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Dyon degeneracies from Mathieu moonshine symmetry

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We study Siegel modular forms associated with the theta lift of twisted elliptic genera of $K3$ orbifolded with g' corresponding to the conjugacy classes of the Mathieu group M_{24}. For this purpose we rederive the explicit expressions for all the twisted elliptic genera for all the classes which belong to $M_{23} \subset M_{24}$. We show that the Siegel modular forms satisfy the required properties for them to be generating functions of $1/4$ BPS dyons of type II string theories compactified on $K3 \times T^2$ and orbifolded by g' which acts as a $\mathbb{Z}_N$ automorphism on $K3$ together with a $1/N$ shift on a circle of T^2. In particular the inverse of these Siegel modular forms admit a Fourier expansion with integer coefficients together with the right sign as predicted from black hole physics. This observation is in accordance with the conjecture by Sen and extends it to the partition function for dyons for all the 7 CHL compactifications. We construct Siegel modular forms corresponding to twisted elliptic genera whose twining character coincides with the class $2B$ and $3B$ of M_{24} and show that they also satisfy similar properties. Apart from the orbifolds corresponding to the 7 CHL compactifications, the rest of the constructions are purely formal.

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I. INTRODUCTION

The partition function of $1/4$ Bogomol'nyi-Prasad-Sommerfield (BPS) dyons for $\mathcal{N} = 4$ string compactifications have been studied extensively. Starting from the original proposal [1] for the degeneracy of dyons in heterotic string theory on T^6 and the study of its asymptotic property [2], it has been generalized to certain Chaudhuri-Hockney-Lykken (CHL) compactifications [3]. The degeneracy of dyons can be obtained from the Fourier coefficients of the inverse of an appropriate $Sp(2,\mathbb{Z})$ Siegel modular forms or its subgroup. For the case of the heterotic string on T^6, it is the Igusa cusp form of weight 10 which is the theta lift or the multiplicative lift of the elliptic genus of $K3$. The elliptic genus of $K3$ plays a role in the degeneracy since the counting of these $1/4$ BPS states is done in the type II picture which is compactified on $K3 \times T^2$ [4,5]. For the case of CHL compactifications [6] considered it turns out that the Siegel modular forms are theta lifts of the twisted elliptic genus of $K3$ [7,8]. This is because the CHL compactifications are dual to $(K3 \times T^2)/\mathbb{Z}_N$ where the orbifold acts as an order $\mathbb{Z}_N$ Nikulin's automorphism [9] on $K3$ together with a $1/N$ shift on one of the circles of T^2 [10,11]. The construction was carried out for the $N = 2$, 3, 5, 7 CHL models, that is 4 out of the 7 CHL models.

With the discovery of Mathieu moonshine in $K3$ [12], it has been seen $K3$ admits 26 twining elliptic genera corresponding to the 26 conjugacy classes of the Mathieu group M_{24}. Before we proceed let us define the twisted elliptic genus of $K3$ by an automorphism g' of order $\mathbb{Z}_N$, given by

$$F^{(r,s)}(\tau,z) = \frac{1}{N}\mathrm{Tr}_{RRg'^r}\left[(-1)^{F_{K3}+\bar{F}_{K3}} g'^s e^{2\pi i z F_{K3}} q^{L_0-c/24}\bar{q}^{\bar{L_0}-\bar{c}/24}\right]. \tag{1.1}$$

Here the trace is taken over the Ramond-Ramond sector of the $\mathcal{N} = (4,,4)$ superconformal field theory of $K3$ with central charge (6,6) and F is the fermion number. The $K3$ CFT is orbifolded by the action of g', a $\mathbb{Z}_N$ automorphism. The values (r,s) run from 0 to $N-1$. For g' belonging to the 26 conjugacy classes of M_{24} only the twining character $F^{(0,1)}$ has been constructed in [13–15]. The names of these classes and the corresponding cycle and the cycle shape are listed in Tables I and II. The order of the symmetry is also listed. For the M_{24} conjugacy classes pA, $p = 2, 3, 5, 7$, the twisted elliptic genera in all the sectors was given earlier in [7]. These genera are obtained by orbifolding the $K3$ by g' which is an order $N = 2, 3, 5, 7$ automorphism. The Siegel modular forms which capture the degeneracy of $1/4$ BPS states in the $\mathcal{N} = 4$ theories obtained by type II compactified on the orbifold $(K3 \times T^2)/\mathbb{Z}_N$ have also been constructed in [7]. The most direct method of constructing these Siegel modular forms is thorough the theta lift of the corresponding twisted elliptic genus of $K3$. For this purpose it is necessary to know the Fourier expansion of the twisted elliptic genus in all its sectors. In this paper we study the properties of these Siegel modular forms in the other conjugacy classes of M_{24}. This is done for all classes in Table I. We demonstrate that the inverse of these Siegel modular forms have the required properties to be generating functions of $1/4$ BPS states of type II string compactified on orbifolds of $K3 \times T^2$ by g' on $K3$ corresponding to

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TABLE I. Conjugacy classes of $M_{23} \subset M_{24}$ (Type 1)

Conjugacy class	Order	Cycle shape	Cycle
1A	1	1^{24}	()
2A	2	$1^8 \cdot 2^8$	(1, 8)(2, 12)(4, 15)(5, 7)(9, 22)(11, 18)(14, 19)(23, 24)
3A	3	$1^6 \cdot 3^6$	(3, 18, 20)(4, 22, 24)(5, 19, 17)(6, 11, 8)(7, 15, 10)(9, 12, 14)
5A	4	$1^4 \cdot 5^4$	(2, 21, 13, 16, 23)(3, 5, 15, 22, 14)(4, 12, 20, 17, 7)(9, 18, 19, 10, 24)
7A	7	$1^3 \cdot 7^3$	(1, 17, 5, 21, 24, 10, 6)(2, 12, 13, 9, 4, 23, 20)(3, 8, 22, 7, 18, 14, 19)
7A	7	$1^3 \cdot 7^3$	(1, 21, 6, 5, 10, 17, 24)(2, 9, 20, 13, 23, 12, 4)(3, 7, 19, 22, 14, 8, 18)
11A	11	$1^2 \cdot 11^2$	(1, 3, 10, 4, 14, 15, 5, 24, 13, 17, 18)(2, 21, 23, 9, 20, 19, 6, 12, 16, 11, 22)
23A	23	$1^1 \cdot 23^1$	(1, 7, 6, 24, 14, 4, 16, 12, 20, 9, 11, 5, 15, 10, 19, 18, 23, 17, 3, 2, 8, 22, 21)
23B	23	$1^1 \cdot 23^1$	(1, 4, 11, 18, 8, 6, 12, 15, 17, 21, 14, 9, 19, 2, 7, 16, 5, 23, 22, 24, 20, 10, 3)
4B	4	$1^4 \cdot 2^2 \cdot 4^4$	(1, 17, 21, 9)(2, 13, 24, 15)(3, 23)(4, 14, 5, 8)(6, 16)(12, 18, 20, 22)
6A	6	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$	(1, 8)(2, 24, 11, 12, 23, 18)(3, 20, 10)(4, 15)(5, 19, 9, 7, 14, 22)(6, 16, 13)
8A	8	$1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$	(1, 13, 17, 24, 21, 15, 9, 2)(3, 16, 23, 6)(4, 22, 14, 12, 5, 18, 8, 20)(7, 11)
14A	14	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1, 12, 17, 13, 5, 9, 21, 4, 24, 23, 10, 20, 6, 2)(3, 18, 8, 14, 22, 19, 7)(11, 15)
14B	14	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1, 13, 21, 23, 6, 12, 5, 4, 10, 2, 17, 9, 24, 20)(3, 14, 7, 8, 19, 18, 22)(11, 15)
15A	15	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2, 13, 23, 21, 16)(3, 7, 9, 5, 4, 18, 15, 12, 19, 22, 20, 10, 14, 17, 24)(6, 8, 11)
15B	15	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2, 23, 16, 13, 21)(3, 12, 24, 15, 17, 18, 14, 4, 10, 5, 20, 9, 22, 7, 19)(6, 8, 11)

TABLE II. Conjugacy classes of $M_{24} \notin M_{23}$ (Type 2)

Conjugacy class	Cycle shape	Cycle
2B	2^{12}	(1, 8)(2, 10)(3, 20)(4, 22)(5, 17)(6, 11)(7, 15)(9, 13)(12, 14)(16, 18)(19, 23)(21, 24)
3B	3^8	(1, 10, 3)(2, 24, 18)(4, 13, 22)(5, 19, 15)(6, 7, 23)(8, 21, 12)(9, 16, 17)(11, 20, 14)
12B	12^2	(1, 12, 24, 23, 10, 8, 18, 6, 3, 21, 2, 7)(4, 9, 11, 15, 13, 16, 20, 5, 22, 17, 14, 19)
6B	6^4	(1, 24, 10, 18, 3, 2)(4, 11, 13, 20, 22, 14)(5, 17, 19, 9, 15, 16)(6, 21, 7, 12, 23, 8)
4C	4^6	(1, 23, 18, 21)(2, 12, 10, 6)(3, 7, 24, 8)(4, 15, 20, 17)(5, 14, 9, 13)(11, 16, 22, 19)
10A	$2^2 \cdot 10^2$	(1, 8)(2, 18, 21, 19, 13, 10, 16, 24, 23, 9)(3, 4, 5, 12, 15, 20, 22, 17, 14, 7)(6, 11)
21A	$3^1 \cdot 21^1$	(1, 3, 9, 15, 5, 12, 2, 13, 20, 23, 17, 4, 14, 10, 21, 22, 19, 6, 7, 11, 16)(8, 18, 24)
21B	$3^1 \cdot 21^1$	(1, 12, 17, 22, 16, 5, 23, 21, 11, 15, 20, 10, 7, 9, 13, 14, 6, 3, 2, 4, 19)(8, 24, 18)
4A	$2^4 \cdot 4^4$	(1, 4, 8, 15)(2, 9, 12, 22)(3, 6)(5, 24, 7, 23)(10, 13)(11, 14, 18, 19)(16, 20)(17, 21)
12A	$2^1 \cdot 4^1 \cdot 6^1 \cdot 12^1$	(1, 15, 8, 4)(2, 19, 24, 9, 11, 7, 12, 14, 23, 22, 18, 5)(3, 13, 20, 6, 10, 16)(17, 21)

these conjugacy classes together with a shift of $1/N$ on one of the circles of T^2. See [16,17] for reviews. Our main objective is to study the Fourier expansion of the resulting Siegel modular forms and observe that their coefficients are integers as well as positive in accordance with the conjecture of [18].

Before we proceed, it is important to stress that the symmetry group M_{24} has not been seen in the nonlinear sigma models of $K3$ [19–21]. Among the conjugacy classes in Table I, the classes pA, $p = 1, 2, 3, 5, 7$ and $4B, 6A, 8A$ correspond to automorphisims of $K3$ which have already been studied in the context of CHL compactifications [6,10,11].[1] However for the rest of the conjugacy classes in Table I it is not clear that symmetries exist at the level of the nonlinear sigma models of $K3$, though a vector space carrying a representation of M_{24} exists [22]. It is known that symmetries of the order 11,14,15 do exist for some $K3$ models [23], however it is not clear that the twining characters corresponding to these symmetries do not correspond to that given by Mathieu moonshine. It is also known that there are no symmetries of order 23 in $K3$ sigma models [19–21]. Therefore our analysis for the classes $11A, 14A/B, 15A/B, 23A/B$ are purely formal. In spite of this we will see that the inverse of the Siegel modular forms corresponding to these twisted elliptic genera do admit Fourier coefficients which satisfy the positivity conjecture of [18].

As we have remarked the first step towards constructing the Siegel modular form obtained as a theta lift of the twisted elliptic genus of $K3$ is the knowledge of the Fourier expansion of the twisted elliptic genus $F^{(r,s)}$. For the conjugacy classes pA, $p = 2, 3, 5, 7$, these were constructed in [7]. The twining character $F^{(0,1)}$ corresponding to each of the classes in Table I is known. The twisted elliptic genus

[1]We will study the low lying hodge numbers of these cases and demonstrate the connection of these classes with CHL compactifications.

