_6 orbifold. At any rate, the deep reason why the T^2/Z_3 orbifold has a different structure is not clear. It is important to study its reason further.

The number of Z_N invariant zero-modes depends on nontrivial Scherk-Schwarz phases and discrete Wilson lines. Thus, our results imply that the number of Z_N invariant zero-modes is universal over all of T^2/Z_N orbifolds if we choose proper conditions on Scherk-Schwarz phases and discrete Wilson lines.

VI. CONCLUSION

We have studied T^2/Z_N orbifold models with magnetic flux. We used the modular transformation to define the orbifolds. Then, we have computed zero-mode wave functions with each eigenvalue of the Z_N twist. We have shown the zero-mode numbers. It is found that the number of the Z_N invariant zero-modes is universal among different T^2/Z_N orbifolds, and it can be obtained by $\lfloor M/N \rfloor + 1$ except one case in the T^2/Z_3 orbifold. The zero-mode number of the Dirac operator on T^2 is given by M. Our result would correspond to such an index.

We can write wave functions analytically for fixed M. Thus, we can compute 3-point couplings and higher order couplings. Hence, our results would be useful to further phenomenological applications. One can also apply our method to not only zero-modes, but also higher modes.

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APPENDIX A: NORMALIZATION OF WAVE FUNCTION AND INNER PRODUCT OF ψ AND χ

In this appendix, we show computations on normalization $\mathcal{N}$ of wave functions and the relations (20) and (21). The computation on normalization is useful for computation of the relations (20) and (21). Now, we compute

$$\int_{T^2} dz d\bar{z} \psi^j (\psi^k)^*, \tag{A1}$$

where

$$\int_{T^2} dz d\bar{z} = \mathcal{A} \int_0^1 d(\mathrm{Re}z) \int_0^1 d\left(\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right), \tag{A2}$$

with $\mathcal{A} = 4\pi^2 R^2 \mathrm{Im}\tau$. The product of the wave functions, $\psi^j (\psi^k)^*$ is written explicitly,

$$\begin{aligned} \psi^j_{\pm} (\psi^k_{\pm})^* &= \psi^{j,qM}(\tau, z) \cdot \psi^{-k,-qM}(\bar{\tau}, \bar{z}) \\ &= \mathcal{N}^2 \cdot e^{-2\pi qM (\mathrm{Im}z)^2/\mathrm{Im}\tau} \cdot \vartheta \begin{bmatrix} \frac{j}{qM} \\ 0 \end{bmatrix} (qMz, qM\tau) \cdot \vartheta \begin{bmatrix} \frac{k}{qM} \\ 0 \end{bmatrix} (-qM\bar{z}, -qM\bar{\tau}). \end{aligned} \tag{A3}$$

The product of theta functions includes the following terms depending on $\mathrm{Re}z$ and $\mathrm{Im}z$:

$$\sum_n \sum_{n'} e^{2\pi i \{(\frac{j}{qM}+n) - (\frac{k}{qM}+n')\} \mathrm{Re}z} \cdot e^{-2\pi \{(\frac{j}{qM}+n) + (\frac{k}{qM}+n')\} \mathrm{Im}z}. \tag{A4}$$

Then, the integration over $\mathrm{Re}z$ leads to the Kronecker delta, $\delta_{j/(qM)+n, k/(qM)+n'}$. Thus, we obtain

$$\int_0^1 d(\mathrm{Re}z)\psi^{j,qM}(\tau,z)\cdot\psi^{-k,-qM}(\bar\tau,\bar z) = \mathcal{N}^2\sum_n e^{-2\pi qM\mathrm{Im}\tau(n+\frac{j}{qM}+\frac{\mathrm{Im}z}{\mathrm{Im}\tau})^2}. \tag{A5}$$

Furthermore, we can find

$$\begin{aligned}\int_0^1 d\left(\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right)\sum_n e^{-2\pi qM\mathrm{Im}\tau(n+\frac{j}{qM}+\frac{\mathrm{Im}z}{\mathrm{Im}\tau})^2} &= \sum_n\int_0^1 d\left(\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right)e^{-2\pi qM\mathrm{Im}\tau(n+\frac{j}{qM}+\frac{\mathrm{Im}z}{\mathrm{Im}\tau})^2}\\ &= \int_{-\infty}^{\infty} dx e^{-2\pi qM\mathrm{Im}\tau x^2}\\ &= \left(\frac{1}{2qM\mathrm{Im}\tau}\right)^{\frac{1}{2}}.\end{aligned} \tag{A6}$$

Then, we find the normalization factor (17).

