Effective potential at three loops

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I present the effective potential at three-loop order for a general renormalizable theory, using the $\overline{\text{MS}}$ renormalization scheme and Landau gauge fixing. As applications and illustrative points of reference, the results are specialized to the supersymmetric Wess-Zumino model and to the standard model. In each case, renormalization group scale invariance provides a consistency check. In the Wess-Zumino model, the required vanishing of the minimum vacuum energy yields an additional check. For the standard model, I carry out the resummation of Goldstone boson contributions, which provides yet more opportunities for nontrivial checks, and obtain the minimization condition for the Higgs vacuum expectation value at full three-loop order. An infrared divergence due to doubled photon propagators appears in the three-loop standard model effective potential, but it does not affect the minimization condition or physical observables and can be eliminated by resummation.

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I. INTRODUCTION

The effective potential [1–3] is a useful tool for understanding spontaneous symmetry breaking in quantum field theories. It can be defined in perturbation theory, and calculated, by expanding the scalar fields appearing in the Lagrangian about constant background values ϕ, and then summing the one-particle-irreducible vacuum (no external legs) Feynman diagrams, using propagator masses and interaction vertices that depend on the background scalar fields. The full two-loop effective potential has been obtained for the standard model in the $\overline{\text{MS}}$ scheme and Landau gauge by Ford, Jack, and Jones in Ref. [4], and in general theories (including softly broken supersymmetric ones, which use a different regulator based on dimensional reduction) in Ref. [5]. The three-loop effective potential for the standard model has been found in the approximation that the QCD and top Yukawa couplings are larger than all other couplings, in Ref. [6], and the four-loop contribution only at leading order in QCD [7].

One application of the effective potential is to study the stability properties of our vacuum state in the standard model [3,8–31] and extensions of it. This has attracted great interest recently due to the apparent proximity of the Higgs boson self-coupling to the critical value associated with metastability.

Another important use of the effective potential is to relate the vacuum expectation value (VEV) of the symmetry breaking scalar field(s) to the Lagrangian parameters, including the negative Higgs squared mass parameter. Note that the VEV can be defined as the value of the constant scalar background fields that minimizes either the tree-level potential or the full effective potential. Choosing the first definition, with the VEV as the minimum of the tree-level potential, has the advantages of providing gauge-invariant running masses, and allowing for checks in subsequent calculations of other quantities by varying the gauge-fixing parameter. However, it requires the inclusion of tadpole diagrams in those calculations (see for example [32–37]), which causes inverse powers of the Higgs coupling to appear in perturbation theory.

By defining the VEV as the minimum of the full effective potential, the sum of all tadpole graphs automatically vanishes, and inverse powers of the Higgs self-coupling do not occur, so that in calculations of other quantities, perturbation theory converges faster. The price to be paid for using this "tadpole-free" scheme is that the resulting VEV is dependent on the gauge-fixing choice, and therefore so are the running $\overline{\text{MS}}$ masses of the particles. This is of course not a real problem, because the VEV and the running masses are not physical observables. Calculations in this approach are simplest in Landau gauge, where there is no mixing between Goldstone bosons and vector gauge bosons and the gauge-fixing parameter is not renormalized. (Reference [38] is a good example of using a tadpole-free scheme but in Feynman gauge.) Although fixing to Landau gauge precludes obtaining checks from varying the gauge-fixing parameter, there are checks of similar power from cancellations in observable quantities between Goldstone bosons and the unphysical components of vector degrees of freedom. The tadpole-free pure $\overline{\text{MS}}$ scheme has been used to calculate the complex pole masses of the Higgs [39], W [40], and Z [41] bosons, and the top quark [42], to full two-loop order, in terms of the $\overline{\text{MS}}$ Lagrangian parameters, using notation and computational methods consistent with the present paper.

In this paper, I will obtain the three-loop effective potential for a general renormalizable quantum field theory in four dimensions, using Landau gauge fixing and the $\overline{\text{MS}}$

scheme [43,44] based on dimensional regularization [45–49]. In the following, $1/16\pi^2$ is used as a loop expansion parameter, so that the effective potential is written as:

$$V_{\rm eff}(\phi) = V^{(0)} + \frac{1}{16\pi^2} V^{(1)} + \frac{1}{(16\pi^2)^2} V^{(2)} + \frac{1}{(16\pi^2)^3} V^{(3)} + \cdots \quad (1.1)$$

As is well-known, the contribution $V^{(\ell)}$ is obtained as the sum of one-particle-irreducible ℓ-loop vacuum Feynman diagrams, using propagator masses and vertices that depend on the constant background scalar field(s) ϕ. First derivatives of the effective potential correspond to tadpole diagrams involving the scalar fields, and so working at the minimum of $V(\phi)$ guarantees that the sum of tree-level and loop-corrected tadpoles vanishes, and therefore tadpoles need not be included in other calculations. The new results for the contributions to $V^{(3)}$ will be presented in Sec. III. As illustrative applications of the general results, I will specialize them to the cases of the supersymmetric Wess-Zumino model and the standard model, in Secs. IV and V respectively. Many of the results obtained below are too lengthy to show in print, and so are presented instead in ancillary electronic files in forms suitable for use with computers.

An important way of checking a calculation of the effective potential is by requiring renormalization group invariance, provided that the pertinent beta functions have already been calculated to the corresponding order by other means. The requirement that $V_{\rm eff}$ does not depend on the choice of the $\overline{\rm MS}$ renormalization scale Q can be written as

$$Q\frac{dV_{\rm eff}}{dQ} = \left(Q\frac{\partial}{\partial Q} + \sum_X \beta_X \frac{\partial}{\partial X}\right) V_{\rm eff} = 0. \quad (1.2)$$

where X runs over all of the independent Lagrangian parameters, including the background scalar field(s). The beta function for a background scalar field ϕ is related to its anomalous dimension γ by $\beta_\phi = -\phi\gamma$. The loop expansions for the beta functions of X can be written:

$$\beta_X = \frac{1}{16\pi^2}\beta_X^{(1)} + \frac{1}{(16\pi^2)^2}\beta_X^{(2)} + \frac{1}{(16\pi^2)^3}\beta_X^{(3)} + \cdots. \quad (1.3)$$

Then it follows that at each loop order $\ell = 1, 2, 3, \ldots$, one must have:

$$Q\frac{\partial}{\partial Q} V^{(\ell)} + \sum_{n=0}^{\ell-1}\left(\sum_X \beta_X^{(\ell-n)} \frac{\partial}{\partial X} V^{(n)}\right) = 0. \quad (1.4)$$

This will be applied below as a check in both the Wess-Zumino model and standard model.

II. CONVENTIONS AND SETUP

The conventions and notations for this paper related to the effective potential and to two-component fermions generally follow Refs. [5,50] respectively, with some minor cosmetic variations. After expansion about the constant background scalar field(s), the Lagrangian can be written without loss of generality in terms of real scalars R_j, real vectors $A^{\mu a}$, and left-handed two-component fermion fields ψ_I, with background-field-dependent masses and interaction couplings. (In many cases, complex bosonic fields with well-defined charges could be used, but in order to present results in a general way, I take advantage of the fact that they can always be decomposed into real and imaginary parts.) For fermion fields that carry conserved charges, it is most convenient to use pairs of 2-component left-handed fields ψ_I and $\psi_{I'}$ with opposite charges and therefore a purely off-diagonal Dirac mass $M^{II'}$, so that the common squared mass for both fields is $M_I^2 = M_{I'}^2 \equiv |M^{II'}|^2$. This means that the two-component fermion fields are always squared mass eigenstates but sometimes not mass eigenstates.

The squared-mass eigenstate fields are therefore labeled by indices j, k, l, m, n, p for real scalars, a, b, c, d, e, f for real vectors, and I, J, K, L for two-component fermions, with the understanding that I', J', K', L' are used to denote the corresponding mass partners when they form a Dirac pair, and with $I' = I$ for a fermion with a Majorana-type mass. As a convention, repeated indices are always taken to be summed over.

The most general interaction Lagrangian for a renormalizable theory can be written in terms of background-field-dependent couplings as (using a metric of signature $-, +, +, +$):

$$\mathcal{L} = -\frac{1}{6}\lambda^{jkl}R_jR_kR_l - \frac{1}{24}\lambda^{jklm}R_jR_kR_lR_m - \frac{1}{2}(Y^{jIJ}R_j\psi_I\psi_J + {\rm c.c.}) + g_I^{aJ}A^{\mu a}\psi^{\dagger I}\bar\sigma_\mu\psi_J - g^{ajk}A^{\mu a}R_j\partial_\mu R_k - \frac{1}{4}g^{abjk}A_\mu^a A^{\mu b}R_jR_k - \frac{1}{2}g^{abj}A_\mu^a A^{\mu b}R_j - g^{abc}A^{\mu a}A^{\nu b}\partial_\mu A_\nu^c - \frac{1}{4}g^{abe}g^{cde}A^{\mu a}A^{\nu b}A_\mu^c A_\nu^d - g^{abc}A^{\mu a}\omega^b\partial_\mu\bar\omega^c. \quad (2.1)$$

Here λ^{jkl} and λ^{jklm} are real scalar interactions that are totally symmetric under interchange of all indices, Y^{jIJ} are Yukawa couplings that are symmetric under interchange of I, J, and g_I^{aJ} are vector interactions with fermions, and g^{ajk}, g^{abjk}, and g^{abj} are vector interactions with scalars, and g^{abc} are vector self-interactions. Note that the sign of g^{abc} has been flipped compared to the notation of Ref. [5]; this has no impact on the results of that reference, because at two-loop order g^{abc} only appears in pairs. Because the scalars and vectors are real, the heights of their indices have no significance, and are chosen for typographical convenience. As a convention, flipping the heights of all fermion indices corresponds to complex conjugation, so that

$$Y_{jIJ} \equiv (Y^{jIJ})^*, \tag{2.2}$$

$$M_{II'} \equiv (M^{II'})^*, \tag{2.3}$$

and

$$g^{aI}_J = (g^{aJ}_I)^*. \tag{2.4}$$

All of the couplings with names involving g have their origin as gauge couplings. In Landau gauge, the ghost fields ω^a and $\bar{\omega}^a$ are massless. Note that the vector cubic, vector quartic, and vector-ghost-antighost interactions are all written in terms of a common, totally antisymmetric, ϕ-dependent, coupling g^{abc}. The vector-vector-scalar couplings g^{abj} are symmetric under interchange of the vector indices a, b, and the vector-scalar-scalar couplings g^{ajk} are antisymmetric under interchange of the scalar indices j, k. Note also that the vector-vector-scalar-scalar couplings are not independent; they can always be written in terms of the vector-scalar-scalar couplings, according to

$$g^{abjk} = g^{ajl}g^{bkl} + g^{akl}g^{bjl}. \tag{2.5}$$

Reference [5] did not mention or exploit this fact, as it leads to only a slight simplification at two-loop order, but it is used extensively below.

In the following, the names of fields or the corresponding indices will be used as synonyms for the corresponding squared mass arguments used in loop integral functions. For example, we can write the well-known 1-loop effective potential in the $\overline{\mathrm{MS}}$ scheme and Landau gauge as simply

$$V^{(1)} = \sum_j f(j) - 2\sum_I f(I) + 3\sum_a f_V(a), \tag{2.6}$$

where

$$f(x) = xA(x)/4 - x^2/8 = \frac{x^2}{4}(\overline{\ln}(x) - 3/2), \tag{2.7}$$

$$f_V(x) = xA(x)/4 + x^2/24 = \frac{x^2}{4}(\overline{\ln}(x) - 5/6). \tag{2.8}$$

Here,

$$\overline{\ln}(x) \equiv \ln(x/Q^2) \tag{2.9}$$

where Q is the $\overline{\mathrm{MS}}$ renormalization scale, x is the squared mass argument, and

$$A(x) = x\overline{\ln}(x) - x \tag{2.10}$$

is a one-loop integral basis function (which was denoted by $J(x)$ in Ref. [5]). The two-loop contribution to the effective potential is

$$\begin{aligned} V^{(2)} = {}& \frac{1}{12}(\lambda^{jkl})^2 f_{SSS}(j,k,l) + \frac{1}{8}\lambda^{jjkk} f_{SS}(j,k) + \frac{1}{2}Y^{jIJ}Y_{jIJ}f_{FFS}(I,J,j) + \frac{1}{4}(Y^{jIJ}Y^{jI'J'}M_{II'}M_{JJ'} + \text{c.c.})f_{\bar{F}\bar{F}S}(I,J,j) \\ & + \frac{1}{4}(g^{ajk})^2 f_{VSS}(a,j,k) + \frac{1}{4}(g^{abj})^2 f_{VVS}(a,b,j) + \frac{1}{2}g^{aJ}_I g^{aI}_J f_{FFV}(I,J,a) + \frac{1}{2}g^{aJ}_I g^{aJ'}_{I'} M^{II'}M_{JJ'}f_{\bar{F}\bar{F}V}(I,J,a) \\ & + \frac{1}{12}(g^{abc})^2 f_{\text{gauge}}(a,b,c), \end{aligned} \tag{2.11}$$

in terms of two-loop integral functions f_{SSS}, f_{SS}, f_{FFS}, $f_{\bar{F}\bar{F}S}$, f_{VSS}, f_{VVS}, f_{FFV}, $f_{\bar{F}\bar{F}V}$, and f_{gauge} that were originally computed (in a different notation) by Ford, Jack, and Jones in Ref. [4] in the context of the standard model. They were given in Ref. [5] in the context of a general renormalizable theory, in the notation of the present paper, with one exception; in Eq. (2.11) above, I have used Eq. (2.5) to combine the terms that involved the functions f_{SSV} and f_{VS} in Ref. [5], by defining a new function

$$f_{VSS}(x,y,z) \equiv f_{SSV}(y,z,x) + f_{VS}(x,y) + f_{VS}(x,z). \tag{2.12}$$

Explicitly,

$$\begin{aligned} f_{VSS}(x,y,z) = {}& [-\lambda(x,y,z)I(x,y,z) + (y-z)^2 I(0,y,z) \\ & + (2x - y + z)A(x)A(y) \\ & + (2x + y - z)A(x)A(z)]/x + A(y)A(z) \\ & + 2(y + z - x/3)A(x) + 2x[A(y) + A(z)] \end{aligned} \tag{2.13}$$

for $x \neq 0$, and

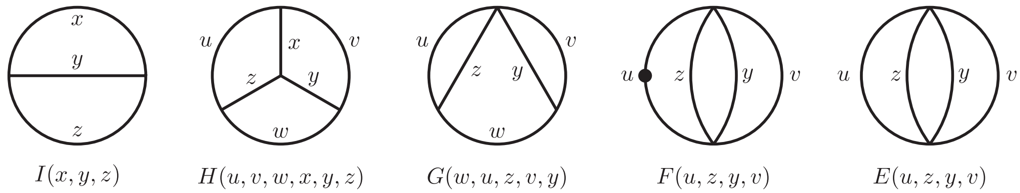


FIG. 1. The topologies for the 2-loop and 3-loop basis vacuum integral functions used in this paper. The large dot in $F(u, z, y, v)$ corresponds to a derivative with respect to the u squared mass argument. The function $\bar{F}(u, z, y, v)$ is the same as $F(u, z, y, v)$ but with a subtraction to render it infrared finite in the limit $u \to 0$. In each case, counterterms have been included to make the integrals finite as the ultraviolet regulator $\epsilon \to 0$. See Ref. [51] for the precise definitions and more information.

$$f_{VSS}(0, y, z) = 3(y + z)I(0, y, z) + 3A(y)A(z) - 2yA(y) - 2zA(z) + (y + z)^2, \tag{2.14}$$

where $I(x, y, z)$ is a two-loop basis integral used in Refs. [4,5], and defined in the latter reference in the notation appropriate for the present paper, and

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \tag{2.15}$$

is the usual triangle function.

In the following, I will present the three-loop contribution to the effective potential in terms of three-loop vacuum integral functions using the notation of Ref. [51], which also provides a computer code 3VIL for their numerical evaluation[1] using the differential equations method. The basis integral functions consist of a 1-loop integral $A(x)$ already given above in Eq. (2.10), the two-loop integral $I(x, y, z)$ mentioned in the previous paragraph, and three-loop integral functions $H(u, v, w, x, y, z)$, $G(w, u, z, v, y)$, and $F(u, z, y, v)$, corresponding to topologies shown in Fig. 1. For convenience, it is also useful to define a related function

$$\bar{F}(u, z, y, v) = F(u, z, y, v) + \overline{\ln}(u)I(v, y, z) \tag{2.16}$$

which is finite in the limit $u \to 0$, and an integral $E(u, z, y, v)$ given by Eq. (2.40) in Ref. [51], which corresponds to the same topology as $F(u, z, y, v)$ but without a derivative with respect to u. In the following, I will employ $F(u, z, y, v)$ instead of $\bar{F}(u, z, y, v)$ when the first argument does not vanish, and otherwise use $\bar{F}(0, z, y, v)$. Technically, $E(u, z, y, v)$ is not a basis integral because it can be written as a linear combination of F (or $\bar{F}$) integrals with the same arguments in different orders, but it is convenient to use E to express some quantities in simplest form. Each of the basis integral functions is defined to include counterterms that make them finite and independent of the ultraviolet dimensional regularization parameter ϵ, but dependent on the $\overline{\text{MS}}$ renormalization scale Q. This simplifies the presentation of results in the $\overline{\text{MS}}$ scheme, as ϵ never appears. Consult Ref. [51] for the precise definitions of the basis integrals, and more information.

