Bulk viscosity of quark-gluon plasma in strong magnetic fields

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We investigate the viscosities of the quark-gluon plasma in strong magnetic fields within the leading-log and lowest Landau level (LLL) approximations. We first show that the bulk viscosity in the direction parallel to the magnetic field is the only component that has a contribution from the quarks occupying the LLL. We then compute the bulk viscosity from the Kubo formula and find an intriguing quark-mass dependence as a consequence of a competition between the suppression of the bulk viscosity by conformal symmetry and an enhancement of the mean-free path by chirality conservation, which governs the behavior in the massless limit. The quark contribution to the viscosity along the magnetic field becomes larger than the one in the absence of a magnetic field. We also briefly estimate the other transport coefficients by considering the contribution of gluons. We show that the shear viscosities are suppressed compared to their values in the absence of a magnetic field.

I. INTRODUCTION

Heavy-ion collisions are the only way to experimentally investigate the quark-gluon plasma (QGP), a form of matter composed of quarks and gluons liberated from color confinement at high temperatures (T), under controlled laboratory conditions. At the same time, such experiments may provide us with an opportunity to investigate QGP matter under the influence of strong magnetic fields (B), since noncentral heavy-ion collisions are thought to generate (via Ampere's law) the strongest magnetic fields ever created in terrestrial experiments [1] (see Refs. [2,3] for recent reviews).

Hydrodynamic simulations have played an important role in the study of phenomenological aspects of heavy-ion collisions. Recent efforts are directed toward applying magnetohydrodynamics (MHD), which takes into account the dynamical coupling of the magnetic field to the fluid in a self-consistent way [4–9]. This is an important progress in the investigation of the QGP in strong magnetic fields. However, these studies have not yet implemented transport coefficients computed in the presence of a magnetic field.

In the studies [10,11], the authors have computed the electrical conductivity in a strong magnetic field. In this case, quarks are confined to the lowest Landau level (LLL). It was shown that the microscopic properties of the LLL dynamics manifest themselves as drastic modifications of the macroscopic transport properties. The key observation was that there is a mismatch between the spatial dimensions in which the quarks reside as compared to gluons: LLL quarks can only propagate in one spatial dimension (parallel to the magnetic field), while gluons can move in all three spatial dimensions. This mismatch of dimensions opens a kinematical window for 1-to-2 scattering [10–12], and this emergent contribution dominates over the conventional leading-order contributions from 2-to-2 scatterings at weak coupling ($g \ll 1$). On the other hand, the chirality-conservation law for one-dimensional quarks strictly prohibits scatterings in the massless limit ($m_f = 0$). Therefore, the parametric dependence of the quark damping rate has been established as $\sim g^2 m_f^2/T$, up to a logarithmic factor (see Refs. [10,11] and Sec. IV), which significantly enhances the electrical conductivity. This dependence is one of the intriguing manifestations of the LLL dynamics,[1] and serves as motivation to investigate other transport coefficients, in order to see whether LLL dynamics has a similarly important influence. Whereas LLL dynamics has been intensively investigated in studies of anomalous transport phenomena (see, e.g., Refs. [2,3,15] for reviews), its manifestation in the transport coefficients of MHD has not been fully explored yet.

In this paper, we evaluate the contribution of the LLL quarks to the transport coefficients of MHD, on the basis of

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[1]See also Refs. [13,14] for a consequence of LLL kinematics, which manifests itself in the drag force.

the aforementioned quark damping rate and the Kubo formulas obtained in Ref. [16]. Since the magnetic field breaks the isotropy of the system, in general MHD has more independent transport coefficients than conventional (isotropic) hydrodynamics. However, we will show that, within the LLL approximation, LLL quarks contribute only to the component of the bulk viscosity parallel to the magnetic field. (The only other transport coefficient with a contribution from LLL quarks is the longitudinal conductivity [10,11].) We find a nontrivial dependence of the bulk viscosity on the current quark mass as a result of the competition between the chirality-conservation law and conformal symmetry. To evaluate the bulk viscosity, we apply the same method that was used to evaluate the electrical conductivity [10,11]. The analyses of the present paper together with those in Refs. [10,11] conclude the computation of the LLL-quark contribution to a certain set of transport coefficients shown in Ref. [16], within the leading-log and LLL approximations. Other related works on the transport coefficients of MHD are, for example, the calculation of the shear viscosities in a weak magnetic field [17,18] and in the holographic setup [19–22], and the calculation of the anisotropic bulk viscosities due to electroweak interactions in dense quark matter [23].

This paper is organized as follows: In the next two sections, we recapitulate the basic equations of MHD and the Kubo formulas, and then identify the relevant components of the viscosities. In Sec. IV, we evaluate the component of the bulk viscosity parallel to the magnetic field within the leading-log and LLL approximations. Section V is devoted to evaluating the quark contribution to the bulk viscosity in heavy-ion collisions. In Sec. VI, we make order-of-magnitude estimates for the gluon contribution to the shear and bulk viscosities. We conclude with a summary of our results in Sec. VII. In Appendix A, we briefly discuss the Landau-level quantization, and, in Appendixes B and C, we evaluate the thermodynamic quantities in the LLL approximation which are necessary for the evaluation of the bulk viscosity. We also derive an expression for the bulk viscosity from the linearized Boltzmann equation in $(1 + 1)$ dimensions which is consistent with our result obtained via the Kubo formula.

II. MAGNETOHYDRODYNAMICS AND KUBO FORMULAS

In this section, we briefly summarize the results of Ref. [16], which comprise the equations of motion of MHD and the constitutive relations. In the presence of a magnetic field, MHD contains two bulk viscosities, five shear viscosities, and three electrical conductivities. We also recapitulate the Kubo formulas for these transport coefficients.

A. Magnetohydrodynamics

The basic equations of MHD consist of the conservation laws of energy, momentum, and electric charge, and the constitutive equations for the energy-momentum tensor ($T^{\mu\nu}$) and the electric-charge current (j^μ).[2] The former ones are given by

$$\partial_\mu j^\mu = 0, \tag{2.1}$$

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} j_\mu, \tag{2.2}$$

where $F^{\mu\nu}$ is the electromagnetic field-strength tensor. If the electric field is much smaller than the magnetic field, the right-hand side of the second equation can be neglected [16]. For the evaluation of the transport coefficients, which is the purpose of this paper, it suffices to consider a static and homogeneous (nondynamical) magnetic field.

The constitutive equations[3] in the Landau frame read [16]

$$j^\mu = n u^\mu + \mathcal{J}^\mu, \tag{2.3}$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P_\perp \Xi^{\mu\nu} + P_\parallel b^\mu b^\nu + \mathcal{T}^{\mu\nu}, \tag{2.4}$$

where u^μ is the flow vector, normalized as $u^2 = 1$, and $b^\mu \equiv \epsilon^{\mu\nu\alpha\beta} F_{\nu\alpha} u_\beta / (2B)$ with $B \equiv \sqrt{-B^\mu B_\mu}$. The tensor which projects onto the three-dimensional space orthogonal to the flow is defined as $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$, while that which projects onto the two-dimensional space orthogonal to both the flow and the magnetic field is $\Xi^{\mu\nu} \equiv \Delta^{\mu\nu} + b^\mu b^\nu$, respectively. We also have the energy density ϵ, the charge density n, the thermodynamic pressure $P_\parallel \equiv P$, and the transverse pressure $P_\perp \equiv P - MB$, including the contribution of the magnetization $M \equiv (\partial P / \partial B)_{T,\mu}$ (μ is the chemical potential associated with the electric charge). To leading order of the derivative expansion the dissipative terms are given by

$$\mathcal{J}^\mu = T(\kappa_\perp \Xi^{\mu\nu} \nabla_\nu \alpha - \kappa_\parallel b^\mu b^\nu \nabla_\nu \alpha - \kappa_\times b^{\mu\nu} \nabla_\nu \alpha), \tag{2.5}$$

$$\begin{aligned} \mathcal{T}^{\mu\nu} = {} & \frac{3}{2} \zeta_\perp \Xi^{\mu\nu} \phi + 3\zeta_\parallel b^\mu b^\nu \psi + 2\eta_0 \left(w^{\mu\nu} - \frac{1}{3} \Delta^{\mu\nu} \theta \right) \\ & + \eta_1 \left(\Delta^{\mu\nu} - \frac{3}{2} \Xi^{\mu\nu} \right) \left(\theta - \frac{3}{2} \phi \right) \\ & + 2[-\eta_2 (b^\mu \Xi^{\nu\alpha} b^\beta + b^\nu \Xi^{\mu\alpha} b^\beta) \\ & - \eta_3 (\Xi^{\mu\alpha} b^{\nu\beta} + \Xi^{\nu\alpha} b^{\mu\beta}) \\ & + \eta_4 (b^{\mu\alpha} b^\nu b^\beta + b^{\nu\alpha} b^\mu b^\beta)] w_{\alpha\beta}, \end{aligned} \tag{2.6}$$

where $\alpha \equiv \beta\mu$, $b^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} b_\alpha u_\beta$, $w^{\mu\nu} \equiv (\nabla^\mu u^\nu + \nabla^\nu u^\mu)/2$, $\phi \equiv \Xi^{\mu\nu} w_{\mu\nu}$, $\psi \equiv b^\mu b^\nu w_{\mu\nu}$, $\theta \equiv \partial^\mu u_\mu$, with $\nabla_\mu \equiv \Delta_{\mu\nu} \partial^\nu$.