Similarly, we compute

$$\int dz d\bar z\chi^{j,M}(z,\tau)\cdot(\psi^{k,M}(z,\tau))^*, \tag{A7}$$

where

$$\chi^{j,M}\cdot\psi^{-k,-M} = \frac{(\mathcal{N})^2}{\sqrt{M}}\cdot e^{-2\pi M\frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau}}\cdot\theta\begin{bmatrix}0\\ \frac{j}{M}\end{bmatrix}\left(z,\frac{\tau}{M}\right)\cdot\theta\begin{bmatrix}\frac{k}{M}\\ 0\end{bmatrix}(-\bar zM,-\bar\tau M). \tag{A8}$$

The product of the theta functions includes the following terms depending on Rez:

$$\sum_l\sum_{l'} e^{2\pi i\{l-M(l'+\frac{k}{M})\}\mathrm{Re}z}. \tag{A9}$$

The integration over Rez leads to the Kronecker delta, $\delta_{\ell,M\ell'+k}$. Thus, we obtain

$$\int_0^1 \mathrm{Re}(z)\chi^{j,M}\cdot\psi^{-k,-M} = e^{2\pi i\frac{jk}{M}}\cdot\mathcal{N}^2\cdot e^{-2\pi M\frac{(\mathrm{Im}z)^2}{\mathrm{Im}\tau}}\sum_l e^{-2\pi M\mathrm{Im}\tau(l+\frac{k}{M})}\cdot e^{-4\pi M(l+\frac{k}{M})\mathrm{Im}z}. \tag{A10}$$

In addition, we can integrate this over Im(z) similar to Eq. (A6). Then, we can derive

$$\int dz d\bar z\chi^{j,M}(z,\tau)\cdot(\psi^{k,M}(z,\tau))^* = e^{2\pi i\frac{jk}{M}}. \tag{A11}$$

That is nothing but the relation (20).

Also we can obtain the complex conjugate of Eq. (A11),

$$\int dz d\bar z\psi^{j,M}(z,\tau)\cdot(\chi^{k,M}(z,\tau))^* = e^{-2\pi i\frac{jk}{M}}, \tag{A12}$$

and this is nothing but the relation (21).

APPENDIX B: COMPUTATION OF $(D)^3$

In this section, we give the computation on $(D^j_{k,M})^3$ for a generic even number M. First, we can obtain

$$\begin{aligned}D^j_{k,M}D^k_{\ell,M} &= \frac{1}{M}\sum_k e^{\frac{2\pi i}{M}(\frac{j^2}{2}+k(j+\ell)+\frac{k^2}{2})}\\ &= \frac{1}{M}\sum_k e^{\frac{\pi i}{M}[(k+j+\ell)^2-\ell(2j+\ell)]}\\ &= \frac{1}{\sqrt{2M}}(1+i)e^{\frac{\pi i}{M}[-\ell(2j+\ell)]}.\end{aligned} \tag{B1}$$

We have used the Landsberg-Schaar relation (54). Then, we can compute

$$\begin{aligned}D^j_{k,M}D^k_{\ell,M}D^\ell_{m,M} &= \frac{1}{\sqrt{2M}}(1+i)\sum_\ell e^{\frac{\pi i}{M}[-\ell(2j+\ell)+\ell^2+2\ell m]}\\ &= \frac{1}{\sqrt{2M}}(1+i)\sum_\ell e^{\frac{2\pi i}{M}\ell(m-j)}\\ &= \frac{1}{\sqrt{2}}(1+i)\delta_{j,m} = e^{\pi i/4}\delta_{j,m}.\end{aligned} \tag{B2}$$

Again, we have used the Landsberg-Schaar relation (54).