$F^{(1,s)}$ has been constructed in the ancillary files provided with [24]. We will rederive them here for our purpose. To obtain the other sectors from the twining character, we use the following transformation property of the twisted elliptic genus under modular transformation

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = \exp\left(2\pi i\frac{cz^2}{c\tau+d}\right)F^{(cs+ar,ds+br)}(\tau,z), \quad (1.2)$$

with

$$a,b,c,d\in\mathbb{Z}, \qquad ad-bc=1. \quad (1.3)$$

In (1.2) the indices $cs+ar$ and $ds+br$ belong to $\mathbb{Z}$ mod N. For example the (0,1) sector on the left-hand side (LHS) of (1.2) is related to the (1,0) sector its arguments is evaluated at $(-1/\tau, z/\tau)$. However this is not sufficient, since we require a Fourier expansion of the (1,0) sector to construct the theta lift, in fact we need further relations to express the expansion in terms of $e^{-2\pi i/\tau}$ in terms ordinary $q=e^{2\pi i\tau}$ expansions. Then identities involving modular forms of $\Gamma_0(N)$ allow one to perform these Fourier expansions. For N prime this procedure is enough to determine all the sectors of the twisted elliptic genus. But, when N is composite it is not possible to relate all the sectors to the (0,1) sector by modular transformation. The various sectors of the twisted elliptic genus break up into suborbits under the action of modular transformations. For example for the class $4B$ with $N=4$ in Table I, the sectors $F^{(0,2)}$, $F^{(2,0)}$, $F^{(2,2)}$ form a suborbit and cannot be related to $F^{(0,1)}$. We determine the twisted elliptic genus in these suborbits using its correspondence with the cycle shape of M_{24}. This correspondence is sufficient to determine the complete twisted elliptic genus for all the classes given in Table I. Note that in this situation unlike the case when N is prime, it is not enough to know the twisted elliptic genus $F^{(1,s)}$ to determine the twisted elliptic genus in all the sectors. We again emphasise that the construction of the twisted elliptic genus for these classes are not new. They can be read out in the ancillary files provided with [24]. We have rederived them and we find that it is useful to provide a list of all the twisted elliptic genera in the Appendix. A detailed comparison with [24] will be made in Sec. 2 3.

Let us now examine the first two conjugacy classes in Table II, $2B$, $3B$. In [25], an explicit rational CFT consisting of 6, $SU(2)$ WZW models at level 1 in which an order 4 orbifold can be performed was introduced. Using this orbifold action the twisted elliptic genus can be evaluated in all the sectors. It was observed that the twining character of this orbifold coincides with that of $2B$. Further more, the twisted elliptic genus exhibits the following property which was called the "quantum symmetry." This essentially means that the sum of the twisted elliptic genus in all its sectors vanishes. For example for the case of $2B$ which is an order 4 action in $K3$ quantum symmetry implies the equality

$$\sum_{r,s=0}^{3} F^{(r,s)}(\tau,z) = 0. \quad (1.4)$$

We emphasize here that this order 4 action does not correspond to the Mathieu moonshine class $2B$ since the order of this action is 2 though their twining character coincides. Nevertheless we investigate the properties of the Siegel modular form corresponding to this class and again demonstrate that the Fourier coefficients of the inverse of this Siegel modular form satisfies the positivity conjecture of [18]. Let us now look at the twining character for class $3B$, which is Jacobi weak form invariant under $\Gamma_0(9)$. Therefore we assume an order 9 orbifold action. Note however the $3B$ orbifold corresponding to the Mathieu moonshine symmetry should in fact be a order 3 action on $K3$. Therefore the twisted ellipic genus we obtain, just as in the case of the $2B$ class, does not correspond to the Mathieu moonshine class $3B$. Assuming an order 9 action and demanding that the orbifold action preserves the holomorphic 2 forms which is necessary to preserve $SU(2)$ holonomy together with the quantum symmetry, we obtain a twisted elliptic genus invariant under $\Gamma_0(9)$ and which coincides with the twining character of the $3B$ class. Thus our construction again is formal. We show that again the that the Fourier coefficients of the inverse of this Siegel modular form corresponding to this twisted elliptic genus satisfies the positivity conjecture.

Once the twisted elliptic genus is known it is straight forward to construct and obtain the weights k of the Siegel modular form $\tilde{\Phi}_k(\rho,\sigma,v)$, obtained from the theta lift of the twisted elliptic genus corresponding to the conjugacy classes in Table I and the first two classes in Table II. As we have mentioned for the classes pA, $p=1,2,3,5,7$, a detailed study of these modular forms has been done in [7,18]. For the rest of the classes in Table I, their formal construction is given in [26] which also studied their factorization property. Our main objective to construct these Siegel modular forms is to study the positivity conjecture of [18]. We also study the factorization property of a closely related modular form $\hat{\Phi}_k(\rho,\sigma,v)$ as $v\to 0$. This enables us to obtain the asymptotic degeneracies of 1/4 BPS black holes of large charges in type II string theory compactified on the orbifold $(K3\times T^2)/\mathbb{Z}_N$ including the subleading corrections. Using the analysis in [27], we see the subleading corrections agree precisely with that obtained using the entropy function method including the Gauss-Bonnet term in these theories. Finally we explicitly evaluate the degeneracies of the low charge dyons in these states using these Siegel modular forms by extracting out the respective Fourier coefficients. This is given by the expression

$$-B_6 = -(-1)^{Q\cdot P}\int_{\mathcal{C}} \mathrm{d}\rho\mathrm{d}\sigma\mathrm{d}v e^{-\pi i(N\rho Q^2+\sigma/NP^2+2vQ\cdot P)} \times \frac{1}{\tilde{\Phi}(\rho,\sigma,v)}, \tag{1.5}$$

where $\mathcal{C}$ is a contour in the complex 3-plane which we will define. Q, P refer to the electric and magnetic charge of the dyons in the heterotic frame. A subset of these Fourier coefficients represent single centered black holes. From the fact that the single centered black holes carry zero angular momentum, it is conjectured that the sign of $-B_6$ is positive [18]. We verify this prediction for low charge dyons. All these properties of $(\tilde{\Phi}_k)^{-1}$ indicate that they capture the degeneracy of dyons in $\mathcal{N}=4$ theories compactified on orbifolds $(K3\times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as g' an order N automorphism in $K3$ together with a $1/N$ shift on one of the circles of T^2. This observation of Siegel modular forms for the cases of composite order $4B, 6A, 8A$ in Table I completes the study of the spectrum of $1/4$ BPS dyons in all the 7 CHL compactifications introduced in [6,10]. The construction for the rest of the conjugacy classes in Table I and the construction for the first two classes in Table II is purely formal as we have mentioned earlier. But it is interesting to note that the Siegel modular form satisfies the positivity conjecture.

To highlight the fact that, it is in fact only a special class of Siegel modular forms which satisfy the positivity conjecture of [18], we examine the Siegel modular forms corresponding to the torus orbifolds introduced recently in [28]. We observe that the Fourier coefficients of the inverse of these modular forms do not satisfy the positivity conjecture.

Again here we remark the construction of the modular forms $\tilde{\Phi}_k$ given the twisted elliptic genus is quite straight forward and our method is the extension of the method first introduced for cyclic orbifolds in [27]. Recently this construction has been extended for noncyclic orbifolds in [26,28,29]. However to our knowledge that the observation of positivity of the Fourier coefficients of the inverse of $\tilde{\Phi}_k$ which is in agreement with the conjecture of [26,28,29] for all the orbifolds considered in this paper is new. It is also important to note that a proof for the positivity conjecture of [18] has been made only for the case of a class of Fourier coefficients of the partition function Φ_{10} by [30]. A general proof of the positivity conjecture for Φ_{10} as well as all the cyclic orbifolds considered in this paper is an open question.

The organization of the paper is as follows: In Sec. II, we re derive the twisted elliptic genus for different orbifolds of $K3$ in each sector, We first discuss the orbifolds of $K3$ corresponding to the classes in Table I and then move on to the twisted elliptic genera whose twining character coincides with that of classes $2B$ and $3B$ of Table II. In Sec. III, we use the twisted elliptic genus to construct the Siegel modular forms that capture degeneracies of $1/4$ BPS dyons of type II theories compactified on $(K3\times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as a order N automorphism on $K3$ together with a $1/N$ shift on one of the circles of T^2. We show that low lying coefficients of the $1/4$ BPS index are positive as expected from black hole considerations in Sec. III A. We also observe that the Fourier coefficients of Siegel modular forms of some torus orbifolds constructed in [28] do not satisfy the positivity conjecture. Finally the Appendix lists the twisted elliptic genus for all the conjugacy classes of Table I.

II. TWISTED ELLIPTIC GENUS

In this section we rederive the twisted elliptic genus of the conjugacy classes in Table I and then examine the classes $2B$ and $3B$ from Table II. Among the classes in Table I, the complete elliptic genus for the classes pA with $p=2,3,5,7$ were given in [7]. To quote the result we first define

$$A = \left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right],$$
$$B(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \tag{2.1}$$

and

$$\mathcal{E}_N(\tau) = \frac{12i}{\pi(N-1)}\partial_\tau[\ln\eta(\tau) - \ln\eta(N\tau)]. \tag{2.2}$$

Under $SL(2,\mathbb{Z})$ transformation $A(\tau,z)$ transforms as a weak Jacobi form of weight 0 and index 1 and $B(\tau,z)$ transforms as a weak Jacobi form of weight -2 and index 1. Now $\mathcal{E}_N(\tau)$ transforms as a modular form of weight 2 under the group $\Gamma_0(N)$. Its transformations under T and S transformations of $SL(2,\mathbb{Z})$ are given by

$$\mathcal{E}_N(\tau+1) = \mathcal{E}_N(\tau), \quad \mathcal{E}_N(-1/\tau) = -\frac{\tau^2}{N}\mathcal{E}_N(\tau/N). \tag{2.3}$$

Then the twisted elliptic genera in all the sectors for the classes pA with $p=2,3,5,7$ are given by

$$F^{(0,0)}(\tau,z) = \frac{8}{N}A(\tau,z),$$
$$F^{(0,s)}(\tau,z) = \frac{8}{N(N+1)}A(\tau,z) - \frac{2}{N+1}B(\tau,z)\mathcal{E}_N(\tau), \quad \text{for } 1\le s\le (N-1),$$
$$F^{(r,rk)} = \frac{8}{N(N+1)}A(\tau,z) + \frac{2}{N(N+1)}\mathcal{E}_N\left(\frac{\tau+k}{N}\right)B(\tau,z),$$
$$\text{for } 1\le r\le (N-1), \quad 1\le k\le (N-1). \tag{2.4}$$

Note that rk is defined up to mod N. Here $N = 2, 3, 5, 7$ corresponding to the classes pA respectively. Let us discuss the low lying coefficients in the expansion of $F^{(0,s)}$ which is given by

$$F^{(0,s)}(\tau, z) = \sum_{j\in\mathbb{Z},n=0}^{\infty} c^{(0,s)}(4n - j^2)e^{2\pi in\tau}e^{2\pi ijz}. \qquad (2.5)$$

Then it is easy to see from (2.4) that the low lying coefficients satisfy the following property

$$\sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2. \qquad (2.6)$$

The above set of equations corresponds to the number of (0,0), (0,2), (2,0), (2,2) forms of the pA orbifold of $K3$. As expected, these are the same as $K3$, since the orbifold preserves these forms [27]. The (2,0) and (0,2) forms are holomorphic forms which are required to be preserved if Type II theory compactified on the pA orbifold $(K3 \times T^2)/\mathbb{Z}_N$ needs to be a $\mathcal{N} = 4$ theory. The orbifold preserves the 0-form as well as the top-form of $K3$. In fact the twisted elliptic genus satisfies the stronger property

$$c^{(0,s)}(\pm 1) = \frac{2}{N}, \qquad s = 0, \cdots N - 1. \qquad (2.7)$$

Now we can also see that

$$\sum_{s=0}^{N-1)} c^{(0,0)}(0) = 2\left(\frac{24}{N+1} - 2\right). \qquad (2.8)$$

The last equation corresponds to the number of the (1,1) forms which are reduced from the $K3$ value of 20 to 12,8,4,2 for $N = 2, 3, 5, 7$ respectively. Finally the orbifold action for all these classes on $K3$ produces another $K3$. Therefore, the elliptic genus of $K3/\mathbb{Z}_N$ should be the same as that of $K3$. This implies that we should obtain

$$\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z). \qquad (2.9)$$

Substituting the expressions for the twisted elliptic genus given in (2.4), we see that this is ensured by the following identity satisfied by $\mathcal{E}_N(\tau)$ for N prime.

$$\sum_{s=0}^{N-1} \mathcal{E}_N\left(\frac{\tau + s}{N}\right) - N\mathcal{E}_N(\tau) = 0. \qquad (2.10)$$

The construction of the twisted elliptic genera for classes $11A$ and $23A$ proceeds identically as the case of pA and it satisfies the properties (2.7) and (2.9). The formula in (2.8) still holds, however note that the right-hand side (RHS) yields 0 and -1 for $11A$ and $23A$ respectively which clearly indicates that these constructions are purely formal. We list out the these twisted elliptic genera in the Appendix. In the next section we briefly discuss the class $4B$, as an example of an orbifold with composite order. We then move on to discuss the order 4 and order 9 twisted elliptic genera whose twining character coincides with that of $2B$ and $3B$ classes respectively.