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[2]For the sake of simplicity, even in the case of multiple flavors we consider only the electric-charge current.

[3]In addition to the terms given in this expression, other terms are generated by the coupling between the vorticity and the magnetic field [24]. However, they are not the subject of this paper, so we have omitted them.

The three κ's are the electrical conductivities, the two ζ's the bulk viscosities, and the five η's the shear viscosities, respectively.

B. Kubo formulas

The transport coefficients in Eqs. (2.5) and (2.6) are given by the following Kubo formulas [16]:

$$\kappa_\parallel = \frac{\partial}{\partial \omega} \mathrm{Im} G^R_{j^3 j^3}|_{\mathbf{p}=\mathbf{0},\omega\to0}, \tag{2.7}$$

$$\zeta_\parallel = \frac{1}{3}\frac{\partial}{\partial \omega}\left(2\mathrm{Im}G^R_{\tilde{P}_\perp \tilde{P}_\parallel} + \mathrm{Im}G^R_{\tilde{P}_\parallel \tilde{P}_\parallel}\right)_{\mathbf{p}=\mathbf{0},\omega\to0}, \tag{2.8}$$

$$\zeta_\perp = \frac{1}{3}\frac{\partial}{\partial \omega}\left(2\mathrm{Im}G^R_{\tilde{P}_\perp \tilde{P}_\perp} + \mathrm{Im}G^R_{\tilde{P}_\parallel \tilde{P}_\perp}\right)_{\mathbf{p}=\mathbf{0},\omega\to0}, \tag{2.9}$$

$$\eta_0 = \frac{\partial}{\partial \omega} \mathrm{Im} G^R_{T^{12}T^{12}}|_{\mathbf{p}=\mathbf{0},\omega\to0}, \tag{2.10}$$

$$\eta_1 = -\frac{4}{3}\eta_0 - 2\frac{\partial}{\partial \omega} \mathrm{Im} G^R_{\tilde{P}_\parallel \tilde{P}_\perp}|_{\mathbf{p}=\mathbf{0},\omega\to0}, \tag{2.11}$$

$$\eta_2 = -\eta_0 + \frac{\partial}{\partial \omega} \mathrm{Im} G^R_{T^{13}T^{13}}|_{\mathbf{p}=\mathbf{0},\omega\to0}, \tag{2.12}$$

$$\eta_3 = \frac{1}{2}\frac{\partial}{\partial \omega} \mathrm{Im} G^R_{\tilde{P}_\perp T^{12}}|_{\mathbf{p}=\mathbf{0},\omega\to0}, \tag{2.13}$$

$$\eta_4 = \frac{\partial}{\partial \omega} \mathrm{Im} G^R_{T^{13}T^{23}}|_{\mathbf{p}=\mathbf{0},\omega\to0}, \tag{2.14}$$

where without loss of generality the direction of the magnetic field is chosen to point along the 3-direction. We have defined $\tilde{P}_\parallel \equiv P_\parallel - \Theta_\beta \epsilon$, $\tilde{P}_\perp \equiv P_\perp - (\Theta_\beta + \Phi_\beta)\epsilon$, with $\Theta_\beta \equiv (\partial P_\parallel/\partial \epsilon)_B$ and $\Phi_\beta \equiv -B(\partial M/\partial \epsilon)_B$. Here, the retarded Green's function is defined[4] as $G^R_{AB}(x) \equiv i\theta(x^0)\langle[A(x), B(0)]\rangle$ with the average in the equilibrium state denoted by angular brackets. Since we focus on the charge-neutral case in this paper, the above Kubo formulas lack some terms that are present in the nonzero-charge case [16].

We also note that the right-hand side of Eq. (2.2) should be maintained to derive the correct Kubo formulas for $\kappa_\perp$ and $\kappa_\times$. However, since they vanish in the LLL approximation, we do not go into this issue in the present analysis. For a discussion of the complete Kubo formulas, see Ref. [24].

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[4]Note that this definition has the opposite sign compared to that in Ref. [16]. Because of this difference, the signs in Eqs. (2.7)–(2.14) are also opposite to those in Ref. [16].

III. CONTRIBUTIONS OF THE LLL QUARKS TO TRANSPORT COEFFICIENTS

Before we compute the transport coefficients in the subsequent sections, we briefly describe the LLL approximation and identify the transport coefficients which have contributions from the quarks in the LLL.

The periodic cyclotron motion in a magnetic field leads to the Landau-level quantization, the energies of which are

$$\epsilon_n = \sqrt{(p^3)^2 + m_f^2 + 2n|B_f|}, \tag{3.1}$$

specified by a non-negative integer $(n \geq 0)$. We have defined $B_f = |e|q_f B$ with $|e|q_f$ being the electric charge, and we explicitly maintain the current quark mass m_f as its dependence turns out to be important later. In this paper, we focus on the strong-field regime that satisfies the hierarchy $|B_f| \gg T^2 \gg m_f^2$. Therefore, the occupation number of quarks in the LLL $(n = 0)$, the energy of which is independent of $|B_f|$, is large in an ensemble at temperature T. On the other hand, the occupation number of the higher Landau levels (hLLs) $[n \geq 1]$, which are separated from the LLL by a large energy gap of the order of $\sqrt{|B_f|}$, is highly suppressed by the Boltzmann factor. Thus, we entirely neglect the hLLs in our calculation.

We work in the Landau gauge specified as $A_2^{\mathrm{ext}} = Bx_1$, with the other components vanishing. Therefore, three components of the quark momentum, which is denoted as $\bar{p}^\mu = (p^0, 0, p^2, p^3)$, are still good quantum numbers in a magnetic field. The final one is provided by the principle quantum number n. The energy eigenstates specified by these quantum numbers are complete and orthogonal. Therefore, by using the eigenfunction shown in Appendix A, the quark field can be expanded as [11,25]

$$\psi(x) = \int_{\bar{p}} e^{-i\bar{p}\cdot x}\mathcal{H}(\tilde{x}_p^f)\mathcal{P}_+\chi(p_L)$$
$$+ (\text{Contributions from } n \geq 1), \tag{3.2}$$

where the explicit expression of the contribution from the hLLs was suppressed as it will not be discussed below. We have introduced the abbreviations $\int_p \equiv \int dp/(2\pi)$, $p_L^\mu \equiv (p^0, 0, 0, p^3)$, and $\tilde{x}_p^f \equiv x^1 - p^2/B_f$ with the second component of the momentum, p^2. The spin projection operator is defined by $\mathcal{P}_\pm \equiv [1 \pm \mathrm{sgn}(B_f)i\gamma^1\gamma^2]/2$ with a sign function $\mathrm{sgn}(B_f)$. The spin of the LLL quark is frozen in a definite direction along the magnetic field due to the Zeeman effect. $\mathcal{H}(x)$ is the normalized Hermite function coming from the quark wave function in the transverse plane in the LLL. In Appendix A, we summarize the properties of $\mathcal{H}(x)$ which will be used below.

The spin projection operator has the useful property $\mathcal{P}_\pm\gamma^\mu\mathcal{P}_\pm = \gamma_L^\mu\mathcal{P}_\pm$, with $\gamma_L^\mu = (\gamma^0, 0, 0, \gamma^3)$. Therefore, the current composed of the LLL quark field is

$$j^\mu(x) = \bar\psi(x)\gamma_L^\mu\psi(x), \qquad (3.3)$$

which has only a temporal as well as a spatial component parallel to the magnetic field. The transverse components demand a spin flip which, however, costs energy for an interlevel transition from $n = 0$ to $n \geq 1$. This is the reason why the LLL quarks contribute only to the longitudinal component of the conductivity shown in Eq. (2.7). The computation of the longitudinal conductivity has been performed in Refs. [10,11].