It is also convenient, when dealing with Feynman diagrams with "doubled propagators" (i.e., two propagators that carry the same momentum) to define functions by:

$$\bar{A}(w,x) = [A(w) - A(x)]/(x - w) \quad (x \neq w), \tag{2.17}$$

$$\bar{I}(w,x,y,z) = [I(w,y,z) - I(x,y,z)]/(x - w) \quad (x \neq w), \tag{2.18}$$

$$K(w,x,u,z,y,v) = [G(w,u,z,y,v) - G(x,u,z,y,v)]/(x - w) \quad (x \neq w). \tag{2.19}$$

In each case, the first two arguments w, x are the squared masses of the doubled propagators. When the squared masses for double propagators coincide, these functions become:

$$\bar{A}(x, x) = -\frac{\partial}{\partial x}A(x), \tag{2.20}$$

$$\bar{I}(x, x, y, z) = -\frac{\partial}{\partial x}I(x, y, z), \tag{2.21}$$

$$K(x, x, u, z, y, v) = -\frac{\partial}{\partial x}G(x, u, z, y, v), \tag{2.22}$$

The necessary derivatives in Eqs. (2.20)–(2.22) can be expressed in terms of the basis functions as:

[1]See also Refs. [52,53] for a different approach to numerical computation of the 3-loop vacuum basis integrals, based on dispersion relations. Also, Refs. [54–73] found a variety of important special analytical cases that have been incorporated into 3VIL.

$$\frac{\partial}{\partial x}A(x) = 1 + A(x)/x, \tag{2.23}$$

$$\begin{aligned}\frac{\partial}{\partial x}I(x,y,z) = \{&(x-y-z)[I(x,y,z) - A(x) - A(y) - A(z) + x + y + z] - 2A(y)A(z)\\ &+ (x-y+z)A(x)A(y)/x + (x+y-z)A(x)A(z)/x\}/\lambda(x,y,z),\end{aligned} \tag{2.24}$$

$$\begin{aligned}\frac{\partial}{\partial x}G(x,u,z,y,v) = &[(x-u-z)/\lambda(x,u,z) + (x-v-y)/\lambda(x,v,y) - 1/x]G(x,u,z,y,v)\\ &+ \{[(x+u-z)A(z) + (x+z-u)A(u) + x(u+z-x)]I(x,v,y)\\ &+ u(u-z-x)A(u)/4 + z(z-u-x)A(z)/4 + x(u+z-x)[A(v)+A(y)]\\ &+ 2x[x^2 + x(2v+2y-u-z) - 2uv - 2uy - 2vz - 2yz - 8uz]/3\\ &+ u(x+z-u)F(u,v,y,z) + z(x+u-z)F(z,u,v,y)\}/x\lambda(x,u,z)\\ &+ \{[(x+v-y)A(y) + (x+y-v)A(v) + x(v+y-x)]I(x,u,z)\\ &+ v(v-y-x)A(v)/4 + y(y-v-x)A(y)/4 + x(v+y-x)[A(u)+A(z)]\\ &+ 2x[x^2 + x(2u+2z-v-y) - 2uv - 2uy - 2vz - 2yz - 8vy]/3\\ &+ v(x+y-v)F(v,u,y,z) + y(x+v-y)F(y,u,v,z)\}/x\lambda(x,v,y)\\ &- (7x+2u+2v+2y+2z)/3x.\end{aligned} \tag{2.25}$$

Special cases that arise when the $\lambda(x,y,z)$ denominators vanish can be obtained as smooth limits of the above. Of particular importance are the following:

$$\frac{\partial}{\partial x}I(x,0,x) = 2\left[\frac{\partial}{\partial x}I(x,0,z)\right]\bigg|_{z=x} = -A(x)^2/x^2, \tag{2.26}$$

$$\begin{aligned}\frac{\partial}{\partial x}G(x,0,z,v,y)\bigg|_{z=x} = &[F(x,0,v,y) - \bar{F}(0,x,v,y)]/2x + \{[(v-y)^2 - x^2]A(x)I(x,v,y)/2x\\ &+ [3x^2 - 4x(v+y) + (v-y)^2]I(x,v,y)/2 + [(v-y-x)A(v) + (y-v-x)A(y)\\ &+ x(x-v-y)]A(x)^2/x + [2A(v)A(y) + 2(x-y)A(y) + 2(x-v)A(v)\\ &+ (7v^2 + 7y^2 + 18vy + 10xv + 10xy - 17x^2)/8]A(x) + x(v+y-x)[A(v)+A(y)]\\ &- 2xA(v)A(y) + x[5x^2 - 4x(v+y) - v^2 - y^2 - 10vy]/3\}/x\lambda(x,v,y),\end{aligned} \tag{2.27}$$

and

$$\frac{\partial}{\partial x}G(x,0,y,0,y)|_{y=x} = 1 - \bar{F}(0,0,x,x)/x - A(x)/x. \tag{2.28}$$

It is often important to have expansions of the integral functions when one or more of the squared mass arguments is small. In the following I will regulate infrared divergences in massless vector bosons by giving the propagator a small squared mass (rather than using dimensional regularization for the infrared divergences, which can cause confusion with the ultraviolet divergences). Goldstone bosons also have squared masses that can be consistently treated as small compared to those of other particles. The expansions of the basis integrals in a small squared mass δ (taking $\delta \ll x, y, z\ldots$, where $x, y, z\ldots$ are other pertinent non-zero squared masses in the diagram) can be accomplished using the differential equations that the basis integrals satisfy, which were given in [51]. As an example, for the 2-loop basis integral function, one can find through order δ^2 that, for $x \neq y$:

$$I(\delta,x,y) = I(0,x,y) + \delta[-(x+y)I(0,x,y) - 2A(x)A(y) + (3x-y)A(x) + (3y-x)A(y) - (x+y)^2]/(x-y)^2 + \delta\overline{\ln}(\delta)\bar{A}(x,y) + \delta^2[-2xyI(0,x,y) - (x+y)A(x)A(y) + (7xy-x^2-2y^2)A(x)/2 + (7xy-2x^2-y^2)A(y)/2 + (x+y)(2xy-5x^2-5y^2)/4]/(x-y)^4 + \delta^2\overline{\ln}(\delta)[(x^2-y^2)/2 + xA(y) - yA(x)]/(x-y)^3 + \mathcal{O}(\delta^3) \tag{2.29}$$

and

$$I(\delta,x,x) = I(0,x,x) + (\delta/x)[4x + 3A(x) + A(x)^2/2x - (x+A(x))\overline{\ln}(\delta)] + (\delta/x)^2[-11x/18 - A(x)/6 + x\overline{\ln}(\delta)/6] + \mathcal{O}(\delta^3), \tag{2.30}$$

$$I(\delta,0,x) = I(0,0,x) + (\delta/x)[\zeta_2 x + 2A(x) + A(x)^2/2x - A(x)\overline{\ln}(\delta)] + (\delta/x)^2[-5x/4 - A(x)/2 + x\overline{\ln}(\delta)/2] + \mathcal{O}(\delta^3), \tag{2.31}$$

$$I(\delta,\delta,x) = I(0,0,x) + (\delta/x)[2\zeta_2 x + 4A(x) + A(x)^2/x - 2A(x)\overline{\ln}(\delta)] + (\delta/x)^2[(2\zeta_2 - 5/2)x + A(x) + A(x)^2/x - (x+2A(x))\overline{\ln}(\delta) + x\overline{\ln}^2(\delta)] + \mathcal{O}(\delta^3), \tag{2.32}$$

$$I(\delta,\delta,\delta) = \delta[3\sqrt{3}\mathrm{Ls}_2 - 15/2 + 6\overline{\ln}(\delta) - 3\overline{\ln}^2(\delta)/2], \tag{2.33}$$

$$I(0,\delta,\delta) = \delta[-5 + 4\overline{\ln}(\delta) - \overline{\ln}^2(\delta)], \tag{2.34}$$

$$I(0,0,\delta) = \delta[-5/2 - \zeta_2 + 2\overline{\ln}(\delta) - \overline{\ln}^2(\delta)/2], \tag{2.35}$$

$$I(0,0,0) = 0, \tag{2.36}$$

where $\mathrm{Ls}_2 = -\int_0^{2\pi/3} dx \ln[2\sin(x/2)] \approx 0.6766277376064358$. A large number of similar expansion formulas for the 3-loop basis integrals, including all of the ones necessary for results below, are given in an ancillary electronic file provided with this paper, called `expzero.anc` [108]. In general, the functions I, $\bar{F}$, G, and H have smooth limits as $\delta \to 0$, with expansion terms that are powers of δ that may be multiplied by polynomials (of up to cubic order) in $\overline{\ln}(\delta)$. The expansion of the function F contains a $\overline{\ln}(\delta)$ as the leading behavior for $\delta \to 0$ if (and only if) the first argument is δ, as can be seen from Eq. (2.16). The limits for small δ of $\bar{A}(\delta,\delta)$ and $\bar{I}(\delta,\delta,x,y)$ and $K(\delta,\delta,u,v,x,y)$ have logarithmic infrared singularities, because they also involve doubled propagators with the same momentum and the same small squared mass δ. Assuming that either x or y and either u or v are large compared to δ, one has:

$$\bar{A}(\delta,\delta) = -\overline{\ln}(\delta), \tag{2.37}$$

$$\bar{I}(\delta,\delta,x,y) = -\overline{\ln}(\delta)\bar{A}(x,y) + \cdots, \tag{2.38}$$

$$K(\delta,\delta,u,v,x,y) = -\overline{\ln}(\delta)\bar{A}(u,v)\bar{A}(x,y) + \cdots, \tag{2.39}$$

where the ellipses refer to terms that are finite as $\delta \to 0$. The expansions needed for the cases that occur in the standard model, through order δ^5 for I, F and $\bar{F}$ functions, through order δ^4 for $\bar{I}$ and G functions, and through order δ^3 for K and H functions, are given in `expzero.anc` [108]. Further expansion cases as may be needed for more general theories can be obtained by using the differential equations given in Ref. [51].

Finally, it is important to note that the loop integral basis functions satisfy certain identities when the squared mass arguments are not generic, either because some of them are equal to each other, or vanish. (These identities can be discovered by requiring smooth limits of derivatives of the integral functions as the arguments approach the non-generic configurations.) Some identities of this type were given in Eqs. (5.79)–(5.80) and (5.82)–(5.86) of Ref. [51]. Other identities that are used in the following are

$$\begin{aligned} F(x,x,y,y) &= (x/y-1)[F(x,0,0,y)+I(0,x,y)+A(y)-2y\zeta_3] \\ &\quad + [A(y)/y - A(x)/x]I(0,x,y) + A(x)[A(y)^2/y + A(x) - 2A(y) \\ &\quad + 3x^2/4y - 9x/2 + 2y]/x - 2x^2/3y + 10x/3 + 2y, \end{aligned} \tag{2.40}$$

$$G(0,0,0,x,y) = -\bar{F}(0,0,x,y) + 2I(0,x,y) + A(x) + A(y) - 4x/3 - 4y/3, \tag{2.41}$$

$$\begin{aligned} G(y,0,0,0,x) &= G(x,0,0,0,y) + (x-y)[F(x,0,0,y) + I(0,x,y) + A(x)A(y)/x \\ &\quad + 2y\zeta_3 - 2(x+5y)/3]/y + A(x)A(y)[A(y)-A(x)]/(2xy) \\ &\quad + [1/4 + 3x/4y + (1+\zeta_2)y/x]A(x) - [2 + x\zeta_2/y]A(y), \end{aligned} \tag{2.42}$$

$$\begin{aligned} G(x,0,y,0,y) &= (y-x)[F(x,0,0,y)/y + \bar{F}(0,0,x,y)/x] + 2A(x)A(y)/x \\ &\quad + [3 - x/y - (1/x + 1/y)A(y)]I(0,x,y) + [y - A(x) - A(y)/3]A(y)^2/xy \\ &\quad + [(3xy/4 - y^2 - 3x^2/4)A(x) - (x-y)^2A(y)]/xy \\ &\quad + 2(x^2 + 2xy - 7y^2)/3y + 2(y^2 + 6xy - 3x^2)\zeta_3/3x. \end{aligned} \tag{2.43}$$

In addition, one can express the following 1-scale integrals in terms of ζ_2 and ζ_3 and powers of $A(x)$, with rational coefficients, using the analytical formulas collected in section V of Ref. [51]: $I(0,0,x)$, $I(0,x,x)$, $F(x,0,0,0)$, $F(x,0,0,x)$, $F(x,x,x,x)$, $\bar{F}(0,0,0,x)$, $\bar{F}(0,0,x,x)$, $G(0,0,0,0,x)$, $G(0,0,0,x,x)$, $G(0,0,x,0,x)$, $G(x,0,0,0,0)$, $G(x,0,0,0,x)$, $G(x,0,x,0,x)$, and $G(0,x,x,x,x)$. The existence of these identities means that the presentation of results for any specific theory (for example, the standard model) in terms of the basis functions is far from unique; the basis is overcomplete when the arguments are not generic.

III. THREE-LOOP CONTRIBUTIONS TO THE EFFECTIVE POTENTIAL

A. Feynman diagrams

In this Sec. I present the results for the three-loop contribution to the effective potential for a general renormalizable quantum field theory. The 1-particle-irreducible Feynman diagrams for the three-loop effective potential have the topologies shown in Fig. 2. To distinguish the different diagrams, each topology is associated with a letter E, G, H, J, K, or L, and then subscripts S, V, F, $\bar{F}$, or g are applied, corresponding respectively to real scalar, real vector, helicity-preserving fermion, helicity-violating fermion, or ghost propagators, in the order designated by the numbering in Fig. 2. The helicity-violating fermion propagators each contain a mass insertion of the type $M_{II'}$ or $M^{II'}$, as described in Ref. [50]. To illustrate this labeling scheme, Fig. 3 shows the Feynman diagrams corresponding to diagrams $H_{F\bar{F}SVF\bar{F}}$ and K_{VVSSFF}. For each diagram, there is a corresponding loop integral function, which one can compute in terms of the basis functions discussed in the previous section after including the $\overline{\text{MS}}$ counterterms. The squared mass arguments are given in the same ordering as the corresponding subscripts.

However, the correspondence between Feynman diagrams and loop integral functions is not one-to-one, because in some cases involving vector bosons it is convenient to define loop integral functions that combine the effects of more than one Feynman diagram, by exploiting the constraints implied by the underlying gauge invariance that is associated with vector fields in renormalizable theories. For example, because of the relation between the vector-scalar-scalar couplings g^{ajk} and the vector-vector-scalar-scalar couplings g^{abjk} given in Eq. (2.5), it is convenient to define a single function $K_{SSSSSV}(u,v,w,x,y,z)$ that combines the effect of the Feynman diagram labeled K_{SSSSSV} with the one labeled J_{SSSSV}. (For this reason, there is no function J_{SSSSV} below.)

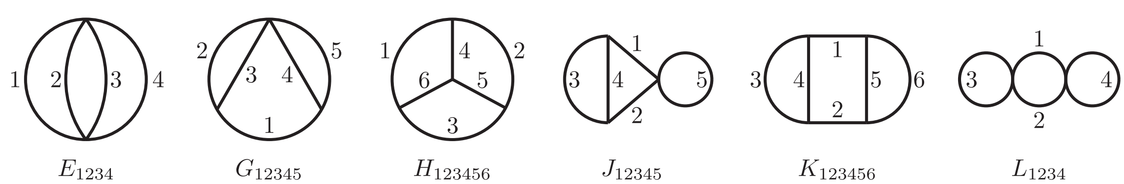
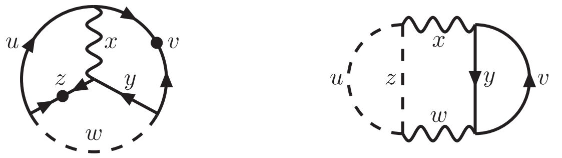


FIG. 2. The Feynman diagram topologies contributing to the 3-loop effective potential, with numerals indicating the ordering of subscripts denoting propagator types (S, F, $\bar{F}$, V, or g) as well as the ordering of the corresponding squared mass arguments.



$H_{F\overline{F}SVF\overline{F}}(u,v,w,x,y,z)$ $\quad$ $K_{VVSSFF}(x,w,u,z,y,v)$

FIG. 3. Examples of the Feynman diagram labeling scheme used in this paper, for the diagrams with loop integral functions denoted $H_{F\bar{F}SVF\bar{F}}(u,v,w,x,y,z)$ and $K_{VVSSFF}(x,w,u,z,y,v)$. Solid lines with arrows represent helicity-preserving fermion propagators. Solid lines with a dot and clashing arrows represent a helicity-violating fermion propagator. Dashed lines indicate a real scalar propagator, and wavy lines stand for real vector propagators. The squared masses are denoted by u,v,w,x,y,z as labeled.

There are numerous similar cases where the contribution of a diagram with a vector-vector-scalar-scalar interaction is combined with the contribution from a related diagram with a pair of vector-scalar-scalar interactions to give a single loop integral function. Furthermore, because of the fact that the vector quartic interaction and the vector-ghost-antighost interaction are determined by the triple vector coupling, as seen in Eq. (2.1), the effects of the diagrams labeled H_{VVVVVV}, H_{VgggVg}, H_{gggVVV}, and parts of G_{VVVVV} and E_{VVVV} can always be combined into a single function that I call $H_{\rm gauge}$. The other parts of diagrams G_{VVVVV} and E_{VVVV}, together with the contributions of diagrams K_{VVVVVV}, K_{VVgggg}, K_{VVVVgg}, K_{gggVVg}, J_{VVVVV}, J_{VVggV}, and L_{VVVV} can always be combined into a single function to be denoted $K_{\rm gauge}$. Similarly, I define a function $H_{{\rm gauge},S}$ that combines the effects from the diagrams H_{VVVVVS} and G_{SVVVV}; a function $K_{{\rm gauge},S}$ that combines the effects of diagrams K_{VVSVVV}, K_{VVSVgg}, and J_{VVVSV}; a function $K_{{\rm gauge},SS}$ that combines the contributions from K_{VVSSVV}, K_{VVSSgg}, J_{VVSSV}, J_{VVVVS}, J_{VVggS}, and L_{VVVS}; a function $K_{{\rm gauge},FF}$ that combines diagrams K_{VVVVFF}, K_{VVFFgg}, and J_{VVFFV}; and a function $K_{{\rm gauge},\bar{F}\bar{F}}$ that combines diagrams $K_{VVVV\bar{F}\bar{F}}$, $K_{VV\bar{F}\bar{F}gg}$, and $J_{VV\bar{F}\bar{F}V}$.

Finally, note that there are two diagrams, G_{VSSSS} and G_{VSSVV}, which one can draw and for which the couplings exist, but for which the corresponding loop integrals vanish identically.