A. Automorphisms g' with composite order and $g' \in M_{23}$

Let us consider automorphisms g' with composite order and those which belong to $M_{23} \subset M_{24}$. Examples of these are the classes $4B$, $6A$, $8A$, $14A$, $15A$ given in Table I. When the order of the automorphism g' is composite, we cannot use the $SL(2,\mathbb{Z})$ modular transformation in (1.2) to arrive at all the sectors of the twisted elliptic genus starting from the twining character. For example for the case of $4B$ which is of the order 4 we cannot reach the sectors (0,2),(2,0),(2,2) starting from the twining character (0,1). We call these sectors suborbits. In general if the order N admits a factorization

$$N = \prod_i n_i \qquad (2.11)$$

then there is a suborbit for each divisor. Since the suborbits are not accessible by modular transformations from the twining character (0,1) one needs to make a choice of a particular character in these sectors. To be more specific, consider the sub-orbit corresponding to the divisor n_i we need to make a choice for the character

$$F^{(0,n_i)}(\tau_i) = \frac{1}{N}\mathrm{Tr}_{RR}[(-1)^{F_{K3}+\bar{F}_{K3}} g'^{n_i} e^{2\pi izF_{K3}} q^{L_0 - c/24} \bar{q}^{\bar{L_0} - \bar{c}/24}]. \qquad (2.12)$$

We will see in the cases for composite orders with $g' \in M_{23} \subset M_{24}$, the cycle shape of g'^{n_i} corresponds to a conjugacy class of order N/n_i. Therefore by appealing to Mathieu moonshine symmetry we can choose for $F^{(0,n_i)}(\tau, z)$, the twining character corresponding to the conjugacy class with the cycle structure of g'^{n_i}. We show that with these choices the construction of the twisted elliptic genera for the remaining conjugacy classes in Table I can be completed. Note again the construction of the twisted ellipitc genera for these conjugacy classes corresponding to composite orders is contained in the general analysis of [24]. We rederive it and outline the methods involved here for our purpose.

1. 4B class

The twining character for the $4B$ conjugacy class is given by

$$F^{(0,1)}(\tau,z)=\frac{A(\tau,z)}{3}-\frac{B(\tau,z)}{4}\left(-\frac{1}{2}\mathcal{E}_2(\tau)+2\mathcal{E}_4(\tau)\right). \tag{2.13}$$

Since the modular forms involved in the twining character is in $\Gamma_0(N)$, the order of the automorphism corresponding to the $4B$ class is 4. Therefore the sectors (0,2), (2,0), (2,2) are not accessible using $SL(2,\mathbb{Z})$ modular transformations. Now the cycle shape of g' in this class is given by $1^4\cdot 2^2\cdot 4^4$ and the cycle shape of g'^2 is given by $1^8\cdot 2^8$. From Table I we see that this cycle shape coincides with the conjugacy class $2A$. Therefore we choose the twisted elliptic genus in the sector (0,2) to be identical to be the $1/2$ the twisting character (0,1) of the $2A$ conjugacy class. The choice of normalization is because we are in an order 4 conjugacy class. We will also show that this normalization results in the expected values for the low lying coefficients of the elliptic genus. Similarly sectors (2,0) and the (2,2) of the $4B$ conjugacy class coincide with $1/2$ the twisted sectors (1,0) and (1,1) of the $2A$ class. The rest of the sectors can be determined by using the relation (1.2) and identities relating expansions in $e^{-2\pi i/\tau}$ to $e^{2\pi i\tau}$. For this we need the following identities

$$\mathcal{E}_2(\tau+1/2)=-\mathcal{E}_2(\tau)+2\mathcal{E}_2(2\tau),$$
$$\mathcal{E}_4(\tau+1/2)=\frac{1}{3}(-\mathcal{E}_2(\tau)+4\mathcal{E}_2(2\tau)). \tag{2.14}$$

One can prove these identities using the definition of $\mathcal{E}_N(\tau)$ in (2.2) together with the equation.

$$\eta\left(\tau+\frac{1}{2}\right)=e^{\pi i/24}\frac{\eta^3(2\tau)}{\eta(\tau)\eta(4\tau)}, \tag{2.15}$$

The identities in (2.14) allow us to obtain the (2,1) or the (2,3) sector from the (1,2). The result for the twisted elliptic genus using these inputs is given by

$$F^{(0,0)}(\tau,z)=2A(\tau,z),$$
$$F^{(0,1)}(\tau,z)=F^{(0,3)}(\tau,z)=\frac{1}{4}\left[\frac{4A}{3}-B\left(-\frac{1}{3}\mathcal{E}_2(\tau)+2\mathcal{E}_4(\tau)\right)\right],$$
$$F^{(1,s)}(\tau,z)=F^{(3,3s)}$$
$$=\frac{1}{4}\left[\frac{4A}{3}+B\left(-\frac{1}{6}\mathcal{E}_2\left(\frac{\tau+s}{2}\right)+\frac{1}{2}\mathcal{E}_4\left(\frac{\tau+s}{4}\right)\right)\right],$$
$$F^{(2,1)}(\tau,z)=F^{(2,3)}=\frac{1}{4}\left(\frac{4A}{3}-\frac{B}{3}(5\mathcal{E}_2(\tau)-6\mathcal{E}_4(\tau)\right),$$
$$F^{(0,2)}(\tau,z)=\frac{1}{4}\left(\frac{8A}{3}-\frac{4B}{3}\mathcal{E}_2(\tau)\right),$$
$$F^{(2,2s)}(\tau,z)=\frac{1}{4}\left(\frac{8A}{3}+\frac{2B}{3}\mathcal{E}_2\left(\frac{\tau+s}{2}\right)\right). \tag{2.16}$$

Note that sector (0,1) is the twining character given by [13–15] for the $4B$ conjugacy class. Using this, the modular transformation property (1.2) and the relations in (2.14) we obtain the sectors (2,1), (2,3). Finally the sectors $(0,2)$, $(2,2s)$ belong to the suborbit which can be identified with the $2A$ class. Note the twisted elliptic genus for this suborbit is $1/2$ of that twisted elliptic genus for the $2A$ class. It is interesting to note that our result in (A6) for the twisted elliptic genus coincides with that obtained in [31]. This was obtained prior to the discovery of the M_{24} symmetry. The approach followed in [31] involved writing down the possible $\Gamma_0(4)$ and $\Gamma_0(2)$ forms allowed in the $(0,s)$ sectors and constraining the coefficients using topological data.

Let us now evaluate the low lying coefficients of the elliptic genus. We have

$$c^{(0,s)}(\pm1)=\frac{1}{2},\qquad s=0,\cdots N-1,$$
$$\sum_{s=0}^{N-1}c^{(0,s)}(\pm1)=2 \tag{2.17}$$

and

$$\sum_{s=0}^{N-1}c^{(0,s)}(0)=6. \tag{2.18}$$

This equation implies that the number of (1,1) forms due to the orbifolding is down to 6 from 20 of the $K3$. This agrees with the analysis of [10] which studies the orbifold of $K3$ dual to the $N=4$ CHL compactification. We can therefore identify the compactification of type II on $(K3\times T^2)/\mathbb{Z}_4$ where $\mathbb{Z}_4$ is the $4A$ automorphism to be dual to the $N=4$ heterotic CHL compactification. Let us now evaluate the full elliptic genus of $K3$ orbifolded by the $4B$ automorphism. This is given by

$$\sum_{r,s=0}^{N-1}F^{(r,s)}(\tau,z)=8A(\tau,z). \tag{2.19}$$

To show this we substitute the twisted elliptic genus given in (A6) along with the identity in (2.10) and finally use the relation

$$\frac{1}{4}\sum_{s=0}^{3}\mathcal{E}_4\left(\frac{\tau+s}{4}\right)=\mathcal{E}_2(\tau). \tag{2.20}$$

B. The conjugacy classes 2*B* and 3*B*

In this section we will consider the twining character $2B$ and $3B$ and write down twisted elliptic genera whose twining character coincides with that of the $2B$ and $3B$ class of Mathieu moonshine. However as mentioned in the introduction, the twisted elliptic genera does not correspond to any orbifold action on $K3$. In fact the twisted elliptic genus we write down whose twining character coincides with that of $2B$ involves a order 4 action. This has been constructed in [25] using an explicit rational conformal field theory consisting of 6 $SU(2)$ Wess-Zumino-Witten (WZW) models at level 1 which realizes $K3$. The action of the orbifold by g' is explicitly realized in this CFT. It was observed that this orbifold satisfied the property called quantum symmetry

$$\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau,z) = 0. \tag{2.21}$$

In this section starting from the twining characters for the $2B$ and $3B$ conjugacy class given in [25] we determine all the sectors of the twisted elliptic genus. This is done by assuming quantum symmetry together with the following condition on the low lying coefficients of the twisted elliptic genus

$$\sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2 \tag{2.22}$$

where order N is 4,9 for the $2B$ and $3B$ conjugacy class respectively. As we have discussed earlier, the above condition on the low lying coefficients of the twisted elliptic genus ensures that the type II theory compactified on $(K_3 \times T^2)/\mathbb{Z}_N$ preserves $\mathcal{N} = 4$ supersymmetry.