Similar to the current, we now identify the nonvanishing components of the energy-momentum tensor in the LLL and the relevant contributions to the viscosities listed in Eqs. (2.8)–(2.14). In the absence of QCD interactions, the quark part of the energy-momentum tensor is written as[5]

$$T^{\mu\nu}(x) = \frac{i}{2}\mathcal{S}\sum_f[\bar\psi\overleftarrow{D}^\mu\gamma^\nu\psi + \bar\psi D^\mu\gamma^\nu\psi], \qquad (3.4)$$

where the sum is taken over the flavor index. The covariant derivatives with the external magnetic field are $D_\mu \equiv \partial_\mu + iq_f|e|A_\mu^{\text{ext}}$ and $\overleftarrow{D}_\mu \equiv -\overleftarrow{\partial}_\mu + iq_f|e|A_\mu^{\text{ext}}$. The symmetrization operator works as $\mathcal{S}f^{\mu\nu} \equiv (f^{\mu\nu} + f^{\nu\mu})/2$.

Consider one of the four terms in Eq. (3.4), e.g.,

$$t^{\mu\nu}(q_L) \equiv \frac{i}{2}\int d^4x e^{-iq_L\cdot x_L}\bar\psi(x)\gamma^\mu D^\nu\psi(x). \qquad (3.5)$$

Since we are interested in the transport coefficients in the static and homogeneous limit, we have taken the transverse momentum to be zero, $q_{1,2} = 0$. (For the moment, we shall keep a finite q_3 for notational simplicity.) Inserting the expansion of the quark field (3.2) into the above term, we have

$$t^{\mu\nu}(q_L) = \frac{1}{2}\int_{\bar p}\bar\chi(p_L + q_L)\gamma_L^\mu\Gamma^\nu(\bar p)\mathcal{P}_+\chi(p_L), \qquad (3.6)$$

where

$$\Gamma^\nu(\bar p) = \int dx^1\mathcal{H}(\tilde x_p^f)(p_L^\nu - \delta^{\nu 2}B_f\tilde x_p^f + i\delta^{\nu 1}\partial^1)\mathcal{H}(\tilde x_p^f). \qquad (3.7)$$

According to the integral formulas given in Appendix A, only the first term survives,

$$\Gamma^\nu(\bar p) = p_L^\nu, \qquad (3.8)$$

which does not depend on the second component p_2 within the LLL approximation. Then, we find

$$t^{\mu\nu}(q_L) = \frac{1}{2}\int_{\bar p}\bar\chi(p_L + q_L)\gamma_L^\mu p_L^\nu\mathcal{P}_+\chi(p_L). \qquad (3.9)$$

An important point is that $t^{\mu\nu}$ has nonvanishing entries only in the longitudinal components specified by μ, $\nu = 0, 3$, and vanishes when either one or both of the indices are 1 or 2. The same conclusion is drawn for the other three terms of the energy-momentum tensor (3.4). This is a natural consequence of the fact that the LLL quarks can carry only an energy and a momentum parallel to the magnetic field.[6]

From the above observation, we find that only the longitudinal pressure $\tilde P_\parallel$ has contributions from the LLL quarks among all Eqs. (2.8)–(2.14). Therefore, the LLL quark carriers mainly contribute to the longitudinal component of the bulk viscosity $\zeta_\parallel$ which is given by the correlator of the diagonal components, $\tilde P_\parallel = T^{\mu\nu}b_\mu b_\nu - \Theta_\beta T^{\mu\nu}u_\mu u_\nu = T^{33} - \Theta_\beta T^{00}$. To reach the above conclusion, note also that, according to Eq. (B8), we have $\tilde P_\perp = P_\perp$ in the LLL which is solely given by the transverse components of the energy-momentum tensor.

All other correlators in Eq. (2.8) also have contributions from gluons, of which we will make order-of-magnitude estimates in Sec. VI.

IV. BULK VISCOSITY

Having identified the contribution of the LLL quarks to the viscosity, we now evaluate the relevant component, that is, the longitudinal component of the bulk viscosity ($\zeta_\parallel$). We also discuss the physical meaning of the result.

A. Calculation

We use the real-time formalism [26,27] for the diagrammatic calculation of the bulk viscosity. First, by using the (12) basis, we write the retarded Green's function in Eq. (2.8) in the low-energy limit as

$$\frac{\partial}{\partial\omega}\text{Im}G_{\tilde P_\parallel\tilde P_\parallel}^R(\omega,\mathbf{0}) \simeq \frac{\beta}{2}G_{\tilde P_\parallel\tilde P_\parallel}^{(12)}(p = 0), \qquad (4.1)$$

where $G_{AB}^{(12)} \equiv \langle T_C A_1(x)B_2(0)\rangle = \langle B_2(0)A_1(x)\rangle$, and T_C is the path-ordering operator on the complex-time path C, which is plotted in Fig. 1. The operator with index 1 (2) is defined on the path C_+ (C_-). As discussed in the previous section, the transverse pressure $\tilde P_\perp$ does not have a contribution from the LLL quarks, so that the other correlator, $G_{\tilde P_\perp\tilde P_\parallel}^R$, in Eq. (2.8) is much smaller than the one in Eq. (4.1). Therefore, we neglect this contribution in the following.

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[5]The trace part vanishes if one uses the equation of motion, which is the on-shell condition for the quark. In our calculation, this condition applies, so we do not write the trace part here.

[6]This conclusion could be modified when the hLLs contribute to the energy-momentum tensor or when the external transverse momentum $q_\perp$ (spatial modulation) is finite.
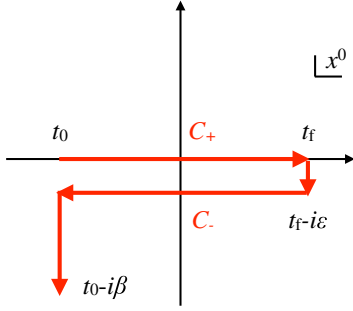
FIG. 1. The contour in the complex-time plane. The part C_+ runs along the real axis and the part C_- runs parallel to this axis, but is displaced by $-i\varepsilon$.

From Eq. (2.4), the relevant pressure component is expressed as $\tilde{P}_\parallel(p) = T^{33}(p) - \Theta_\beta T^{00}(p)$. Thus, Green's function with distinct external momenta reads

$$G^{(12)}_{\tilde{P}_\parallel \tilde{P}_\parallel}(p, p') = \langle T_C \tilde{P}_\parallel(p) \tilde{P}_\parallel(p') \rangle$$

$$= \int_{\bar{k}} \int_{\bar{k}'} \langle \bar{\chi}_2(p_L + k_L) \not{k}_\Theta \mathcal{P}_+ \chi_2(k)$$

$$\times \bar{\chi}_1(p'_L + k'_L) \not{k}'_\Theta \mathcal{P}_+ \chi_1(k') \rangle, \quad (4.2)$$

where we have defined $\not{k}_\Theta \equiv k^3 \gamma^3 - \Theta_\beta k^0 \gamma^0$ and inserted the expression of the energy-momentum tensor for vanishing transverse momentum $p_\perp = p'_\perp = 0$ which was discussed in the previous section. All four of the terms in Eq. (3.4) result in the same expression up to differences which will vanish in the end when we take the limit $p, p' \to 0$.