Taking into account the above considerations, I find that the three-loop contributions to the $\overline{\rm MS}$ renormalized Landau gauge effective potential for a general renormalizable theory can be expressed in terms of only 89 distinct loop integral functions:

$$
\begin{aligned}
&H_{SSSSSS},\quad K_{SSSSSS},\quad J_{SSSSS},\quad G_{SSSSS},\quad L_{SSSS},\quad E_{SSSS},\quad H_{FF\bar{F}SSS},\quad H_{\bar{F}\bar{F}\bar{F}SSS},\quad H_{FFSSFF},\\
&H_{FFSS\bar{F}\bar{F}},\quad H_{F\bar{F}SSF\bar{F}},\quad H_{\bar{F}\bar{F}SS\bar{F}\bar{F}},\quad K_{SSSSFF},\quad K_{SSSS\bar{F}\bar{F}},\quad K_{FFFSSF},\quad K_{FF\bar{F}SS\bar{F}},\\
&K_{\bar{F}\bar{F}FSSF},\quad K_{\bar{F}F\bar{F}SSF},\quad K_{\bar{F}\bar{F}\bar{F}SS\bar{F}},\quad K_{SSFFFF},\quad K_{SSFF\bar{F}\bar{F}},\quad K_{SS\bar{F}\bar{F}\bar{F}\bar{F}},\quad J_{SSFFS},\quad J_{SS\bar{F}\bar{F}S},\\
&H_{SSSSSV},\quad H_{VVSSSS},\quad H_{SSVVSS},\quad H_{VVVSSS},\quad H_{SSSVVV},\quad H_{VVSSVS},\quad H_{SSVVVV},\\
&H_{SVVVSV},\quad K_{SSSSSV},\quad K_{SSSSVV},\quad K_{SSSVVS},\quad K_{VVSSSS},\quad K_{SSSVVV},\quad K_{VVSSVS},\\
&K_{SSVVVV},\quad K_{VVSVVS},\quad J_{SSVSS},\quad J_{SSVVS},\quad G_{VSVVS},\quad H_{{\rm gauge},S},\quad K_{{\rm gauge},S},\quad K_{{\rm gauge},SS},\\
&H_{FFVVFF},\quad H_{FFVV\bar{F}\bar{F}},\quad H_{F\bar{F}VVF\bar{F}},\quad H_{\bar{F}\bar{F}VV\bar{F}\bar{F}},\quad H_{FFFVVV},\quad H_{F\bar{F}\bar{F}VVV},\quad K_{FFFVVF},\\
&K_{FF\bar{F}VV\bar{F}},\quad K_{\bar{F}\bar{F}FVVF},\quad K_{\bar{F}F\bar{F}VVF},\quad K_{\bar{F}\bar{F}\bar{F}VV\bar{F}},\quad K_{VVFFFF},\quad K_{VVFF\bar{F}\bar{F}},\quad K_{VV\bar{F}\bar{F}\bar{F}\bar{F}},\\
&K_{{\rm gauge},FF},\quad K_{{\rm gauge},\bar{F}\bar{F}},\quad H_{FFSVFF},\quad H_{FFSV\bar{F}\bar{F}},\quad H_{F\bar{F}SV\bar{F}F},\quad H_{F\bar{F}SVF\bar{F}},\quad H_{\bar{F}\bar{F}SV\bar{F}\bar{F}},\\
&H_{FFFVSS},\quad H_{F\bar{F}\bar{F}VSS},\quad H_{\bar{F}\bar{F}FVSS},\quad H_{F\bar{F}FSVV},\quad H_{FF\bar{F}SVV},\quad H_{\bar{F}\bar{F}\bar{F}SVV},\quad K_{FFFSVF},\\
&K_{FF\bar{F}SV\bar{F}},\quad K_{\bar{F}\bar{F}FSVF},\quad K_{\bar{F}F\bar{F}SVF},\quad K_{\bar{F}FFSV\bar{F}},\quad K_{\bar{F}\bar{F}\bar{F}SV\bar{F}},\quad K_{SSSVFF},\quad K_{SSSV\bar{F}\bar{F}},\\
&K_{SSVVFF},\quad K_{SSVV\bar{F}\bar{F}},\quad K_{VVSSFF},\quad K_{VVSS\bar{F}\bar{F}},\quad K_{VVSVFF},\quad K_{VVSV\bar{F}\bar{F}},\quad H_{\rm gauge},\quad K_{\rm gauge}.
\end{aligned}
\tag{3.1}
$$

It remains to give $V^{(3)}$ by providing expressions for these 89 functions in terms of the basis functions described in the previous section, with arguments that are $\overline{\rm MS}$ squared masses (and, implicitly, the renormalization scale Q), and also to provide the coefficients of these 89 functions in $V^{(3)}$ in terms of the $\overline{\rm MS}$ couplings appearing in Eq. (2.1). Regarding the first task, many of the expressions for the 89 loop integral functions in terms of basis integrals are extremely complicated and not of much use to the human eye. All of these results are therefore presented in an ancillary electronic file called `functions.anc` [108] distributed with this paper, suitable for inclusion in symbolic manipulation code or numerical computer programs. Only the first 24, relatively simple, functions in Eq. (3.1), corresponding to the diagrams that do not involve vector propagators will be given in the text below.

The formula for $V^{(3)}$ in terms of the 89 functions listed in Eq. (3.1) is split up below as:

$$V^{(3)} = V_S^{(3)} + V_{SF}^{(3)} + V_{SV}^{(3)} + V_{FV}^{(3)} + V_{SFV}^{(3)} + V_V^{(3)}, \tag{3.2}$$

where $V_S^{(3)}$ contains only scalar interactions, $V_{SF}^{(3)}$ contains scalars and fermions only, $V_{SV}^{(3)}$ contains only scalars and vectors (and ghosts), $V_{FV}^{(3)}$ contains only fermions and vectors (and ghosts), $V_{SFV}^{(3)}$ contains scalars, fermions, and vectors (and ghosts), and $V_V^{(3)}$ contains only vectors and ghosts.

B. Pure scalar contributions

The pure scalar contributions to the three-loop effective potential can be written as:

$$\begin{aligned} V_S^{(3)} = {} & \frac{1}{24}\lambda^{jkm}\lambda^{kln}\lambda^{jlp}\lambda^{mnp}H_{SSSSSS}(j,k,l,m,n,p) + \frac{1}{16}\lambda^{jlm}\lambda^{klm}\lambda^{jnp}\lambda^{knp}K_{SSSSSS}(j,k,l,m,n,p) \\ & + \frac{1}{8}\lambda^{jknn}\lambda^{jlm}\lambda^{klm}J_{SSSSS}(j,k,l,m,n) + \frac{1}{8}\lambda^{jkl}\lambda^{jmn}\lambda^{klmn}G_{SSSSS}(j,k,l,m,n) \\ & + \frac{1}{16}\lambda^{jkll}\lambda^{jkmm}L_{SSSS}(j,k,l,m) + \frac{1}{48}\lambda^{jklm}\lambda^{jklm}E_{SSSS}(j,k,l,m), \end{aligned} \tag{3.3}$$

where the scalar field indices j, k, l, m, n, p are also used to represent the $\overline{\text{MS}}$ background-field-dependent squared masses. The loop integral functions appearing in Eq. (3.3) are easy to write in terms of the basis integrals:

$$H_{SSSSSS}(u,v,w,x,y,z) = -H(u,v,w,x,y,z), \tag{3.4}$$

$$K_{SSSSSS}(u,v,w,x,y,z) = -K(u,v,w,x,y,z), \tag{3.5}$$

$$J_{SSSSS}(w,x,v,y,u) = A(u)\bar{I}(w,x,v,y), \tag{3.6}$$

$$G_{SSSSS}(w,u,z,y,v) = G(w,u,z,y,v), \tag{3.7}$$

$$L_{SSSS}(w,x,u,v) = -A(u)A(v)\bar{A}(w,x), \tag{3.8}$$

$$E_{SSSS}(u,z,y,v) = -E(u,z,y,v). \tag{3.9}$$

C. Scalar and fermion contributions

The contributions involving scalars and fermions (but not vectors or ghosts) can be written as:

$$\begin{aligned} V_{SF}^{(3)} = {} & \frac{1}{2}(\lambda^{jkl}Y^{jIJ}Y_{kJK}Y_{lIK'}M^{KK'} + \text{c.c.})H_{FF\bar{F}SSS}(I,J,K,j,k,l) \\ & + \frac{1}{6}(\lambda^{jkl}Y^{jIJ}Y^{kJ'K}Y^{lI'K'}M_{II'}M_{JJ'}M_{KK'} + \text{c.c.})H_{\bar{F}\bar{F}\bar{F}SSS}(I,J,K,j,k,l) + \frac{1}{4}Y^{kIJ}Y^{kKL}Y_{jIL}Y_{jJK}H_{FFSSFF}(I,J,j,k,K,L) \\ & + \frac{1}{2}(Y^{kIJ}Y_{kKL}Y_{jIL'}Y_{jJK'}M^{KK'}M^{LL'} + \text{c.c.})H_{FFSS\bar{F}\bar{F}}(I,J,j,k,K,L) \\ & + \frac{1}{2}Y^{kIJ}Y_{kKL}Y_{jIL'}Y^{jJ'K}M_{JJ'}M^{LL'}H_{F\bar{F}SSF\bar{F}}(I,J,j,k,K,L) \\ & + \frac{1}{8}(Y^{kIJ}Y^{kKL}Y^{jI'L'}Y^{jJ'K'}M_{II'}M_{JJ'}M_{KK'}M_{LL'} + \text{c.c.})H_{\bar{F}\bar{F}SS\bar{F}\bar{F}}(I,J,j,k,K,L) + \frac{1}{4}\lambda^{jlm}\lambda^{klm}Y^{jIJ}Y_{kIJ}K_{SSSSFF}(j,k,l,m,I,J) \\ & + \frac{1}{8}(\lambda^{jlm}\lambda^{klm}Y^{jIJ}Y^{kI'J'}M_{II'}M_{JJ'} + \text{c.c.})K_{SSSS\bar{F}\bar{F}}(j,k,l,m,I,J) + \frac{1}{2}Y^{jIK}Y_{jJK}Y_{kIL}Y^{kJL}K_{FFFSSF}(I,J,K,j,k,L) \\ & + \frac{1}{2}Y^{jIK}Y^{jJK'}Y_{kIL}Y_{kJL'}M_{KK'}M^{LL'}K_{FF\bar{F}SS\bar{F}}(I,J,K,j,k,L) + \frac{1}{2}Y^{jIK}Y_{jJK}Y^{kI'L}Y_{kJ'L}M_{II'}M^{JJ'}K_{\bar{F}\bar{F}FSSF}(I,J,K,j,k,L) \\ & + (Y^{jIK}Y^{jJK'}Y^{kI'L}Y_{kJL}M_{II'}M_{KK'} + \text{c.c.})K_{\bar{F}F\bar{F}SSF}(I,J,K,j,k,L) \\ & + \frac{1}{4}(Y^{jIK}Y^{jJK'}Y^{kI'L}Y^{kJ'L'}M_{II'}M_{JJ'}M_{KK'}M_{LL'} + \text{c.c.})K_{\bar{F}\bar{F}\bar{F}SS\bar{F}}(I,J,K,j,k,L) \\ & + \frac{1}{8}Y^{jIJ}Y_{kIJ}(Y^{jKL}Y_{kKL} + \text{c.c.})K_{SSFFFF}(j,k,I,J,K,L) + \frac{1}{4}Y^{jIJ}Y_{kIJ}(Y^{jKL}Y^{kK'L'}M_{KK'}M_{LL'} + \text{c.c.})K_{SSFF\bar{F}\bar{F}}(j,k,I,J,K,L) \\ & + \frac{1}{16}(Y^{jIJ}Y^{kI'J'}M_{II'}M_{JJ'} + \text{c.c.})(Y^{jKL}Y^{kK'L'}M_{KK'}M_{LL'} + \text{c.c.})K_{SS\bar{F}\bar{F}\bar{F}\bar{F}}(j,k,I,J,K,L) \\ & + \frac{1}{4}\lambda^{jkll}Y^{jIJ}Y_{kIJ}J_{SSFFS}(j,k,I,J,l) + \frac{1}{8}(\lambda^{jkll}Y^{jIJ}Y^{kI'J'}M_{II'}M_{JJ'} + \text{c.c.})J_{SS\bar{F}\bar{F}S}(j,k,I,J,l). \end{aligned} \tag{3.10}$$

I find that the loop integral functions appearing in Eq. (3.10) are, in terms of the basis integrals:

$$H_{FF\bar{F}SSS}(u,v,w,x,y,z) = (u+v-x)H(u,v,w,x,y,z) + G(w,u,z,v,y) - G(y,v,w,x,z) - G(z,u,w,x,y), \quad (3.11)$$

$$H_{\bar{F}\bar{F}\bar{F}SSS}(u,v,w,x,y,z) = 2H(u,v,w,x,y,z), \quad (3.12)$$

$$\begin{aligned} H_{FFSSFF}(u,v,w,x,y,z) &= (uy+vz-wx)H(u,v,w,x,y,z) - uG(u,v,x,w,z) - vG(v,u,x,w,y) + wG(w,u,z,v,y) \\ &\quad + xG(x,u,v,y,z) - yG(y,v,w,x,z) - zG(z,u,w,x,y) - E(u,v,y,z) + E(v,w,x,z) \\ &\quad + E(u,w,x,y), \end{aligned} \quad (3.13)$$

$$H_{FFSS\bar{F}\bar{F}}(u,v,w,x,y,z) = (u+v-x)H(u,v,w,x,y,z) + G(w,u,z,v,y) - G(y,v,w,x,z) - G(z,u,w,x,y), \quad (3.14)$$

$$\begin{aligned} H_{F\bar{F}SSF\bar{F}}(u,v,w,x,y,z) &= (v-w-x+z)H(u,v,w,x,y,z) - G(v,u,x,w,y) + G(w,u,z,v,y) + G(x,u,v,y,z) \\ &\quad - G(z,u,w,x,y), \end{aligned} \quad (3.15)$$

$$H_{\bar{F}\bar{F}SS\bar{F}\bar{F}}(u,v,w,x,y,z) = 2H(u,v,w,x,y,z), \quad (3.16)$$

$$\begin{aligned} K_{SSSSFF}(x,w,u,z,y,v) &= [v+y-(w+x)/2]K(x,w,u,z,y,v) + G(w,u,z,y,v)/2 + G(x,u,z,y,v)/2 \\ &\quad - [A(v)+A(y)]\bar{I}(x,w,u,z), \end{aligned} \quad (3.17)$$

$$K_{SSSS\bar{F}\bar{F}}(x,w,u,z,y,v) = 2K(x,w,u,z,y,v), \quad (3.18)$$

$$\begin{aligned} K_{FFFSSF}(x,w,u,z,y,v) &= (x^2+w^2+2uv-2uy-2vz+2yz+uw+vw \\ &\quad + ux+vx-wy-xy-wz-xz)K(x,w,u,z,y,v)/4 \\ &\quad + (y+z-u-v-w-x)[G(w,u,z,y,v)+G(x,u,z,y,v)]/4 \\ &\quad + (u+w-z)[A(v)-A(y)]\bar{I}(x,w,u,z)/2 \\ &\quad + (v+w-y)[A(u)-A(z)]\bar{I}(x,w,v,y)/2 + E(u,v,y,z)/2 \\ &\quad + [A(y)-A(v)]I(u,x,z)/2 + [A(z)-A(u)]I(v,x,y)/2 \\ &\quad + \bar{A}(x,w)[A(z)-A(u)][A(y)-A(v)]/2, \end{aligned} \quad (3.19)$$

$$K_{FF\bar{F}SS\bar{F}}(x,w,u,z,y,v) = (w+x)K(x,w,u,z,y,v) - G(w,u,z,y,v) - G(x,u,z,y,v), \quad (3.20)$$

$$\begin{aligned} K_{\bar{F}\bar{F}FSSF}(x,w,u,z,y,v) &= \{(w^2x+wx^2+2uwx+2vwx-2wxy+uvw+uvx-uwy \\ &\quad - uxy-vwz-vxz-2wxz+wyz+xyz)K(x,w,u,z,y,v) \\ &\quad + (uv-wx-uy-vz+yz)[G(x,u,z,y,v)+G(w,u,z,y,v)] \\ &\quad + 2(v-y)[uF(u,v,y,z)-zF(z,u,v,y)+[A(u)-A(z)]I(v,x,y)] \\ &\quad + 2(u-z)[vF(v,u,y,z)-yF(y,u,v,z)+[A(v)-A(y)]I(u,x,z)] \\ &\quad + 2x(u+w-z)[A(v)-A(y)]\bar{I}(w,x,u,z) \\ &\quad + 2x(v+w-y)[A(u)-A(z)]\bar{I}(w,x,v,y) \\ &\quad + 2[x\bar{A}(w,x)+A(x)][A(v)-A(y)][A(u)-A(z)] \\ &\quad + (u-z)[yA(y)-vA(v)]/2 + (v-y)[zA(z)-uA(u)]/2 \\ &\quad + 4(v-y)(u-z)(u+v+y+z)/3\}/4wx, \end{aligned} \quad (3.21)$$

$$\begin{aligned} K_{\bar{F}F\bar{F}SSF}(x,w,u,z,y,v) &= [v-y+(w+x)/2]K(x,w,u,z,y,v) - G(w,u,z,v,y)/2 - G(x,u,z,v,y)/2 \\ &\quad + [A(v)-A(y)]\bar{I}(x,w,u,z), \end{aligned} \quad (3.22)$$

$$K_{\bar{F}\bar{F}\bar{F}SS\bar{F}}(x,w,u,z,y,v) = 2K(x,w,u,z,y,v), \tag{3.23}$$

$$\begin{aligned} K_{SSFFFF}(x,w,u,z,y,v) = {} & [(w+x)(u+v+y+z) - 2(u+z)(v+y) - w^2 - x^2]K(x,w,u,z,y,v)/2 - E(u,v,y,z) \\ & + (w+x-u-v-y-z)[G(w,u,z,y,v) + G(x,u,z,y,v)]/2 \\ & + (u-w+z)[A(v)+A(y)]\bar{I}(w,x,u,z) + (v-w+y)[A(u)+A(z)]\bar{I}(w,x,v,y) \\ & + [A(v)+A(y)]I(u,x,z) + [A(u)] + A(z)]I(v,x,y) \\ & - \bar{A}(w,x)[A(u)+A(z)][A(v)+A(y)], \end{aligned} \tag{3.24}$$

$$\begin{aligned} K_{SSFF\bar{F}\bar{F}}(x,w,u,z,y,v) = {} & (w+x-2u-2z)K(x,w,u,z,y,v) - G(w,u,z,v,y) - G(x,u,z,v,y) \\ & + 2[A(u)+A(z)]\bar{I}(x,w,y,v), \end{aligned} \tag{3.25}$$

$$K_{SS\bar{F}\bar{F}\bar{F}\bar{F}}(x,w,u,z,y,v) = -4K(x,w,u,z,y,v), \tag{3.26}$$

$$J_{SSFFS}(x,w,y,v,u) = A(u)\{(w-v-y)\bar{I}(x,w,y,v) - I(v,x,y) + \bar{A}(x,w)[A(v)+A(y)]\}, \tag{3.27}$$

$$J_{SS\bar{F}\bar{F}S}(x,w,y,v,u) = -2A(u)\bar{I}(x,w,y,v). \tag{3.28}$$

These can also be found in the ancillary electronic file `functions.anc` [108].

