PHYSICAL REVIEW D **96,** 096002 (2017)

Magnetic confinement and the Linde problem

Yu. A. Simonov

State Research Center, Institute of Theoretical and Experimental Physics, 117218, Moscow, B. Cheremushkinskaya 25, Russia

(Received 27 December 2016; published 10 November 2017)

Perturbation theory of thermodynamic potentials in QCD at $T > T_c$ is considered with the nonperturbative background vacuum taken into account. It is shown that the colormagnetic confinement in the QCD vacuum prevents the infrared catastrophe of the perturbation theory present in the case of the free vacuum (the "Linde problem"). A short discussion is given of the applicability of the nonperturbative formalism at large T and of the relation with hard-thermal-loop theory. The observation of Linde, that the terms $O(g^n), n > 6$ contribute to the order $O(g^6)$, is confirmed also with the account of the colormagnetic confinement, and it is shown that the latter ensures that these terms are IR convergent. To make these terms summable, an integral equation is formulated for the ladder graphs, which allows us to define the sum of the $O(g^6)$ terms via a nonsingular kernel. It is argued that the purely nonperturbative term without gluon exchanges may dominate for $T < 600$ Mev.

DOI: 10.1103/PhysRevD.96.096002

I. INTRODUCTION

The Linde problem in thermal QCD has a long history; see [1,2] and references therein. It has occurred in thermal perturbative QCD, where similarly to QED, the originally massless constituents (gluons) acquire effective perturbative mass operators $m(T)$, which regulate the convergence of g^n terms and of the whole perturbative series. Correspondingly, the colorelectric screening mass $m_D(T)$, obtained from $\Pi_{00}(T)$ (similarly to the QED case), starts from gT; however, the colormagnetic screening mass does not exist perturbatively [1,2] (again, as in QED), and if introduced effectively as $O(g^2T)$, the perturbative series is not defined at the order g^6 [problem (1)]. Linde also states that the higher order diagrams contribute to the same order [problem (2)].

Meanwhile the effective perturbative theory of thermal QCD [hard-thermal-loop (HTL) theory] was developed in [3,4], using the colorelectric $m_D(T)$ and the resummation technique through order g^2, g^3, g^4, g^5, which appears to be quite successful; see [5] for a review. The nonperturbative nature of the magnetic scale g^2T, which appears necessarily at $O(g^6)$, can be connected to the 3D Yang Mills theory; see, e.g., [6].

A natural question arises, how can this situation be explained and treated in a 4D approach to QCD, where nonperturbative (NP) physics (including confinement) is taken into account?

In what follows we shall consider the NP approach to QCD, developed in [7–12]. For an alternative approach see [13,14].

Most effects in QCD at low and intermediate energies cannot be explained without NP dynamics, which enters in the theory, e.g., via the string tension σ, or a mass of some meson (ρ, K), or else the constant $\Lambda_{\rm QCD}$, entering in $\alpha_s(Q)$. The approach of the NP vacuum, ensuring confinement and stabilizing perturbation theory, was developed in [7] for QCD at zero temperature T, and in [8,9] for $T > 0$; see reviews [10,11] for $T = 0$ and [12] for $T > 0$. The problem of the confinement and deconfinement is treated in our approach, called the field correlator method (FCM), taking into account two kinds of colorelectric correlators $D^E(z), D_1^E(z) \sim \langle E_i(x)E_i(y)\rangle$, and two kinds of colormagnetic, $D^H(z), D_1^H(z) \sim \langle H_i(x)H_i(y)\rangle$, where the first ones (D^E, D^B) are of purely non-Abelian character, while D_1^E, D_1^H exist also in QED.

Assumed in [8] and later confirmed on the lattice [15], the non-Abelian colorelectric correlator $D^E(z)$ vanishes together with confinement at T_c, while all others stay nonzero for $T > T_c$, in particular the non-Abelian colorelectric correlator D_1^E, responsible for the nonzero Polyakov lines, while the non-Abelian colormagnetic correlator $D^H(z)$ ensures the magnetic confinement for the motion in the spatial planes. This property was studied in the FCM formalism in [16], and analytically and numerically in a different approach in [17]; see also [18] for later developments.

As a result of magnetic confinement there appears the spatial string tension, which defines the area law of the spatial projection of any Wilson loop in 4D,

$$\langle W(C)\rangle = \exp\left(-\sigma_s A_{\rm 3D}(C)\right), \quad \sigma_s = \frac{1}{2}\int d^2z D^H(z). \quad (1)$$

Moreover, $D^H(z)$ can be calculated via the gluelumps [19], known both analytically [20] and on the lattice [21], which yields the relation

$$\sqrt{\sigma_s(T)} = c_\sigma g^2(T)T, \quad (2)$$

where c_σ is of a NP origin, as shown in the Appendix A. This coincides with the lattice data results [22], where $c_\sigma = 0.566 \pm 0.013$.

The two-loop approximation is generally used for $g^2(T)$ [23].

One can now consider any QCD diagram and the whole perturbative series as being immersed in the NP vacuum, so that all closed loops in the 3D space are covered by the confining film, and for 4D loop the confinng film covers the 3D projection of the loop. This fact as we shall argue, defines the main features of thermal QCD dynamics.

Namely, as we show below, the spatial Wilson loops not only serve as a cutoff factor at the distance $X_{\max} \sim \frac{1}{\sqrt{\sigma_s}}$, but due to Eq. (2) this cutoff depends on $g(T)$ and converts the perturbative $O(g^n)$ term into $O(g^6)$ (problem 2 of Linde). Exactly the same situation would occur if instead of spatial confinement one introduces the magnetic mass of gluons.

Then again Eq. (2) implies that the magnetic mass $m_D^H \sim \sqrt{\sigma_s}$ as a cutoff parameter makes the four-loop integral convergent, which resolves what can be called problem 1 of Linde, as will be clarified below. This is a purely NP result, irrespective of the appearing g^2 factor. Problem (2) of Linde [see (iii) in [1] on p. 290], that the sum of the infinite ladder of gluon loops with $n > 4$ contributes to the same order g^6T^4, also occurs in this case of magnetic mass.

However, the notion of magnetic mass (or any other effective mass) is irrelevant in the case of confinement, since gluons are connected by the confining string, which constitutes the greater part of the total energy (mass) of the system in contrast to the free motion of a gluon with any effective mass.