APPENDIX C: EIGENVECTORS IN Z_4 ORBIFOLD MODELS

In this section, we give explicitly Z_4 eigenvectors for $M = 2, \ldots, 12$. These eigenvectors are represented in the basis $\sum_{k=0}^{M-1} a_k \psi^{k,M}$. The Z_2 even states, $(\psi^{j,M} + \psi^{M-j,M})$, correspond to the Z_4 eigenstates with eigenvalues $\gamma = \pm 1$, while Z_2 odd states, $(\psi^{j,M} - \psi^{M-j,M})$, correspond to the Z_4 states with eigenvalues $\gamma = \pm i$. Thus, $\psi^{0,M}$ does not correspond to eigenstates with Z_4 eigenvalues $\gamma = \pm i$, but always appears as eigenstates with Z_4 eigenvalues $\gamma = \pm 1$. Similarly, when M is even, $\psi^{M/2,M}$ corresponds to only Z_4 eigenvalues $\gamma = \pm 1$. The other modes appear in all of the eigenstates with eigenvalues $\gamma = \pm 1, \pm i$. For all the cases with $M = 4n, 4n+1, 4n+2, 4n+3$, there are $(n+1)$ independent eigenstates with $\gamma = 1$. It seems convenient to use the basis such that only one of $a_1, a_2, \cdots a_{n+1}$ is nonvanishing in $\sum_{k=0}^{M-1} a_k \psi^{k,M}$, that is,

$$
\begin{aligned}
&(a_0, 1, 0, \ldots, 0, 0, a_{n+1}, a_{n+2}, \ldots, a_{M-1}),\\
&(a_0, 0, 1, 0, \ldots, 0, a_{n+1}, a_{n+2}, \ldots, a_{M-1}),\\
&\quad \cdots\cdots\cdots\\
&(a_0, 0, \ldots, 0, 1, a_{n+1}, a_{n+2}, \ldots, a_{M-1}),
\end{aligned}
\tag{C1}
$$

where the other coefficients, a_0 and $a_{n+2}, \ldots, a_{M-1}$ are determined by eigenvector equations.

Similarly, when there are m independent modes, it seems convenient to use the basis such that only one of $a_1, a_2, \cdots a_m$ is nonvanishing. The fluxes can be classified as $M = 4n, 4n+1, 4n+2, 4n+3$. For such classes, we show explicitly eigenvectors in the basis $\sum_{k=0}^{M-1} a_k \psi^{k,M}$ in what follows. As said above, the coefficients a_k other than $a_1, a_2, \cdots a_m$ can be written by linear combinations of $a_1, a_2, \cdots a_m$. The eigenvectors for $M =$ even are relatively simple, while some of eigenvectors for $M =$ odd are written by lengthy linear combinations. In such cases, we omit writing them explicitly and just denote $LP_i(a_1, a_2, \cdots a_m)$. At any rate, $LP_i(a_1, a_2, \cdots a_m)$ can be computed by use of eigenvector equations.

1. $M = 4n+2, n \in \mathbb{Z}$

eigenvalue: $+1$
$M = 2$

$$(a_0, a_1,) \propto ((\sqrt{2}+1)a_1, a_1),$$

$M = 6$

$$(a_0, a_1, a_2, a_3, a_4, a_5) \propto \left(\frac{1}{2}(\sqrt{6}a_1 + (2+\sqrt{6})a_2), a_1, a_2, \frac{1}{2}((2-\sqrt{6})a_1 + \sqrt{6}a_2), a_2, a_1\right),$$

$M = 10$

$$
\begin{aligned}
(a_0, \ldots, a_9) \propto \Bigg(&\frac{1}{2}((1-\sqrt{2}-\sqrt{5}+\sqrt{10})a_1 + (2+\sqrt{10})a_2 + (-1+\sqrt{2}+\sqrt{5})a_3),\\
&a_1, a_2, a_3, \frac{1}{2+\sqrt{10}}((-2+2\sqrt{2}+2\sqrt{5}-\sqrt{10})a_1 + (-2-\sqrt{10})a_2 + (2-2\sqrt{2}-2\sqrt{5}+\sqrt{10})a_3),\\
&\frac{1}{2(2+\sqrt{10})}((-2+5\sqrt{2}+2\sqrt{5}-\sqrt{10})a_1 + (-10-2\sqrt{10})a_2 + (8-5\sqrt{2}-2\sqrt{5}+\sqrt{10})a_3),\\
&\frac{1}{2+\sqrt{10}}((-2+2\sqrt{2}+2\sqrt{5}-\sqrt{10})a_1 + (-2-\sqrt{10})a_2 + (2-2\sqrt{2}-2\sqrt{5}+\sqrt{10})a_3), a_3, a_2, a_1\Bigg).
\end{aligned}
$$

eigenvalue: -1
$M = 2$

$$(a_0, a_1,) \propto ((-\sqrt{2}+1)a_1, a_1),$$

$M = 6$

$$(a_0, a_1, a_2, a_3, a_4, a_5) \propto \left(\frac{1}{2}(-\sqrt{6}a_1 + (2-\sqrt{6})a_2), a_1, a_2, \frac{1}{2}((2+\sqrt{6})a_1 - \sqrt{6}a_2), a_2, a_1\right),$$