1. An order 4 orbifold

An explicit realization of an orbifold action of order 4 on $K3$ was given in [25] in which $K3$ is realized a rational CFT consisting of 6 $SU(2)$ WZW models at level 1. Rather than use this realization, we will start from the twining character given in [13–15] for the $2B$ conjugacy class

$$F^{(0,1)}(\tau,z) = \frac{B(\tau,z)}{2}(\mathcal{E}_2(\tau) - \mathcal{E}_4(\tau)) \tag{2.23}$$

Note that this is distinct from the classes belonging to Table I in that it does not have any component of the weak Jacobi form $A(\tau,z)$. It is clear form the structure of the twining character, the $2B$ automorphism is of the order 4. Using the modular transformations (1.2) together with the identities in (2.14), we can determine the elliptic genus in the following sectors to be given by

$$\begin{aligned}
F^{(0,1)}(\tau,z) &= F^{(0,3)}(\tau,z),\\
F^{(1,s)}(\tau,z) &= F^{(3,3s)}(\tau,z)\\
&= -\frac{B(\tau,z)}{4}\left(\mathcal{E}_2\left(\frac{\tau+s}{2}\right) - \mathcal{E}_4\left(\frac{\tau+s}{4}\right)\right),\\
F^{(2,1)}(\tau,z) &= F^{(2,3)}(\tau,z) = \frac{B(\tau,z)}{2}\left(-\frac{1}{6}\mathcal{E}_2(\tau) + \frac{2}{3}\mathcal{E}_2(2\tau)\right).
\end{aligned} \tag{2.24}$$

The remaining sectors (0,2), (2,0), (2,2) belong to a suborbit. To determine the structure of the elliptic genus in this suborbit let us first focus on the (0,2) sector. We assume that is a $\Gamma_0(2)$ weak Jacobi form. Thus it can be written as

$$F^{(0,2)}(\tau,z) = \alpha A(\tau,z) + \beta B(\tau,z)\mathcal{E}_2(\tau), \tag{2.25}$$

where α, β are undetermined constants. Now the sectors (2,0) and (2,2) can be determined using the modular transformations (1.2) to be

$$F^{(2,2s)}(\tau,z) = \alpha A(\tau,z) - \frac{\beta}{2}\mathcal{E}_2\left(\frac{\tau+s}{2}\right). \tag{2.26}$$

Imposing the Eqs. (2.21) and (2.22) we obtain

$$\alpha = \beta = -\frac{2}{3}. \tag{2.27}$$

To summarize the twisted elliptic genus for the order 4 orbifold is given by

$$\begin{aligned}
F^{(0,0)}(\tau,z) &= 2A; \qquad F^{(0,1)}(\tau,z) = F^{(0,3)}(\tau,z),\\
F^{(0,1)}(\tau,z) &= \frac{B(\tau,z)}{2}(\mathcal{E}_2(\tau) - \mathcal{E}_4(\tau)),\\
F^{(0,2)}(\tau,z) &= -\frac{2A(\tau,z)}{3} - \frac{2B(\tau,z)}{3}\mathcal{E}_2(\tau),\\
F^{(1,s)}(\tau,z) &= F^{(3,3s)}\\
&= -\frac{B(\tau,z)}{4}\left(\mathcal{E}_2\left(\frac{\tau+s}{2}\right) - \mathcal{E}_4\left(\frac{\tau+s}{4}\right)\right),\\
F^{(2,1)}(\tau,z) &= F^{(2,3)} = \frac{B(\tau,z)}{2}\left(-\frac{1}{6}\mathcal{E}_2(\tau) + \frac{2}{3}\mathcal{E}_2(2\tau)\right),\\
F^{(2,2s)}(\tau,z) &= -\frac{2A(\tau,z)}{3} + \frac{B(\tau,z)}{3}\mathcal{E}_2\left(\frac{\tau+s}{2}\right).
\end{aligned} \tag{2.28}$$

We have also evaluated the complete twisted elliptic genus using the explicit rational CFT realization of this orbifold in [25] and have verified that it agrees with that given in (2.28). Note that the twining character of the orbifold coincides with the $2B$ conjugacy class. Evaluating the low lying coefficient corresponding to the invariant (1,1) forms of $K3$ we obtain

$$\sum_{s=0}^{3} c^{(0,s)}(0) = 0. \tag{2.29}$$

Therefore, as expected this twisted ellipitc genus is purely a formal construction.

2. An order 9 orbifold

Let us consider the twining character of the $3B$ conjugacy class which is given by

$$F^{(0,1)}(\tau,z) = -\frac{2B(\tau,z)}{9}\frac{\eta^6(\tau)}{\eta^2(3\tau)} \tag{2.30}$$

Note that this is invariant under $\Gamma_0(9)$, therefore we look for a twisted elliptic genus corresponding to an order 9 orbifold. Thus the following sectors

$$\begin{aligned}&(0,3),(0,6);(3,0),(3,3),(3,6);\\&(6,0),(6,3),(6,6)\end{aligned} \tag{2.31}$$

forms a suborbit under $Sl(2,\mathbb{Z})$ modular transformations. The remaining sectors can be obtained from the twining character in (2.30) by using the transformation (1.2), together with the modular properties of the η function. Once that is obtained we assume the following Jacobi weak form of $\Gamma_0(3)$ for the (0,3) sector of the (2.31).

$$F^{(0,3)}(\tau,z) = \alpha A(\tau,z) + \beta B(\tau,z)\mathcal{E}_3(\tau). \tag{2.32}$$

Here α, β are undetermined constants. Then using modular transformation (1.2) and the identities satisfied by $\Gamma_0(3)$ forms we can obtain the twisted elliptic genus in the suborbit. Finally imposing the conditions (2.21) and (2.22) we determine the constants α, β as

$$\alpha = -\frac{1}{9}, \qquad \beta = \frac{1}{4}. \tag{2.33}$$

Using all these steps we obtain the twisted elliptic genus for the $3B$ conjugacy class to be given by

$$\begin{aligned}
F^{(0,0)}(\tau,z) &= \frac{8A(\tau,z)}{9},\\
F^{(0,1)}(\tau,z) &= F^{(0,2)} = F^{(0,4)} = F^{(0,5)} = F^{(0,7)} = F^{(0,8)};\\
F^{(0,1)}(\tau,z) &= -\frac{2B(\tau,z)}{9}\frac{\eta^6(\tau)}{\eta^2(3\tau)},\\
F^{(0,3)}(\tau,z) &= -\frac{A(\tau,z)}{9} - \frac{B(\tau,z)}{4}\mathcal{E}_3(\tau),\\
F^{(r,rs)}(\tau,z) &= \frac{2B(\tau,z)}{3}\frac{\eta^6(\tau+s)}{\eta^2(\frac{\tau+s}{3})}, \qquad r = 1,2,4,5,7,8\\
F^{(3,1)}(\tau,z) &= -\frac{2B(\tau,z)}{9}e^{2\pi i/3}\frac{\eta^6(\tau)}{\eta^2(3\tau)},\\
&= F^{(3,4)} = F^{(3,7)} = F^{(6,2)} = F^{(6,8)} = F^{(6,5)};\\
F^{(3,2)}(\tau,z) &= -\frac{2B(\tau,z)}{9}e^{4\pi i/3}\frac{\eta^6(\tau)}{\eta^2(3\tau)},\\
&= F^{(3,5)} = F^{(3,8)} = F^{(6,1)} = F^{(6,7)} = F^{(6,4)};\\
F^{(3r,3rk)}(\tau,z) &= -\frac{A(\tau,z)}{9} + \frac{B(\tau,z)}{12}\mathcal{E}_3\left(\frac{\tau+k}{3}\right).
\end{aligned} \tag{2.34}$$

The number of (1,1) forms is given by consider the following low lying coefficients of the twisted elliptic genus

$$\sum_{s=0}^{8} c^{(0,s)}(0) = -2 \tag{2.35}$$

Again the orbifold of $K3$ by the order 9 action is formal.

The rest of the conjugacy classes in Table II have more than one suborbit. Quantum symmetry given in (2.21) and the supersymmetry condition (2.22) is not enough to determine the unknown constants in these suborbits. It will be interesting to determine the twisted elliptic genera for all the remaining conjugacy classes of Table II.

C. Comparision with literature

As remarked earlier the work of [24] provides the mathematical justification for the construction of the twisted elliptic genus over for all the cyclic orbifolds considered in this paper. To compare with the rederivation of the twisted elliptic genera in this paper, let us briefly review their construction. Let g' be the cyclic orbifold corresponding to the conjugacy class of M_{24} with order N. Then the twisted elliptic genus admits the following decomposition in terms of the characters of the $\mathcal{N}=4$ superconformal algebra with central charge $c=6$

$$F^{(r,s)}(\tau,z) = \sum_{k=n+\frac{r}{N}\geq 0}^{\infty} \mathrm{Tr}_{\mathcal{H}_{g'^r,k}}(\rho_{g'^r,k}(g'^s))\mathrm{ch}_{h=\frac{1}{4}+k,l}(\tau,z) \tag{2.36}$$

Note that $l=\frac{1}{2}$ except when $h=1/4$ for which both $l=\frac{1}{2}$ and $l=0$ are understood to be present in the sum. The vector space $\mathcal{H}_{g'^r,k}$ is finite dimensional and is the projective representation of the centralizer $C_{M_{24}}(g')$ which satisfies properties detailed in [24]. Thus the problem of determining the twisted twining elliptic genera reduces to determing characters of the projective representations. Though not easy to extract from the ancillary files provided along with [24], a careful examination of the files lists out some of the twisted twining elliptic genera for the orbifolds considered in the paper. The $F^{(1,s)}$ sector which is listed out in the ancillary files. The files also enable the evaluation of the characters of the projective representations and a verification of the expansion of the twisted elliptic genera as given in (2.36). Though explicit expressions are not listed in the main body of the paper, this is sufficient to construct the twisted elliptic genera in all the sectors easily for the case when the order the orbifold is prime.[2]

Note that explicit formulas for the twisted elliptic genera for pA orbifolds with $p = 2, 3, 5, 7$ were known even before the discovery of moonshine symmetry in [7] and before the work of [24]. Since the latter paper as well the present work uses modular transformations (1.2) to obtain

[2]We have been informed by Mathias Gaberdiel that these explicit expressions were known to the authors of [24] however they did not write them out in the body of their paper.

the twisted elliptic genera we are assured that both the constructions agree. In our discussion in the Appendix we have also explicitly compared the low lying coefficients of the twisted elliptic genera for the case of $4B$, $6A$, $8A$ orbifolds with the Hodge numbers of the CHL compactifictaions discussed in [6,10] and found agreement. The check that sum over all the sectors of the orbifolds when $g' \in M_{23}$ yields back the elliptic genus of $K3$ was also performed in [24] as can be read from the discussion in the text.[3] In this work this is assured by the identities of the kind given (2.10).

As we have discussed earlier, when the order of the orbifold is composite, there are suborbits in the twisted sectors which cannot be reached by modular transformations from the twining character. We have used moonshine symmetry to determine the twisted elliptic genus in these sectors. The treatment of such situations in [24] is more general. Their discussion also encompases orbifolds by non-cyclic groups. Though not treated explicitly, the case of the cyclic orbifolds is implicit in their discussion.[4]

III. 1/4 BPS DYON PARTITION FUNCTIONS

Given the twisted elliptic genus one can construct a Siegel modular form as follows [27]. The twisted elliptic genus can be expanded as

$$F^{(r,s)}(\tau,z) = \sum_{b=0}^{1} \sum_{j\in 2\mathbb{Z}+b,\, n\in \mathbb{Z}/N} c_b^{(r,s)}(4n - j^2) e^{2\pi i n\tau + 2\pi i j z}. \tag{3.1}$$

Then a Siegel modular form associated with the twisted elliptic genus is given by

$$\begin{aligned}\tilde{\Phi}(\rho,\sigma,v) &= e^{2\pi i(\tilde{\alpha}\rho+\tilde{\beta}\sigma+v)} \\ &\times \prod_{b=0,1}\prod_{r=0}^{N-1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},\, l\in\mathbb{Z},\\ j\in 2\mathbb{Z}+b\\ k',l\geq 0,\, j<0\, k'=l=0}} (1-e^{2\pi i(k'\sigma+l\rho+jv)})^{\sum_{s=0}^{N-1} e^{2\pi i s l/N} c_b^{r,s}(4k'l-j^2)}.\end{aligned} \tag{3.2}$$

where

$$\tilde{\beta} = \frac{1}{24N}\chi(M),$$
$$\tilde{\alpha} = \frac{1}{24N}\chi(M) - \frac{1}{2N}\sum_{s=0}^{N-1} Q_{0,s}\frac{e^{-2\pi i s/N}}{1-e^{2\pi i s/N}},$$
$$Q_{r,s} = N(c_0^{r,s}(0) + 2c_1^{r,s}(-1)). \tag{3.3}$$

[3]See discussion in the beginning of Sec. IV of [24].

[4]We thank Mathias Gaberdiel for correspondence which enabled us to compare our work with [24].