Coming back to the results of the perturbation theory and comparing HTL results with the lattice calculations, one can conclude that the $O(g^6)$ term is basically important for $T \le 0.5$ GeV (see, e.g., Fig. 1 of [6]). As a result a new HTL version appeared in [24], called "the $O(g^6)$ fitted" HTL contribution, as well as "the $O(g^6)$ fitted + nonpert." version. As we shall show below, the $O(g^6)$ terms indeed contain the whole series $O(g^n), n > 6$, as was shown by Linde [1], but in addition the colormagnetic confinement makes these terms finite and summable via the solution of an integral equation. All this makes our analysis and discussion of the Linde problems even more timely and relevant.

The paper is organized as follows. In the next section we describe qualitatively the possible solution of the problem. In Sec. III we write the general background field formalism for the thermodynamic potential, define its perturbation series, and study the gluonic multiloop diagram with spatial (magnetic) confinement. We define its infrared and ultraviolet properties, showing that indeed the presence of σ_s prevents the IR divergence of any diagram, and formulate the integral equation for the sum of ladder graphs contributing to the order $O(g^6)$.

Section IV is devoted to the summary and prospectives. In Appendix A a detailed derivation is given of σ_s in terms of gluon propagators and gluelumps, while Appendix B contains a detailed version of the kernel with colormagnetic confinement.

II. QUALITATIVE ANALYSIS OF THE PERTURBATIVE DIAGRAMS

Coming back to Linde problem (1), the standard perturbation theory (without nonperturbative background), which proceeds essentially in 3D, becomes infrared divergent, starting with the sixth order in g [1,2]. In essence, the problem occurs due to very weak falloff of the gluon propagator in 3D without σ_s, e.g., in the x-space

$$G(x,y) \sim \frac{T}{\pi|\mathbf{x}-\mathbf{y}|}, \qquad |\mathbf{x}-\mathbf{y}| \gg 1/T. \tag{3}$$

Let us now consider an nth order diagram of the thermal perturbation theory; an example of this diagram for $n = 8$ is shown in Fig. 1. One can count the number of gluon propagators, Eq. (3), in the diagram: $N_{\rm prop} = \frac{3n}{2}$.

The number of vertices with derivatives $\frac{\partial}{\partial x_i}$ at each vertex is n, and the number of space integrals $\frac{d^3x^{(i)}}{T}$ in each vertex is n; however, one integral yields the overall volume, so that the amplitude can be written as

$$A_n \sim g^n \prod_{i=1}^{n} \int \frac{d^3x^{(i)}}{T} \frac{\partial}{\partial x^{(i)}} \prod_{k=1}^{N_{\rm prop}} G^{(k)}(x^{(i)} - x^{(j)}) \sim \frac{V_3}{T} \bar{A}_n. \tag{4}$$

As a result one obtains the spatial dimension of the amplitude $\bar{A}_n$ in terms of an overall upper limit of 3D coordinate X,

$$\bar{A}_n = g^n X^{\frac{n}{2}-3} T^{\frac{n}{2}+1}. \tag{5}$$

It is clear from (5) that $\bar{A}_n$ is IR divergent for $n \ge 6$, in agreement with Linde problem 1 [1].

Now let us take into account the spatial (colormagnetic) confinement in 3D, which can be introduced in (4) in the form of the area law factor $\langle W(C) \rangle = \exp(-\sigma_s S_{\min})$, as

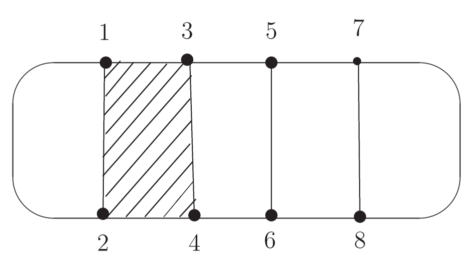

FIG. 1. The eighth order graph with the crossed rectangle under study.

in (1) with the minimal surface $S_{\min} = A_{3D}(C)$, covering all diagrams in Fig. 1. For us it is only important that $S_{\min}$ be quadratic in coordinates $x^{(i)}, x^{(j)}$, and consequently it behaves at large X as $S_{\min} \sim X^2$. Being positive definite, it makes the $\langle W(C)\rangle$ the real cutoff function, as proposed in [25], and can make the spatial integrals converge, namely,

$$\bar{A}_n^{(\mathrm{conf})} = g^n T^{\frac{n}{2}+1} \int (dX)^{\frac{n}{2}-3} \exp(-\sigma_s |X|^2). \qquad (6)$$

It is clear that by introducing the dimensionless coordinate $Y = \sqrt{\sigma_s} X$, one obtains the following representation for $\bar{A}_n^{(\mathrm{conf})}$,

$$\bar{A}_n^{(\mathrm{conf})} = g^n T^{\frac{n}{2}+1} (\sqrt{\sigma_s})^{-(\frac{n}{2}-3)} J_n, \qquad (7)$$

where J_n is a dimensionless converging integral.

Now taking into account Eq. (2), $\sqrt{\sigma_s} \sim g^2(T)T$, one obtains finally

$$\bar{A}_n^{(\mathrm{conf})} = g^6 T^4 C_n, C_n = J_n (c_\sigma)^{3-\frac{n}{2}}. \qquad (8)$$

Equation (8) exemplifies the second part of the Linde problem: all the series with $n \geq 6$ contribute to the $O(g^6)$ term.

Note that we have not introduced above the magnetic or any other mass parameters for gluons, since in the case of confinement the notion of mass can be ascribed only to the given string state, containing two (or more) gluons, connected by the adjoint string.

Nevertheless, if we introduce the effective mass of the gluon instead of confinement, $m_{\mathrm{mag}}(T)$, then the gluon Green's function acquires a factor $\exp(-m_{\mathrm{mag}}|\mathbf{x}-\mathbf{y}|)$, and these factors can be assembled in the total factor $\exp(-m_{\mathrm{mag}} \sum_{i,j} |\mathrm{x}_i - \mathrm{x}_j|)$, which would replace $\exp(-\sigma_s |X|^2)$ in (6).