$M = 10$

$$(a_0, \ldots, a_9) \propto \bigg(\frac{1}{2}(-(1+\sqrt{2})(-1+\sqrt{5})a_1 - (-2+\sqrt{10})a_2 + (-1-\sqrt{2}+\sqrt{5})a_3),$$
$$a_1, a_2, a_3, (1+\sqrt{2})a_1 + a_2 - (1+\sqrt{2})a_3, \frac{1}{2}((1-\sqrt{5})a_1 - \sqrt{10}a_2 + (1+\sqrt{5}+\sqrt{10})a_3),$$
$$(1+\sqrt{2})a_1 + a_2 - (1+\sqrt{2})a_3, a_3, a_2, a_1\bigg),$$

eigenvalue: $+i$
$M = 2$ nothing,
$M = 6$

$$(a_0, a_1, a_2, a_3, a_4, a_5) \propto (0, a_1, (-1+\sqrt{2})a_1, 0, -(-1+\sqrt{2})a_1, -a_1),$$

$M = 10$

$$(a_0, \ldots, a_9) \propto \bigg(0, a_1, a_2, \Big(-2+\sqrt{5+\sqrt{5}}\Big)a_1 + \Big(-\sqrt{5}+\sqrt{5-\sqrt{5}}\Big)a_2,$$
$$\frac{1}{\sqrt{5-\sqrt{5}}}\Big(\Big(-5+\sqrt{5}+\sqrt{25-5\sqrt{5}}\Big)a_1 + 2\Big(-\sqrt{5}+\sqrt{5-\sqrt{5}}\Big)a_2\Big), 0,$$
$$-\bigg(\frac{1}{\sqrt{5-\sqrt{5}}}\Big(\Big(-5+\sqrt{5}+\sqrt{25-5\sqrt{5}}\Big)a_1 + 2\Big(-\sqrt{5}+\sqrt{5-\sqrt{5}}\Big)a_2\Big)\bigg),$$
$$-\Big(\Big(-2+\sqrt{5+\sqrt{5}}\Big)a_1 + \Big(-\sqrt{5}+\sqrt{5-\sqrt{5}}\Big)a_2\Big), -a_2, -a_1\bigg),$$

eigenvalue: $-i$
$M = 2$ nothing,
$M = 6$

$$(a_0, a_1, a_2, a_3, a_4, a_5) \propto (0, a_1, (-1-\sqrt{2})a_1, 0, -(-1-\sqrt{2})a_1, -a_1),$$

$M = 10$

$$z(a_0, \ldots, a_9) \propto \bigg(0, a_1, a_2, -\Big(2+\sqrt{5+\sqrt{5}}\Big)a_1 - \Big(\sqrt{5}+\sqrt{5-\sqrt{5}}\Big)a_2,$$
$$\frac{1}{\sqrt{5-\sqrt{5}}}\Big(\Big(5-\sqrt{5}+\sqrt{25-5\sqrt{5}}\Big)a_1 + 2\Big(\sqrt{5}+\sqrt{5-\sqrt{5}}\Big)a_2\Big), 0,$$
$$-\bigg(\frac{1}{\sqrt{5-\sqrt{5}}}\Big(\Big(5-\sqrt{5}+\sqrt{25-5\sqrt{5}}\Big)a_1 + 2\Big(\sqrt{5}+\sqrt{5-\sqrt{5}}\Big)a_2\Big)\bigg),$$
$$-\Big(-\Big(2+\sqrt{5+\sqrt{5}}\Big)a_1 - \Big(\sqrt{5}+\sqrt{5-\sqrt{5}}\Big)a_2\Big), -a_2, -a_1\bigg),$$