We start with the one-loop approximation. In this approximation, we can evaluate Green's function by using Wick's theorem (the corresponding diagram is drawn in Fig. 2). The thermal averages of the spinors are replaced by the thermal LLL propagator $S^{(ij)}(k_L, k'_L) = \langle \chi_i(k_L) \bar{\chi}_j(k'_L) \rangle = \delta^{(3)}(\bar{k} - \bar{k}') S^{(ij)}(k_L)$. Note that there is three-dimensional momentum conservation in the Landau gauge. Inserting the propagators into Green's function, we will, therefore, get a delta function, $\delta^{(3)}(p + p')$, for the overall conservation of external momenta, which in turn becomes the (three-dimensional) system volume in the limit $p, p' \to 0$. Dividing Green's function (4.2) by the volume $V_4 = L_1 \delta^{(3)}(0)$ with L_1 being the length in the residual dimension, we have
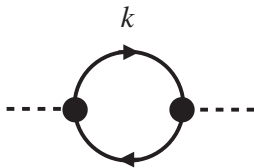

FIG. 2. The one-loop diagram for $G^{(12)}_{\tilde{P}_\parallel \tilde{P}_\parallel}$. The solid line is the quark propagator, and the blob is the vertex for $\tilde{P}_\parallel$, which is $k^3 \gamma^3 - \Theta_\beta k^0 \gamma^0$.

$$G^{(12)}_{\tilde{P}_\parallel \tilde{P}_\parallel}(p = 0) = \frac{1}{V_4} G^{(12)}_{\tilde{P}_\parallel \tilde{P}_\parallel}(p = 0, p' = 0), \quad (4.3)$$

and the one-loop expression is found to be

$$G^{(12)}_{\tilde{P}_\parallel \tilde{P}_\parallel}(p = 0)$$

$$= -N_c \sum_f \frac{|B_f|}{2\pi} \int_{k_L} \text{Tr}[S^{(21)}(k_L) \not{k}_\Theta \mathcal{P}_+ S^{(12)}(k_L) \not{k}_\Theta]$$

$$= N_c \sum_f \frac{|B_f|}{2\pi} \int_{k_L} \text{Tr}[\{(\not{k}_L + m_f) \not{k}_\Theta\}^2 \mathcal{P}_+]$$

$$\times n_F(k^0)[1 - n_F(k^0)][\rho^S(k_L)]^2. \quad (4.4)$$

In the first line, we obtained the density of states $|B_f|/(2\pi)$ in the transverse plane as explained in Appendix A. In the last line, we have inserted $S^{(12)}(k_L) = -(\not{k}_L + m_f) n_F(k^0) \rho^S(k_L)$ and $S^{(21)}(k_L) = (\not{k}_L + m_f)[1 - n_F(k^0)] \rho^S(k_L)$ with the quark spectral function $\rho^S(k_L)$ and the Fermi distribution function $n_F(k^0) \equiv [e^{\beta k^0} + 1]^{-1}$.

The trace in Eq. (4.4) is evaluated as

$$\text{Tr}[\{(\not{k}_L + m_f) \not{k}_\Theta\}^2 \mathcal{P}_+] = 4[(\epsilon_k^L)^2 X - m_f^2]^2, \quad (4.5)$$

where we have introduced $X = 1 - \Theta_\beta$ and used the on-shell condition $k^0 = \pm \epsilon_k^L$ ($\epsilon_k^L \equiv \sqrt{(k^3)^2 + m_f^2}$), which will be justified later. According to Appendix B, X vanishes at $m_f = 0$, and so does the trace in the massless limit.

The square of the spectral function needs to be treated with care: If we naively use the noninteracting form for the spectral function, $\rho_0^S(k_L) = 2\pi \text{sgn}(k^0) \delta(k_L^2 - m_f^2)$, it diverges on account of a pinch singularity. Physically, the viscosity is indeed expected to diverge in a free theory and becomes finite due to interactions. Therefore, the spectral function resummed with a finite damping rate (ξ_k) gives a finite result [28–31]. By using $\rho^S(k_L) = 4\xi_k k^0 / [(k_L^2 - m_f^2)^2 + (2\xi_k k^0)^2]$, it is approximated as [11,32]

$$[\rho^S(k_L)]^2 \simeq \frac{\rho_0^S(k_L)}{2\xi_k k^0}, \quad (4.6)$$

where we have neglected terms that vanish after the k^0 integration.

The spectral function (4.6) and the trace in Eq. (4.5) allow us to express Green's function (4.4) in terms of the damping rate. Plugging these expressions into the Kubo formula (2.8), the bulk viscosity can be written as

$$\zeta_\parallel = \frac{2\beta}{3} N_c \sum_f \frac{|B_f|}{2\pi} m_f^4 \int_{k_L} \left[\frac{3}{\pi^2 T^2} (\epsilon_k^L)^2 - 1 \right]^2$$

$$\times \frac{\rho_0^S(k_L)}{2\xi_k k^0} [1 - n_F(k^0)] n_F(k^0), \quad (4.7)$$

where the expression of X in Eq. (B5) has been used.

The quark damping rate arising from 1-to-2 scattering was already evaluated at the leading-log accuracy in Ref. [11] in the following two cases: One is the case $M_g \ll m_f \ll T$, with M_g being the Schwinger mass of the gluon [33,34],

$$M_g^2 \equiv \frac{1}{2} \frac{g^2}{\pi} \sum_f \frac{|B_f|}{2\pi}, \tag{4.8}$$

where g is the QCD coupling constant. The result in this case reads

$$\epsilon_k^L \xi_k \simeq \frac{g^2 C_f m_f^2}{4\pi} \left[\frac{1}{2} + n_B(\epsilon_k^L) \right] \ln\left(\frac{T}{m_f} \right), \tag{4.9}$$

at the leading-log accuracy, with $C_f \equiv (N_c^2 - 1)/(2N_c)$ and $n_B(k^0) \equiv [e^{\beta k^0} - 1]^{-1}$. The factor of m_f^2 can be explained in terms of chirality conservation [11,35]. In the other case, we have $m_f \ll M_g \ll T$. The expression in this case can be obtained by replacing the log in Eq. (4.9) with $\ln(T/M_g)$.

B. Results

By combining Eqs. (4.7) and (4.9), we can evaluate $\zeta_\parallel$. In the case $M_g \ll m_f$ the result reads

$$\zeta_\parallel = \frac{8\beta}{3} N_c \sum_f \frac{|B_f|}{2\pi} \frac{m_f^2}{g^2 C_f \ln(T/m_f)}$$

$$\times \int_0^\infty dk^3 \frac{1}{\epsilon_k^L} \left[\frac{3}{\pi^2 T^2} (\epsilon_k^L)^2 - 1 \right]^2 \bar{N}(\epsilon_k^L)$$

$$\simeq N_c \sum_f \frac{|B_f|}{2\pi} \frac{m_f^2}{g^2 C_f T \ln(T/m_f)} \left[\frac{4}{\pi^2} - \frac{56}{3} \zeta'(-2) \right]$$

$$\sim eBT^2 \left(\frac{m_f^2}{T^2} \right)^2 \frac{T}{g^2 m_f^2 \ln(T/m_f)}, \tag{4.10}$$

where we have defined $\bar{N}(\epsilon_k^L) \equiv [1 - n_F(\epsilon_k^L)] n_F^2(\epsilon_k^L) / n_B(\epsilon_k^L)$ and assumed $m_f \ll T$. Useful integration formulas are given by $\int_0^\infty d\epsilon \epsilon^3 \bar{N}(\epsilon) = T^4 \pi^2/2$, $\int_0^\infty d\epsilon \epsilon \bar{N}(\epsilon) = T^2/2$, and $\int_0^\infty d\epsilon \epsilon^{-1} \bar{N}(\epsilon_k^L) = -7\zeta'(-2)$ with the first derivative of the zeta function $\zeta'(-2) \simeq -0.0304$. Expression (4.10) is one of the central results of our paper, which is valid when the thermal excitations are well activated ($m_f \ll T$) and the LLL approximation works ($T \ll \sqrt{|B_f|}$).[7] Although the above result was obtained by the diagrammatic method applied to a Kubo formula, we can obtain the same result by using the Boltzmann equation in $(1+1)$ dimensions, as is shown in Appendix C. We note that this quantity is proportional to m_f^2, so the s quark contributes more than

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[7]When $m_f \ll M_g$, the log factor is replaced by $\ln(T/M_g)$ as indicated in the previous section. The former condition is also replaced by $M_g \ll T$, accordingly.

the u and d quarks. This is in striking contrast to the longitudinal component of the electrical conductivity [10,11], which is proportional to m_f^{-2}, and thus the u and d quarks dominate over the contribution from the s quark.

Several remarks on the parametric behavior are in order: To this end, we first recapitulate the behavior at $B = 0$ [36], which reads

$$\zeta_{B=0} \sim (\text{typical momentum})^4 \frac{(\text{conformal breaking factor})^2}{(\text{mean-free path})^{-1}}$$

$$\sim T^4 \left(\frac{m_f^2}{T^2} \right)^2 \frac{1}{g^4 T \ln(1/g)}. \tag{4.11}$$

The physical origin of the two conformal breaking factors are clear in the calculation with the Boltzmann equation [36], which is done in Appendix C. In comparison, we find the following points in our results:

(i) An overall factor of $|B_f|$ appears. It originates from the Landau degeneracy of the quarks in the transverse plane, which are carriers of the pressure. This factor replaces one factor of T^2 in the expression for $B = 0$, resulting in an enhancement of the bulk viscosity in a strong magnetic field, $|B_f| \gg T^2$.