Evaluating $\tilde{\alpha}, \tilde{\beta}$ for the twisted elliptic genus corresponding to all the conjugacy classes considered in the previous section as well as the pA classes with $p = 1, 2, 3, 5, 7$ we obtain

$$\tilde{\alpha} = 1, \qquad \tilde{\beta} = \frac{1}{N}. \tag{3.4}$$

Here N is the order of the orbifold action. This Siegel modular form in (3.2) transforms as a weight k form under appropriate subgroups of $Sp(2,\mathbb{Z})$. The weight k is related to the low lying coefficients of the twisted elliptic genus and is given by

$$k = \frac{1}{2}\sum_{0}^{N-1} c_0^{0,s}(-1). \tag{3.5}$$

The weights of the Siegel modular forms corresponding to the twisted elliptic genera constructed in this paper is listed in Tables III and IV. Now consider type II theory compactified on $(K3\times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as the automorphism g' belonging to any of the conjugacy classes together with a $1/N$ shift along one of the circles of T^2, S^1. Then by the analysis in [27], the generating function of the index of 1/4 BPS states in this theory is given by $1/\tilde{\Phi}(\rho,\sigma,v)$. Let us work in the dual heterotic frame in which the orbifolded heterotic theory is compactified in general on T^6. For example the cases of the pA orbifolds of $K3\times T^2$ with $p = 2, 3, 4, 5, 6, 7, 8$ corresponds to the $N = 2, 3, 4, 5, 6, 7, 8$ CHL compactifications on the heterotic side. Let us label the charges of the 1/4 BPS state by (Q, P) corresponding to the electric and magnetic charge of the dyon. Let Q^2, P^2 and $Q\cdot P$ denote the continuous T-duality invariants in this duality frame. Then the 1/4 BPS index in this frame is given by

$$-B_6(Q,P) = \frac{1}{N}(-1)^{Q\cdot P+1}\int_C d\rho d\sigma dv e^{-\pi i(N\rho Q^2 \sigma P^2/N + 2vQ\cdot P)} \times \frac{1}{\tilde{\Phi}(\rho,\sigma,v)}. \tag{3.6}$$

TABLE III. Weight of Siegel modular forms corresponding to classes in M_{23}.

Type 1	pA	4B	6A	8A	14A	15A
Weight	$\frac{24}{p+1} - 2$	3	2	1	0	0

TABLE IV. Weight of Siegel modular forms corresponding to the classes $\notin M_{23}$.

Type 2	2B	3B
Weight	0	−1

TABLE V. Factorization of $\tilde{\Phi}_k(\rho, \sigma, v)$ as $\lim v \to 0$ as shown in (3.10), $p \in \{1, 2, 3, 5, 7, 11, 23\}$.

Conjugacy class	k	$f^{(k+2)}(\rho)$	$g^{(k+2)}(\sigma)$
pA	$\frac{24}{p+1} - 2$	$\eta^{k+2}(\rho)\eta^{k+2}(p\rho)$	$\eta^{k+2}(\sigma)\eta^{k+2}(\sigma/p)$
4B	3	$\eta^4(4\rho)\eta^2(2\rho)\eta^4(\rho)$	$\eta^4(\frac{\sigma}{4})\eta^2(\frac{\sigma}{2})\eta^4(\sigma)$
6A	2	$\eta^2(\rho)\eta^2(2\rho)\eta^2(3\rho)\eta^2(6\rho)$	$\eta^2(\sigma)\eta^2(\frac{\sigma}{2})\eta^2(\frac{\sigma}{3})\eta^2(\frac{\sigma}{6})$
8A	1	$\eta^2(\rho)\eta(2\rho)\eta(4\rho)\eta^2(8\rho)$	$\eta^2(\sigma)\eta(\frac{\sigma}{2})\eta(\frac{\sigma}{4})\eta^2(\frac{\sigma}{8})$
14A	0	$\eta(\rho)\eta(2\rho)\eta(7\rho)\eta(14\rho)$	$\eta(\sigma)\eta(\frac{\sigma}{2})\eta(\frac{\sigma}{7})\eta(\frac{\sigma}{14})$
15A	0	$\eta(\rho)\eta(3\rho)\eta(5\rho)\eta(15\rho)$	$\eta(\sigma)\eta(\frac{\sigma}{3})\eta(\frac{\sigma}{5})\eta(\frac{\sigma}{15})$
2B	0	$\frac{\eta^8(\rho)}{\eta^4(2\rho)}$	$\frac{\eta^8(\sigma)}{\eta^4(\frac{\sigma}{2})}$
3B	−1	$\frac{\eta^3(\rho)}{\eta(3\rho)}$	$\frac{\eta^3(\sigma)}{\eta(\frac{\sigma}{3})}$

The contour $\mathcal{C}$ is defined over a 3 dimensional subspace of the 3 complex dimensional space ($\rho{=}\rho_1{+}i\rho_2, \sigma{=}\sigma_1{+}i\sigma_2, v{=}v_1{+}iv_2$).

$$\begin{aligned} &\rho_2 = M_1, \qquad \sigma_2 = M_2, \qquad v_2 = -M_3, \\ &0 \le \rho_1 \le 1, \qquad 0 \le \sigma_1 \le N, \qquad 0 \le v_1 \le 1. \end{aligned} \tag{3.7}$$

The choice of (M_1, M_2, M_3) is determined by the domain in which one needs to evaluate the index $-B_6$ [32,33]. We pick up the Fourier coefficients by expanding $1/\tilde{\Phi}$ in powers of $e^{2\pi i\rho}$, $e^{2\pi i\sigma}$ and $e^{-2\pi iv}$. For this expansion to make sense we must have [27,32]

$$M_1, M_2 \gg 0, \qquad M_3 \ll 0, \qquad |M_3| \ll M_1, M_2. \tag{3.8}$$

Since this is an index, the Fourier coefficient $-B_6$ must be an integer. Let us now focus on 1/4 BPS states which are single centered black holes. Then from the fact that the single centered black holes carry zero angular momentum, it is predicted that the index $-B_6$ for these black holes is positive [18]. The argument for this goes as follows. Given the domain (3.8), these 1/4 BPS black states have regular event horizons and are single centered only if the charges satisfy the condition [18]

$$Q \cdot P \ge 0, \qquad (Q \cdot P)^2 < Q^2 P^2, \qquad Q^2, P^2 > 0. \tag{3.9}$$

Thus if we can show that the index $-B_6$ is positive for states satisfying the condition (3.9), then it will imply the $-B_6$ is positive for single centered 1/4 BPS dyons as predicted from black hole considerations. In the next section we show that for low lying charges satisfying (3.9), $-B_6$ is indeed positive for all the Siegel modular forms associated with the twisted elliptic genera constructed in this paper. This is the generalization of the observation seen first in [18] for the pA conjugacy classes with $p = 1, 2, 3, 5, 7$. For $p = 1$ and for a special class of charges it was proved that the coefficient $-B_6$ is positive [30].

Before we proceed we will study two properties of the Siegel modular forms which are theta lifts of the twisted elliptic genera constructed in this paper. First the Siegel modular forms factorize in the $v \to 0$ limit as

$$\lim_{v\to 0} \tilde{\Phi}_k(\rho, \sigma, v) \sim v^2 f^{(k+2)}(\rho) g^{(k+2)}(\sigma). \tag{3.10}$$

where $f^{(k+2)}$, $g^{(k+2)}$ are weight $k + 2$ modular forms transforming under $\Gamma_0(N)$. The explicit modular forms on which the $\Phi_k' s$ factorize are given in Table V. The function $1/f^{(k+2)}(\rho)$ is the partition function of purely electric states while $1/g^{(k+2)}(\sigma)$ is the partition function of purely magnetic states. In fact $f^{(k+2)}$ and $g^{(k+2)}$ are related to each other by a S transformation.

The second property we discuss is the asymptotic property of the index in (3.6) when the charges Q, P are equally large. The procedure to obtain the asymptotic behaviour has been developed in [4,27,34], which we summarize briefly.[5] Consider another Siegel modular form $\hat{\Phi}(\rho, \sigma, v)$ of weight k associated with the twisted elliptic genus defined by

$$\hat{\Phi}(\rho, \sigma, v) = e^{2\pi i(\hat{\alpha}\rho + \hat{\beta}\sigma + v)} \prod_{b=0,1} \prod_{r=0}^{N-1} \prod_{\substack{k', l \in \mathcal{Z}, \\ j \in 2\mathcal{Z}+b \\ k', l \ge 0, j<0 k'=l=0}} (1 - e^{2\pi ir/N} e^{2\pi i(k'\sigma + l\rho + jv)})^{\sum_{s=0}^{N-1} e^{-2\pi isr/N} c_b^{0,s}(4k'l - j^2)}. \tag{3.11}$$

Here we have,

$$\hat{\beta} = \hat{\alpha} = \hat{\gamma} = \frac{1}{24}\chi(M) = 1. \tag{3.12}$$

Under $v \to 0$, this modular form factorizes symmetrically in ρ and σ as

[5] We follow the discussion in [27].

TABLE VI. Factorization of $\hat{\Phi}_k(\rho,\sigma,v)$ as $\lim v\to 0$ as shown in (3.13), $p\in\{1,2,3,5,7,11,23\}$.

Conjugacy class	$h^{(k+2)}(\rho)$
pA	$\eta^{k+2}(\rho)\eta^{k+2}(p\rho)$
4B	$\eta^4(4\rho)\eta^2(2\rho)\eta^4(\rho)$
6A	$\eta^2(\rho)\eta^2(2\rho)\eta^2(3\rho)\eta^2(6\rho)$
8A	$\eta^2(\rho)\eta(2\rho)\eta(4\rho)\eta^2(8\rho)$
14A	$\eta(\rho)\eta(2\rho)\eta(7\rho)\eta(14\rho)$
15A	$\eta(\rho)\eta(3\rho)\eta(5\rho)\eta(15\rho)$
2B	$\frac{\eta^8(4\rho)}{\eta^4(2\rho)}$
3B	$\frac{\eta^3(9\rho)}{\eta(3\rho)}$

$$\lim_{v\to 0}\hat{\Phi}(\rho,\sigma,v)\sim v^2 h^{(k+2)}(\rho)h^{(k+2)}(\sigma). \tag{3.13}$$

Then the leading behaviour of the index $-B_6$ is given by

$$-B_6(Q,P)\sim\exp(-S(Q,P)) \tag{3.14}$$

where $S(Q,P)$ is obtained by minimizing the function

$$-S(Q,P)=\frac{\pi}{2\tau_2}|Q^2+\tau P^2|^2-\ln(h^{(k+2)}(\tau))\\ -\ln(h^{(k+2)}(-\bar{\tau})-(k+2)\ln(2\tau_2) \tag{3.15}$$

with respect to τ_1, τ_2. The minimum lies at

$$\tau_1=\frac{Q\cdot P}{P^2},\qquad \tau_2=\frac{1}{P^2}\sqrt{Q^2P^2-(Q\cdot P)^2}. \tag{3.16}$$

Substituting the above values for τ_1, τ_2 results in the asymptotic behavior of the index $-B_6$. The list of the $\Gamma_0(N)$ modular forms for the models constructed in this paper is provided in Table VI.

Let us now compare this to the behavior of the entropy of single centered large charge $1/4$ BPS dyons in these $\mathcal{N}=4$ theories obtained compactifying type II theory on $(K3\times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as the automorphisms on $K3$ together with a $1/N$ shift on one of the circles of T^2. Apart from the usual 2 derivative terms in the effective action, a one loop computation shows that the coefficient of the Gauss-Bonnet term is given by

$$\Delta\mathcal{L}=\phi(a,S)(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2). \tag{3.17}$$

where a, S is the axion and dilaton moduli in the heterotic frame. The function $\phi(a,S)$ is given by

$$\phi(a,S)=-\frac{1}{64\pi^2}((k+2)\ln S+\ln h^{(k+2)}(a+iS)\\ +\ln h^{(k+2)}(-a+iS)). \tag{3.18}$$

It is important to note that the $\Gamma_0(N)$ modular form $h^{(k+2)}(\tau)$ for each of the compactifications is identical to the $\Gamma_0(N)$ form that occurs in the factorization (3.13) [27]. Now evaluating the Hawking-Bekenstein-Wald entropy including the correction due to the Gauss-Bonnet term using the entropy function formalism leads to the following minimizing problem. The entropy is given by minimizing the function

$$\mathcal{E}(Q,P)=\frac{\pi}{2\tau_2}|Q^2+\tau P^2|^2-\ln h^{(k+2)}(\tau)\\ -\ln h^{(k+2)}(-\tau)-(k+2)\ln(2\tau_2). \tag{3.19}$$

Here $\tau=a+iS$. The entropy function is identical to the statistical entropy function (3.15) which occurred while obtaining the asymptotic behavior of $-B_6$. Thus the partition function $1/\tilde{\Phi}(\rho,\sigma,v)$ captures the degeneracy of large charge single centered $1/4$ BPS black holes in these class of $\mathcal{N}=4$ compactifications including the correction from the Gauss-Bonnet term.