As a result one obtains instead of (7) the representation

$$\bar{A}_n^{(\mathrm{mass})} = g^2 T^{\frac{n}{2}+1} (m_{\mathrm{mag}})^{-(\frac{n}{2}-3)} J_n^{(\mathrm{mass})}, \qquad (9)$$

and assuming for m_{mag} the form of magnetic mass $m_{\mathrm{mag}} = c_m g^2 T$, one again comes to the result (8). In this way one obtains that both spatial confinement and magnetic mass yield the same qualitative result: the sum of all g^n terms with $n \geq 6$ contributes to the $O(g^6)$ term, in agreement with the problem 2 of Linde [1], and in both cases the space integrals converge. In the next section we shall make our arguments more concise, developing a special representation for a four-point (or three-point) diagram with confinement taken into account.

III. BACKGROUND PERTURBATION THEORY IN MAGNETIC CONFINEMENT

In this section we exploit the background perturbation theory, developed in [8,9], to study soft and hard regimes of the internal integrations and to demonstrate the role that is played in this process by the magnetic confinement. Since we are mostly interested in the high T gluon contributions, we confine ourselves to the case of pure gluodynamics.

We split the gluonic field A_μ into the NP background B_μ and the perturbative part a_μ,

$$A_\mu = B_\mu + a_\mu, \qquad (10)$$

and the partition function Z can be written as a double average, using the 't Hooft identity [8,9]

$$Z \equiv \langle \langle \exp(-S(B+a)) \rangle_a \rangle_B, \qquad (11)$$

where the action S contains the standard gluon, ghost, and gauge-fixing terms and in particular the triple vertices a^3, $a^2 B$.

The inverse gluon propagator can be written as

$$G^{-1} = -D^2(B)_{ab} \cdot \delta_{\mu\nu} - 2g F^c_{\mu\nu}(B) f^{acb}, \qquad (12)$$

where

$$(D_\lambda)_{ca} = \partial_\lambda \delta_{ca} - i g T^b_{ca} B^b_\lambda. \qquad (13)$$

In what follows we shall for simplicity neglect the gluon spin term—the last term on the rhs of (12) [the latter gives a correction to spatial (magnetic) confinement], and then the gluon propagator can be written as

$$(-D^2)^{-1}_{xy} = \langle x| \int_0^\infty dt e^{tD^2(B)} |y\rangle = \int_0^\infty dt (Dz)^w_{xy} e^{-K} \Phi(x,y), \qquad (14)$$

where

$$K = \frac{1}{4} \int_0^s d\tau \left(\frac{dz_\mu}{d\tau}\right)^2, \qquad \Phi(x,y) = P \exp ig \int_y^x B_\mu dz_\mu, \qquad (15)$$

and a winding path measure is

$$(Dz)^w_{xy} = \lim_{N\to\infty} \prod_{m=1}^N \frac{d^4\zeta(m)}{(4\pi\varepsilon)^2} \times \sum_{n=0,\pm1,..}^{\infty} \int \frac{d^4p}{(2\pi)^4} e^{ip(\sum \zeta(m) - (x-y) - n\beta\delta_{\mu 4})}. \qquad (16)$$

In the free case, $B_\mu \equiv 0$, one obtains the gluon propagator

$$G(x,y) \to (\partial^2)^{-1}_{xy} = \sum_{k=0,\pm1,\ldots} \int \frac{T d^3 p}{(2\pi)^3} \frac{e^{-i\vec{p}(\vec{x}-\vec{y})-i2\pi kT(x_4-y_4)}}{(\vec{p}^2+(2\pi kT)^2)}. \quad (17)$$

At large distances the zero mode ($k=0$) yields the behavior shown in (3) and this is the origin of the IR divergence of higher order g^n contributions to the free energy, as was shown in [1], while magnetic confinement, contained in $\Phi(x,y)$, cuts off all divergences, as will be demonstrated below.

One can easily find the lowest order (one-loop) NP contribution to the free energy

$$F_0^{gl}(B) = T\left\{\frac{1}{2}\log\det G^{-1} - \log\det\left(-D^2(B)\right)\right\}, \quad (18)$$

which can be written as

$$\langle F_0^{gl}(B)\rangle = -T\int_0^\infty \frac{ds}{s}\xi(s)d^4x(Dz)^w_{xx}e^{-K}\langle \mathrm{tr}_a\Phi(x,x)\rangle_B \quad (19)$$

and finally for the pressure $P_{gl} = -\frac{1}{V_3}\langle F_0^{gl}(B)\rangle$,

$$P_{gl} = (N_c^2-1)\int_0^\infty \frac{ds}{s}\sum_{n\neq0} G^{(n)}(s), \quad (20)$$

with

$$G^{(n)}(s) = \int (Dz)^w_{on}e^{-K}\langle \mathrm{tr}_a\Phi(x,x)\rangle_B. \quad (21)$$

$\Phi(x,x)$ contains colorelectric fields $B_4(x)$, which produce Polyakov lines $L_{\mathrm{adj}}(T)$ [9], and in addition also color-magnetic fields, which are contained in the spatial Wilson loop, $\langle \mathrm{tr}_a\Phi_s(C_n)\rangle_B \equiv \langle W_s(C_n)\rangle$, which can be written in terms of field correlators [7], as an integral over minimal surface inside the loop C,

$$\langle W_s(C_n)\rangle = \mathrm{tr}_a\left\langle \exp\left(ig\int_C A_\mu dz_\mu\right)\right\rangle = \mathrm{tr}_a\left\langle \exp\left(ig\int ds_{\mu\nu}F_{\mu\nu}\right)\right\rangle, \quad (22)$$

and using the cumulant expansion [7,11,19] and dropping all cumulants except for the quadratic, one has

$$\langle W_s(C_n)\rangle = \exp\left(-\frac{1}{2}\int_S\int_S ds_{\mu\nu}(u)ds_{\lambda\sigma}(v)\langle F_{\mu\nu}(u)F_{\lambda\sigma}(v)\rangle\right). \quad (23)$$

The form (23) contains only quadratic terms in $F_{\mu\nu}$ (the so-called Gaussian approximation). It can be argued that higher order terms $O(\langle F^n\rangle)$, $n>4$, contain a small parameter $(\langle F\rangle^2\lambda^4)^{n/2}$, where $\langle F\rangle$ is the gluonic condensate $\langle F^2\rangle \approx O(0,1\ \mathrm{Gev}^4)$, and $\lambda = 0.1$ fm is the vacuum correlation length [19]. Moreover, the Gaussian term (23) is proportional to the quadratic Casimir operator $C_2(j)$. The presence of higher $C_n(j)$ was not found on the lattice with $O(1\%)$ accuracy; thus the Casimir scaling supports the validity of the Gaussian approximation. For more discussion see [10,11]. One may wonder, what kind of physics can be obtained in principle from higher order terms? It is shown in [10,11] that in the stochastic vacuum with convergent cumulative series (for $F\lambda^2 \ll 1$) the higher order terms also contribute to the linear confinement.