D. Scalar and vector (and ghost) contributions

The contributions that involve both scalars and vectors, but not fermions, are written as:

$$\begin{aligned} V^{(3)}_{SV} = {} & \frac{1}{4}\lambda^{jkm}\lambda^{kln}g^{alj}g^{amn}H_{SSSSSV}(j,k,l,m,n,a) + \frac{1}{2}\lambda^{klm}g^{abk}g^{ajm}g^{blj}H_{VVSSSS}(a,b,j,k,l,m) \\ & + \frac{1}{8}g^{ajm}g^{bjk}g^{akl}g^{blm}H_{SSVVSS}(j,k,a,b,l,m) + \frac{1}{6}\lambda^{jkl}g^{abj}g^{bck}g^{acl}H_{VVVSSS}(a,b,c,j,k,l) \\ & + \frac{1}{6}g^{abc}g^{ajk}g^{bkl}g^{clj}H_{SSSVVV}(j,k,l,a,b,c) + \frac{1}{2}g^{abk}g^{bcj}g^{ajl}g^{clk}H_{VVSSVS}(a,b,j,k,c,l) \\ & + \frac{1}{2}g^{ack}g^{adj}g^{bcd}g^{bjk}H_{SSVVVV}(j,k,a,b,c,d) + \frac{1}{8}g^{acj}g^{abk}g^{bdj}g^{cdk}H_{SVVVSV}(j,a,b,c,k,d) \\ & + \frac{1}{4}\lambda^{jlm}\lambda^{klm}g^{ajn}g^{ank}K_{SSSSSV}(j,k,l,m,n,a) + \frac{1}{8}\lambda^{jlm}\lambda^{klm}g^{abj}g^{abk}K_{SSSSVV}(j,k,l,m,a,b) \\ & + \frac{1}{4}g^{ajl}g^{alk}g^{bjm}g^{bmk}K_{SSSVVS}(j,k,l,a,b,m) + \frac{1}{16}g^{ajk}g^{bkj}g^{aml}g^{blm}K_{VVSSSS}(a,b,j,k,l,m) \\ & + \frac{1}{4}g^{ajl}g^{alk}g^{bcj}g^{bck}K_{SSSVVV}(j,k,l,a,b,c) + \frac{1}{4}g^{ajk}g^{bkj}g^{acl}g^{bcl}K_{VVSSVS}(a,b,j,k,c,l) \\ & + \frac{1}{16}g^{abj}g^{abk}g^{cdj}g^{cdk}K_{SSVVVV}(j,k,a,b,c,d) + \frac{1}{4}g^{acj}g^{bcj}g^{adk}g^{bdk}K_{VVSVVS}(a,b,j,c,d,k) \\ & + \frac{1}{4}\lambda^{jkmm}g^{ajl}g^{alk}J_{SSVSS}(j,k,a,l,m) + \frac{1}{8}g^{abj}g^{abk}\lambda^{jkll}J_{SSVVS}(j,k,a,b,l) + \frac{1}{2}g^{abj}g^{ack}g^{bjl}g^{ckl}G_{VSVVS}(a,j,b,c,k) \\ & + \frac{1}{4}g^{abd}g^{bce}g^{acj}g^{dej}H_{\mathrm{gauge},S}(a,b,c,d,e,j) + \frac{1}{4}g^{acd}g^{bcd}g^{aej}g^{bej}K_{\mathrm{gauge},S}(a,b,c,d,e,j) \\ & + \frac{1}{8}g^{acd}g^{bcd}g^{ajk}g^{bkj}K_{\mathrm{gauge},SS}(a,b,c,d,j,k). \end{aligned} \tag{3.29}$$

The loop integral functions appearing here are presented explicitly in the ancillary electronic file `functions.anc` [108], in computer-readable form. Many of them are quite lengthy.

E. Fermion and vector (and ghost) contributions

The contributions involving fermions and vectors (but not scalars) are written as:

$$
\begin{aligned}
V^{(3)}_{FV} ={}& \frac{1}{4} g^{aL}_I g^{bK}_L g^{aJ}_K g^{bI}_J H_{FFVVFF}(I,J,a,b,K,L) + g^{aL}_I g^{bL'}_K g^{aJ}_{K'} g^{bI}_J M^{KK'} M_{LL'} H_{FFVV\bar F\bar F}(I,J,a,b,K,L) \\
&+ \frac{1}{2} g^{aL}_I g^{bL'}_K g^{aK}_J g^{bI}_{J'} M^{JJ'} M_{LL'} H_{F\bar F VVF\bar F}(I,J,a,b,K,L) + \frac{1}{4} g^{aI}_L g^{bK}_{L'} g^{aK'}_J g^{bI'}_{J'} M_{II'} M^{JJ'} M_{KK'} M^{LL'} H_{\bar F\bar F VV\bar F\bar F}(I,J,a,b,K,L) \\
&+ \frac{i}{3} g^{abc} g^{aI}_J g^{cK}_I g^{bJ}_K H_{FFFVVV}(I,J,K,a,b,c) + i g^{abc} g^{aI}_J g^{cK}_I g^{bK'}_{J'} M^{JJ'} M_{KK'} H_{F\bar F\bar F VVV}(I,J,K,a,b,c) \\
&+ \frac{1}{2} g^{aK}_I g^{aJ}_K g^{bL}_J g^{bI}_L K_{FFFVVF}(I,J,K,a,b,L) + \frac{1}{2} g^{aK}_I g^{aK'}_J g^{bJ}_L g^{bI}_{L'} M_{KK'} M^{LL'} K_{FF\bar F VV\bar F}(I,J,K,a,b,L) \\
&+ \frac{1}{2} g^{aK}_I g^{aJ}_K g^{bJ'}_L g^{bL}_{I'} M^{II'} M_{JJ'} K_{\bar F\bar F FVVF}(I,J,K,a,b,L) + (g^{aK}_I g^{aK'}_J g^{bJ}_L g^{bL}_{I'} M^{II'} M_{KK'} + \text{c.c.}) K_{\bar F F\bar F VVF}(I,J,K,a,b,L) \\
&+ \frac{1}{4} (g^{aK}_I g^{aK'}_J g^{bL}_{J'} g^{bL'}_{I'} M^{II'} M^{JJ'} M_{KK'} M_{LL'} + \text{c.c.}) K_{\bar F\bar F\bar F VV\bar F}(I,J,K,a,b,L) + \frac{1}{4} g^{aI}_J g^{bJ}_I g^{aK}_L g^{bL}_K K_{VVFFFF}(a,b,I,J,K,L) \\
&+ \frac{1}{2} g^{aI}_J g^{bJ}_I g^{aK}_L g^{bK'}_{L'} M_{KK'} M^{LL'} K_{VVFF\bar F\bar F}(a,b,I,J,K,L) + \frac{1}{4} g^{aI}_J g^{bI'}_{J'} g^{aK}_L g^{bK'}_{L'} M_{II'} M^{JJ'} M_{KK'} M^{LL'} K_{VV\bar F\bar F\bar F\bar F}(a,b,I,J,K,L) \\
&+ \frac{1}{4} g^{acd} g^{bcd} g^{aI}_J g^{bJ}_I K_{\text{gauge},FF}(a,b,c,d,I,J) + \frac{1}{4} g^{acd} g^{bcd} g^{aI}_J g^{bI'}_{J'} M_{II'} M^{JJ'} K_{\text{gauge},\bar F\bar F}(a,b,c,d,I,J). \qquad (3.30)
\end{aligned}
$$

Again the loop integral functions appearing here are presented explicitly in the ancillary electronic file `functions.anc` [108].

F. Scalar, fermion, and vector contributions

The contributions that involve all three of scalars, fermions, and vectors are

$$
\begin{aligned}
V^{(3)}_{SFV} ={}& \frac{1}{2} g^{aI}_J g^{aL}_K Y_{jIL} Y^{jJK} H_{FFSVFF}(I,J,j,a,K,L) + g^{aI}_J g^{aK}_L Y_{jIL'} Y^{jJK'} M_{KK'} M^{LL'} H_{FFSV\bar F\bar F}(I,J,j,a,K,L) \\
&+ \frac{1}{2} (g^{aI}_J g^{aL}_K Y_{jIL} Y_{jJ'K'} M^{JJ'} M^{KK'} + \text{c.c.}) H_{F\bar F SV\bar F F}(I,J,j,a,K,L) \\
&+ \frac{1}{2} (g^{aI}_J g^{aK}_L Y_{jIL'} Y_{jJ'K} M^{JJ'} M^{LL'} + \text{c.c.}) H_{F\bar F SVF\bar F}(I,J,j,a,K,L) \\
&+ \frac{1}{2} g^{aI}_J g^{aL}_K Y^{jI'L'} Y_{jJ'K'} M_{II'} M^{JJ'} M^{KK'} M_{LL'} H_{\bar F\bar F SV\bar F\bar F}(I,J,j,a,K,L) + i g^{ajk} g^{aI}_J Y_{kIK} Y^{jJK} H_{FFFVSS}(I,J,K,a,j,k) \\
&+ (i g^{ajk} g^{aI}_J Y_{kIK} Y_{jJ'K'} M^{JJ'} M^{KK'} + \text{c.c.}) H_{F\bar F\bar F VSS}(I,J,K,a,j,k) + i g^{ajk} g^{aI}_J Y^{kI'K} Y_{jJ'K} M_{II'} M^{JJ'} H_{\bar F\bar F FVSS}(I,J,K,a,j,k) \\
&+ (g^{abj} g^{aJ}_K g^{bK}_I Y^{jIJ'} M_{JJ'} + \text{c.c.}) H_{F\bar F FSVV}(I,J,K,j,a,b) + \frac{1}{2} (g^{abj} g^{aK}_J g^{bK'}_I Y^{jIJ} M_{KK'} + \text{c.c.}) H_{FF\bar F SVV}(I,J,K,j,a,b) \\
&+ \frac{1}{2} (g^{abj} g^{aJ}_K g^{bI}_{K'} Y^{jI'J'} M_{II'} M_{JJ'} M^{KK'} + \text{c.c.}) H_{\bar F\bar F\bar F SVV}(I,J,K,j,a,b) + g^{aL}_I g^{aJ}_L Y^{jIK} Y_{jJK} K_{FFFSVF}(I,J,K,j,a,L) \\
&+ \frac{1}{2} (g^{aL}_I g^{aL'}_J Y^{jIK} Y^{jJK'} M_{KK'} M_{LL'} + \text{c.c.}) K_{FF\bar F SV\bar F}(I,J,K,j,a,L) + g^{aI}_L g^{aL}_J Y^{jI'K} Y_{jJ'K} M_{II'} M^{JJ'} K_{\bar F\bar F FSVF}(I,J,K,j,a,L) \\
&+ (g^{aI}_L g^{aL}_J Y^{jI'K} Y^{jJK'} M_{II'} M_{KK'} + \text{c.c.}) K_{\bar F F\bar F SVF}(I,J,K,j,a,L) \\
&+ (g^{aI}_L g^{aJ}_{L'} Y^{jI'K} Y_{jJK} M_{II'} M^{LL'} + \text{c.c.}) K_{\bar F FFSV\bar F}(I,J,K,j,a,L) \\
&+ \frac{1}{2} (g^{aI}_L g^{aJ}_{L'} Y^{jI'K} Y^{jJ'K'} M_{II'} M_{JJ'} M_{KK'} M^{LL'} + \text{c.c.}) K_{\bar F\bar F\bar F SV\bar F}(I,J,K,j,a,L) + \frac{1}{2} g^{ajl} g^{alk} Y^{jIJ} Y_{kIJ} K_{SSSVFF}(j,k,l,a,I,J) \\
&+ \frac{1}{4} (g^{ajl} g^{alk} Y^{jIJ} Y^{kI'J'} M_{II'} M_{JJ'} + \text{c.c.}) K_{SSSV\bar F\bar F}(j,k,l,a,I,J) + \frac{1}{4} g^{abj} g^{abk} Y^{jIJ} Y_{kIJ} K_{SSVVFF}(j,k,a,b,I,J) \\
&+ \frac{1}{8} (g^{abj} g^{abk} Y^{jIJ} Y^{kI'J'} M_{II'} M_{JJ'} + \text{c.c.}) K_{SSVV\bar F\bar F}(j,k,a,b,I,J) + \frac{1}{4} g^{ajk} g^{bkj} g^{aI}_J g^{bJ}_I K_{VVSSFF}(a,b,j,k,I,J) \\
&+ \frac{1}{4} g^{ajk} g^{bkj} g^{aI}_J g^{bI'}_{J'} M_{II'} M^{JJ'} K_{VVSS\bar F\bar F}(a,b,j,k,I,J) + \frac{1}{2} g^{acj} g^{bcj} g^{aI}_J g^{bJ}_I K_{VVSVFF}(a,b,j,c,I,J) \\
&+ \frac{1}{2} g^{acj} g^{bcj} g^{aI}_J g^{bI'}_{J'} M_{II'} M^{JJ'} K_{VVSV\bar F\bar F}(a,b,j,c,I,J). \qquad (3.31)
\end{aligned}
$$

As before, the loop integral functions appearing here are presented explicitly in the ancillary electronic file `functions.anc` [108].

G. Pure vector and ghost contributions

Finally, the contributions that involve only vector bosons and ghost fields can be written in terms of a couple of loop integral functions:

$$V_V^{(3)} = \frac{1}{24} g^{abd} g^{bce} g^{acf} g^{def} H_{\text{gauge}}(a,b,c,d,e,f) + \frac{1}{16} g^{acd} g^{bcd} g^{aef} g^{bef} K_{\text{gauge}}(a,b,c,d,e,f). \tag{3.32}$$

These contributions vanish except when the gauge symmetry associated with the vectors is non-Abelian and (at least partly) spontaneously broken by the scalar background fields. The loop integral functions appearing here are again presented explicitly in the ancillary electronic file `functions.anc` [108].

H. Comments on the general results

Equations (3.3)–(3.32) constitute the complete $\overline{\text{MS}}$ three-loop effective potential contributions for a generic renormalizable quantum field theory with Landau gauge fixing. However, for the specialization to any particular theory with massless gauge bosons, there is still a little processing to do in order to obtain the effective potential in practice. This is because each loop integral function involving a vector field with squared mass x will contain a factor of $1/x$, which naively might appear to be have a pole singularity in the massless limit. This is due to the structure of the Landau gauge vector propagator proportional to

$$\frac{\eta^{\mu\nu}}{p^2+x} - \frac{p^\mu p^\nu}{p^2(p^2+x)}, \tag{3.33}$$

(using a metric of signature $-, +, +, +$) where the second term has a partial fraction decomposition proportional to

$$\frac{1}{x}\left(\frac{1}{p^2+x} - \frac{1}{p^2}\right). \tag{3.34}$$

Massless gauge bosons, with their potential infrared problems, are treated here by putting $x = \delta$ and taking the limit $\delta \to 0$. The factors of $1/\delta$ actually always cancel in the limit, leaving behind either a finite result or singularities in each diagram that are at most logarithmic in δ. However, demonstrating this starting from the general loop integral functions appearing in the ancillary file `functions.anc` [108], and finding the limits, requires using the expansions of the basis integrals in small squared masses, as given in the ancillary file `expzero.anc` [108]. This can be performed systematically on a case-by-case basis, as will be done below for the example of the standard model. While the pole singularities in δ always cancel at the level of the loop integral functions, logarithmic singularities as $\delta \to 0$ can occur, but only when there is a doubled propagator, which means the diagram topology is K, J, or L (see Fig. 2) with the first two squared mass arguments both equal to δ. The $\overline{\ln}(\delta)$ singularities can then be obtained with the help of expansion formulas of the type in the ancillary file `expzero.anc` [108]. Cancellations of the infrared singularities associated with massless vector bosons in the full effective potential occurs after summing the contributions of distinct diagrams, as will be illustrated below for the standard model.

Note also that the function $K_{\bar{F}\bar{F}FSSF}(x,w,u,z,y,v)$ in Eq. (3.21) contains a factor $1/wx$, which naively might appear to be singular when either w or x approaches 0. However, this is illusory; $K_{\bar{F}\bar{F}FSSF}(\delta,w,u,z,y,v)$ and $K_{\bar{F}\bar{F}FSSF}(x,\delta,u,z,y,v)$ and $K_{\bar{F}\bar{F}FSSF}(\delta,\delta,u,z,y,v)$ are each finite as $\delta \to 0$, as one can check by using the expansions given in the ancillary file `expzero.anc` [108]. Furthermore, this function appears in $V^{(3)}$ multiplied by $\sqrt{wx}$ [because of the fermion mass insertions multiplying it in Eq. (3.10)]. Therefore, it does not contribute at all when w and/or x is zero. More generally, the contribution from every integral function with an $\bar{F}$ subscript listed in Eq. (3.1) vanishes when the corresponding fermion squared mass is taken to 0. Also, $K_{FFFSSF}(\delta,\delta,u,z,y,v)$ and $K_{FF\bar{F}SS\bar{F}}(\delta,\delta,u,z,y,v)$, etc., have no $\overline{\ln}(\delta)$ singularities. There are no infrared problems associated with massless fermions.

For checking purposes, it is useful to be able to take derivatives of the loop integral functions with respect to the $\overline{\text{MS}}$ renormalization scale Q. First, for the basis functions and related functions, one has from Ref. [51]:

$$Q\frac{\partial}{\partial Q} A(x) = -2x, \tag{3.35}$$

$$Q\frac{\partial}{\partial Q} \bar{A}(x,y) = 2, \tag{3.36}$$

$$Q\frac{\partial}{\partial Q} I(x,y,z) = 2[A(x) + A(y) + A(z) - x - y - z], \tag{3.37}$$

$$Q\frac{\partial}{\partial Q} \bar{I}(w,x,y,z) = 2 + 2\bar{A}(w,x), \tag{3.38}$$

$$Q\frac{\partial}{\partial Q}E(w,x,y,z) = 2[A(w)A(x) + A(w)A(y) + A(w)A(z) + A(x)A(y) + A(x)A(z) + A(y)A(z) + wx + wy + wz + xy + xz + yz] + (w-2x-2y-2z)A(w) + (x-2w-2y-2z)A(x) + (y-2w-2x-2z)A(y) + (z-2w-2x-2y)A(z) - 9(w^2+x^2+y^2+z^2)/4, \tag{3.39}$$

$$Q\frac{\partial}{\partial Q}F(w,x,y,z) = 2[-A(x) - A(y) - A(z) + x + y + z - w]A(w)/w + 7w/2, \tag{3.40}$$

$$Q\frac{\partial}{\partial Q}\bar{F}(w,x,y,z) = 2[A(x) + A(y) + A(z) - A(w) - x - y - z - I(x,y,z)] + 7w/2, \tag{3.41}$$

$$Q\frac{\partial}{\partial Q}G(v,w,x,y,z) = 2[I(v,w,x) + I(v,y,z) + A(w) + A(x) + A(y) + A(z) + v] - 4(w+x+y+z), \tag{3.42}$$

$$Q\frac{\partial}{\partial Q}H(u,v,w,x,y,z) = 12\zeta_3, \tag{3.43}$$

$$Q\frac{\partial}{\partial Q}K(u,v,w,x,y,z) = 2[\bar{I}(u,v,w,x) + \bar{I}(u,v,y,z) - 1], \tag{3.44}$$

These results can now be applied to obtain the Q derivatives of the 89 integral functions of Eq. (3.1). Again the results are rather lengthy, and so are consigned to an ancillary electronic file `QdQ.anc` [108] provided with this paper.