The construction of the Siegel modular form given the coefficient of the twisted ellipitc genus of $K3$ is quite straightforward and for cyclic orbifolds, this was first given in [27]. Recently the Refs. [26,28,29] extend it for noncyclic orbifolds and also study its $v\to 0$ limit. It is important to emphasize the there are 2 modular forms associated with the twisted ellipitc genus of $K3$. The $\tilde{\Phi}_k$ and the $\hat{\Phi}_k$ are constructed in equations (3.2) and (3.11) respectively. The Fourier expansion of the inverse of $\tilde{\Phi}_k$ capture the degeneracy of the $1/4$ BPS dyon and its zeros at $v\to 0$ are associated with the walls of marginal stability of the dyon. The zero's of $\hat{\Phi}_k$ are however associated with the asymptotic growth of the degeneracies for large charges. We mention that $\hat{\Phi}_k$ has not been constructed for the orbifolds listed in this paper in the Refs. [26,28,29]. We emphasize that our objective in constructing the Siegel modular form $\tilde{\Phi}_k$ in particular is to verify that the Fourier expansions of the inverse of these forms are integers and positive as predicted by the conjecture of [18]. This was verified earlier by [18] for the Siegel modular forms $\tilde{\Phi}_k$, associated with the pA, $p=1$, 2, 3, 5, 7 orbifolds. In this next section we extend this observation for all the orbifolds discussed in this paper. To our knowledge, this observation has not been seen in the works of [26,28,29]. We also emphasize that to obtain this observation the explicit construction of the twisted elliptic genus in all its sectors together with the normalizations as discussed earlier is important.

A. Positivity of the $1/4$ BPS index

In this subsection we provide the list for the index $-B_6$ for low lying charges for all the Siegel modular forms $\tilde{\Phi}_k$ associated with the twisted elliptic genera constructed. From the expansion of $\tilde{\Phi}_k$ in Fourier coefficients in the domain (3.8) together with the expression for $-B_6$ in (3.6) we see that the electric charge Q^2 is quantized in units of

TABLE VII. Some results for the index $-B_6$ for the $4B$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(1/2, 2)	−512	176	8	0	0
(1/2, 4)	−1536	896	80	0	0
(1/2, 6)	−4544	3616	480	0	0
(1/2, 8)	11 752	12 848	2176	24	0
(1,4)	−4592	5024	832	16	0
(1,6)	−13408	22 464	36 786	224	0
(1,8)	−33568	88 320	26 176	1760	0
(3/2, 6)	−37330	1 12 316	36 786	2998	38
(3/2, 8)	−80896	4 91 920	1 96 960	23 616	592

TABLE VIII. Some results for the index $-B_6$ for the $6A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(1/3, 2)	−98	40	1	0	0
(1/3, 4)	−224	148	12	0	0
(1/3, 6)	−546	478	49	0	0
(1/3,8)	−1120	1352	186	0	0
(2/3, 4)	−512	592	92	0	0
(2/3, 6)	−1240	2080	436	8	0
(2/3, 8)	−2504	6416	1676	0	0
(1, 6)	−2926	7880	2172	116	0
(1, 10)	−2450	81 380	32 300	3494	49
(1, 12)	−4696	2 34 900	1 04 176	13 856	316

$(2/N)\mathbb{Z}$, while the magnetic charge P^2 is quantized in units of $2\mathbb{Z}$ and the angular momentum $Q \cdot P$ is an integer. We see that the index $-B_6$ for the low lying charges examined is always an integer. Furthermore for charges satisfying the condition (3.9) it is positive. This property is a sufficient condition which ensures that single centered black holes carry zero angular momentum. One important point to emphasize is that it is possible to obtain the Fourier expansion of the Siegel modular forms for low lying charges only after the explicit construction of the twisted elliptic genus. The index $-B_6$ is listed in Tables VII–XV. As a check on our *Mathematica* routines to obtain these Fourier coefficients, we have verified that our routine reproduces all the tables given in [18] for the pA orbifold of $K3$ with $p = 1, 2, 3, 5, 7$.

TABLE IX. Some results for the index $-B_6$ for the $8A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(1/4, 2)	−60	20	0	0	0
(1/4, 4)	−120	68	2	0	0
(1/4, 6)	−280	196	10	0	0
(1/4, 8)	−520	504	40	0	0
(1/2, 6)	−560	724	96	0	0
(1/2,8)	−1038	1998	352	2	0
(3/4, 6)	−1114	2280	450	6	0
(3/4, 8)	−2024	6704	1728	56	0
(3/4, 10)	−3860	18 256	5564	300	0
(3/4, 12)	−6168	46 456	16 296	1192	4

TABLE X. Some results for the index $-B_6$ for the $11A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(2/11, 2)	−50	10	0	0	0
(2/11, 4)	−100	30	0	0	0
(2/11, 6)	−200	82	1	0	0
(4/11, 6)	−400	276	18	0	0
(6/11, 6)	−800	806	83	0	0
(6/11, 8)	−1438	2064	314	2	0
(6/11, 10)	−2584	4962	937	16	0
(6/11, 12)	−4328	11 132	2558	72	0
(6/11, 22)	−34000	3 66 378	1 39 955	12 760	114

TABLE XI. Some results the index $-B_6$ for the $14A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(1/7, 2)	−18	4	0	0	0
(1/7, 4)	−24	10	0	0	0
(1/7, 6)	−54	24	0	0	0
(2/7, 6)	−72	70	5	0	0
(2/7, 8)	−96	156	16	0	0
(3/7, 8)	−216	406	65	0	0
(3/7, 10)	−412	890	165	2	0
(4/7, 12)	−710	4682	1443	58	0
(5/7, 12)	−1180	11 512	4156	292	0
(5/7, 14)	−1622	24 744	9816	908	5

It is interesting to note that the nongeometric orbifolds $11A, 14A/B, 15A/B, 23A/B$ and the order 4 and order 9 orbifold also satisfy the positivity constraints conjectured by [18], see Tables X–XV. We have attached the *Mathematica* files which generate the Fourier coefficients for the $11A$ and $3B$ orbifolds as ancillary files.

1. Torus orbifolds

Let us examine if the Fourier coefficients of the inverse of the Siegel modular forms corresponding to the torus orbifolds constructed recently in [28] also satisfy the positivity conjecture of [18]. These orbifolds realize quantum symmetry. The twisted ellipitc genus of the $N = 2$ orbifold is given by

$$F^{(0,0)} = 4A(\tau, z), \qquad F^{(0,1)} = -\frac{4}{3}A(\tau, z) - \frac{4}{3}\mathcal{E}_2(\tau)B(\tau, z),$$

$$F^{(1,0)} = -\frac{4}{3}A(\tau, z) + \frac{2}{3}\mathcal{E}_2\left(\frac{\tau}{2}\right)B(\tau, z),$$

$$F^{(1,1)} = -\frac{4}{3}A(\tau, z) + \frac{2}{3}\mathcal{E}_2\left(\frac{\tau+1}{2}\right)B(\tau, z). \tag{3.20}$$

TABLE XII. Some results for the index $-B_6$ for the 15A orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(2/15, 2)	−8	4	0	0	0
(2/15, 4)	−16	8	0	0	0
(2/15, 6)	−24	20	0	0	0
(2/5, 8)	−120	274	45	0	0
(2/5, 10)	−203	578	113	1	0
(4/15, 6)	−48	50	4	0	0
(4/15, 8)	−80	102	13	0	0
(8/15, 12)	−440	2844	898	40	0
(2/3, 12)	−638	6818	2498	178	0
(4/5, 18)	8236	1 41 252	73 651	12 124	419

TABLE XIII. Some results for the index $-B_6$ for the 23A orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(2/23, 2)	−8	1	0	0	0
(2/23, 4)	−12	3	0	0	0
(2/23, 6)	−20	7	1	0	0
(4/23, 6)	−30	53	6	0	0
(4/23, 8)	−42	91	11	0	0
(6/23, 6)	−48	103	23	2	0
(6/23, 8)	−66	190	47	4	0
(6/23, 10)	−104	312	74	6	0

TABLE XIV. Some results for the index $-B_6$ for the order 4 orbifold whose twining character equals 2B Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(1/2, 2)	320	288	24	0	0
(1/2, 4)	0	512	256	0	0
(1/2, 6)	−752	1120	888	48	0
(1/2, 8)	384	3328	2048	384	0
(1,4)	32	4416	2240	32	0
(1,6)	−2304	22 464	13 248	224	0
(1,8)	5920	42 944	27 328	5920	64
(3/2, 6)	−2008	1 02 380	66 172	9032	28
(3/2, 8)	59 392	3 72 736	2 43 712	59 392	2048

Here we have rewritten the twisted elliptic genus given in terms of Jacobi θ functions given in [28][6] in the standard form involving $\Gamma_0(2)$ modular forms. We have also normalized the twisted elliptic genus so that the low lying coefficients of the elliptic genus satisfy (2.6). Using the expression (3.2) to write down the corresponding Seigel modular forms, we can extract the low lying coefficients of the putative dyon degeneracies. These are listed in Table XVI.

From the analysis of [18], for $N = 2$ orbifolds, the Fourier coefficients of the putative dyon degeneracies are expected to be positive when

[6]See Eqs. (6.1) to (6.4) in [28].

TABLE XV. Some results for the index $-B_6$ for the order 9 orbifold whose twining character equals 3B for different values of Q^2, P^2 and $Q \cdot P$.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2	3
(2/9, 2)	0	18	0	0	0
(2/9, 4)	18	27	0	0	0
(2/9, 6)	0	78	21	0	0
(4/9, 4)	42	150	33	0	0
(4/9, 6)	0	270	81	0	0
(4/9, 8)	0	378	162	0	0
(2/3, 6)	0	918	297	0	0
(2/3, 8)	0	2460	1239	93	0

$$\begin{aligned} &Q\cdot P \geq 0, \quad Q\cdot P \leq 2Q^2, \quad Q\cdot P \leq P^2, \\ &3Q\cdot P \leq 2Q^2 + P^2, \\ &\qquad Q^2, \quad P^2, \quad \{Q^2P^2 - (Q\cdot P)^2\} > 0. \end{aligned} \tag{3.21}$$

In Table XVI, the bold face entries indicate the violation of this conjecture. This indicates either the Siegel modular forms corresponding to the twisted elliptic genus given in (3.20) do not capture degeneracies of black holes or one must redo the analysis in [18] and figure out if the conditions necessary for black holes to be single centered needs to be modified for these orbifolds. In either case, this analysis highlights the nontrivial feature of the modular forms constructed for all the $K3$ orbifolds geometric or formal constructed in this paper.

Let us finally examine the $N = 3$ torus orbifold given in [28]. The twisted elliptic genus is given by

$$\begin{aligned} F^{(0,0)} &= \frac{8}{3}A(\tau,z), \\ F^{(0,s)} &= -\frac{1}{3}A(\tau,z) - \frac{3}{4}\mathcal{E}_3(\tau)B(\tau,z), \quad \text{for } 1 \leq s \leq 2, \\ F^{(r,rk)} &= -\frac{1}{3}A(\tau,z) + \frac{1}{4}\mathcal{E}_3\left(\frac{\tau+k}{3}\right)B(\tau,z), \\ &\quad \text{for } 1 \leq r \leq 2, \quad 1 \leq k \leq 2. \end{aligned} \tag{3.22}$$

Going through the procedure of constructing the Siegel modular form and obtaining the putative dyon degeneracies

TABLE XVI. Some results for the index $-B_6$ for the order 2 torus orbifold for different values of Q^2, P^2 and $Q \cdot P$. The bold face entries indicate the violation of the positivity conjecture.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2
(1, 2)	−1152	**−224**	96	0
(1,6)	−13008	**−3392**	1376	**−224**
(1,10)	−101440	**−30336**	13 152	**−3392**
(2, 2)	−10380	**−1248**	1968	−12
(2,4)	39 456	18 240	840	2592
(3,2)	−69344	1728	22 528	**−224**
(2,6)	−113344	**−10320**	**−1376**	2432

TABLE XVII. Some results for the index $-B_6$ for the order 3 torus orbifold of $K3$ for different values of Q^2, P^2, and $Q \cdot P$. The bold face entries indicate the violation of the positivity conjecture.