Considering only spatial loops C and surface areas S for $k=0$, i.e., the term without higher Matsubara frequencies, one needs with colormagnetic correlators only,

$$\frac{g^2}{N_c}\langle H_i(u)H_j(v)\rangle = \delta_{ij}D^H(u-v) + O(D_1^H). \quad (24)$$

To find σ_s in (1) one can use the connection of D^H with the gluelump Green's function [19], which, as shown in the Appendix, can be written as

$$D^H(z) = \frac{g^4(N_c^2-1)}{2}T^2G^{(2g)}_{3\mathrm{D}}(z), \quad (25)$$

where $G^{(2g)}_{3\mathrm{D}}$ is the two-gluon NP Green's function in 3D. As a result using (1) one can write the T-dependent part of σ_s as

$$\sigma_s(T) = g^4T^2c_\sigma^2, \qquad c_\sigma^2 = \frac{N_c^2-1}{4}\int d^2zG^{(2g)}_{3\mathrm{D}}(z), \quad (26)$$

where c_σ is a dimensionless number and a fully NP quantity.

Insertion of $\langle \mathrm{tr}_a\Phi\rangle$ in (21) as an area law (1) yields a loop graph of a gluon, where the string tension σ_s controls the area inside the loop, so that the gluon cannot go far from the initial point, the maximal distance being $R \lesssim \frac{1}{\sqrt{\sigma_s}}$.

To this end one can explicitly calculate the integral $(Dz_4)^w_{on}$ in (21) and express $G^{(n)}$ via the 3D gluon Green's function $G_3(s)$,

$$G^{(n)}(s) = \frac{1}{\sqrt{4\pi s}}e^{-\frac{n^2}{4T^2s}}G_3(s)L^n_{\mathrm{adj}}, \quad (27)$$

where L_{adj} is the Polyakov loop in the adjoint representation, so that the gluon pressure is (see [26] for more details)

$$P_{gl} = \frac{(N_c^2-1)}{\sqrt{4\pi}}\sum_{n=0,1,2,\ldots}\int_0^\infty \frac{ds}{s^{3/2}}e^{-\frac{n^2}{4T^2s}}G_3(s)L^n_{\mathrm{adj}}. \quad (28)$$

In general, $G_3(s)$ can be expanded in the series of 3D eigenvalues in the CM confinement $M_\nu^2 = a_\nu\sigma_s$,

$$G_3(s) = \frac{1}{\sqrt{\pi s}} \sum_{\nu=0,1,2} \psi_\nu^2(0) e^{-M_\nu^2 s}. \tag{29}$$

In the free limit, $\sigma_s \to 0$, one obtains $G_3^{(0)}(s) = \frac{1}{(4\pi s)^{3/2}}$, and the gluon pressure tends to the Stephan-Boltzmann limit, $P_{gl}^{SB} = \frac{(N_c^2-1)}{45} T^4 \pi^2$. An approximate form for $G_3(s)$ was found in [26] for the case of the linear confinement:

$$G_3^{(\rm lin)}(s) \cong \frac{1}{(4\pi s)^{3/2}} \left(\frac{M_0^2 s}{sh(M_0^2 s)} \right)^{1/2}, \qquad M_0 = 2\sqrt{\sigma_s}. \tag{30}$$

Here M_0 is the screening mass found in [16,27]. Insertion of (30) into (28) yields the gluon pressure $P_{gl}^{(\rm lin)}$. This gluon pressure was compared in [26] with accurate lattice data from [24], showing a good agreement. This fact implies that the perturbative series for P_{gl} in powers of g^n, which we have disregarded up to now, should give a numerical correction to the nonperturbative result. Indeed, $P_{gl}^{(\rm lin)}$ can be considered as the zeroth order term in this series, which depends only on $\sigma_s(T)$ and which reproduces the Stefan-Boltzmann pressure in the limit $\sigma_s \to 0, L_{\rm adj} \to 1$, while higher order terms are the result of the perturbation theory in the confining background.

One can now generalize this picture to the higher terms in the perturbative series $O(g^n)$, where these terms are formed by applying the term L_3 in the original QCD Lagrangian

$$L_3 = g \partial_\mu a_\nu^a f^{abc} a_\mu^b a_\nu^c \tag{31}$$

on any gluon line. As a result one obtains, e.g., the diagram of Fig. 1 of the order g^8. It is essential that each gluon propagator $G_{\mu\nu}^a(x^{(i)}, x^{(k)}) \equiv \langle a_\mu^a(x^{(i)}) a_\nu^b(x^{(k)}) \rangle$ be proportional to $\Phi(x^{(i)}, x^{(k)})$, and the latter, after averaging over background fields B_μ, in the product together with all the other gluon propagators, forms the total Wilson loop with the same outer contour C_n, but now with inner lines dissecting it into a sum of pieces of area $\Delta A^{(i)}$, $\mathcal{A} \to \sum_i \Delta \mathcal{A}^{(i)}$. Each piece is subject to the area law with the same σ_s, so that one obtains the factor $\exp(-\sigma_s \sum_i \Delta \mathcal{A}^{(i)})$, which prevents the escape of all gluons from the center of the area, and in this way ensures infrared stability.

One can say that each gluon is interacting with the closest neighbor via linear confining interaction and therefore the distance between them is of the order of $(\sqrt{\sigma_s})^{-1}$.