(ii) Since the dominant scattering process is 1-to-2 [10–12] instead of 2-to-2, the g dependence of the quark damping rate ($\sim$ inverse of the mean-free path) is $\sim g^2$, not $\sim g^4$. Therefore, the denominator of the bulk viscosity is proportional to g^2.

(iii) The m_f dependence is also different because of chirality conservation. It yields a factor of m_f^2/T in the inverse of the mean-free path, and this factor partially cancels the m_f dependence coming from the two conformal breaking factors $(m_f^2/T^2)^2$ in the numerator. Thus, the m_f dependence of $\zeta_\parallel$ becomes a quadratic one in the end, meaning that the bulk viscosity decreases with a decreasing mass more slowly than the one computed without the effects of the strong magnetic field. This is a consequence of the competition between the two constraints which govern the behavior in the massless limit.

These points are shown clearly in the final lines of Eqs. (4.10) and (4.11).

Finally, we comment on the effect of higher-loop orders. In general, ladder-diagram contributions could be of the same order of magnitude as the one-loop diagram when a pinch singularity appears [28–31]. For this reason, one may wonder if one needs to resum all-order ladder diagrams to obtain the correct leading-order result. However, it is easy to show that the resummation is not required at the leading-log accuracy, meaning that our result (4.10) is correct at this order.

The proof follows the line shown in Sec. 5 of Ref. [11]. There are two steps in the proof. The first step is to identify the terms which have the maximum number of pinch singularities and are potentially as large in magnitude as the one-loop contribution. As in the case without external magnetic field, one indeed finds, by using the r/a basis, the terms in which all pairs of fermion propagators facing each other in the ladder diagram have pinch singularities. In the second step, one obtains the Bethe-Salpeter equation for the ladder resummation, which results in a gauge-invariant integral equation. Inserting the explicit forms of the quark and gluon spectral functions, one finds that the iterative correction vanishes[8] when $m_f \gg M_g$ and is suppressed by an inverse log factor $1/\ln(T/M_g)$ when $m_f \ll M_g$. Therefore, in both cases the one-loop result is correct within the leading-log approximation, which is valid when the inverse log factor is small, i.e., when $T/m_f \gg 1$ and $T/M_g \gg 1$, respectively.

V. ESTIMATE OF THE BULK VISCOSITY

In this section, we compute the value of $\zeta_\parallel$ for some choices of the parameters, which appear to be realistic for heavy-ion collisions, and compare it with the value at $B = 0$. We use the following values for the parameters:

$$\alpha_s \equiv \frac{g^2}{4\pi} = 0.3,$$
$$N_c = 3,$$
$$eB = 10m_\pi^2 = (443 \text{ MeV})^2,$$
$$m_f = 100 \text{ MeV} \quad (s \text{ quark}), \tag{5.1}$$

where we have assumed a strong magnetic field, with $m_\pi = 140$ MeV being the pion mass. For the case $N_f = 3$, the parameters above yield $M_g \simeq 160$ MeV. Because it is larger than m_f, we use the expression for the bulk viscosity in the case $M_g \gg m_f$,

$$\zeta_\parallel = N_c \frac{|B_f|}{2\pi} \frac{m_f^2}{g^2 C_f T \ln(T/M_g)} \left[\frac{4}{\pi^2} - \frac{56}{3} \zeta'(-2) \right]$$
$$\simeq 0.031 \frac{|eB| m_f^2}{T \ln(T/M_g)}, \tag{5.2}$$

where we have taken only the contribution from the s quark, since it is dominating over the contributions from the other flavors.

Let us compare this estimate with the one at $B = 0$. The contribution from the s quark is estimated as [36]

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[8]Nevertheless, the quark damping rate is computed within the leading-log approximation.
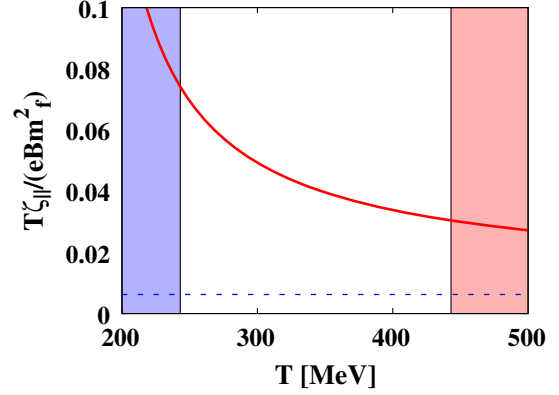

FIG. 3. The solid (red) line is our result ($\zeta_\parallel$), and the dotted (blue) line is $\zeta_{B=0}$. The red (blue) area on the right (left) is the temperature region $\sqrt{eB} < T$ ($T < \sqrt{\alpha_s eB}$) in which the LLL (leading-log) approximation is not reliable.

$$\zeta_{B=0} \simeq 0.011 \frac{m_f^4}{\alpha_s^2 T} \simeq 0.12 \frac{m_f^4}{T}, \tag{5.3}$$

where we used the same parameters (5.1). Equations (5.2) and (5.3) are plotted as functions of T in Fig. 3. To show the limit of our LLL and leading-log approximations, we have colored the temperature regions, $\sqrt{eB} < T$ and $T < \sqrt{\alpha_s eB}$, in which the two approximations are not justified. The two approximations are reliable only in the window between these areas. This plot suggests that the presence of a strong magnetic field enhances the longitudinal component of the bulk viscosity in a wide temperature range. This can actually be understood by looking at the parametric behavior: Neglecting log factors, we have $\zeta_\parallel \sim eB m_f^2/(g^2 T)$ and $\zeta_{B=0} \sim m_f^4/(g^4 T)$, and their ratio is $\zeta_\parallel/\zeta_{B=0} \sim g^2 eB/m_f^2 \sim (M_g/m_f)^2$. From the values estimated around Eq. (5.1), this ratio is larger than one with the current values of the parameters. The temperature dependence in Fig. 3 comes from the logarithmic factor in $\zeta_\parallel$, because of the temperature-independent infrared cutoff.

VI. GLUON CONTRIBUTION TO SHEAR AND BULK VISCOSITIES

In this section, we estimate the order of magnitude of the contribution of gluons to the shear and bulk viscosities. All estimates are performed in the LLL approximation assuming that $|B_f| \gg T^2$. There is no gluon contribution to the conductivities, since the gluon does not carry electric charge, so we do not discuss them here.

A. Shear viscosities

We begin with the five shear viscosities, $\eta_{0,\dots,4}$. In this case, the order-of-magnitude estimate can be performed using the schematic expression,

$$\eta_i \sim \frac{(\text{typical momentum})^4}{(\text{mean-free path})^{-1}}, \qquad (6.1)$$

which can be finite without any conformal breaking factor, in contrast to the bulk viscosity. The typical momentum is apparently of the order of T, while the mean-free path needs further consideration. Let us consider the following three scattering processes: (1) 1-to-2 scattering, (2) 2-to-2 gluon-quark t-channel scattering, and (3) 2-to-2 gluon-gluon t-channel scattering. They were estimated in Sec. V and Appendix B of Ref. [10] in the context of the color conductivity.

It was discussed that process (1) gives a gluon damping rate ($\sim$ inverse of the mean-free path), which is of the order of $g^2 m_f^2 eB/T^3$. A naive estimate of the contribution from process (2) to the damping rate is of the order of $g^4 TeB/\Lambda_{\text{IR}}^2$, where the dominant infrared (IR) cutoff $\Lambda_{\text{IR}} \sim (g^2 eB/m_f^2)^{\frac{2}{3}} T^2$ arises from Landau damping [see discussions below Eq (B.9) of Ref. [10]]. The color randomization can be achieved without a momentum exchange, so that it is not suppressed in the IR regime. However, for the randomization of the gluon momentum, which is relevant for the computation of viscosities, one finds a smaller IR enhancement, because the difference between the thermal distribution functions in the initial and final states also vanishes in the numerator of the collision integral as the momentum transfer decreases. An appropriate treatment of this point gives a modification of the parametric estimate as $g^4 TeB/\Lambda_{\text{IR}}^2 \times (\Lambda_{\text{IR}}/T)^2 \sim g^4 eB/T$ (up to a possible logarithmic factor), as was discussed in Appendix B of Ref. [10]. In the same way, the contribution from (3) is expected to be $g^4 T^3/\Lambda_{\text{IR}}^2 \times (\Lambda_{\text{IR}}/T)^2 \sim g^4 T$ with $\Lambda_{\text{IR}} \sim gT$. This is smaller than the contribution from (2), so we do not need to take this contribution into account.