IV. THE WESS-ZUMINO MODEL

In this section, we consider as an example (and a confidence-building consistency check) the supersymmetric Wess-Zumino model [74,75], with superpotential (for a review, see [76]):

$$W = \frac{m}{2}\Phi^2 + \frac{y}{6}\Phi^3, \tag{4.1}$$

with real mass and coupling parameters m and y. The chiral superfield Φ contains a 2-component fermion ψ and a complex scalar field that one can write as

$$\phi + (R + iI)/\sqrt{2}, \tag{4.2}$$

where ϕ is a constant background field and R, I are canonically normalized real scalar fields. In the following, depending on context, the names of the component quantum fields will also be used as synonyms for their field-dependent squared masses:

$$R = m^2 + 3ym\phi + 3y^2\phi^2/2, \tag{4.3}$$

$$I = m^2 + ym\phi + y^2\phi^2/2, \tag{4.4}$$

$$\psi = (m + y\phi)^2. \tag{4.5}$$

The nonvanishing interaction couplings of the fields R, I, ψ are given by

$$\lambda^{RRRR} = \lambda^{IIII} = 3\lambda^{RRII} = 3y^2/2, \tag{4.6}$$

$$\lambda^{RRR} = 3\lambda^{RII} = 3y(m + y\phi)/\sqrt{2}, \tag{4.7}$$

$$Y^{R\psi\psi} = y/\sqrt{2}, \tag{4.8}$$

$$Y^{I\psi\psi} = iy/\sqrt{2}, \tag{4.9}$$

and permutations $\lambda^{RIRI} = \lambda^{RIIR} = \lambda^{IRRI} = \lambda^{IRIR} = \lambda^{IIRR} = \lambda^{RRII}$, and $\lambda^{IRI} = \lambda^{IIR} = \lambda^{RII}$. There are no vector fields in the Wess-Zumino model.

The tree-level potential for the background field ϕ is

$$V^{(0)} = \phi^2(m + y\phi/2)^2. \tag{4.10}$$

Plugging into the results of Eqs. (2.6), (2.11), (3.3), and (3.10) above gives the effective potential contributions at one, two, and three-loop orders for the Wess-Zumino model:

$$V^{(1)} = f(R) + f(I) - 2f(\psi), \tag{4.11}$$

$$\begin{aligned} V^{(2)} = {}& y^2[3f_{SS}(R,R)/16 + 3f_{SS}(I,I)/16 + f_{SS}(I,R)/8 + 3\psi f_{SSS}(R,R,R)/8 + \psi f_{SSS}(I,I,R)/8 + f_{FFS}(\psi,\psi,R)/4 \\ & + f_{FFS}(\psi,\psi,I)/4 + \psi f_{\bar{F}\bar{F}S}(\psi,\psi,R)/4 - \psi f_{\bar{F}\bar{F}S}(\psi,\psi,I)/4], \end{aligned} \tag{4.12}$$

$$\begin{aligned} V^{(3)} = {}& y^4\psi^2[27H_{RRRRRR}/32 + H_{IIRRII}/32 + H_{IIIRRR}/8 + 81K_{RRRRRR}/64 + 9K_{RRIIRR}/32 + K_{RRIIII}/64 + K_{IIIRIR}/16 \\ & + H_{\bar\psi\bar\psi RR\bar\psi\bar\psi}/16 - H_{\bar\psi\bar\psi IR\bar\psi\bar\psi}/8 + H_{\bar\psi\bar\psi II\bar\psi\bar\psi}/16 + H_{\bar\psi\bar\psi\bar\psi RRR}/4 - H_{\bar\psi\bar\psi\bar\psi IIR}/4 + 9K_{RRRR\bar\psi\bar\psi}/16 + K_{RRII\bar\psi\bar\psi}/16 \\ & - K_{IIIR\bar\psi\bar\psi}/8 + K_{RR\bar\psi\bar\psi\bar\psi\bar\psi}/16 + K_{II\bar\psi\bar\psi\bar\psi\bar\psi}/16 + K_{\bar\psi\bar\psi\bar\psi RR\bar\psi}/8 - K_{\bar\psi\bar\psi\bar\psi IR\bar\psi}/4 + K_{\bar\psi\bar\psi\bar\psi II\bar\psi}/8] \\ & + y^4\psi[27G_{RRRRR}/32 + 3G_{RIIRR}/16 + G_{IIRIR}/8 + 3G_{RIIII}/32 + 27J_{RRRRR}/32 + 9J_{RRRRI}/32 + 3J_{RRIIR}/32 \\ & + J_{RRIII}/32 + 3J_{IIIRI}/16 + J_{IIIRR}/16 + H_{\psi\psi RR\bar\psi\bar\psi}/4 + H_{\psi\psi RI\bar\psi\bar\psi}/4 - H_{\psi\psi IR\bar\psi\bar\psi}/4 - H_{\psi\psi II\bar\psi\bar\psi}/4 + H_{\psi\bar\psi RR\psi\bar\psi}/8 \\ & + H_{\psi\bar\psi IR\psi\bar\psi}/4 + H_{\psi\bar\psi II\psi\bar\psi}/8 + 3H_{\psi\psi\bar\psi RRR}/4 - H_{\psi\psi\bar\psi RII}/4 + H_{\psi\psi\bar\psi IIR}/2 + 9K_{RRRR\psi\psi}/16 + K_{RRII\psi\psi}/16 \\ & + K_{IIIR\psi\psi}/8 + K_{RR\psi\psi\bar\psi\bar\psi}/8 - K_{II\psi\psi\bar\psi\bar\psi}/8 + K_{\psi\psi\bar\psi RR\bar\psi}/8 - K_{\psi\psi\bar\psi IR\bar\psi}/4 + K_{\psi\psi\bar\psi II\bar\psi}/8 + K_{\bar\psi\bar\psi\psi RR\psi}/8 \\ & + K_{\bar\psi\bar\psi\psi IR\psi}/4 + K_{\bar\psi\bar\psi\psi II\psi}/8 + K_{\bar\psi\psi\bar\psi RR\psi}/2 + K_{\bar\psi\psi\bar\psi RI\psi}/2 - K_{\bar\psi\psi\bar\psi IR\psi}/2 - K_{\bar\psi\psi\bar\psi II\psi}/2 + 3J_{RR\bar\psi\bar\psi R}/16 \\ & + J_{RR\bar\psi\bar\psi I}/16 - J_{II\bar\psi\bar\psi R}/16 - 3J_{II\bar\psi\bar\psi I}/16] + y^4[H_{\psi\psi RR\psi\psi}/16 - H_{\psi\psi IR\psi\psi}/8 + H_{\psi\psi II\psi\psi}/16 + K_{RR\psi\psi\psi\psi}/16 \\ & + K_{II\psi\psi\psi\psi}/16 + K_{\psi\psi\psi RR\psi}/8 + K_{\psi\psi\psi IR\psi}/4 + K_{\psi\psi\psi II\psi}/8 + 9L_{RRRR}/64 + 9L_{IIII}/64 + 3L_{IIIR}/32 + L_{IIRR}/64 \\ & + L_{RRII}/64 + 3L_{RRIR}/32 + 3E_{RRRR}/64 + 3E_{IIII}/64 + E_{IIRR}/32 + 3J_{RR\psi\psi R}/16 + J_{RR\psi\psi I}/16 + J_{II\psi\psi R}/16 \\ & + 3J_{II\psi\psi I}/16]. \end{aligned} \tag{4.13}$$

In the latter equation, I have used a short-hand notation, such that, for example, $H_{IIRRII} \equiv H_{SSSSSS}(I,I,R,R,I,I)$ and $K_{\bar\psi\psi\bar\psi RI\psi} \equiv K_{\bar{F}F\bar{F}SSF}(\psi,\psi,\psi,R,I,\psi)$.

As a nontrivial consistency check, consider the renormalization group scale invariance condition for the effective potential, as expressed by Eq. (1.4), with $X = y, m, \phi$. From[2] Refs. [77,78],

$$\beta_y^{(1)}/3y = \beta_m^{(1)}/2m = -\beta_\phi^{(1)}/\phi = y^2/2, \tag{4.14}$$

$$\beta_y^{(2)}/3y = \beta_m^{(2)}/2m = -\beta_\phi^{(2)}/\phi = -y^4/2, \tag{4.15}$$

$$\beta_y^{(3)}/3y = \beta_m^{(3)}/2m = -\beta_\phi^{(3)}/\phi = (3\zeta_3/2 + 5/8)y^6. \tag{4.16}$$

Using Eqs. (4.14)–(4.16), one finds from Eq. (4.10):

$$\sum_X \beta_X^{(1)} \frac{\partial}{\partial X} V^{(0)} = y^2\phi^2(m + y\phi/2)^2, \tag{4.17}$$

$$\sum_X \beta_X^{(2)} \frac{\partial}{\partial X} V^{(0)} = -y^4\phi^2(m + y\phi/2)^2, \tag{4.18}$$

$$\sum_X \beta_X^{(3)} \frac{\partial}{\partial X} V^{(0)} = (3\zeta_3 + 5/4)y^6\phi^2(m + y\phi/2)^2, \tag{4.19}$$

and from Eq. (4.11), also using Eqs. (2.23):

$$\sum_X \beta_X^{(1)} \frac{\partial}{\partial X} V^{(1)} = y^2(m^2 + 3ym\phi + 3y^2\phi^2/2)A(R) + y^2(m^2 + ym\phi + y^2\phi^2/2)A(I) - 2y^2(m + y\phi)^2 A(\psi), \tag{4.20}$$

$$\sum_X \beta_X^{(2)} \frac{\partial}{\partial X} V^{(1)} = -y^4(m^2 + 3ym\phi + 3y^2\phi^2/2)A(R) - y^4(m^2 + ym\phi + y^2\phi^2/2)A(I) + 2y^4(m + y\phi)^2 A(\psi), \tag{4.21}$$

[2] Some other references had given incorrect results for the 3-loop beta functions of the Wess-Zumino model.

and from Eq. (4.12), also using Eqs. (2.23) and (2.24):

$$\sum_X \beta_X^{(1)} \frac{\partial}{\partial X} V^{(2)} = \frac{7y^4}{8}[-3(m+y\phi)^2 I(R,R,R) + (6m^2 + 10ym\phi + 5y^2\phi^2) I(\psi,\psi,R) - (m+y\phi)^2 I(R,I,I) - (2m^2 + 2ym\phi + y^2\phi^2) I(\psi,\psi,I) + 3A(R)^2/2 + 3A(I)^2/2 + A(R)A(I) - 4A(R)A(\psi) - 4A(I)A(\psi) + 4A(\psi)^2] + y^4(m^2 + 3ym\phi + 3y^2\phi^2/2)A(R) + y^4(m^2 + ym\phi + y^2\phi^2/2)A(I) - 2y^4(m+y\phi)^2 A(\psi) - y^6\phi^2(m+y\phi/2)^2. \tag{4.22}$$

Meanwhile, from Eqs. (4.11), (4.12), and (4.13), using Eqs. (3.35)–(3.44), one obtains:

$$Q\frac{\partial}{\partial Q} V^{(1)} = -y^2\phi^2(m+y\phi/2)^2, \tag{4.23}$$

$$Q\frac{\partial}{\partial Q} V^{(2)} = -y^2(m^2 + ym\phi + y^2\phi^2/2)A(I) - y^2(m^2 + 3ym\phi + 3y^2\phi^2/2)A(R) + 2y^2(m+y\phi)^2 A(\psi) + y^4\phi^2(m+y\phi/2)^2, \tag{4.24}$$

$$Q\frac{\partial}{\partial Q} V^{(3)} = \frac{7y^4}{8}[(2m^2 + 2ym\phi + y^2\phi^2) I(\psi,\psi,I) - (6m^2 + 10ym\phi + 5y^2\phi^2) I(\psi,\psi,R) + 3(m+y\phi)^2 I(R,R,R) + (m+y\phi)^2 I(R,I,I) - 3A(R)^2/2 - 3A(I)^2/2 - A(R)A(I) - 4A(\psi)^2 + 4A(\psi)A(R) + 4A(\psi)A(I)] - y^6\phi^2(m+y\phi/2)^2(3\zeta_3 + 1/4). \tag{4.25}$$

Now Eqs. (4.17)–(4.25) can be plugged in to verify that Eq. (1.4) indeed holds for each of $\ell = 1, 2, 3$.

As another check, recall that at a supersymmetric minimum of the tree-level potential, the full effective potential must vanish at each order in perturbation theory [79]. There are two supersymmetric minima of $V^{(0)}$, namely $\phi = 0$ and $\phi = -2m/y$. At each of these, one has equality of the field-dependent squared masses: $x \equiv R = I = \psi = m^2$. It is now straightforward to plug this into Eqs. (4.11), (4.12), and (4.13), to verify that each of $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$ also vanishes at the supersymmetric minima. This relies on non-trivial cancellations between the different loop integral functions defined in Eqs. (3.4)–(3.9) and (3.11)–(3.28), which become apparent upon putting everything in terms of the basis integral functions $H(x,x,x,x,x,x)$, $K(x,x,x,x,x,x)$, $G(x,x,x,x,x)$, $F(x,x,x,x)$, $I(x,x,x)$, $\bar{I}(x,x,x,x)$, $A(x)$, and $\bar{A}(x,x)$.

V. THE STANDARD MODEL

A. Standard model effective potential at three-loop order

In this section, I will consider the complete three-loop effective potential for the standard model as another application of the general results. The full two-loop effective potential for the Standard Model was found in Ref. [4]. The leading three-loop parts, in the limit that the QCD coupling and the top-quark Yukawa coupling are large compared to all other couplings, were found in Ref. [6]. The four-loop contribution at leading order in QCD is also known [7].

The tree-level potential for the standard model is given by

$$V = \Lambda + m^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2, \tag{5.1}$$

where Λ is a field-independent constant energy density necessary for renormalization scale invariance, m^2 is a negative Higgs squared mass parameter, and λ is the Higgs self-coupling. The Higgs complex doublet scalar field is

$$\Phi = \begin{pmatrix} [\phi + H + iG^0]/\sqrt{2} \\ G^+ \end{pmatrix}, \tag{5.2}$$

where ϕ is the constant background field, H is the Higgs boson field, and G^0 and $G^\pm$ are Goldstone bosons. The ϕ-dependent squared masses of the Higgs boson and the Goldstone bosons (both neutral and charged) are

$$H = m^2 + 3\lambda\phi^2, \tag{5.3}$$

$$G = m^2 + \lambda\phi^2, \tag{5.4}$$

and the other nonzero squared masses are

$$t = y_t^2\phi^2/2, \tag{5.5}$$

$$W = g^2\phi^2/4, \tag{5.6}$$

$$Z = (g^2 + g'^2)\phi^2/4, \tag{5.7}$$

where y_t is the top-quark Yukawa coupling and g, g' are the electroweak gauge couplings. The Yukawa couplings of the bottom quark and other fermions are quite negligible, and are therefore taken to vanish.

The field content of the electroweak standard model with n_G generations, compatible with the conventions of Sec. II, is

$$\text{Real scalars}: H, G^0, G_R, G_I \tag{5.8}$$

$$\text{2-component fermions}: t, \bar{t}, b, \bar{b}, \tau, \bar{\tau}, \nu_\tau + (n_G - 1) \times (u, \bar{u}, d, \bar{d}, e, \bar{e}, \nu_e), \tag{5.9}$$

$$\text{Real vectors}: \gamma, Z, W_R, W_I \tag{5.10}$$

Here we have written the complex Goldstone scalar bosons and W vector bosons in terms of real components as $G^\pm = (G_R \pm iG_I)/\sqrt{2}$ and $W^\pm = (W_R \pm iW_I)/\sqrt{2}$. The unbarred fermion fields are $SU(2)_L$ doublets, and the barred fermion fields are $SU(2)_L$ singlets. (Not shown explicitly are the color multiplicity for quarks, or the massless gluon vector fields.)

To facilitate an automated calculation of the 3-loop effective potential, it is useful to have at hand a list of the nonvanishing field-dependent couplings of these mass eigenstate fields. There are scalar cubic interactions of the type λ^{jkl}:

$$\lambda^{HHH} = 6\lambda\phi, \tag{5.11}$$

$$\lambda^{HG^0G^0} = \lambda^{HG_RG_R} = \lambda^{HG_IG_I} = 2\lambda\phi, \tag{5.12}$$

and scalar quartic couplings of the type λ^{jklm}:

$$\lambda^{HHHH} = \lambda^{G^0G^0G^0G^0} = \lambda^{G_RG_RG_RG_R} = \lambda^{G_IG_IG_IG_I} = 6\lambda, \tag{5.13}$$

$$\lambda^{HHG^0G^0} = \lambda^{HHG_RG_R} = \lambda^{HHG_IG_I} = \lambda^{G^0G^0G_RG_R} = \lambda^{G^0G^0G_IG_I} = \lambda^{G_RG_RG_IG_I} = 2\lambda, \tag{5.14}$$

as well as permutations determined by the symmetry under interchange of any two scalars. The nonvanishing Yukawa couplings of the type Y^{jIJ} are given by

$$Y^{Ht\bar{t}} = -Y^{G_Rb\bar{t}} = -iY^{G^0t\bar{t}} = iY^{G_Ib\bar{t}} = y_t/\sqrt{2}, \tag{5.15}$$

which are symmetric under interchange of the last two (fermionic) indices. (All Yukawa couplings other than for the top-quark are neglected.) The interactions of the electroweak vector bosons with the quarks and leptons are given by couplings of the type g_I^{aJ}:

$$g_f^{Zf} = (-I_f g^2 + Y_f g'^2)\Big/\sqrt{g^2 + g'^2}, \tag{5.16}$$

$$g_{\bar{f}}^{Z\bar{f}} = -Q_f g'^2\Big/\sqrt{g^2 + g'^2}, \tag{5.17}$$

$$g_f^{\gamma f} = -g_{\bar{f}}^{\gamma\bar{f}} = -Q_f e, \tag{5.18}$$

where

$$e = gg'\Big/\sqrt{g^2 + g'^2}, \tag{5.19}$$

and $Q_u = 2/3$ and $Q_d = -1/3$ and $Q_\nu = 0$ and $Q_e = -1$, and $I_u = I_\nu = 1/2$ and $I_d = I_e = -1/2$, and $Y_f = Q_f - I_f$ for each f, and