Consider now the diagram of Fig. 1 without magnetic confinement, where for each gluon line one can write the free gluon Green's function $G_0(x_i, x_j) = \frac{T}{|x_i - x_j|}$, and at each vertex x_k the factor $\Gamma^{abc}(x_k) = g f^{abc} \frac{\partial}{\partial x_k} +$ perm, so that the loop graph of Fig. 1 of the nth order can be written in a shorthand manner as

$$\begin{aligned} A_n(T) &= \prod_{i<j=1}^{n} \int G_0(x_i - x_j) G_0(x_1 - x_2) G_0(x_n - x_{n-1}) \\ &\quad \times \prod_{i=1}^{n} \frac{\Gamma_i d^3 x_i}{T} \\ &= \int S_n(x_{n-1}, x_n) G_0(x_{n-1}, x_n) \frac{d^3 x_{n-1}}{T} \frac{d^3 x_n}{T}. \end{aligned} \tag{32}$$

For the set of ladder diagrams of this type, one can write the relation

$$\begin{aligned} &S_{n+2}(x_n, x_{n+1}) \\ &\quad = \int S_n(x_{n-1}, x_n) G_0 \Gamma(x_n) G_0 \Gamma(x_{(n+1)}) G_0 \frac{d^3 x_n}{T} \frac{d^3 x_{n+1}}{T}, \end{aligned} \tag{33}$$

and in general the total sum $\bar{S}$ satisfies the integral equation

$$\bar{S}(x, y) = S_0(x, y) + \int \bar{S}(x', y') K(x' y' | x, y) \frac{d^3 x' d^3 y'}{T^2}, \tag{34}$$

where S_0 refers to the first few terms, while the kernel in the pure perturbation theory $K_{\rm pert}$ from (33) is

$$\begin{aligned} &K_{\rm pert}(x', y' | x, y) \\ &\quad = G_0(x - x') \Gamma(x) G_0(x - y) \Gamma(y) G_0(y - y'). \end{aligned} \tag{35}$$

The overall coordinate integration in (34) has the form

$$\int K_{\rm pert} \frac{d^3 x' d^3 y'}{T^2} \sim g^2 T \int \frac{d^3 X}{X^2} \tag{36}$$

and is IR diverging at large X [Linde problem (1)]. Introducing, as in Sec. II, the cutoff factor due to magnetic mass $\exp(-m_{\rm mag}|X|)$ or due to confinement $\exp(-\sigma_s |X|^2)$, one obtains the convergent kernel $K_{\rm conv} \sim K \exp(-\sigma_s |X|^2)$, which yields in (33) the result

$$K_{\rm conv} \frac{d^3 x' d^3 y'}{T^2} \sim g^2 T \int e^{-\sigma_s |X|^2} \frac{d^3 X}{|X|^2} \sim \frac{g^2 T}{\sqrt{\sigma_s}} \sim O(1), \tag{37}$$

where we have used (26). This is problem (2) of Linde.

However, as we shall show our integral Eq. (30) with the kernel $K_{\rm pert}$, replaced by its convergent form, $K_{\rm conv}$, is well defined and can be solved, e.g., numerically.

At this point it is necessary to specify the kernel $K = K_{\rm conv}$ in (34) and the term $S_0(x, y)$). We start with the contribution of $\bar{S}$ to the gluon pressure, which can be written as

$$P_{gl} \sim \frac{T}{V_3} \mathrm{tr}\left(\bar{S}(x,y) G_0(x,y) \frac{d^4 x d^3 y}{T^2}\right). \tag{38}$$

To specify the exact form of the confinement factor in the $K_{\rm conv}$ written as $K_{\rm conv}(x'y'|x,y) = K_{\rm pert}(x',y'|x,y)$ $\mathcal{L}(x',y'|x,y)$, i.e., one should realize, as discussed above, that confinement enters in each loop of the leader diagram as a piece of the area law factor $\exp(-\sigma_s \Delta \mathcal{A}_3)$.

As a result one obtains for each internal loop in the ladder a converging factor $\Delta W = \exp(-\sigma_s \Delta \mathcal{A}^{(i)})$, where $\Delta \mathcal{A}^{(i)}$ is a piece of surface, bounded by the gluon propagators, connecting for points of this piece. This yields for the converging factor $\mathcal{L}$ in $K_{\rm conv}$ the form

$$\mathcal{L}(x',y'|x,y) = \exp(-\sigma_s \Delta \mathcal{A}(x',y'|x,y). \tag{39}$$

It is clear that the form (39) is not easy to implement numerically, and therefore we derive in Appendix B another form with more explicit expressions.

Now one should take into account that the first terms of S_n can be conveniently taken into account in (30) in S_0, $S_0 = \sum_{n=2,4,6} S_n$, so that the formal solution of (30) can be written in the operator form as $\bar{S} = S_2 + S_4 + \frac{S_6}{1-\hat{K}}$, so that the total ladder sum $\bar{S}$ and the pressure

$$P_{gl} = P_{gl}^{(0)} + P_6 (1-\hat{K})^{-1},$$
$$P_{gl}^{(0)} = P_{gl}^{(\rm lin)} + P_2 + P_4 \tag{40}$$

formally contain terms only up to $O(g^6)$. Note that from the calculation of the terms $O(g^2)$, $O(g^4)$, and $O(g^6)$ in [6] one can see alternating signs of these terms, which might imply that the norm of $\hat{K}$ is negative, and hence better convergence in the last term of (40) (we have neglected the terms that are odd in g for simplicity).

Note that the resulting form for $\bar{S}$ or P_{gl} is the same, as in the case of scattering amplitude for the strong interaction kernel, when the perturbative series makes no sense.

Concluding this section we remark that the convergent factor in the kernel $\exp(-\sigma_s \Delta \mathcal{A}_3)$ was not formulated explicitly in terms of integration parameters, and therefore we define it in Appendix B in convenient variables.

IV. SUMMARY AND CONCLUSION

We have considered above gluon thermodynamics with nonperturbative background fields, which ensure spatial confinement due to colormagnetic correlators (24). As a result one obtains the area law of the spatial Wilson loop with the nonzero spatial string tension. Qualitatively it is clear that all multigluon diagrams in 3D would be convergent at large spatial distances, and this property was used in [25] to argue that the Linde problem is absent in the confining vacuum. In the present paper this qualitative argument was given a more quantitative foundation.

Indeed, the explicit account of the spatial confinement not only formally solves the Linde problem, but it is also vitally important in the thermodynamics of the quark gluon plasma (QGP), as was shown recently in the case of the SU(3) [26,28], as well as in the case of $n_f = 2+1$ thermodynamics in the deconfined phase [29]. It was demonstrated there that by taking into account correlator D_1^E (which generates Polyakov lines) and D^H for spatial confinement one obtains good agreement with the most accurate lattice data.

As we argued above, qualitatively one can exploit the universal effective gluon mass $m_D^H(T) \cong 2\sqrt{\sigma_s(T)}$ instead of spatial confinement with $\sigma_s(T)$ as a first approximation in the effective perturbation theory up to the g^6 order.