The relative magnitude of contributions (1) and (2) depends on that of m_f and gT. But actually, whichever the larger contribution is, contribution (2) determines the shear viscosity,[9] as can be seen in the following: The Coulomb-gauge gluon propagator was shown to have two orthogonal components around the mass shell ($p^2 \simeq 0$) [34],

$$D^{\mu\nu} \simeq -\frac{P_T^{\mu\nu} - P_\perp^{\mu\nu}}{p^2 - (\Pi_T + \Pi_\parallel)} - \frac{P_\perp^{\mu\nu}}{p^2 - (\Pi_T + \Pi_\perp)}. \qquad (6.2)$$

The first term has a damping rate given by $\text{Im}(\Pi_T + \Pi_\parallel)$ while the other term has one determined by $\text{Im}(\Pi_T + \Pi_\perp)$. Here, Π_i are the coefficients of the tensor decomposition of the retarded gluon self-energy $\Pi^{\mu\nu}$, namely $\Pi^{\mu\nu} = \sum_{i=T,L,\parallel,\perp} \Pi_i P_i^{\mu\nu}$. We refer to Ref. [34] for the definitions of the projection tensors $P_i^{\mu\nu}$.

The scattering process (1) only contributes to $\text{Im}\Pi_\parallel$ [14], while we can show by explicit calculation that (2) contributes to $\text{Im}\Pi_T$. The perpendicular component $\Pi_\perp$ is absent. Therefore, when the former contribution is much larger than the latter one ($\text{Im}\Pi_\parallel \gg \text{Im}\Pi_T$), the contribution to the shear viscosity from the second mode in Eq. (6.2) is much larger than that from the first mode. In the opposite case ($\text{Im}\Pi_\parallel \ll \text{Im}\Pi_T$), the damping rates are determined by $\text{Im}\Pi_T$ and both terms in Eq. (6.2) contribute to a similar order of magnitude.

Thus, the inverse of the mean-free path is of order $g^4 eB/T$. Combining these order-of-magnitude estimates, we have

$$\eta_i \sim T^4 \times \frac{T}{g^4 eB} = \frac{T^5}{g^4 eB}. \qquad (6.3)$$

This is suppressed compared to the value at $B = 0$, $\eta \sim T^3/g^4$,[10] by a factor of T^2/eB. Physically, this originates from the fact that the gluon damping rate is enhanced by the abundance of the quark scatterers, the density of which increases with eB in the transverse plane.

B. Bulk viscosities

The order-of-magnitude estimate of the gluon contribution to the bulk viscosity is more complicated.

Let us start with the Kubo formula for $\zeta_\parallel$, Eq. (2.8), and see how one can recover the expression in the absence of the magnetic field. At $B = 0$, the expressions of the two components of the pressure in terms of $T^{\mu\nu}$ reduce to

$$\tilde{P}_\parallel = T^{33} - \Theta_\beta T^{00}, \qquad (6.4)$$

$$\tilde{P}_\perp = \frac{1}{2}(T^{11} + T^{22}) - \Theta_\beta T^{00}, \qquad (6.5)$$

so that $2\tilde{P}_\perp + \tilde{P}_\parallel = \sum_i T^{ii} - 3\Theta_\beta T^{00}$. In this case, we note that $\Theta_\beta = 1/3 - X$, where $X \sim g^4$ or m^2/T^2. The former contribution to X comes from the conformal anomaly, while the latter one is from the explicit breaking of the conformal symmetry by the current quark mass. Then, Eq. (2.8) gives

$$\zeta_\parallel = \frac{1}{3}\frac{\partial}{\partial\omega}\text{Im}G^R_{(2\tilde{P}_\perp + \tilde{P}_\parallel)\tilde{P}_\parallel}(\mathbf{p} = \mathbf{0}, \omega \to 0)$$
$$= \frac{1}{3}\frac{\beta}{2}\left\langle (-T_\mu^\mu + 3XT^{00})_2 \left(T^{33} - \frac{1}{3}T^{00} + XT^{00}\right)_1 \right\rangle (p = 0).$$
$$(6.6)$$

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[9]This point was not correctly considered in Ref. [10].

[10]If the contribution of the second term in Eq. (6.2) vanishes due to the tensor structure or any other reason, the shear viscosities will be even more suppressed with only the contribution from the first term, when $\text{Im}\Pi_\parallel \gg \text{Im}\Pi_T$.

The first factor $(-T^\mu_\mu + 3XT^{00})$ vanishes in the conformal case, so it yields one conformal breaking factor. In the other factor $(T^{33} - \frac{1}{3}T^{00} + XT^{00})$, one can replace T^{33} by $\frac{1}{3}\sum_i T^{ii}$ because of rotation symmetry in the absence of a magnetic field. Therefore, this factor also yields the same conformal breaking factor. In total, the bulk viscosity at $B = 0$ contains two conformal breaking factors as elaborated in Refs. [36,37], resulting in the parametric estimate shown in Eq. (4.11). The evaluation of $\zeta_\perp$ can be done in the same way. Here, we could recover the expression at $B = 0$ because of the rotation symmetry and the three-dimensional equation of state (EoS), $P = \frac{1}{3}\epsilon + o(m_f^2) + o(g^2)$.

Now, let us check what happens in the presence of strong magnetic fields. In the same way as above, we have

$$\zeta_\parallel = \frac{1}{3}\frac{\beta}{2}\langle(-T^\mu_\mu + 3XT^{00})_2(T^{33} - T^{00} + XT^{00})_1\rangle(p = 0).$$

(6.7)

Notice the difference to the $B = 0$ case: Namely, the factor in front of T^{00} in the second factor is unity instead of $1/3$. This factor of unity originates from the one-dimensional EoS in a strong magnetic field, $P_\parallel = \epsilon + o(m_f^2) + o(g^2)$, and makes a big difference in the order-of-magnitude estimate, as we will see below.

We focus on the product of $3XT^{00}$ and $T^{33} - T^{00}$ taken from the first and second factors in Eq. (6.7), respectively. At one-loop order we have

$$\zeta_\parallel \sim -\frac{\beta}{2}X\langle T_C(F^{a0\alpha}F^{a0}{}_\alpha)_2$$
$$\times (-F^{b3\beta}F^{b3}{}_\beta + F^{b0\beta}F^{b0}{}_\beta)_1\rangle(p = 0)$$
$$= -\frac{\beta}{2}X\langle T_C([\partial^0 A^{a\alpha}][\partial^0 A^a_\alpha])_2$$
$$\times (-[\partial^3 A^{b\beta}][\partial^3 A^b_\beta] + [\partial^0 A^{b\beta}][\partial^0 A^b_\beta])_1\rangle(p = 0)$$
$$= -\beta X(N_c^2 - 1)\int \frac{d^4k}{(2\pi)^4}(k^0)^2[(k^0)^2 - (k^3)^2]$$
$$\times n_B(k^0)[1 + n_B(k^0)]\rho^{\alpha\beta}(k)\rho_{\alpha\beta}(k),$$

(6.8)

where $\rho^{\mu\nu}(k)$ is the spectral function of the gluon. We have used the energy-momentum tensor of the gluons,

$$T^{\mu\nu} = \frac{g^{\mu\nu}}{4}F^{a\alpha\beta}F^a_{\alpha\beta} - F^{a\mu\alpha}F^{a\nu}{}_\alpha,$$

(6.9)

and the on-shell condition $k^2 \simeq 0$, and adopted the Coulomb gauge, in which A^0 and k^iA^i do not contribute at this order. Here, the field-strength tensor is $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$ with the $SU(N_c)$ structure constant f^{abc}.