$$g_d^{W_Ru} = g_u^{W_Rd} = g_e^{W_R\nu} = g_\nu^{W_Re} = -g/2, \tag{5.20}$$

$$g_u^{W_Id} = -g_d^{W_Iu} = g_\nu^{W_Ie} = -g_e^{W_I\nu} = -ig/2. \tag{5.21}$$

There are also vector-vector-scalar couplings of the type g^{abj},

$$g^{\gamma W_RG_R} = g^{\gamma W_IG_I} = ge\phi/2, \tag{5.22}$$

$$g^{ZW_RG_R} = g^{ZW_IG_I} = -g'e\phi/2, \tag{5.23}$$

$$g^{W_RW_RH} = g^{W_IW_IH} = g^2\phi/2, \tag{5.24}$$

$$g^{ZZH} = (g^2 + g'^2)\phi/2, \tag{5.25}$$

with symmetry under interchange of the first two (vector) indices. The vector-scalar-scalar couplings of the type g^{ajk} are

$$g^{\gamma G_RG_I} = e, \tag{5.26}$$

$$g^{ZG^0H} = \sqrt{g^2 + g'^2}/2, \tag{5.27}$$

$$g^{ZG_RG_I} = (g^2 - g'^2)\Big/\left(2\sqrt{g^2 + g'^2}\right), \tag{5.28}$$

$$g^{W_RG_RG^0} = g^{W_RHG_I} = g^{W_IG_IG^0} = g^{W_IG_RH} = g/2, \tag{5.29}$$

with others determined by the antisymmetry with respect to interchange of the last two (scalar) indices. There are also vector-vector-scalar-scalar couplings of the type g^{abjk}, determined in terms of these by Eq. (2.5). Finally there are the totally antisymmetric vector-vector-vector couplings defined by:

$$g^{\gamma W_R W_I} = e, \tag{5.30}$$

$$g^{Z W_R W_I} = g^2/\sqrt{g^2 + g'^2}. \tag{5.31}$$

The tree-level and one-loop contributions to the effective potential are

$$V^{(0)} = \Lambda + m^2\phi^2/2 + \lambda\phi^4/4, \tag{5.32}$$

$$V^{(1)} = 3f(G) + f(H) - 12f(t) + 6f_V(W) + 3f_V(Z), \tag{5.33}$$

with functions $f(x)$ and $f_V(x)$ defined in Eqs. (2.7) and (2.8). The two-loop contribution is given by [4]:

$$\begin{aligned}
V^{(2)} = {}& \frac{3}{4}\lambda[f_{SS}(H,H) + 2f_{SS}(G,H) + 5f_{SS}(G,G)] + 3\lambda^2\phi^2[f_{SSS}(H,H,H) + f_{SSS}(G,G,H)] \\
& + \frac{3y_t^2}{2}[f_{FFS}(t,t,H) + tf_{\bar{F}\bar{F}S}(t,t,H) + f_{FFS}(t,t,G) - tf_{\bar{F}\bar{F}S}(t,t,G) + 2f_{FFS}(0,t,G)] + \frac{g^2+g'^2}{8}f_{VSS}(Z,G,H) \\
& + \frac{(g^2-g'^2)^2}{8(g^2+g'^2)}f_{VSS}(Z,G,G) + \frac{g^2}{4}[f_{VSS}(W,G,H) + f_{VSS}(W,G,G)] + \frac{e^2}{2}f_{VSS}(0,G,G) + \frac{g^4\phi^2}{8}f_{VVS}(W,W,H) \\
& + \frac{(g^2+g'^2)^2\phi^2}{16}f_{VVS}(Z,Z,H) + \frac{e^2\phi^2}{4}[g'^2 f_{VVS}(W,Z,G) + g^2 f_{VVS}(0,W,G)] - [4g_3^2 + 4e^2/3]tf_{\bar{F}\bar{F}V}(t,t,0) \\
& + g^2[3f_{FFV}(0,t,W) + (4n_G - 3)f_{FFV}(0,0,W)]/2 + [(9g^4 - 6g^2g'^2 + 17g'^4)f_{FFV}(t,t,Z) \\
& + 8g'^2(3g^2 - g'^2)tf_{\bar{F}\bar{F}V}(t,t,Z) + ((24n_G - 9)g^4 + 6g^2g'^2 + (40n_G - 17)g'^4)f_{FFV}(0,0,Z)]/24(g^2+g'^2) \\
& + \frac{e^2}{2}f_{\text{gauge}}(W,W,0) + \frac{g^4}{2(g^2+g'^2)}f_{\text{gauge}}(W,W,Z),
\end{aligned} \tag{5.34}$$

where $n_G = 3$ is the number of quark and lepton generations. This result for $V^{(2)}$ for the standard model is an example of the application of the general result in Eq. (2.11), using the couplings listed above. It is written here in terms of the functions $f_{SS}(x,y)$, $f_{SSS}(x,y,z)$, $f_{FFS}(x,y,z)$, $f_{\bar{F}\bar{F}S}(x,y,z)$, $f_{VVS}(x,y,z)$, $f_{FFV}(x,y,z)$, $f_{\bar{F}\bar{F}V}(x,y,z)$, and $f_{\text{gauge}}(x,y,z)$ defined in Ref. [5], and the function $f_{VSS}(x,y,z)$ defined in Eqs. (2.12)–(2.13) of the present paper, replacing the functions f_{SSV} and f_{VS} of Ref. [5].

The three-loop effective potential contribution in the standard model can now be obtained by applying the couplings given above in Eqs. (5.11)–(5.29) to the general forms of Eqs. (3.4)–(3.32). The resulting expression contains 536 integral functions of the 89 types in Eq. (3.1) with specific assignments of squared mass arguments H, G, t, W, Z, and δ (used as an infrared regulator for the squared masses of gluons, photons, and quarks and leptons other than the top quark). The 536 functions are given, expanded in δ to retain the $\overline{\ln}(\delta)$ terms, but dropping all terms of order δ, in an ancillary electronic file distributed with this paper, called `SMV3functions.anc` [108]. Of these 536 functions, the following 23 vanish identically in the limit $\delta \to 0$:

$$\begin{gathered}
H_{FFVVFF}(0,0,0,0,0,0), \quad K_{FFFVVF}(0,0,0,0,0,0), \quad K_{VVFFFF}(0,0,0,0,0,0), \\
K_{\bar{F}\bar{F}FSVF}(t,t,t,G,0,t), \quad K_{\bar{F}\bar{F}FSVF}(t,t,t,H,0,t), \quad K_{\bar{F}\bar{F}FSVF}(t,t,0,G,0,t), \\
K_{\bar{F}F\bar{F}SVF}(t,t,t,G,0,t), \quad K_{\bar{F}F\bar{F}SVF}(t,t,t,H,0,t), \quad K_{FFFSVF}(t,t,t,G,0,t), \\
K_{FFFSVF}(t,t,t,H,0,t), \quad K_{FFFSVF}(t,t,0,G,0,t), \quad K_{FFFSVF}(0,0,t,G,0,0), \\
K_{\bar{F}\bar{F}FVVF}(t,t,t,Z,0,t), \quad K_{\bar{F}\bar{F}FVVF}(t,t,t,0,W,0), \quad K_{\bar{F}\bar{F}FVVF}(t,t,t,0,0,t), \\
K_{\bar{F}F\bar{F}VVF}(t,t,t,Z,0,t), \quad K_{\bar{F}F\bar{F}VVF}(t,t,t,0,0,t), \quad K_{FFFVVF}(t,t,t,Z,0,t), \\
K_{FFFVVF}(t,t,t,0,W,0), \quad K_{FFFVVF}(t,t,t,0,0,t), \quad K_{FFFVVF}(0,0,t,W,0,0), \\
K_{FFFVVF}(0,0,0,W,0,0), \quad K_{FFFVVF}(0,0,0,Z,0,0).
\end{gathered} \tag{5.35}$$

The coefficients of the nonvanishing 513 functions that remain, and thus the expression for $V^{(3)}$, are given in another ancillary electronic file called SMV3.anc [108]. These coefficients are built out of couplings g_3, g, g', y_t, λ, and the background Higgs field ϕ. In the text below, I will discuss explicitly the parts of $V^{(3)}$ that are leading order in QCD, and also the parts involving infrared logarithms $\overline{\ln}(\delta)$, where δ is used for the gluon and photon squared masses. (There are no $\overline{\ln}(\delta)$ infrared divergences due to massless fermions, as discussed in subsection III H.)

Consider the contribution proportional to g_3^4. It is

$$\begin{aligned} V^{(3)}_{g_3^4} = g_3^4 N_c C_F \Big\{ &(C_F - C_G/2)\Big[\frac{1}{2}H_{FFVVFF}(t,t,0,0,t,t) - 2tH_{FFVV\bar{F}\bar{F}}(t,t,0,0,t,t) + tH_{F\bar{F}VVF\bar{F}}(t,t,0,0,t,t) \\ &+ \frac{t^2}{2}H_{\bar{F}\bar{F}VV\bar{F}\bar{F}}(t,t,0,0,t,t)\Big] + C_F[tK_{FF\bar{F}VV\bar{F}}(t,t,t,0,0,t) + t^2K_{\bar{F}\bar{F}\bar{F}VV\bar{F}}(t,t,t,0,0,t)] \\ &+ C_G\Big[\frac{1}{3}H_{FFFVVV}(t,t,t,0,0,0) - tH_{F\bar{F}\bar{F}VVV}(t,t,t,0,0,0) + \frac{1}{2}K_{\text{gauge},FF}(0,0,0,0,t,t) - \frac{1}{2}tK_{\text{gauge},\bar{F}\bar{F}}(0,0,0,0,t,t)\Big] \\ &+ T_F[K_{VVFFFF}(\delta,\delta,t,t,t,t) - 2tK_{VVFF\bar{F}\bar{F}}(\delta,\delta,t,t,t,t) + t^2K_{VV\bar{F}\bar{F}\bar{F}\bar{F}}(\delta,\delta,t,t,t,t)] \\ &+ 2(2n_G - 1)T_F[K_{VVFFFF}(0,0,0,0,t,t) - tK_{VVFF\bar{F}\bar{F}}(0,0,0,0,t,t)]\Big\}, \end{aligned} \tag{5.36}$$

where $C_G = N_c = 3$ and $C_F = 4/3$ and $T_F = 1/2$ for QCD, and $n_G = 3$ for the standard model. In the following, I will leave n_G arbitrary, to allow for more informative comparisons. In writing Eq. (5.36), I have taken advantage of the fact that $K_{FFFVVF}(t,t,t,0,0,t)$ and $K_{\bar{F}F\bar{F}VVF}(t,t,t,0,0,t)$ and $K_{\bar{F}\bar{F}FVVF}(t,t,t,0,0,t)$ happen to vanish, even though those functions do not vanish for general squared-mass arguments. The remaining loop integral functions for the special squared mass arguments appearing in Eq. (5.36) are also quite simple. The ones without infrared gluon divergences (therefore setting $\delta = 0$) are

$$\begin{aligned} H_{FFVVFF}(t,t,0,0,t,t) = {}&(225/2 - 208\zeta_3)t^2 - 85tA(t) \\ &+ 6A(t)^2 - 10t^2H(0,t,t,t,0,t), \end{aligned} \tag{5.37}$$

$$\begin{aligned} H_{FFVV\bar{F}\bar{F}}(t,t,0,0,t,t) = {}&(140/3 - 80\zeta_3)t - 40A(t) \\ &+ 12A(t)^2/t - 6tH(0,t,t,t,0,t), \end{aligned} \tag{5.38}$$

$$\begin{aligned} H_{F\bar{F}VVF\bar{F}}(t,t,0,0,t,t) = {}&(32\zeta_3 - 16)t + 10A(t) \\ &+ 6A(t)^2/t + 6tH(0,t,t,t,0,t), \end{aligned} \tag{5.39}$$

$$\begin{aligned} H_{\bar{F}\bar{F}VV\bar{F}\bar{F}}(t,t,0,0,t,t) = {}&16/3 + 16\zeta_3 - 8A(t)/t \\ &- 10H(0,t,t,t,0,t), \end{aligned} \tag{5.40}$$

$$\begin{aligned} K_{FF\bar{F}VV\bar{F}}(t,t,t,0,0,t) = {}&146t/3 - 60A(t) + 18A(t)^2/t \\ &- 18A(t)^3/t^2, \end{aligned} \tag{5.41}$$

$$\begin{aligned} K_{\bar{F}\bar{F}\bar{F}VV\bar{F}}(t,t,t,0,0,t) = {}&-38/3 + 48\zeta_3 + 10A(t)/t \\ &- 12A(t)^2/t^2 - 6A(t)^3/t^3, \end{aligned} \tag{5.42}$$

$$\begin{aligned} H_{FFFVVV}(t,t,t,0,0,0) = {}&(24\zeta_3 - 233/8)t^2 + 117tA(t)/4 \\ &- 27A(t)^2/2, \end{aligned} \tag{5.43}$$

$$\begin{aligned} H_{F\bar{F}\bar{F}VVV}(t,t,t,0,0,0) = {}&(48\zeta_3 - 136/3)t + 45A(t) \\ &- 27A(t)^2/t + 3A(t)^3/t^2, \end{aligned} \tag{5.44}$$

$$K_{\text{gauge},FF}(0,0,0,0,t,t) = -283t^2/6 + 40tA(t) - 13A(t)^2, \tag{5.45}$$

$$\begin{aligned} K_{\text{gauge},\bar{F}\bar{F}}(0,0,0,0,t,t) = {}&[(-296 - 208\zeta_3)t + 281A(t) \\ &- 97A(t)^2/t + 26A(t)^3/t^2]/3, \end{aligned} \tag{5.46}$$

$$K_{VVFFFF}(0,0,0,0,t,t) = 49t^2/6 - 7tA(t) + 2A(t)^2, \tag{5.47}$$

$$\begin{aligned} K_{VVFF\bar{F}\bar{F}}(0,0,0,0,t,t) = {}&[(56 + 32\zeta_3)t - 52A(t) \\ &+ 20A(t)^2/t - 4A(t)^3/t^2]/3, \end{aligned} \tag{5.48}$$

while the ones that do individually have gluon infrared divergences are

$$K_{VVFFFF}(\delta,\delta,t,t,t,t) = -(5+56\zeta_3)t^2 - 62tA(t) - 8A(t)^2 - 8A(t)^3/t + 12t^2(1+A(t)/t)^2\overline{\ln}(\delta), \tag{5.49}$$

$$K_{VVFF\bar{F}\bar{F}}(\delta,\delta,t,t,t,t) = [-(8+280\zeta_3)t - 340A(t) - 52A(t)^2/t - 28A(t)^3/t^2]/3 + 12t(1+A(t)/t)^2\overline{\ln}(\delta), \tag{5.50}$$

$$K_{VV\bar{F}\bar{F}\bar{F}\bar{F}}(\delta,\delta,t,t,t,t) = -152/3 - 56\zeta_3 - 96A(t)/t - 36A(t)^2/t^2 - 8A(t)^3/t^3 + 12(1+A(t)/t)^2\overline{\ln}(\delta). \tag{5.51}$$

It is now clear that the $\overline{\ln}(\delta)$ contributions successfully cancel when these results are put into Eq. (5.36). The result is

$$\begin{aligned} V^{(3)}_{g_3^4} &= g_3^4 t^2[8131/9 - 84n_G + (320 - 256n_G/3)\zeta_3 + (248n_G/3 - 2834/3)A(t)/t \\ &\quad + (428 - 112n_G/3)A(t)^2/t^2 + (32n_G/3 - 216)A(t)^3/t^3 - 16H(0,t,t,t,0,t)/3] \end{aligned} \tag{5.52}$$

$$\begin{aligned} &= g_3^4 t^2[22429/9 - 644n_G/3 - 512\mathrm{Li}_4(1/2)/3 + 64\ln^2(2)[\pi^2 - \ln^2(2)]/9 \\ &\quad + 176\pi^4/135 + (288 - 256n_G/3)\zeta_3 + (32\zeta_3 + 568n_G/3 - 7346/3)\overline{\ln}(t) \\ &\quad + (1076 - 208n_G/3)\overline{\ln}^2(t) + (32n_G/3 - 216)\overline{\ln}^3(t)], \end{aligned} \tag{5.53}$$

where the analytical results for $A(t)$ and $H(0,t,t,t,0,t)$ have been used to obtain the last expression. This agrees with the result found (using different methods, and in particular with dimensional regularization of infrared divergences) in Ref. [6].

Having demonstrated that the infrared divergences associated with doubled massless gluon propagators cancel, let us now consider those coming from the massless photon. First, from the results given in the ancillary electronic file `SMV3.anc` and `SMV3functions.anc` [108], one finds that $V^{(3)}$ contains QED contributions exactly analogous to the QCD ones mentioned above:

$$\frac{16e^4}{9}[K_{VVFFFF}(\delta,\delta,t,t,t,t) - 2tK_{VVFF\bar{F}\bar{F}}(\delta,\delta,t,t,t,t) + t^2 K_{VV\bar{F}\bar{F}\bar{F}\bar{F}}(\delta,\delta,t,t,t,t)], \tag{5.54}$$

where δ is now the infrared regulator squared mass of the photon. The $\overline{\ln}(\delta)$ parts of this cancel in the same way as in QCD.

There are also contributions (given in `SMV3.anc` [108]) to $V^{(3)}$ from diagrams involving the top quark, the W boson, the charged Goldstone bosons, and doubled photon propagators, which individually behave like $\overline{\ln}(\delta)$ as $\delta \to 0$. They can be grouped as:

$$\frac{4e^4}{3}[K_{\mathrm{gauge},FF}(\delta,\delta,W,W,t,t) - tK_{\mathrm{gauge},\bar{F}\bar{F}}(\delta,\delta,W,W,t,t)], \tag{5.55}$$

and

$$\frac{2e^4g^2\phi^2}{3}[K_{VVSVFF}(\delta,\delta,G,W,t,t) - tK_{VVSV\bar{F}\bar{F}}(\delta,\delta,G,W,t,t)]. \tag{5.56}$$

The $\overline{\ln}(\delta)$ infrared divergent parts of these contributions can be extracted from the results given in the ancillary electronic file `SMV3functions.anc` [108]:

$$K_{\mathrm{gauge},FF}(\delta,\delta,W,W,t,t) \sim \frac{3}{2}[W + 6A(W)][t + A(t)]\overline{\ln}(\delta), \tag{5.57}$$

$$K_{\mathrm{gauge},\bar{F}\bar{F}}(\delta,\delta,W,W,t,t) \sim \frac{3}{2t}[W + 6A(W)][t + A(t)]\overline{\ln}(\delta), \tag{5.58}$$

$$K_{VVSVFF}(\delta,\delta,G,W,t,t) \sim -\frac{3}{4}[1 - 6\bar{A}(G,W)][t + A(t)]\overline{\ln}(\delta), \tag{5.59}$$

$$K_{VVSV\bar{F}\bar{F}}(\delta,\delta,G,W,t,t) \sim -\frac{3}{4t}[1 - 6\bar{A}(G,W)][t + A(t)]\overline{\ln}(\delta). \tag{5.60}$$

The $\overline{\ln}(\delta)$ contributions in each of Eqs. (5.55) and (5.56) again are seen to cancel.