From this point of view we have stressed the existence of the effective screening mass parameter, which is of NP origin and occurs due to magnetic confinement string tension σ_s—this is the answer to what we call problem 1 of Linde. The second problem of Linde, the infinite g^n series with $n \geq 6$ contributing to the order $g^6 T^4$, is confirmed above in the NP approach.

As mentioned above, the whole dynamics of diagrams with $n \geq 6$ lies in the soft NP region, where the magnetic confinement and not the gluon exchange mechanism plays the most important role. It is an open question what is the physical meaning of the sum of $n > 6$ NP terms with magnetic confinement. One can connect it to the gg amplitude in the case of two interactions in 3D, confining $V_{\rm conf}$ and gluon exchange $V_{\rm OGE}$, but the answer is possible to obtain using the integral Eq. (34) and its generalization.

Indeed, higher orders in $V_{\rm QGE}$, as we have seen, do not involve higher orders in g^n, but nevertheless this fact does not imply the divergence of the sum, as is known from two-potential scattering series, leading to the finite result. In the literature there are other methods to deal with the resummation problem; see, e.g., the review in the second reference in [4], where the introduction of the gluon mass helps to deal with the IR problem and to improve the convergence. However, the physical essence of the Linde problem from our point of view is connected to the lack of spatial confinement and should be treated by its correct inclusion.

ACKNOWLEDGMENTS

The author is grateful to M. A. Andreichikov, B. O. Kerbikov, and M. S. Lukashov for useful discussions. This work was done in the framework of the scientific project, supported by the Russian Science Fund, Grant No. 16-12-10414.

APPENDIX A: CALCULATION OF THE SPATIAL STRING TENSION

Calculation of the spatial string tension via two-gluon Green's function

To calculate $D^H(z)$ one can use the technique developed in [19] for $D^E(z)$, which allows us to express it via the two-gluon Green's function $G^{(2g)}_{4D}(z) = G^{(g)}_{4D} \otimes G^{(g)}_{4D}$, where two gluons interact nonperturbatively.

The starting point for the gluon propagator $G^{(g)}_{4D}$ is the integration in the fourth direction in (14) with the exponent $K_4 = \frac{1}{4}\int_0^s d\tau (\frac{dz_4}{d\tau})^2$, which gives for the spatial loop with $x_4 = y_4$

$$J_4 \equiv \int (Dz_4)_{x_4x_4} e^{-K_4} = \sum_{n=0,\pm1,\dots} \frac{1}{2\sqrt{\pi s}} e^{-\frac{(n\beta)^2}{4s}} = \frac{1}{2\sqrt{\pi s}}\left(1 + \sum_{n=\pm1,\pm2} e^{-\frac{(n\beta)^2}{4s}}\right). \quad \text{(A1)}$$

The second term in (A1) at large $T \gg \frac{1}{2\sqrt{s}}$ yields $2\sqrt{\pi s}T$, which gives $J_4 = \frac{1}{2\sqrt{\pi s}} + T$.

The same term that is linear in T is obtained using the Poisson relation [8,9]. As a result the 4D gluon propagator is reduced to the 3D one,

$$G^{(g)}_{4D}(z) = TG^{(g)}_{3D}(z) + K_{3D}(z), \quad \text{(A2)}$$

where $K_{3D}(z)$ does not depend on T. In what follows we consider only the first term in (A2), keeping in mind that $G^{(g)}_{4D}(z)$ at small T has a nonzero limit. Substituting this term in the general expression for $D^E(z)$ ($D^H(z)$) obtained in [19], one has

$$D^H(z) = \frac{g^4(N_c^2-1)}{2}\langle G^{(2g)}_{4D}(z)\rangle \to \frac{g^4(N_c^2-1)T^2}{2}\langle G^{(2g)}_{3D}(z)\rangle, \quad \text{(A3)}$$

where $G^{(2g)}_{3D}$ is the two-gluon Green's function in 3D with all interaction between gluons taken into account:

$$\langle G^{(2g)}_{3D}\rangle = \langle G^{(g)}_{3D}(x,y) G^{(g)}_{3D}(x,y)\rangle_B. \quad \text{(A4)}$$

In terms of the gluelump phenomenology, studied in [20,21], (A4) is called the two-gluon gluelump, computed on the lattice in [21] and analytically in [20]. In our case we are interested in the 3D version of the corresponding Green's function. Choosing in 3D the $x_3 \equiv t$ axis as the Euclidean time, we proceed as in [16], exploiting the path integral technique [7,30], which yields

$$G^{(2g)}_{3D}(x-y) = \frac{t}{8\pi}\int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}}\int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}}(D^2z_1)_{xy}(D^2z_2)_{xy} \times e^{-K_1(\omega_1)-K_2(\omega_2)-Vt}, \quad \text{(A5)}$$

where V includes a spatial confining interaction between the three objects: gluon 1, gluon 2, and the fixed straight line of the parallel transporter, which makes all construction gauge invariant (see [19,20] for details). In (A5), $t = |x-y| \equiv |w|$, and finally

$$\sigma_s(T) = \frac{g^4(N_c^2-1)T^2}{4}\int \langle G^{(2g)}_{3D}(w)\rangle d^2w. \quad \text{(A6)}$$

Constructing in the exponent of (A5) the three-body Hamiltonian in the 2d spatial coordinates

$$H(\omega_1,\omega_2) = \frac{\omega_1^2 + \mathbf{p}_1^2}{2\omega_1} + \frac{\omega_2^2 + \mathbf{p}_2^2}{2\omega_2} + V(\mathbf{z}_1,\mathbf{z}_2), \quad \text{(A7)}$$

one can rewrite (A5) as follows (see [30]):

$$G^{(2g)}_{3D}(t) = \frac{t}{8\pi}\int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}}\int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} \times \sum_{n=0}^\infty |\psi_n(0,0)|^2 e^{-M_n(\omega_1,\omega_2)t}. \quad \text{(A8)}$$

Here $\Psi_n(0,0) \equiv \Psi_n(\mathbf{z}_1,\mathbf{z}_2)|_{\mathbf{z}_1=\mathbf{z}_2=0}$, and M_n is the eigenvalue of $H(\omega_1,\omega_2)$. The latter was studied in [20] in three spatial coordinates. For our purpose here we only mention that $G^{(2g)}_{3D}(z)$ has the dimension of the mass squared and the integral in (A6) is therefore dimensionless. Hence one obtains $\sqrt{\sigma_s(T)} = g^2Tc_\sigma + \text{const}$, as was stated above in (3), where

$$c_\sigma^2 = \frac{(N_c^2-1)}{4}\int d^2w \langle G^{(2g)}_{3D}(w)\rangle, \quad \text{(A9)}$$

and const is obtained from the second term in (A2).