In the last line of Eq. (6.8), we do not see a reason for the factor $(k^0)^2 - (k^3)^2$ to vanish. This is in contrast to the case

of the LLL quark contribution which has a $(1 + 1)$-dimensional conformal symmetry: We have seen that this factor indeed vanishes in the massless limit because the LLL quark has a $(1 + 1)$-dimensional dispersion relation, leaving a small factor of m_f^2.[11] As a result of the mismatch between the dimensions of the quark-dominant EoS and of the gluon dispersion relation, the gluon contribution to the bulk viscosities is expected to have only one conformal breaking factor. Therefore, we get the estimate

$$\zeta_{\parallel,\perp} \sim (\text{typical momentum})^4 \frac{(\text{conformal breaking factor})}{(\text{mean-free path})^{-1}}$$

$$\sim T^4 \times \frac{T}{g^4eB} \times X \sim \frac{T^3m_f^2}{g^4eB},$$

(6.10)

up to a possible logarithmic factor. When the dimensionless combination $M_g\sqrt{eB}/T^2$ is much larger than unity, the gluon contribution is subdominant compared to the quark contribution. Also, compared to the bulk viscosity at $B = 0$, the gluon contribution is suppressed in a strong magnetic field such that $eB \gg T^2(T/m_f)^2$, where the ratio T/m_f is of order one because of the important contribution from s quarks.

VII. SUMMARY

We have evaluated the longitudinal component of the bulk viscosity $\zeta_\parallel$ of the QGP in a strong magnetic field, by using the LLL approximation and the Kubo formula. Together with the longitudinal component of the electrical conductivity [10,11], this completes the evaluation of the first-order transport coefficients to which the LLL quarks contribute. We found that the current quark mass dependence significantly changes compared with the one at $B = 0$ [36], and explained this behavior as a result of the competition between conformal symmetry and chirality conservation. We also estimated the gluon contribution to the shear and bulk viscosities which is suppressed by a large value of the gluon damping rate, which is enhanced by the density of the LLL quark scatterers $\sim eB$.

When the LLL quark contribution is larger than the gluon contribution, the longitudinal component $\zeta_\parallel$ is larger than $\zeta_\perp$. This anisotropy may lead to modifications of the hydrodynamic expansion of the QGP in heavy-ion collisions. Indeed, a large value of $\zeta_\parallel$ has the tendency to suppress the hydrodynamic expansion in the direction of the magnetic field, as is schematically sketched in Fig. 4. Therefore, our result suggests that a strong magnetic field (which is in general orthogonal to the reaction plane)

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[11]As seen in Eq. (3.1), the dispersion relations of the hLLs explicitly depend on the field strength B, which, thus, will serve as the dominant conformal symmetry breaking factor at moderate magnetic field strengths, $m_f^2 \lesssim eB \lesssim T^2$, where populations of thermal excitations in the hLLs are as large as that in the LLL.
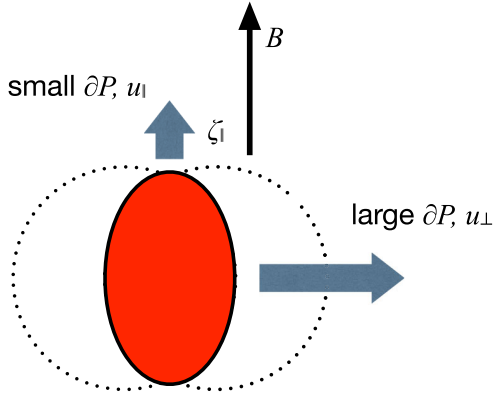
FIG. 4. Schematic picture of the elliptic flow and the effect of the anisotropic bulk viscosity on it.

potentially induces a positive contribution to the elliptic flow measured in heavy-ion collisions.

To complete the full evaluation of the shear and bulk viscosities in strong magnetic fields and to make a more quantitative estimate for phenomenological implications, we need to calculate the gluon contributions in more detail. We leave this interesting task to future work.

APPENDIX A: DIRAC EQUATION IN A MAGNETIC FIELD

We briefly summarize the solution for the Dirac equation in a magnetic field,

$$(i\not{D} - m_f)\psi = 0. \tag{A1}$$

We introduce the operators $a, a^\dagger = i[D^1 \pm \mathrm{sgn}(B_f)iD^2]/\sqrt{2|B_f|}$ where the upper and lower signs are for a and $a^\dagger$, respectively. These operators satisfy $[a, a^\dagger] = 1$. Then, the Dirac operator is cast into the form

$$i\not{D} - m_f = i\not{\partial}_L - m_f - \sqrt{2|B_f|}\gamma^1(a\mathcal{P}_+ + a^\dagger\mathcal{P}_-). \tag{A2}$$

Spin eigenstates $\psi_\pm = \mathcal{P}_\pm\psi$ should satisfy

$$[\partial_t^2 - \partial_z^2 + (2a^\dagger a + 1 \mp 1)|q_f B| + m_f^2]\psi_\pm = 0. \tag{A3}$$

Therefore, the solution for the ground state, the LLL, is $\psi_+ = e^{-ip_L \cdot x}\phi(x_\perp)\mathcal{P}_+\chi(p_L)$ with a Dirac spinor $\chi(p_L)$ and an eigenfunction such that $a\phi(x_\perp) = 0$. We find the LLL dispersion relation to be $(\epsilon_p^L)^2 = (p^3)^2 + m_f^2$.

Inserting ψ_+ of the LLL into the Dirac equation, one finds that the Dirac spinor $\chi(p_L)$ obeys the $(1+1)$-dimensional "free" Dirac equation

$$(\not{p}_L - m_f)\chi(p_L) = 0. \tag{A4}$$

This equation indicates that the LLL spinor depends only on the longitudinal momentum p_L, and the LLL fermions have a $(1+1)$-dimensional "free" propagator. From the condition $a\phi(x_\perp) = 0$, the normalized wave function in the Landau gauge is obtained as

$$\phi(x_\perp) = e^{ip^2 x^2}\mathcal{H}\left(x^1 - \frac{p^2}{B_f}\right), \tag{A5}$$

where

$$\mathcal{H}(x) = \left(\frac{|B_f|}{\pi}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2}|B_f|}. \tag{A6}$$

We use the following properties of the Hermite function in the LLL:

$$\int dx \mathcal{H}(x)\mathcal{H}(x) = 1, \tag{A7}$$

$$\int dx x \mathcal{H}(x)\mathcal{H}(x) = 0, \tag{A8}$$

$$\int dx \mathcal{H}(x)\partial_x\mathcal{H}_n(x) = 0. \tag{A9}$$

Finally, we count the density of states in a finite box [25,38]. To this end, note that the second component p^2 of the canonical momentum serves as a label of the degenerate states, and the center coordinate of the cyclotron motion is given by $x_c^1 = p^2/B_f$. Accordingly, when the cyclotron center is located within the length of the system $0 \leq x_c^1 \leq L_1$, we have $0 \leq p_y \leq B_f L_1$ when $\mathrm{sgn}(B_f) > 0$, and $-|B_f|L_1 \leq p^2 \leq 0$ when $\mathrm{sgn}(B_f) < 0$. Therefore, we get the density of states in the transverse plane as

$$\frac{1}{L_1}\int_0^{|B_f|L_1} \frac{dp^2}{2\pi} = \frac{|B_f|}{2\pi}. \tag{A10}$$

APPENDIX B: THERMODYNAMIC QUANTITIES IN THE LLL APPROXIMATION

In this appendix, we first evaluate Θ_β, which is required for the calculation of the bulk viscosity in the LLL approximation. Before we do so, we need to obtain the energy density and the pressure in the direction of B: The contribution from the quarks in the LLL reads

$$\epsilon = \frac{|B_f|}{2\pi} N_c \frac{2}{\pi} \int_0^\infty dk^3 \frac{(\epsilon_k^L)^2}{\epsilon_k^L} n_F(\epsilon_k^L), \qquad (B1)$$

$$P_\parallel = \frac{|B_f|}{2\pi} N_c \frac{2}{\pi} \int_0^\infty dk^3 \frac{(k^3)^2}{\epsilon_k^L} n_F(\epsilon_k^L). \qquad (B2)$$

In the massless limit, both quantities are equal, as $\epsilon = P_\parallel = |B_f| N_c T^2 / 12$. In this case, $\Theta_\beta = 1$, which corresponds to $X = 0$. This result can be understood by remembering that, at $B = 0$, conformal symmetry requires that $\Theta_\beta = 1/3$, since the number of spatial dimensions is three. In our case, the number of spatial dimensions is effectively reduced to one thanks to the strong magnetic field, so Θ_β is unity.