All other possible $\overline{\ln}(\delta)$ contributions are found to vanish at the level of the functions given in `SMV3functions.anc` [108], except for the following contributions involving doubled photon propagators, W bosons, and charged Goldstone bosons:

$$\frac{e^4}{4}[K_{\rm gauge}(\delta,\delta,W,W,W,W)+4WK_{{\rm gauge},S}(\delta,\delta,W,W,W,G)$$
$$+4W^2K_{VVSVVS}(\delta,\delta,G,W,W,G)]. \tag{5.61}$$

The relevant infrared behaviors can again be extracted from `SMV3functions.anc` [108]:

$$K_{\rm gauge}(\delta,\delta,W,W,W,W)\sim\frac{3}{16}[W+6A(W)]^2\overline{\ln}(\delta) \tag{5.62}$$

$$K_{{\rm gauge},S}(\delta,\delta,W,W,W,G)$$
$$\sim-\frac{3}{32}[W+6A(W)][1-6\bar{A}(W,G)]\overline{\ln}(\delta) \tag{5.63}$$

$$K_{VVSVVS}(\delta,\delta,G,W,W,G)\sim\frac{3}{64}[1-6\bar{A}(W,G)]^2\overline{\ln}(\delta). \tag{5.64}$$

It follows that the infrared divergences from doubled photon propagators do *not* cancel. The final result for the infrared divergence, which comes entirely from Eq. (5.61), can be simplified to:

$$V^{(3)}\sim\frac{27e^4}{16}\left(\frac{WG\ln(W/G)}{W-G}\right)^2\overline{\ln}(\delta). \tag{5.65}$$

While it might at first seem surprising that there is an uncanceled QED infrared divergence in the three-loop effective potential, it is important to remember that the effective potential itself is not a physical observable. (Recall that it is not even gauge invariant.) What is important is that this infrared divergence does not infect physical observables and closely related quantities. The key property that guarantees this is that Eq. (5.65) is of second order in G. As we will see in the next section, after the necessary resummation of Goldstone boson contributions, terms of higher-than-linear order in G do not affect the minimization condition of the effective potential, nor contribute to the value of the effective potential at its minimum, and so can simply be dropped.

As an aside, one could also eliminate the infrared divergence in Eq. (5.65) by resumming photon self-energies. Note that Eq. (5.65) comes from the 3-loop representatives of the family of ℓ-loop Feynman diagrams that involve a ring of $\ell-1$ photon propagators that carry the same momentum and connect $\ell-1$ one-loop subdiagrams with either W^+, W^- or $W^\pm$, $G^\mp$ internal lines. These diagrams scale like $1/\delta^{\ell-3}$ for $\ell>3$. Resumming these diagrams to all orders in ℓ would yield a contribution to $V_{\rm eff}$ that cancels the three-loop infrared divergence in Eq. (5.65) in the limit $\delta\to 0$:

$$\Delta V_{\rm eff}\sim\frac{3}{16\pi^2}f_V(\Delta_\gamma/16\pi^2)$$
$$-\frac{1}{(16\pi^2)^3}\frac{3}{4}\Delta_\gamma^2[\overline{\ln}(\delta)+2/3]+\cdots \tag{5.66}$$

where the ellipses represent contributions from four loops and beyond, and

$$\Delta_\gamma=3e^2WG\ln(W/G)/2(W-G). \tag{5.67}$$

However, this resummation of photon self-energies is actually an unnecessary complication. The key fact is that with or without the photon ring resummation the contribution is second order in G, and so has no effect on physical quantities and can be dropped after Goldstone boson resummation. Therefore, I prefer not to resum the photon self-energies, for simplicity. In the next subsection, I will discuss the expansion and resummation of the Goldstone boson contributions, and explicitly derive the resulting Higgs VEV minimization condition and find that it has no infrared divergences [or spurious imaginary parts associated with $\overline{\ln}(G)$ when G is negative] of any kind.

As a check of the standard model result, consider the renormalization group scale invariance conditions, which take the form of Eq. (1.4) at each loop order $\ell=1,2,3$, where X is summed over $\Lambda, m^2, \lambda, g_3, g, g', y_t, \phi$, with $\beta_\phi^{(\ell)}=-\gamma^{(\ell)}\phi$ where γ is the scalar field anomalous dimension. The necessary Q derivatives of the three-loop integral functions are given in the ancillary electronic file `QdQ.anc` [108]. The necessary three-loop beta functions and anomalous dimension have been found in Refs. [80–86],[3] except for the field-independent vacuum energy Λ. From the ϕ-independent parts of Eq. (1.4), I find that:

$$\beta_\Lambda^{(1)}=2(m^2)^2, \tag{5.68}$$

$$\beta_\Lambda^{(2)}=[12g^2+4g'^2-12y_t^2](m^2)^2, \tag{5.69}$$

$$\begin{aligned}\beta_\Lambda^{(3)}=&[(192\zeta_3-204)y_t^2g_3^2+153y_t^4/4\\&+(189/8-108\zeta_3)y_t^2g^2-(73/8+20\zeta_3)y_t^2g'^2\\&+(4215/32-51n_G/2-18\zeta_3)g^4\\&+(36\zeta_3-441/16)g^2g'^2+(6\zeta_3-233/32-85n_G/6)g'^4\\&+18\lambda^2](m^2)^2.\end{aligned} \tag{5.70}$$

With this included, I have checked, using the Q derivatives found in `QdQ.anc` [108], that Eq. (1.4) is indeed satisfied by the result for $V^{(3)}$ given in `SMV3.anc` and `SMV3functions.anc` [108]. [Note that the coefficient

[3]Extensions to QCD 4-loop and 5-loop order will not be needed here, but can be found in Refs. [7,87–93].

of $\overline{\ln}(\delta)$ in Eq. (5.65) is independent of Q, so that while Eq. (5.65) does contribute nontrivially to $Q\partial V^{(3)}/\partial Q$, it does so without producing a $\overline{\ln}(\delta)$ part.] This constitutes an important consistency check.

B. Goldstone boson resummation of the standard model effective potential

The three-loop standard model effective potential given in the previous subsection suffers from two related problems associated with the Goldstone boson contributions. First, the squared mass G can easily be negative at the minimum of the (real part of the) effective potential, depending on the choice of renormalization scale Q at which it is evaluated. Due to the presence of $\overline{\ln}(G)$, the usual effective potential then has an imaginary part even at one-loop order. This imaginary part is spurious because it does not correspond to a genuine physical instability. Second, if one chooses a reasonable renormalization scale such that $G \to 0$, then there are $\overline{\ln}(G)$ singularities in the three-loop effective potential and in the derivative of the two-loop effective potential, and $1/G^{L-3}$ singularities in the L-loop effective potential and the derivatives of the $(L-1)$-loop effective potential for $L > 3$. This was noted in the context of the standard model at leading order in the top-quark Yukawa coupling in Ref. [6] (see also [10,15]), where it was somewhat melodramatically referred to as the "Goldstone boson catastrophe." In practice, one can usually simply ignore the problem while maintaining good numerical accuracy, by choosing a renormalization scale such that $|G|$ is not too tiny, and dropping the imaginary part if G is negative. However, this is clearly sub-optimal, and a principled solution was given in Refs. [94,95], where the problem was shown to be resolved by resumming the leading Goldstone boson contributions to all orders, treating G as small compared to the other squared mass parameters of the theory.

The basic idea is very simple: the effects of Goldstone bosons with propagators with squared masses G are reexpanded about a different squared mass $G + \Delta$, which vanishes at the minimum of the full effective potential; this is the pole squared mass of the Goldstone boson in Landau gauge, and therefore a good expansion point. This resolution by resummation has the added benefit that it actually makes it simpler in practice to implement the minimization condition that relates the Higgs VEV to the Lagrangian parameters. It can, and should, also be applied to other calculations such as the pole squared masses of the physical particles. For important further developments and related perspectives, see Refs. [96–102].

To apply the Goldstone boson resummation procedure to the full three-loop[4] standard model effective potential, consider the ordinary effective potential in the form:

$$V_{\rm eff} = V^{(0)} + \frac{1}{16\pi^2} V^{(1)}(G) + \frac{1}{(16\pi^2)^2} V^{(2)}(G) + \frac{1}{(16\pi^2)^3} V^{(3)}(G). \tag{5.71}$$

Here the dependence of each term on the Goldstone boson squared mass G has now been indicated explicitly, with the dependences on the other independent parameters $(g_3, g, g', y_t, \lambda, \phi^2, Q)$ left implicit.[5] Now one can resum the contributions to all loop orders from diagrams that consist of single rings of Goldstone boson propagators punctuated by one-particle-irreducible subdiagrams that feature larger masses, by writing

$$V^{(1)}(G) \to V^{(1)}(G+\Delta) - \Delta \frac{\partial}{\partial G} V^{(1)}(G) - \frac{1}{2}\Delta^2 \frac{\partial^2}{\partial G^2} V^{(1)}(G) \tag{5.72}$$

where the quantity

$$\Delta = \frac{1}{16\pi^2}\Delta_1 + \frac{1}{(16\pi^2)^2}\Delta_2 + \frac{1}{(16\pi^2)^3}\Delta_3 + \cdots \tag{5.73}$$

will be given below, and is defined[6] by the properties that $G + \Delta$ vanishes at the minimum of the full effective potential, and each Δ_ℓ does not depend on G. Now we can write, through three-loop order:

$$V_{\rm eff} \to V^{(0)} + \frac{1}{16\pi^2} V^{(1)}(G+\Delta) + \frac{1}{(16\pi^2)^2} \hat{V}^{(2)}(G) + \frac{1}{(16\pi^2)^3}\Big[V^{(3)}(G) - \Delta_2 \frac{\partial}{\partial G} V^{(1)}(G) - \frac{1}{2}(\Delta_1)^2 \frac{\partial^2}{\partial G^2} V^{(1)}(G)\Big], \tag{5.74}$$

where we have defined

$$\hat{V}^{(2)}(G) \equiv V^{(2)}(G) - \Delta_1 \frac{\partial}{\partial G} V^{(1)}(G), \tag{5.75}$$

Now one can continue the resummation procedure by making the replacement:

[4]The extension of this whole procedure to any given higher loop order should be clear from the following.

[5]Note that $H = 2\lambda\phi^2 + G$, so that it is not an independent parameter, and in the following $\partial H/\partial G = 1$. The other squared mass parameters t, W, Z are independent of G.

[6]A warning about a notational switch: the Δ_ℓ in the present paper are equal to what I called $\hat{\Delta}_\ell$ in Ref. [94]. The following discussion could be equally well reformulated in terms of the quantities called Δ_ℓ in Ref. [94], with results that are consistent up to four-loop order contributions. However, that alternative formulation is complicated slightly by the fact that the Δ_ℓ in the notation of Ref. [94] depend on G, through their dependences on H, so I omit it for simplicity.

$$\hat{V}^{(2)}(G) \to \hat{V}^{(2)}(G+\Delta) - \Delta_1 \frac{\partial}{\partial G}\hat{V}^{(2)}(G) = \hat{V}^{(2)}(G+\Delta) - \Delta_1 \frac{\partial}{\partial G}V^{(2)}(G) + (\Delta_1)^2 \frac{\partial^2}{\partial G^2}V^{(1)}(G), \tag{5.76}$$

with the result

$$V_{\rm eff} \to V^{(0)} + \frac{1}{16\pi^2}V^{(1)}(G+\Delta) + \frac{1}{(16\pi^2)^2}\hat{V}^{(2)}(G+\Delta) + \frac{1}{(16\pi^2)^3}\hat{V}^{(3)}(G), \tag{5.77}$$

where

$$\hat{V}^{(3)}(G) \equiv V^{(3)}(G) - \Delta_1 \frac{\partial}{\partial G}V^{(2)}(G) - \Delta_2 \frac{\partial}{\partial G}V^{(1)}(G) + \frac{1}{2}(\Delta_1)^2 \frac{\partial^2}{\partial G^2}V^{(1)}(G). \tag{5.78}$$

Finally, we can replace G by $G+\Delta$ in the three-loop term, since the difference is of four-loop order. Thus, the resummed effective potential at three-loop order is

$$V_{\rm eff}^{\rm resummed} = V^{(0)} + \frac{1}{16\pi^2}V^{(1)}(G+\Delta) + \frac{1}{(16\pi^2)^2}\hat{V}^{(2)}(G+\Delta) + \frac{1}{(16\pi^2)^3}\hat{V}^{(3)}(G+\Delta), \tag{5.79}$$

where the functions $\hat{V}^{(2)}$ and $\hat{V}^{(3)}$ are defined in terms of the usual perturbatively calculated (non-resummed) quantities by Eqs. (5.75) and (5.78), respectively.

In order to construct the functions $\hat{V}^{(2)}(G)$ and $\hat{V}^{(3)}(G)$ from the results given in the previous subsection, one needs Δ_1, Δ_2, $\frac{\partial}{\partial G}V^{(1)}(G)$, $\frac{\partial^2}{\partial G^2}V^{(1)}(G)$, and $\frac{\partial}{\partial G}V^{(2)}(G)$, which are all straightforward to obtain from the one-loop and two-loop order effective potentials. The results for Δ_1 and Δ_2 have already been given in Eqs. (4.19) and (4.20) of Ref. [94], and are also provided in an ancillary electronic file of the present paper called `SMDeltas.anc` [108]. For example,

$$\Delta_1 = 3\lambda A(h) - 6y_t^2 A(t) + \frac{3g^2}{2}A(W) + \frac{3(g^2+g'^2)}{4}A(Z) + (3g^4 + 2g^2g'^2 + g'^4)\phi^2/8, \tag{5.80}$$

where

$$h \equiv H - G = 2\lambda\phi^2. \tag{5.81}$$

Also, one has the simple one-loop results:

$$\frac{\partial}{\partial G}V^{(1)}(G) = \frac{3}{2}A(G) + \frac{1}{2}A(H), \tag{5.82}$$

$$\frac{\partial^2}{\partial G^2}V^{(1)}(G) = \frac{3}{2}[1 + A(G)/G] + \frac{1}{2}[1 + A(H)/H]. \tag{5.83}$$

The expression for $\partial V^{(2)}/\partial G$ is more complicated, and is given in an ancillary electronic file `SMdV2dG.anc` [108].

A crucial feature of $V_{\rm eff}^{\rm resummed}$ is that in the expansions of $V^{(1)}(G+\Delta)$, $\hat{V}^{(2)}(G+\Delta)$, and $\hat{V}^{(3)}(G+\Delta)$ for small $G+\Delta$, terms with $\overline{\ln}(G+\Delta)$ do not appear until quadratic order in $G+\Delta$. This can be seen by performing the expansions for small G for basis integral functions that have G as an argument, using the tools in the ancillary electronic file `expzero.anc` [108]. The results can be written in the form

$$V^{(1)}(G) = V^{(1)}(0) + GV^{(1)\prime}(0) + \mathcal{O}(G^2), \tag{5.84}$$

$$\hat{V}^{(2)}(G) = \hat{V}^{(2)}(0) + G\hat{V}^{(2)\prime}(0) + \mathcal{O}(G^2), \tag{5.85}$$

$$\hat{V}^{(3)}(G) = \hat{V}^{(3)}(0) + G\hat{V}^{(3)\prime}(0) + \mathcal{O}(G^2), \tag{5.86}$$

where $V^{(1)}(0)$, $V^{(1)\prime}(0)$, $\hat{V}^{(2)}(0)$, $\hat{V}^{(2)\prime}(0)$, $\hat{V}^{(3)}(0)$, and $\hat{V}^{(3)\prime}(0)$ do not depend on G. In particular, the cancellations of the $G\overline{\ln}(G)$ terms in $\hat{V}^{(2)}(G)$, and the $\overline{\ln}(G)$, $G\overline{\ln}(G)$, and $G\overline{\ln}^2(G)$ terms in $\hat{V}^{(3)}(G)$, provide an important check. Because of the absence of these terms in Eqs. (5.85) and (5.86), the resummed effective potential $V_{\rm eff}^{\rm resummed}$ defined by Eq. (5.79), and its first derivatives with respect to arbitrary parameters, are finite and real at its minimum.

Note also that the expansions to linear order given in Eqs. (5.84)–(5.86), applied to Eq. (5.79), are sufficient to produce the minimization condition for the Higgs VEV valid through full three-loop order, because first derivatives of terms of order $(G+\Delta)^2$ or higher will vanish there. Since the quadratic terms have been dropped, the QED infrared divergence of Eq. (5.65) does not appear.