APPENDIX B: ONE LOOP DIAGRAM WITH CONFINEMENT

We now turn to the more formal procedure to define the properties of the one-loop part of the complicated diagrams, shown in Fig. 1 as a crossed rectangle. At each vertex of this diagram the operator (31) enters, which generates 3g vertex Γ_i with momentum operator p_i, so that the quadratic loop diagram in the 3D space can be written as

$$G(p^{(1)},p^{(2)},p^{(3)},p^{(4)}) = \prod_{i=1}^4 \Gamma_i \int_0^\infty ds_i (Dz^{(i)})_{x^{(i)},x^{(i-1)}} e^{-K_i}\Phi^{(i)} e^{ip^{(i)}x^{(i)}} dx^{(i)}. \quad \text{(B1)}$$

Here we have introduced the phase factors

$$\Phi^{(i)}(x^{(i)},x^{(i-1)}) = P_A \exp\left(ig\int_{x^{(i-1)}}^{x^{(i)}} A_\mu dz_\mu\right), \quad \text{(B2)}$$

omitting for simplicity the gluon spin phase factor, originating from the last term in (24), since it is inessential in the asymptotic limit for large $|x^{(i)} - x^{(i-1)}|$. Here $(Dz^{(i)})$ is

$$(Dz^{(i)})_{xy} = \lim_{N\to\infty} \prod_{k=1}^{N} \frac{d^3\xi^{(i)}(k)}{(4\pi\varepsilon)^2} \frac{d^3q^{(i)}}{(2\pi)^4} e^{iq^{(i)}(\sum_k \xi^{(i)}(k)-(x-y))},$$
$$N\varepsilon = s. \tag{B3}$$

It is essential that the product of all phase factors $\Phi^{(i)}$ in the whole diagram of Fig. 1 should be averaged over vacuum configurations, yielding 3D confinement, and each gluon line is in adjoint representation, and can be represented as the double fundamental line in the simple case of the large N_c limit, so that one finally has a product of independent closed fundamental lines, circumvented by a common line in the outer contour. In the same large N_c limit the average of this product can be represented as the product of averages of individual loops times the average of the outer contour, which yields the overall confining factor. In what follows we shall be interested in the properties of one rectangular loop and demonstrate its spatial convergence, while the overall confining loop will exhibit additional convergence.

The rather complicated calculations, given in Appendix B of the paper [30] for the case $d = 4$, can be done in an analogous manner for the case $d = 3$, and one obtains the following form of the rectangular of Fig. 1 by taking into account the spatial confinement,

$$G_4(p_i) = (2\pi)^3\delta^{(3)}\left(\sum p^{(i)}\right)\prod_{i=1}^{4}\int \frac{d^3q_i\Gamma_i}{q_i^2} I_4(b), \tag{B4}$$

where

$$I_4(b) = \int \frac{d^3\mathcal{P}}{(2\pi)^3}\left(\frac{4\pi}{\sigma}\right)^6 \exp\left(-\frac{2}{\sigma}\sqrt{b_1^2b_2^2 - (\mathbf{b}_1\mathbf{b}_2)^2}\right)$$
$$\times \exp\left(-\frac{2}{\sigma}\sqrt{b_3^2b_4^2 - (\mathbf{b}_3\mathbf{b}_4)^2}\right), \tag{B5}$$

and b_i are

$$b_1 = q_1 - p_2 - p_3 + \mathcal{P}, \qquad b_2 = q_2 - p_3 + \mathcal{P},$$
$$b_3 = q_3 + \mathcal{P}, \qquad b_4 = q_4 + p_4 + \mathcal{P}. \tag{B6}$$

One can check that at large momenta (the hard regime) when $b_i^2 \gg \sigma, i = 1, 2, 3, 4$,

$$I_4(b) \to \prod_{i=1,2,4} \delta^{(3)}(b_i), \tag{B7}$$

and the product of four factors d^3q_i is reduced to a single integration d^3q_3, as it should be in the free case without confinement.

As a result one has in (B4) for one loop in Fig. 1 the combination $d^3q\prod_{i=1}^4 \frac{\Gamma_i}{q_i^2}$, and for the whole chain of n loops, as in Fig. 1, one obtains an estimate (see [1]):

$$\mathcal{M}_n(T) \sim g^{2(n-1)}\left(T\int_{a\sqrt{\sigma}}^{T} d^3q\right)^n \frac{q^{2(n-1)}}{(q^2)^{3(n-1)}}. \tag{B8}$$

Here we have used the hard limit condition, $q \geq a\sqrt{\sigma}, a \gg 1$.

Integration in (B8) yields the result $(n > 4)$

$$\mathcal{M}_n^{\rm hard}(T) \sim \frac{g^{2(n-1)}T^n\sigma_s^{\frac{4-n}{2}}}{a^{n-4}} \sim \frac{g^6T^4}{(c_\sigma a)^{n-4}}, \tag{B9}$$

where we have exploited (2). This, apart from the $c_\sigma a$ factor, is problem (2) of Linde [1]: all terms with $n > 4$ contribute to the order g^6T^4, however with decreasing magnitude for $c_\sigma a \gg 1$.

To complete our study we consider now the soft regime: all momenta q_i, p_i in (B6) are small, q_i, $P \lesssim \sqrt{\sigma}$. In this case every loop integration d^3q in (B8) is replaced by

$$d^3q \to \prod_{i=1}^{4} d^3q_iI_4(b) = \sigma_s^{3/2}f\left(\frac{q_i}{\sqrt{\sigma}}\right), \tag{B10}$$

where f in (B5) yields a cutoff in the d^3q integration in (B8), and as a result one obtains in the soft regime

$$\mathcal{M}_n^{\rm soft}(T) \sim g^{2(n-1)}T^n\sigma_s^{\frac{4-n}{2}}\varphi_n \sim g^6T^4\varphi_n, \tag{B11}$$

where φ_n is a converging integral of dimensionless ratios $q_i/\sqrt{\sigma}$. One can see that (B11) yields qualitatively the same result as in (B9), for the order of magnitude estimates. However quantitatively one should calculate nonperturbatively the whole series $n \geq 4$ to recover the $O(g^6)$ contribution. This situation is similar to the solution of the relativistic problem of two potentials: one confining and another gluon exchange but without small parameters, and one should sum up the series, or rather solve the corresponding relativistic Hamiltonian equation [31].