As we have seen in Eq. (4.5), we need to evaluate the deviation from the $m_f = 0$ case. Therefore, what we should evaluate is

$$\epsilon - P_\parallel = m_f^2 \frac{|B_f|}{2\pi} N_c \frac{2}{\pi} \int_0^\infty dk^3 \frac{1}{\epsilon_k^L} n_F(\epsilon_k^L). \qquad (B3)$$

We note that this integral has a logarithmic infrared divergence in the massless limit. At the leading-log accuracy in terms of $\ln(T/m_f)$, we get

$$\epsilon - P_\parallel \simeq m_f^2 \frac{|B_f|}{2\pi} N_c \frac{1}{\pi} \ln \frac{T}{m_f}. \qquad (B4)$$

Now, we can obtain Θ_β as

$$\Theta_\beta = 1 - \frac{\partial}{\partial \epsilon} (\epsilon - P_\parallel) \simeq 1 - \frac{3m_f^2}{\pi^2 T^2}, \qquad (B5)$$

in the temperature region $m_f \ll T$. Here we have used the property $df/d\epsilon = (df/dT)/(d\epsilon/dT)$ for a fixed value of B. This result corresponds to $X = 3m_f^2/(\pi^2 T^2)$.

Next, we evaluate Φ_β. To this end, we need to evaluate M first. From the definition of M and Eq. (B2), we have

$$M = \frac{P_\parallel}{B}, \qquad (B6)$$

which is quite different from the expression at weak B. From this expression, we get

$$\Phi_\beta = -\left(\frac{\partial P_\parallel}{\partial \epsilon}\right)_B = -1 + \left(\frac{\partial(\epsilon - P_\parallel)}{\partial \epsilon}\right)_B. \qquad (B7)$$

By comparison to Eq. (B5), we have

$$\Phi_\beta = -\Theta_\beta. \qquad (B8)$$

Finally, let us comment on the interpretation of Eq. (B6): In the case of a weak magnetic field, the effect of B on the energy or pressure comes mainly from the Zeeman effect, namely the fact that flipping spins of the quarks costs energy. Therefore, the magnetization is proportional to the total number of spins of the quarks, as M is the pressure difference when we increase B. In contrast, in the presence of a strong magnetic field, the spins of the quarks in the LLL are always pointing in the direction of B and they cannot be flipped. Instead, the main effect of B on the energy or pressure is to create a large quark density in the transverse plane, due to the degeneracy of the LLL. For this reason, the magnetization in a strong magnetic field B agrees with the pressure due to the quarks, divided by the degeneracy of the LLL, as can be see from the above result.

APPENDIX C: EQUIVALENCE TO LINEARIZED BOLTZMANN EQUATION

In this appendix, we show that the linearized Boltzmann equation can reproduce the result for the bulk viscosity (4.10) obtained from the diagrammatic calculation. We follow the strategy of Ref. [36]: We consider the situation that the system is at equilibrium and at rest in the beginning, so that the distribution functions are given by the standard Fermi and Bose ones. Then, the system is disturbed by an expansion in the direction of B. Linear-response theory requires one to evaluate the change in the pressure, in order to get information on the bulk viscosity.

The time evolution of the system is described by the Boltzmann equation for the quark distribution function (f) in the LLL, which is effectively a $(1 + 1)$-dimensional equation,

$$(\partial_t + v^3 \partial_z) f(k^3, t, z) = C[f], \qquad (C1)$$

where $v^3 \equiv k^3/\epsilon_k^L$ is the velocity in the direction of B, (t, z) are the space-time coordinates, and $C[f]$ is the collision term for the 1-to-2 process, the expression of which is given in Refs. [10,11].

In general, the distribution function can be written in terms of the deviation from the equilibrium value, namely

$$f(k^3, t, z) = f_{eq}(k^3, t, z) + \delta f(k^3, t, z), \qquad (C2)$$

where

$$f_{eq}(k^3, t, z) \equiv (\exp\{\beta(t)\gamma_u[\epsilon_k^L - k^3 u^3(z)]\} + 1)^{-1} \qquad (C3)$$

is the distribution function in equilibrium in the presence of the flow (u^3). Here $\gamma_u \equiv [1 - (u^3)^2]^{-1/2}$ is the gamma factor. For $u^3 = 0$, f_{eq} reduces to n_F. We note that the temperature depends on time, since the expansion decreases the energy density of the system.

We consider the linear response regime, so u^3 and δf are assumed to be small. Then, the left-hand side of Eq. (C1) is approximated as

$$
-n_F(\epsilon_k^L)[1 - n_F(\epsilon_k^L)][\epsilon_k^L \partial_t \beta - \beta v^3 k^3 \theta(z)]
$$
$$
= -\beta n_F(\epsilon_k^L)[1 - n_F(\epsilon_k^L)]\theta(z)(\epsilon_k^L \Theta_\beta - v^3 k^3), \quad \text{(C4)}
$$

where $\theta(z) \equiv \partial_z u^3(z)$ represents the magnitude of the expansion. In the last line, we have used $\partial_t \beta = \beta \Theta_\beta \theta(z)$ [36]. The right-hand side of Eq. (C1) vanishes in equilibrium, so that it is of linear order in δf. In the relaxation-time approximation, the collision term is given by $-\tau_k^{-1} \delta f(k^3, t, z)$ with the parameter τ_k being the relaxation time. Combining both sides, the solution of the Boltzmann equation reads

$$
\delta f = \tau_k \beta n_F(\epsilon_k^L)[1 - n_F(\epsilon_k^L)]\theta(z)(\epsilon_k^L \Theta_\beta - v^3 k^3). \quad \text{(C5)}
$$

We note that, in the massless limit where the system becomes conformal[12] in the classical limit, the quantity in the bracket vanishes [see the definition of v^3 and ϵ_k^L, and Eq. (B5)]. This means that the system still persists to be at equilibrium even in the presence of the expansion, which is a natural consequence of the conformal invariance.

The bulk viscosity appears in the constitutive relation (2.6) as

---

[12]Here, the conformal symmetry is defined not in $(3 + 1)$ but in $(1 + 1)$ dimensions.

$$
\delta P_\parallel = -3\zeta_\parallel \theta, \quad \text{(C6)}
$$

where $\delta P_\parallel$ is the deviation of $P_\parallel$ from the equilibrium value. Here, we have omitted terms proportional to other transport coefficients which do not have contributions from the LLL quarks. Thus, we need to evaluate $\delta P_\parallel$. Naively, it is given by

$$
\delta P_\parallel = \frac{|B_f|}{2\pi} N_c \frac{2}{\pi} \int_0^\infty dk^3 \frac{1}{\epsilon_k^L}(k^3)^2 \delta f(k), \quad \text{(C7)}
$$

where we replaced n_F by δf in Eq. (B2). However, we note that, even when $\delta f = 0$, the pressure changes since the temperature decreases in time due to the expansion. Therefore, we need to subtract this effect, which is found to require subtraction of $\Theta_\beta T^{00}$ from T^{33} [36]. T^{00} can be expressed in terms of δf as in Eq. (C7) with the replacement of $(k^3)^2$ by $(\epsilon_k^L)^2$. The subtracted result is found to be

$$
\delta[P_\parallel - \Theta_\beta \epsilon] = \frac{|B_f|}{2\pi} N_c \frac{2}{\pi} \int_0^\infty dk^3 \frac{\delta f(k)}{\epsilon_k^L}[(k^3)^2 - \Theta_\beta(\epsilon_k^L)^2]
$$
$$
= -\frac{|B_f|}{2\pi} N_c \frac{2}{\pi} \beta \int_0^\infty dk^3 \frac{[(k^3)^2 - \Theta_\beta(\epsilon_k^L)^2]^2}{(\epsilon_k^L)^2}
$$
$$
\times \tau_k n_F(\epsilon_k^L)[1 - n_F(\epsilon_k^L)]\theta(z). \quad \text{(C8)}
$$

We see that another conformal breaking factor $(k^3)^2 - \Theta_\beta(\epsilon_k^L)^2$ appears in addition to the one in Eq. (C7). By identifying $\tau_k^{-1} = 2\xi_k$, we find that the expression for the bulk viscosity obtained from this equation and Eq. (C6) is identical to Eq. (4.7) from the diagrammatic method.

The equivalence beyond the relaxation-time approximation can also be shown, as was done in Ref. [11] in the case of the electrical conductivity.

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