2. $M = 4n$

eigenvalue: $+1$
$M = 4$

$$(a_0, a_1, a_2, a_3) \propto (2a_1 + a_2, a_1, a_2, a_1),$$

$M = 8$

$$(a_0, \ldots, a_7) \propto \left(\frac{1}{\sqrt{2}}a_1 + (1+\sqrt{2})a_2 + \frac{1}{\sqrt{2}}a_3, a_1, a_2, a_3, -\frac{1}{\sqrt{2}}a_1 + (1+\sqrt{2})a_2 - \frac{1}{\sqrt{2}}a_3, a_3, a_2, a_1\right),$$

$M = 12$

$$(a_0, \ldots, a_{11}) \propto \left(\frac{1}{2}((3+\sqrt{3})a_2 + 2\sqrt{3}a_3 + (-1+\sqrt{3})a_4), a_1, a_2, a_3, a_4, -a_1 + \sqrt{3}a_2 + 2a_3 - \sqrt{3}a_4, \frac{1}{2}((-1+\sqrt{3})a_2 - 2\sqrt{3}a_3 + (3+\sqrt{3})a_4), -a_1 + \sqrt{3}a_2 + 2a_3 - \sqrt{3}a_4, a_4, a_3, a_2, a_1\right).$$

eigenvalue: -1
$M = 4$

$$(a_0, a_1, a_2, a_3) \propto (-a_1, a_1, a_1, a_1),$$

$M = 8$

$$(a_0, \ldots, a_7) \propto (-\sqrt{2}a_1 + (1-\sqrt{2})a_2, a_1, a_2, a_1, \sqrt{2}a_1 + (1-\sqrt{2})a_2, a_1, a_2, a_1),$$

$M = 12$

$$(a_0, \ldots, a_{11}) \propto \left(\left(-1 - \frac{1}{\sqrt{3}}\right)a_1 + (1-\sqrt{3})a_2 + \left(1 - \frac{2}{\sqrt{3}}\right)a_3, a_1, a_2, a_3, \frac{1}{3}(2\sqrt{3}a_1 + 3a_2 - 2\sqrt{3}a_3), a_1, (-1+\sqrt{3})a_1 + (1-\sqrt{3})a_2 + a_3, a_1, \frac{1}{3}(2\sqrt{3}a_1 + 3a_2 - 2\sqrt{3}a_3), a_3, a_2, a_1\right).$$

eigenvalue: $+i$
$M = 4$

$$(a_0, a_1, a_2, a_3) \propto (0, a_1, 0, -a_1),$$

$M = 8$

$$(a_0, \ldots, a_7) \propto (0, a_1, a_2, a_1 - \sqrt{2}a_2, 0, -(a_1 - \sqrt{2}a_2), -a_2, -a_1),$$

$M = 12$

$$(a_0, \ldots, a_{11}) \propto (0, a_1, a_2, a_3, 2a_1 - a_2 - (1+\sqrt{3})a_3, -a_1 - (-1-\sqrt{3})a_3, 0, a_1 + (-1-\sqrt{3})a_3, -2a_1 + a_2 + (1+\sqrt{3})a_3, -a_3, -a_2, -a_1).$$

eigenvalue: $-i$
$M = 4$ nothing,
$M = 8$

$$(a_0, \ldots, a_7) \propto (0, a_1, -\sqrt{2}a_1, -a_1, 0, a_1, \sqrt{2}a_1, -a_1),$$

$M = 12$

$$(a_0, \ldots, a_{11}) \propto (0, a_1, a_2, -(1+\sqrt{3})a_1 - (1+\sqrt{3})a_2,\\ a_2, a_1 + 2a_2, 0, -a_1 - 2a_2, -a_2, (1+\sqrt{3})a_1 + (1+\sqrt{3})a_2, -a_2, -a_1).$$

3. $M = 4n + 3$

eigenvalue: $+1$
$M = 3$

$$(a_0, a_1, a_2) \propto ((\sqrt{3}+1)a_1, a_1, a_1),$$

$M = 7$

$$(a_0, \ldots, a_6) \propto (LP_0(a_1, a_2), a_1, a_2, LP_3(a_1, a_2), LP_3(a_1, a_2), a_2, a_1),$$