$(Q^2, P^2)\backslash Q \cdot P$	−2	0	1	2
(1,4)	−252	**−36**	45	0
(1,6)	504	216	**−9**	18
(1,10)	−1692	**−324**	378	**−36**
(2,2)	0	324	162	0
(2,4)	1458	540	864	54
(3,6)	23 808	55 986	31 332	10 200

we obtain the Table XVII for the low lying coefficients. From the analysis of [18], for $N = 3$ orbifolds $-B_6$ is expected to be positive for charges satisfying the constraints

$$Q \cdot P \geq 0, \qquad Q \cdot P \leq 3Q^2, \qquad Q \cdot P \leq P^2,$$
$$5Q \cdot P \leq 6Q^2 + P^2, \qquad 5Q \cdot P \leq 3Q^2 + 2P^2,$$
$$7Q \cdot P \leq 6Q^2 + 2P^2, \qquad Q^2, \qquad P^2,$$
$$\{Q^2P^2 - (Q \cdot P)^2\} > 0. \tag{3.23}$$

However note that there are violations to this expectation in Table XVII which are indicated by the bold face entries.

IV. CONCLUSIONS

We studied the Siegel modular forms associated with the twisted elliptic genera that capture the degeneracy of $1/4$ BPS states in $\mathcal{N} = 4$ theories obtained by compactifying type II theory on $(K3 \times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as a order N automorphism associated with the conjugacy class of M_{24} on $K3$ together with a $1/N$ shift on one of the circles of T^2. We show that the dyon partition function satisfied the required properties expected from black hole physics. In particular the Fourier coefficients of the $1/4$ BPS index are integers and certain low lying charges are positive in agreement with the conjecture of [18]. This is a sufficient condition predicted from the fact that single centered black holes carry zero angular momentum. The observation that the dyon partition function associated with the $4A, 6A, 8A$ classes satisfies the positivity conjecture along with the earlier studied cases of pA with $p = 2, 3, 5, 7$ completes this analysis for all the CHL models. We also observed that the positivity conjecture of [18] was satisfied even when the orbifold as well as construction of the Siegel modular form were purely formal. This result provides some evidence for the conjecture that these symmetries which act as in Mathieu moonshine do exist in actual $K3$ sigma models[7]

It is worthwhile to complete this analysis of this paper for the remaining 9 conjugacy classes of Table II. The construction for the twisted elliptic genera corresponding to these classes would required new ingredients. One possible direction is to use positivity and integrality of the low lying coefficients in the associated Siegel modular form to determine the twisted elliptic genera in the sectors which form suborbits under $SL(2, \mathbb{Z})$. These conjugacy classes have more than one suborbit. One can also verify if the Siegel modular forms constructed from the twisted elliptic genera for these classes provided in the ancillary files associated with [24] is in agreement with the positivity conjecture of [18].

References [26,28,29] have studied more general non-cyclic twisted twining elliptic genera of $K3$ than considered in this paper. It is important to check if the more general twining elliptic genera considered in these references admit a $1/4$ BPS dyon partition function with integral Fourier coefficients and obey the positivity constraints as expected from black hole physics. Recently multiplicative lifts of more general weak Jacobi forms[8] as well as the Siegel modular forms of $Sp(2, \mathbb{Z})$ of weight 35 and 12 were studied and were shown to have properties which make them candidates for partition of black holes [35]. It will be interesting to check if the Fourier coefficients of these Siegel modular forms also satisfy the positivity constraints required from black hole physics. As a preliminary investigation we observed that two of the torus orbifolds of [28] do not satisfy the positivity conjecture of [18].

The discovery of the Mathieu moonshine symmetry has provided useful insights in string compactifications [36–38] as well as provided new examples where precision microscopic counting of black holes is possible as seen in this paper. It is certainly worthwhile to explore the implication of this symmetry further.

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APPENDIX: LIST OF TWISTED ELLIPTIC GENERA

In this appendix we provide the list of the twisted elliptic genera used in this paper.

[7]We thank the anonymous referee for pointing this.

[8]These were Jacobi forms of weight 0 but index > 1.

1. Conjugacy class 11*A*

Going through these steps we obtain the following formula for the twisted elliptic genus for 11*A*.

$$\begin{aligned} F^{(0,0)} &== \frac{8}{11}A(\tau,z),\\ F^{(0,s)} &= \frac{2}{33}A(\tau,z) - B(\tau,z)\left(\frac{1}{6}\mathcal{E}_{11}(\tau) - \frac{2}{5}\eta^2(\tau)\eta^2(11\tau)\right),\\ F^{(r,rs)} &= \frac{2}{33}A(\tau,z) + B(\tau,z)\left(\frac{1}{66}\mathcal{E}_{11}\left(\frac{\tau+s}{11}\right)\right.\\ &\quad\left. - \frac{2}{55}\eta^2(\tau+s)\eta^2\left(\frac{\tau+s}{11}\right)\right). \end{aligned} \tag{A1}$$

2. Conjugacy class 23*A*/*B*

The twining characters for the conjugacy classes 23A and 23B are identical and was determined in [13–15]. It is given by

$$\begin{aligned} F^{(0,0)} &= \frac{8}{23}A(\tau,z),\\ F^{(0,1)} &= \frac{1}{69}A - B\left(\frac{1}{12}\mathcal{E}_{23} - \frac{1}{22}f_{23,1}(\tau) - \frac{7}{22}\eta^2(\tau)\eta^2(23\tau)\right). \end{aligned} \tag{A2}$$

We can use the same procedure as discussed for the class 11*A* in the previous section to determine the twisted elliptic genus in all the sectors. Essentially we use the transformation law given in (1.2) to move to twisted elliptic genus in the other sectors from the (0,1) sector. As discussed in the previous section we need identities satisfied by the modular forms $\mathcal{E}_{23}(\tau), \eta^2(\tau)\eta^2(23\tau)$ and $f_{23,1}(\tau)$ to express the expansion in terms of $e^{-2\pi i/\tau}$ in terms of a the usual q expansion. Note that all these transform as modular forms under $\Gamma_0(23)$. The new form $f_{23,1}(\tau)$ under $\Gamma_0(23)$ has been constructed in [39,40] which involves Hecke eigenforms. A closed formula for $f_{23,1}(\tau)$ in terms of η functions is provided in the ancillary files associated with [24]. This is given by

$$\begin{aligned} f_{23,1}(\tau) &= 2\frac{\eta^3(\tau)\eta^3(23\tau)}{\eta(2\tau)\eta(46\tau)} + 8\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau)\\ &\quad + 8\eta^2(2\tau)\eta^2(46\tau) + 5\eta^2(\tau)\eta^2(23\tau). \end{aligned} \tag{A3}$$

It can be seen that from (A3) that the S transformation of $f_{23,1}(\tau)$ is given by

$$f_{23,1}\left(-\frac{1}{\tau}\right) = -\frac{\tau^2}{23}f_{23,1}\left(\frac{\tau}{23}\right). \tag{A4}$$

we obtain the twisted elliptic genus of the conjugacy class 23*A*. The result is given by

$$\begin{aligned} F^{(0,k)}(\tau,z) &= \frac{1}{23}\left(\frac{1}{3}A - B\left(\frac{23}{12}\mathcal{E}_{23}(\tau) - \frac{23}{22}f_{23,1}(\tau) - \frac{161}{22}\eta^2(\tau)\eta^2(23\tau)\right)\right),\\ F^{(r,rk)}(\tau,z) &= \frac{1}{23}\left[\frac{1}{3}A + B\left(\frac{1}{12}\mathcal{E}_{23}\left(\frac{\tau+k}{23}\right) - \frac{1}{22}f_{23,1}\left(\frac{\tau+k}{23}\right) - \frac{7}{22}\eta^2(\tau+k)\eta^2\left(\frac{\tau+k}{23}\right)\right)\right]. \end{aligned} \tag{A5}$$

3. Conjugacy class 4*B*

The twisted elliptic genus for this class is given by

$$\begin{aligned} F^{(0,0)}(\tau,z) &= 2A(\tau,z),\\ F^{(0,1)}(\tau,z) = F^{(0,3)}(\tau,z) &= \frac{1}{4}\left[\frac{4A}{3} - B\left(-\frac{1}{3}\mathcal{E}_2(\tau) + 2\mathcal{E}_4(\tau)\right)\right],\\ F^{(1,s)}(\tau,z) = F^{(3,3s)} &= \frac{1}{4}\left[\frac{4A}{3} + B\left(-\frac{1}{6}\mathcal{E}_2\left(\frac{\tau+s}{2}\right) + \frac{1}{2}\mathcal{E}_4\left(\frac{\tau+s}{4}\right)\right)\right],\\ F^{(2,1)}(\tau,z) = F^{(2,3)} &= \frac{1}{4}\left(\frac{4A}{3} - \frac{B}{3}(5\mathcal{E}_2(\tau) - 6\mathcal{E}_4(\tau)\right),\\ F^{(0,2)}(\tau,z) &= \frac{1}{4}\left(\frac{8A}{3} - \frac{4B}{3}\mathcal{E}_2(\tau)\right),\\ F^{(2,2s)}(\tau,z) &= \frac{1}{4}\left(\frac{8A}{3} + \frac{2B}{3}\mathcal{E}_2\left(\frac{\tau+s}{2}\right)\right). \end{aligned} \tag{A6}$$

4. Conjugacy class 6A

The twisted elliptic genus for 6A are given by

$$F^{(0,0)} = \frac{4}{3}A; \qquad F^{(0,1)} = F^{(0,5)}; \qquad F^{(0,2)} = F^{(0,4)};$$
$$F^{(0,1)} = \frac{1}{6}\left[\frac{2A}{3} - B\left(-\frac{1}{6}\mathcal{E}_2(\tau) - \frac{1}{2}\mathcal{E}_3(\tau) + \frac{5}{2}\mathcal{E}_6(\tau)\right)\right],$$
$$F^{(0,2)} = \frac{1}{6}\left[2A - \frac{3}{2}B\mathcal{E}_3(\tau)\right],$$
$$F^{(0,3)} = \frac{1}{6}\left[\frac{8A}{3} - \frac{4}{3}B\mathcal{E}_2(\tau)\right]. \tag{A7}$$

$$F^{(1,k)} = F^{(5,5k)} = \frac{1}{6}\left[\frac{2A}{3} + B\left(-\frac{1}{12}\mathcal{E}_2\left(\frac{\tau+k}{2}\right) - \frac{1}{6}\mathcal{E}_3\left(\frac{\tau+k}{3}\right) + \frac{5}{12}\mathcal{E}_6\left(\frac{\tau+k}{6}\right)\right)\right], \tag{A8}$$

$$F^{(2,2k+1)} = \frac{A}{9} + \frac{B}{36}\left[\mathcal{E}_3\left(\frac{\tau+2+k}{3}\right) + \mathcal{E}_2(\tau) - \mathcal{E}_2\left(\frac{\tau+k+2}{3}\right)\right],$$
$$F^{(4,4k+1)} = \frac{A}{9} + \frac{B}{36}\left[\mathcal{E}_3\left(\frac{\tau+1+k}{3}\right) + \mathcal{E}_2(\tau) - \mathcal{E}_2\left(\frac{\tau+k+1}{3}\right)\right],$$
$$F^{(3,1)} = F^{(3,5)} = \frac{A}{9} - \frac{B}{12}\mathcal{E}_3(\tau) - \frac{B}{72}\mathcal{E}_2\left(\frac{\tau+1}{2}\right) + \frac{B}{8}\mathcal{E}_2\left(\frac{3\tau+1}{2}\right),$$
$$F^{(3,2)} = F^{(3,4)} = \frac{A}{9} - \frac{B}{12}\mathcal{E}_3(\tau) - \frac{B}{72}\mathcal{E}_2\left(\frac{\tau}{2}\right) + \frac{B}{8}\mathcal{E}_2\left(\frac{3\tau}{2}\right), \tag{A9}$$

$$F^{(2r,2rk)} = \frac{1}{6}\left[2A + \frac{1}{2}B\mathcal{E}_3\left(\frac{\tau+k}{3}\right)\right],$$
$$F^{(3,3k)} = \frac{1}{6}\left[\frac{8A}{3} + \frac{2}{3}B\mathcal{E}_2\left(\frac{\tau+k}{2}\right)\right]. \tag{A10}$$

The low lying coefficients of the 6A twisted elliptic genus is given by

$$c^{(0,s)}(\pm 1) = \frac{1}{3}, \qquad s = 0, \cdots 5,$$
$$\sum_{s=0}^{5} c^{(0,s)}(\pm 1) = 2. \tag{A11}$$

and

$$\sum_{s=0}^{5} c^{(0,s)}(0) = 4. \tag{A12}$$

Therefore the number of (1,1) forms is 4. This agrees with [10] which studies the orbifold of $K3$ dual to the $N = 6$ CHL compactification. We therefore identify the compactification of type II on $(K3 \times T^2)/\mathbb{Z}_6$ where $\mathbb{Z}_6$ is the 6A automorphism to be dual to the $N = 4$ heterotic CHL compactification. The full elliptic genus of $K3$ orbifolded by the 6A automorphism. is given by

$$\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z). \tag{A13}$$

Thus the result of the 6A orbifold of $K3$ is $K3$ itself.