The explicit one-loop and two-loop order results are

$$V^{(1)}(0) = f(h) - 12f(t) + 6f_V(W) + 3f_V(Z), \tag{5.87}$$

$$V^{(1)\prime}(0) = A(h)/2, \tag{5.88}$$

and

$$\begin{aligned}
\hat{V}^{(2)}(0) = {} & \frac{3}{4}\lambda f_{SS}(h,h) + 3\lambda^2\phi^2[f_{SSS}(h,h,h) + f_{SSS}(0,0,h)] + \frac{3y_t^2}{2}[f_{FFS}(t,t,h) \\
& + t f_{\bar{F}\bar{F}S}(t,t,h) + f_{FFS}(t,t,0) - t f_{\bar{F}\bar{F}S}(t,t,0) + 2f_{FFS}(0,t,0)] \\
& + \frac{g^2+g'^2}{8} f_{VSS}(Z,0,h) + \frac{(g^2-g'^2)^2}{8(g^2+g'^2)} f_{VSS}(Z,0,0) \\
& + \frac{g^2}{4}[f_{VSS}(W,0,h) + f_{VSS}(W,0,0)] + \frac{g^4\phi^2}{8} f_{VVS}(W,W,h) \\
& + \frac{(g^2+g'^2)^2\phi^2}{16} f_{VVS}(Z,Z,h) + \frac{e^2\phi^2}{4}[g'^2 f_{VVS}(W,Z,0) + g^2 f_{VVS}(0,W,0)] \\
& - [4g_3^2 + 4e^2/3] t f_{\bar{F}\bar{F}V}(t,t,0) + g^2[3f_{FFV}(0,t,W) + (4n_G - 3)f_{FFV}(0,0,W)]/2 \\
& + [(9g^4 - 6g^2g'^2 + 17g'^4) f_{FFV}(t,t,Z) + 8g'^2(3g^2 - g'^2) t f_{\bar{F}\bar{F}V}(t,t,Z) \\
& + ((24n_G - 9)g^4 + 6g^2g'^2 + (40n_G - 17)g'^4) f_{FFV}(0,0,Z)]/24(g^2+g'^2) \\
& + \frac{e^2}{2} f_{\text{gauge}}(W,W,0) + \frac{g^4}{2(g^2+g'^2)} f_{\text{gauge}}(W,W,Z) - \Delta_1 A(h)/2, \qquad (5.89)
\end{aligned}$$

$$\begin{aligned}
\hat{V}^{(2)\prime}(0) = {} & 3y_t^2(y_t^2/4\lambda - 1)I(h,t,t) + 3[g^2/4 - \lambda - g^4/32\lambda + g^4/16(g^2 - 2\lambda)]I(h,W,W) \\
& + 3[(g^2+g'^2)/8 - \lambda/2 - (g^2+g'^2)^2/64\lambda + (g^2+g'^2)^2/32(g^2+g'^2-2\lambda)]I(h,Z,Z) \\
& - 3\lambda I(h,h,h)/2 + (6\lambda + 3g^2/4)I(0,h,W) + [3\lambda + 3(g^2+g'^2)/8]I(0,h,Z) \\
& + [3(2g^2+g'^2)^3/4(g^2+g'^2)^2]I(0,W,Z) + \{3[-(g^2+g'^2)/16\lambda \\
& - (g^2+g'^2)/8(g^2+g'^2-2\lambda) + (g'^4 - g^4 + 6g^2g'^2)/4(g^2+g'^2)^2]A(Z)^2 \\
& - 3[g^2/8\lambda + g^2/4(g^2-2\lambda) + (g^4 + 6g^2g'^2 + 4g'^4)/2(g^2+g'^2)^2]A(W)^2 \\
& + [3(8g^4 + 8g^2g'^2 + g'^4)/(g^2+g'^2)^2]A(W)A(Z) - (3y_t^2/2\lambda)A(h)A(t) \\
& + [3(g^2+g'^2)^2/16\lambda(g^2+g'^2-2\lambda)]A(h)A(Z) + [3g^4/8\lambda(g^2-2\lambda)]A(h)A(W) \\
& + (9 + 3y_t^2/2\lambda)A(t)^2 + 3A(h)^2/4\}/\phi^2 - (3y_t^4/2\lambda)A(t) \\
& + 3[(g^2+g'^2)^2(g^2+g'^2-4\lambda)/32\lambda(g^2+g'^2-2\lambda) \\
& - g^2(7g^2+10g'^2)/4(g^2+g'^2)]A(Z) \\
& + 3g^2[g^2/16\lambda - g^2/8(g^2-2\lambda) - (6g^4 + 6g^2g'^2 + g'^4)/4(g^2+g'^2)^2]A(W) \\
& + [-3y_t^4/4\lambda + 3y_t^2/2 - 3\lambda - 3g^2/8 - g'^2/8 + (21g^4 + 14g^2g'^2 + 7g'^4)/64\lambda \\
& - 3(g^2+g'^2)^2/32(g^2+g'^2-2\lambda) - 3g^4/16(g^2-2\lambda)]A(h) \\
& + [3y_t^4 + 6\lambda^2 - 9g^4/8 + 3g^2g'^2/8 + 3g^6/8(g^2+g'^2) + 3g^8/16(g^2+g'^2)^2]\phi^2\zeta_2 \\
& + 3[y_t^6/4\lambda + y_t^4 - \lambda y_t^2 + \lambda(3g^2+g'^2)/6 - (3g^6 + 3g^4g'^2 + 3g^2g'^4 + g'^6)/128\lambda \\
& + 3g^6/32(g^2-2\lambda) + 3(g^2+g'^2)^3/64(g^2+g'^2-2\lambda) + (81g^8 + 158g^6g'^2 \\
& + 110g^4g'^4 + 28g^2g'^6 + g'^8)/96(g^2+g'^2)^2]\phi^2. \qquad (5.90)
\end{aligned}$$

These are included, along with the much more complicated results for $\hat{V}^{(3)}(0)$ and $\hat{V}^{(3)\prime}(0)$, in an ancillary electronic file `SMVresummedGexp.anc` [108]. The results are given in terms of basis loop integral functions, with squared mass arguments $h, t, W, Z, 0$ and with coefficients built out of g_3, g, g', y_t, λ, and ϕ^2.

C. The standard model Higgs VEV at three-loop order

In this subsection, I discuss the application of the standard model three-loop effective potential to obtain the minimization condition for the Higgs VEV $v = \phi_{\text{min}}$, given the Higgs squared mass parameter m^2, or vice versa. This condition is

$$\frac{1}{\phi}\frac{\partial}{\partial\phi}V_{\rm eff}^{\rm resummed}|_{\phi=v} = 0, \tag{5.91}$$

which can be written as

$$G = m^2 + \lambda v^2 = -\sum_{\ell=1}^{\infty}\frac{1}{(16\pi^2)^\ell}\Delta_\ell. \tag{5.92}$$

This result can also be expressed as the relation between the $\overline{\rm MS}$ tree-level VEV

$$v_{\rm tree} \equiv \sqrt{-m^2/\lambda} \tag{5.93}$$

and the VEV v defined as the minimum of the full effective potential. One has:

$$v_{\rm tree}^2 = v^2 + \frac{1}{\lambda}\sum_{\ell=1}^{\infty}\frac{1}{(16\pi^2)^\ell}\Delta_\ell. \tag{5.94}$$

Using the expansions of Eq. (5.84)–(5.86) in Eq. (5.79) and the fact that, by definition, $G + \Delta$ vanishes at the minimum, we have through three-loop order:

$$\Delta_1 = \frac{1}{\phi}\frac{\partial}{\partial\phi}V^{(1)}(0) + V^{(1)\prime}(0)\frac{1}{\phi}\frac{\partial G}{\partial\phi}, \tag{5.95}$$

$$\Delta_2 = \frac{1}{\phi}\frac{\partial}{\partial\phi}\hat{V}^{(2)}(0) + \hat{V}^{(2)\prime}(0)\frac{1}{\phi}\frac{\partial G}{\partial\phi} + V^{(1)\prime}(0)\frac{1}{\phi}\frac{\partial\Delta_1}{\partial\phi}, \tag{5.96}$$

$$\Delta_3 = \frac{1}{\phi}\frac{\partial}{\partial\phi}\hat{V}^{(3)}(0) + \hat{V}^{(3)\prime}(0)\frac{1}{\phi}\frac{\partial G}{\partial\phi} + \hat{V}^{(2)\prime}(0)\frac{1}{\phi}\frac{\partial\Delta_1}{\partial\phi} + V^{(1)\prime}(0)\frac{1}{\phi}\frac{\partial\Delta_2}{\partial\phi}, \tag{5.97}$$

with $\phi = v$. Now one can use:

$$\frac{1}{\phi}\frac{\partial G}{\partial\phi} = 2\lambda, \tag{5.98}$$

$$\frac{1}{\phi}\frac{\partial\Delta_1}{\partial\phi} = [6\lambda A(h) - 12y_t^2A(t) + 3g^2A(W) + 3(g^2+g'^2)A(Z)/2]/\phi^2 + 12\lambda^2 - 6y_t^4 + 15g^4/8 + 5g^2g'^2/4 + 5g'^4/8 \tag{5.99}$$

together with the derivatives of the two-loop and three-loop basis functions as given in Ref. [51], to iteratively evaluate Δ_1, Δ_2, and Δ_3. As mentioned above, the first two were already given above in Eqs. (4.19) and (4.20) of Ref. [94], and all three are given in the ancillary electronic file `SMDeltas.anc` [108] distributed with the present paper. These results are given in terms of basis functions with arguments $h, t, W, Z, 0$ and $\overline{\rm MS}$ renormalization scale Q. The complete lists of the specific one-loop, two-loop, and three-loop basis functions needed are

$$\mathcal{I}^{(1)} = \{A(h), A(t), A(W), A(Z)\}, \tag{5.100}$$

$$\begin{aligned}\mathcal{I}^{(2)} = \{&\zeta_2, I(0,h,W), I(0,h,Z), I(0,t,W), I(0,W,Z), I(h,h,h),\\ &I(h,t,t), I(h,W,W), I(h,Z,Z), I(t,t,Z), I(W,W,Z)\},\end{aligned} \tag{5.101}$$

$$\begin{aligned}\mathcal{I}^{(3)} = \{&\zeta_3, F(h,0,0,t), F(h,0,0,W), F(h,0,0,Z), F(h,0,h,W), F(h,0,h,Z), F(h,0,t,t),\\
&F(h,0,t,W), F(h,0,W,W), F(h,0,W,Z), F(h,0,Z,Z), F(h,h,t,t), F(h,h,W,W),\\
&F(h,h,Z,Z), F(h,t,t,Z), F(h,W,W,Z), F(t,0,0,W), F(t,0,0,Z), F(t,0,h,W),\\
&F(t,0,t,W), F(t,0,W,Z), F(t,h,t,Z), F(t,t,W,W), F(t,t,Z,Z), F(W,0,0,Z),\\
&F(W,0,h,h), F(W,0,h,t), F(W,0,h,Z), F(W,0,t,t), F(W,0,t,Z), F(W,0,W,W),\\
&F(W,0,Z,Z), F(W,h,W,Z), F(Z,0,h,h), F(Z,0,h,W), F(Z,0,t,t), F(Z,0,t,W),\\
&F(Z,0,W,W), F(Z,0,W,Z), F(Z,0,Z,Z), F(Z,h,t,t), F(Z,h,W,W), \bar{F}(0,0,h,t),\\
&\bar{F}(0,0,h,W), \bar{F}(0,0,h,Z), \bar{F}(0,0,t,W), \bar{F}(0,0,t,Z), \bar{F}(0,0,W,Z), \bar{F}(0,h,t,t),\\
&\bar{F}(0,h,W,W), \bar{F}(0,t,t,Z), \bar{F}(0,W,W,Z), G(0,0,0,h,t), G(0,0,0,t,Z),\\
&G(0,t,t,W,W), G(h,0,0,0,W), G(h,0,0,0,Z), G(h,0,0,h,h), G(h,0,0,t,t),\\
&G(h,0,0,W,W), G(h,0,0,Z,Z), G(h,0,W,0,Z), G(h,0,W,h,h), G(h,0,W,t,t),\end{aligned}$$

$$
\begin{aligned}
&G(h,0,W,W,W), G(h,0,W,Z,Z), G(h,0,Z,h,h), G(h,0,Z,t,t), G(h,0,Z,W,W),\\
&G(h,0,Z,Z,Z), G(h,h,h,h,h), G(h,h,h,t,t), G(h,h,h,W,W), G(h,h,h,Z,Z),\\
&G(h,t,t,W,W), G(h,t,t,Z,Z), G(h,W,W,Z,Z), G(t,0,0,0,W), G(t,0,0,h,t),\\
&G(t,0,0,t,Z), G(t,0,W,h,t), G(t,0,W,t,Z), G(t,h,t,t,Z), G(W,0,0,0,Z),\\
&G(W,0,0,h,W), G(W,0,0,W,Z), G(W,0,h,0,t), G(W,0,h,0,Z), G(W,0,h,h,W),\\
&G(W,0,h,W,Z), G(W,0,t,0,Z), G(W,0,t,h,W), G(W,0,t,W,Z), G(W,0,Z,h,W),\\
&G(W,0,Z,W,Z), G(W,h,W,W,Z), G(Z,0,0,h,Z), G(Z,0,0,t,t), G(Z,0,0,W,W),\\
&G(Z,0,h,0,W), G(Z,0,h,h,Z), G(Z,0,h,t,t), G(Z,0,h,W,W), G(Z,0,W,h,Z),\\
&G(Z,0,W,t,t), G(Z,0,W,W,W), G(Z,h,Z,t,t), G(Z,h,Z,W,W), G(Z,t,t,W,W),\\
&H(0,0,0,0,0,h), H(0,0,0,0,0,t), H(0,0,0,0,0,W), H(0,0,0,0,0,Z),\\
&H(0,0,0,0,h,W), H(0,0,0,0,t,t), H(0,0,0,0,t,W), H(0,0,0,0,W,W),\\
&H(0,0,0,0,W,Z), H(0,0,0,h,t,t), H(0,0,0,h,W,W), H(0,0,0,h,Z,Z),\\
&H(0,0,0,t,t,Z), H(0,0,0,W,W,Z), H(0,0,h,0,W,W), H(0,0,h,h,W,W),\\
&H(0,0,h,h,Z,Z), H(0,0,h,W,0,0), H(0,0,h,W,0,Z), H(0,0,h,W,W,Z),\\
&H(0,0,h,Z,0,0), H(0,0,h,Z,0,W), H(0,0,h,Z,W,W), H(0,0,t,0,t,W),\\
&H(0,0,t,0,W,W), H(0,0,t,h,t,t), H(0,0,t,W,0,t), H(0,0,t,Z,0,0),\\
&H(0,0,t,Z,0,W), H(0,0,t,Z,W,W), H(0,0,W,0,W,Z), H(0,0,W,h,h,h),\\
&H(0,0,W,h,Z,Z), H(0,0,W,t,h,t), H(0,0,W,t,t,Z), H(0,0,W,W,0,0),\\
&H(0,0,W,W,0,h), H(0,0,W,W,0,W), H(0,0,W,W,0,Z), H(0,0,W,W,h,W),\\
&H(0,0,W,W,W,Z), H(0,0,W,Z,0,0), H(0,0,W,Z,0,h), H(0,0,W,Z,h,Z),\\
&H(0,0,W,Z,t,t), H(0,0,Z,h,h,h), H(0,0,Z,h,W,W), H(0,0,Z,W,h,W),\\
&H(0,0,Z,Z,0,0), H(0,0,Z,Z,0,W), H(0,0,Z,Z,W,W), H(0,h,h,W,h,W),\\
&H(0,h,h,Z,h,Z), H(0,h,t,Z,t,t), H(0,h,W,W,W,h), H(0,h,W,W,W,Z),\\
&H(0,h,Z,W,Z,W), H(0,h,Z,Z,Z,h), H(0,t,t,t,0,t), H(0,t,t,t,h,t),\\
&H(0,t,t,t,Z,t), H(0,t,t,W,0,W), H(0,t,t,W,h,W), H(0,t,t,W,Z,W),\\
&H(0,W,W,W,0,W), H(0,W,W,W,h,W), H(0,W,W,W,Z,W), H(0,W,W,Z,h,Z),\\
&H(0,W,Z,Z,W,W), H(h,h,h,h,h,h), H(h,h,t,h,t,t), H(h,h,W,h,W,W),\\
&H(h,h,Z,h,Z,Z), H(h,t,t,t,h,t), H(h,t,t,t,Z,t), H(h,t,Z,t,t,Z),\\
&H(h,W,W,W,h,W), H(h,W,W,W,Z,W), H(h,W,Z,W,W,Z),\\
&H(h,Z,Z,Z,h,Z), H(t,t,Z,Z,t,t), H(W,W,Z,Z,W,W)\}.
\end{aligned}
\tag{5.102}
$$

(Recall that this choice of basis functions is not unique, because of the identities mentioned at the end of Sec. II.) The form of the result is then

$$
\begin{aligned}
\Delta_3 = {}& \sum_i c_i^{(3)} \mathcal{I}_i^{(3)} + \sum_{i,j} c_{i,j}^{(2,1)} \mathcal{I}_i^{(2)} \mathcal{I}_j^{(1)} + \sum_i c_i^{(2)} \mathcal{I}_i^{(2)} + \sum_{i,j,k} c_{i,j,k}^{(1,1,1)} \mathcal{I}_i^{(1)} \mathcal{I}_j^{(1)} \mathcal{I}_k^{(1)} \\
&+ \sum_{i,j} c_{i,j}^{(1,1)} \mathcal{I}_i^{(1)} \mathcal{I}_j^{(1)} + \sum_i c_i^{(1)} \mathcal{I}_i^{(1)} + c^{(0)},
\end{aligned}
\tag{5.103}
$$

with coefficients that are built out of g_3, g, g', y_t, λ, and ϕ^2, and are given in `SMDeltas.anc` [108]. The result for Δ_3 extends the partial result (in the approximation that $g_3^2, y_t^2 \gg g^2, g'^2, \lambda$) given in Eq. (4.21) of Ref. [94]. The expression for Δ_4 is known at leading order in QCD only, and was given in Eq. (5.5) [see also Eqs. (4.39) and (4.40)] of Ref. [7].

As a check, one can demand that Eq. (5.92) satisfies renormalization group scale invariance. This condition takes the form, for each loop order ℓ:

$$Q\frac{\partial}{\partial Q}\Delta_\ell = -\phi^2\beta_\lambda^{(\ell)} - 2\lambda\phi\beta_\phi^{(\ell)} + \lambda\phi^2(\beta_{m^2}^{(\ell)}/m^2) + \sum_{n=1}^{\ell-1}\left[(\beta_{m^2}^{(n)}/m^2) - \sum_X \beta_X^{(n)}\frac{\partial}{\partial X}\right]\Delta_{\ell-n} \tag{5.104}$$

where $\phi = v$ at the minimum of the potential, and X is summed over g_3, g, g', y_t, λ, and ϕ. I have verified Eq. (5.104) for each of $\ell = 1, 2, 3$, using the results above.

VI. OUTLOOK

In this paper I have provided the results for the effective potential at full three-loop order for a general renormalizable theory, in the $\overline{\rm MS}$ scheme and using Landau gauge fixing. The results for the standard model provided in Sec. V allow the most accurate theoretical determination possible at this time for the relationship between the Higgs VEV and the Lagrangian parameters, including the negative Higgs squared mass parameter m^2. In practice, this can be used to eliminate m^2 and G in favor of v (and $H = 3\lambda v^2 + m^2$ in favor of $h = 2\lambda v^2$), from other calculations in which they appear. A study of the numerical impact of the three-loop contributions is not given here, but will appear in future work. This is also part of a larger program, as begun in Refs. [39–42], to obtain high-precision results for the pole masses of the standard model particles, and other observables, in the tadpole-free pure $\overline{\rm MS}$ scheme.

In general, three-loop order contributions to the effective potential can suffer from various kinds of infrared divergences that arise due to doubled propagators carrying the same momentum and small squared masses. The problematic contributions associated with Goldstone bosons are eliminated by resummation. The infrared divergences associated with doubled gluon propagators cancel completely after including all diagrams at three-loop order. I also found an uncanceled infrared divergence from doubled photon propagators in the three-loop standard model effective potential; this can be eliminated by resummation of photon self-energies, but it is actually benign even without doing so, provided that one resums the Goldstone boson contributions.

One might also worry about the case of doubled massless or light fermion propagators, for example in models of supersymmetry breaking such as the O'Raifeartaigh model [103] that feature massless goldstino fermions. However, the results above show explicitly that, as suggested by power counting arguments, there are no such infrared divergences from massless fermions (no "goldstino catastrophe") at three-loop order.

The $\overline{\rm MS}$ renormalization scheme based on dimensional regularization does not respect supersymmetry when there are gauge fields present. Therefore, the results given here are not of direct applicability to softly broken supersymmetric gauge theories, such as realistic supersymmetric extensions of the standard model. Further work will be needed in order to obtain the three-loop effective potential in the $\overline{\rm DR}'$ renormalization scheme [104] based on regularization by dimensional reduction [105–107], which is appropriate for such theories.

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