$M = 11$

$$(a_0, \ldots, a_{10}) \propto (LP_0(a_1, a_2, a_3), a_1, a_2, a_3, LP_4(a_1, a_2, a_3),\\ LP_5(a_1, a_2, a_3), LP_5(a_1, a_2, a_3), LP_4(a_1, a_2, a_3), a_3, a_2, a_1).$$

eigenvalue: -1
$M = 3$

$$(a_0, a_1, a_2) \propto ((-\sqrt{3}+1)a_1, a_1, a_1),$$

$M = 7$

$$(a_0, \ldots, a_6) \propto (LP_0(a_1, a_2), a_1, a_2, LP_3(a_1, a_2), LP_3(a_1, a_2), a_2, a_1),$$

$M = 11$

$$(a_0, \ldots, a_{10}) \propto (LP_0(a_1, a_2, a_3), a_1, a_2, a_3, LP_4(a_1, a_2, a_3),\\ LP_5(a_1, a_2, a_3), LP_5(a_1, a_2, a_3), LP_4(a_1, a_2, a_3), a_3, a_2, a_1).$$

eigenvalue: $+i$
$M = 3$

$$(a_0, a_1, a_2) \propto (0, a_1, -a_1),$$

$M = 7$

$$(a_0, \ldots, a_6) \propto (0, a_1, a_2, LP_3(a_1, a_2), -LP_3(a_1, a_2), -a_2, -a_1),$$

$M = 11$

$$(a_0, \ldots, a_{10}) \propto (0, a_1, a_2, a_3, LP_4(a_1, a_2, a_3), LP_5(a_1, a_2, a_3), -LP_5(a_1, a_2, a_3), -LP_4(a_1, a_2, a_3), -a_3, -a_2, -a_1).$$

eigenvalue: $-i$
$M = 3$ nothing,
$M = 7$

$$(a_0, \ldots, a_6) \propto (0, a_1, LP_2(a_1), LP_3(a_1), -LP_3(a_1), -LP_2(a_1), -a_1),$$

$M = 11$

$$(a_0, \ldots, a_{10}) \propto (0, a_1, a_2, LP_3(a_1, a_2), LP_4(a_1, a_2), LP_5(a_1, a_2), -LP_5(a_1, a_2), -LP_4(a_1, a_2), -LP_3(a_1, a_2), -a_2, -a_1),$$

4. $M = 4n + 1$

eigenvalue: $+1$
$M = 5$

$$(a_0, a_1, a_2, a_3, a_4) \propto \left(\frac{\sqrt{5}+1}{2}(a_1 + a_2), a_1, a_2, a_2, a_1\right),$$

$M = 9$

$$(a_0, \ldots, a_8) \propto (LP_0(a_1, a_2, a_3), a_1, a_2, a_3, LP_4(a_1, a_2), LP_4(a_1, a_2), a_3, a_2, a_1).$$

eigenvalue: -1
$M = 5$

$$(a_0, a_1, a_2, a_3, a_4) \propto ((-\sqrt{5}+1)a_1, a_1, a_1, a_1, a_1),$$

$M = 9$

$$(a_0, \ldots, a_8) \propto (LP_0(a_1, a_2), a_1, a_2, LP_3(a_1, a_2), LP_4(a_1, a_2), LP_4(a_1, a_2), a_3, a_2, a_1).$$

eigenvalue: $+i$
$M = 5$

$$(a_0, a_1, a_2, a_3, a_4) \propto (0, a_1, LP_2(a_1), -LP_2(a_1), -a_1),$$

where

$$LP_2(a_1) = -\frac{2}{1+\sqrt{5}-\sqrt{2(5+\sqrt{5})}}a_1,$$

$M = 9$

$$(a_0, \ldots, a_8) \propto (0, a_1, a_2, LP_3(a_1, a_2), LP_4(a_1, a_2), -LP_4(a_1, a_2), -LP_3(a_1, a_2), -a_2, -a_1).$$

eigenvalue: $-i$
$M = 5$

$$(a_0, a_1, a_2, a_3, a_4) \propto (0, a_1, LP_2(a_1), -LP_2(a_1), -a_1),$$

where

$$LP_2(a_1) = -\frac{2}{1+\sqrt{5}+\sqrt{2(5+\sqrt{5})}}a_1,$$

$M = 9$

$$(a_0, \ldots, a_8) \propto (0, a_1, a_2, LP_3(a_1, a_2), LP_4(a_1, a_2), -LP_4(a_1, a_2), -LP_3(a_1, a_2), -a_2, -a_1).$$

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