[1] A. D. Linde, Phys. Lett. **96B**, 289 (1980); Rep. Prog. Phys. **42**, 389 (1979).
[2] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
[3] E. Braaten and R. D. Pisarski, Nucl. Phys. **B337**, 569 (1990); J. O. Andersen, E. Braaten, and M. Strickland, Phys. Rev. Lett. **83**, 2139 (1999); J. O. Andersen and M. Strickland, Ann. Phys. (Amsterdam) **317**, 281 (2005); U. Kraemmer and A. Rebhan, Rep. Prog. Phys. **67**, 351 (2004).
[4] J. O. Andersen, L. E. Leganger, M. Strickland, and N. Su, J. High Energy Phys. 08 (2011) 053; S. Mogliacci, J. O. Andersen, M. Strickland, N. Su, and A. Vuorinen, J. High Energy Phys. 12 (2013) 055; N. Haque, J. O. Andersen, M. G. Mustafa, M. Strickland, and N. Su, Phys. Rev. D **89**, 061701 (2014); N. Haque, A. Bandyopadhyay, J. O. Andersen, M. G. Mustafa, M. Strickland, and N. Su, J. High Energy Phys. 05 (2014) 027.
[5] N. Su, Int. J. Mod. Phys. A **30**, 1530025 (2015).
[6] K. Kajantie, M. Laine, K. Rummukainen, and Y. Schröder, Phys. Rev. D **67**, 105008 (2003).
[7] H. G. Dosch, Phys. Lett. B **190**, 177 (1987); H. G. Dosch and Yu. A. Simonov, Phys. Lett. B **205**, 339 (1988); Yu. A. Simonov, Nucl. Phys. **B307**, 512 (1988).
[8] Yu. A. Simonov, JETP Lett. **54**, 249 (1991); **55**, 627 (1992); Yu. A. Simonov, Phys. At. Nucl. **58**, 309 (1995); N. O. Agasian, JETP Lett. **57**, 208 (1993); **71**, 43 (2000); H. G. Dosch, H. J. Pirner, and Yu. A. Simonov, Phys. Lett. B **349**, 335 (1995).
[9] Yu. A. Simonov, Ann. Phys. (NY) **323**, 783 (2008); E. V. Komarov and Yu. A. Simonov, Ann. Phys. (NY) **323**, 1230 (2008); Yu. A. Simonov and M. A. Trusov, JETP Lett. **85**, 598 (2007); Phys. Lett. B **650**, 36 (2007).
[10] Yu. A. Simonov, Phys. Usp. **39**, 313 (1996).
[11] A. Di Giacomo, H. G. Dosch, V. I. Shevchenko, and Yu. A. Simonov, Phys. Rep. **372**, 319 (2002).
[12] A. V. Nefediev, Yu. A. Simonov, and M. A. Trusov, Int. J. Mod. Phys. E **18**, 549 (2009).
[13] D. Zwanziger, Phys. Rev. D **76**, 125014 (2007); K. Lichtenegger and D. Zwanziger, Phys. Rev. D **78**, 034038 (2008).
[14] K. Fukushima and N. Su, Phys. Rev. D **88**, 076088 (2013); N. Su and K. Tywoniuk, Phys. Rev. Lett. **114**, 161601 (2015).
[15] A. Di Giacomo, E. Meggiolaro, and H. Panagopoulos, Nucl. Phys. **B483**, 371 (1997); M. D'Elia, A. Di Giacomo, and E. Meggiolaro, Phys. Rev. D **67**, 114504 (2003).
[16] E. L. Gubankova and Yu. A. Simonov, Phys. Lett. **360**, 93 (1995).
[17] V. Koch, E. V. Shuryak, G. E. Born, and A. D. Jackson, Phys. Rev. D **46**, 3169 (1992); C. Bernard, T. DeGrand, C. DeTar, S. Gottlieb, A. Krasnitz, M. Ogilvie, R. Sugar, and D. Toussaint, Phys. Lett. **68**, 2125 (1992).
[18] A. V. Nefediev and Yu. A. Simonov, Phys. At. Nucl. **71**, 171 (2008).
[19] Yu. A. Simonov, Phys. At. Nucl. **69**, 528 (2006); Yu. A. Simonov and V. I. Shevchenko, Adv. High Energy Phys. **2009**, 1 (2009).
[20] Yu. A. Simonov, Nucl. Phys. **B592**, 350 (2001).
[21] I. Jorysz and C. Michael, Nucl. Phys. **B302**, 448 (1988); N. Campbell, I. Jorysz, and C. Michael, Phys. Lett. **167B**, 91 (1986).
[22] G. Boyd, O. Kaczmarek, and F. Zantow, Nucl. Phys. **B469**, 419 (1996); F. Karsch, E. Laermann, and M. Lütgemeier, Phys. Lett. B **346**, 94 (1995).
[23] L. McLerran, Rev. Mod. Phys. **58**, 1021 (1986); J. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, 1989).
[24] Sz. Borsanyi, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, J. High Energy Phys. 07 (2012) 056.
[25] Yu. A. Simonov, in *Varenna 1995, Selected Topics in Non-perturbative QCD*, edited by A. Di Giacomo and D. Diakonov (IOS Press, Amsterdam, 1998), p. 319; arXiv:hep-ph/9509404.
[26] N. O. Agasian, M. S. Lukashov, and Yu. A. Simonov, Eur. Phys. J. A **53**, 138 (2017).
[27] N. O. Agasian and Yu. A. Simonov, Phys. Lett. B **639**, 82 (2006).
[28] N. O. Agasian, M. S. Lukashov, and Yu. A. Simonov, Mod. Phys. Lett. A **31**, 1650222 (2016).
[29] M. S. Lukashov and Yu. A. Simonov, JETP Lett. **105**, 691 (2017).
[30] Yu. A. Simonov, Int. J. Mod. Phys. A **31**, 1650016 (2016).
[31] Yu. A. Simonov, Phys. Rev. D **88**, 025028 (2013).