5. Conjugacy class 8A

$$F^{(0,0)}(\tau, z) = A(\tau, z),$$
$$F^{(0,1)} = F^{(0,3)} = F^{(0,5)} = F^{(0,7)},$$
$$= \frac{1}{8}\left[\frac{2A}{3} - B\left(-\frac{1}{2}\mathcal{E}_4(\tau) + \frac{7}{3}\mathcal{E}_8(\tau)\right)\right]. \tag{A14}$$

$$F^{(r,rk)}(\tau, z) = \frac{1}{8}\left[\frac{2A}{3} + \frac{B}{8}\left(-\mathcal{E}_4\left(\frac{\tau+k}{4}\right) + \frac{7}{3}\mathcal{E}_8\left(\frac{\tau+k}{8}\right)\right)\right]. \tag{A15}$$

where $r = 1, 3, 5, 7$.

$$F^{(2,1)} = F^{(6,3)} = F^{(2,5)} = F^{(6,7)},$$
$$= \frac{1}{8}\left[\frac{2A}{3} + \frac{B}{3}\left(-\mathcal{E}_2(2\tau) + \frac{3}{2}\mathcal{E}_4\left(\frac{2\tau+1}{4}\right)\right)\right];$$
$$F^{(2,3)} = F^{(6,5)} = F^{(2,7)} = F^{(6,1)},$$
$$= \frac{1}{8}\left[\frac{2A}{3} + \frac{B}{3}\left(-\mathcal{E}_2(2\tau) + \frac{3}{2}\mathcal{E}_4\left(\frac{2\tau+3}{4}\right)\right)\right]. \tag{A16}$$

$$F^{(0,2)} = F^{(0,6)} = \frac{1}{8}\left(\frac{4A}{3} - B\left(-\frac{1}{3}\mathcal{E}_2(\tau) + 2\mathcal{E}_4(\tau)\right)\right),$$
$$F^{(0,4)} = \frac{1}{8}\left(\frac{8A}{3} - \frac{4B}{3}\mathcal{E}_2(\tau)\right),$$
$$F^{(2,2s)} = F^{(6,6s)} = \frac{1}{8}\left(\frac{4A}{3} + B\left(-\frac{1}{6}\mathcal{E}_2\left(\frac{\tau+s}{2}\right) + \frac{1}{2}\mathcal{E}_4\left(\frac{\tau+s}{4}\right)\right)\right),$$
$$F^{(4,4s)} = \frac{1}{8}\left(\frac{8A}{3} + \frac{2B}{3}\mathcal{E}_2\left(\frac{\tau+s}{2}\right)\right),$$
$$F^{(4,2)} = F^{(4,6)} = \frac{1}{8}\left(\frac{4A}{3} - \frac{B}{3}(3\mathcal{E}_2(\tau) - 4\mathcal{E}_2(2\tau)\right),$$
$$F^{(4,2k+1)} = \frac{1}{8}\left(\frac{2A}{3} + B\left(\frac{4}{3}\mathcal{E}_2(4\tau) - \frac{2}{3}\mathcal{E}_2(2\tau) - \frac{1}{2}\mathcal{E}_4(\tau)\right)\right). \tag{A17}$$

Finally the low lying coefficients of this orbifold satisfy

$$c^{(0,s)}(\pm 1) = \frac{1}{4}, \qquad s = 0, \cdots 7,$$
$$\sum_{s=0}^{7} c^{(0,s)}(\pm 1) = 2. \tag{A18}$$

and

$$\sum_{s=0}^{7} c^{(0,s)}(0) = 2. \tag{A19}$$

The above equation implies that the number of (1,1) forms is 2 which agrees with the $K3$ orbifold dual to the $N = 8$ CHL compactification [10]. The full elliptic genus of $K3$ orbifolded by the $8A$ automorphism. is given by

$$\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z) \tag{A20}$$

Thus the elliptic genus of the $8A$ orbifold of $K3$ is $K3$ itself.

6. Conjugacy class 14A

$$F^{(0,1)}(\tau, z) = F^{(0,3)} = F^{(0,5)} = F^{(0,9)} = F^{(0,11)} = F^{(0,13)};$$
$$= \frac{1}{14}\left[\frac{A}{3} - B\left(-\frac{1}{36}\mathcal{E}_2(\tau) - \frac{7}{12}\mathcal{E}_7(\tau) + \frac{91}{36}\mathcal{E}_{14}(\tau) - \frac{14}{3}\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)\right)\right]; \tag{A21}$$

$$F^{(r,rk)} = \frac{1}{14}\left[\frac{A}{3} + B\left(-\frac{1}{72}\mathcal{E}_2\left(\frac{\tau+k}{2}\right) - \frac{1}{12}\mathcal{E}_7\left(\frac{\tau+k}{7}\right) + \frac{13}{72}\mathcal{E}_{14}\left(\frac{\tau+k}{14}\right) - \frac{1}{3}\eta(\tau+k)\eta\left(\frac{\tau+k}{2}\right)\eta\left(\frac{\tau+k}{7}\right)\eta\left(\frac{\tau+k}{14}\right)\right)\right]; \tag{A22}$$

where $r = 1,3,5,9,11,13$ and rk is Mod 14.

The even twisted sectors with odd twining characters can be found by similar manipulations as discussed in detail for the case of the $11A$ conjugacy class. This leads to the following equalities.

$$F^{(2,13)} = F^{(12,1)} = F^{(6,11)} = F^{(8,3)} = F^{(4,5)} = F^{(10,9)}. \tag{A23}$$

Combining all these results into a single formula we obtain

$$F^{(2r,2rk+7)} = \frac{1}{14}\left[\frac{A}{3} + B\left(-\frac{1}{6}\mathcal{E}_2(\tau) - \frac{1}{12}\mathcal{E}_7\left(\frac{\tau+k}{7}\right) + \frac{1}{3}\mathcal{E}_7\left(\frac{2\tau+2k}{7}\right) - \frac{2}{3}\eta(\tau+k)\eta(2\tau+2k)\eta\left(\frac{\tau+k}{7}\right)\eta\left(\frac{2\tau+2k}{7}\right)\right)\right]; \tag{A24}$$

where k runs from 0 to 6 and except 3 and r from 1 to 6. Next the following sectors are given by

$$F^{(7,2k+1)} = \frac{1}{14}\left[\frac{A}{3} + B\left(-\frac{7}{12}\mathcal{E}_7(\tau) + \frac{49}{72}\mathcal{E}_2\left(\frac{7\tau+1}{2}\right) - \frac{1}{72}\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\right.\right.$$
$$\left.\left. + \frac{7}{3}e^{i\pi 11/12}\eta(\tau)\eta(7\tau)\eta\left(\frac{\tau+1}{2}\right)\eta\left(\frac{7\tau+1}{2}\right)\right)\right];$$
$$F^{(7,2k)} = \frac{1}{14}\left[\frac{A}{3} + B\left(-\frac{7}{12}\mathcal{E}_7(\tau) + \frac{49}{72}\mathcal{E}_2\left(\frac{7\tau}{2}\right) - \frac{1}{72}\mathcal{E}_2\left(\frac{\tau}{2}\right) + \frac{7}{3}\eta(\tau)\eta(7\tau)\eta\left(\frac{\tau}{2}\right)\eta\left(\frac{7\tau}{2}\right)\right)\right]. \tag{A25}$$

Finally the sectors belonging to the $2A$ and $7A$ suborbits are given by

$$F^{(0,0)} = \frac{4}{7}A. \tag{A26}$$

$$F^{(0,2k)} = \frac{1}{14}\left[A - \frac{7}{4}B\mathcal{E}_7(\tau)\right] \qquad k \text{ runs from 1 to 6,}$$
$$F^{(2r,2rk)} = \frac{1}{14}\left[A + \frac{1}{4}B\mathcal{E}_7\left(\frac{\tau+k}{7}\right)\right]; \qquad k \text{ runs from 0 to 6.} \tag{A27}$$

$$F^{(0,7)} = \frac{1}{14}\left[\frac{8}{3}A - \frac{4}{3}B\mathcal{E}_2(\tau)\right],$$
$$F^{(7,7k)} = \frac{1}{14}\left[\frac{8}{3}A + \frac{2}{3}B\mathcal{E}_2\left(\frac{\tau+k}{2}\right)\right] \qquad k \text{ runs from 0 to 1.} \tag{A28}$$

7. Conjugacy class 15*A*

$$F^{(0,1)}(\tau,z) = F^{(0,2)} = F^{(0,4)} = F^{(0,7)} = F^{(0,8)} = F^{(0,11)} = F^{(0,13)} = F^{(0,14)};$$
$$= \frac{1}{15}\left[\frac{A}{3} - B\left(-\frac{1}{16}\mathcal{E}_3(\tau) - \frac{5}{24}\mathcal{E}_5(\tau) + \frac{35}{16}\mathcal{E}_{15}(\tau) - \frac{15}{4}\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)\right)\right]. \tag{A29}$$

$$F^{(r,rk)} = \frac{1}{15}\left[\frac{A}{3} + B\left(-\frac{1}{48}\mathcal{E}_3\left(\frac{\tau+k}{3}\right) - \frac{1}{24}\mathcal{E}_5\left(\frac{\tau+k}{5}\right) + \frac{7}{48}\mathcal{E}_{15}\left(\frac{\tau+k}{15}\right) - \frac{1}{4}\eta(\tau+k)\eta\left(\frac{\tau+k}{3}\right)\eta\left(\frac{\tau+k}{5}\right)\eta\left(\frac{\tau+k}{15}\right)\right)\right]; \tag{A30}$$

where $r = 1,2,4,7,8,11,13,14$ and rk is mod 15. The sectors belonging to the 5*A* and 3*A* sub-orbits are given by

$$F^{(0,0)} = \frac{8}{15}A. \tag{A31}$$

$$F^{(0,3k)} = \frac{1}{15}\left(\frac{4}{3}A - \frac{5}{3}B\mathcal{E}_5(\tau)\right) \qquad k \text{ runs from 1 to 4;}$$
$$F^{(3r,3rk)} = \frac{1}{15}\left(\frac{4}{3}A + \frac{1}{3}B\mathcal{E}_5\left(\frac{\tau+k}{5}\right)\right); \qquad k \text{ runs from 0 to 4.} \tag{A32}$$

$$F^{(0,5k)} = \frac{1}{15}\left(2A - \frac{3}{2}B\mathcal{E}_3(\tau)\right);$$
$$F^{(5r,5rk)} = \frac{1}{15}\left(2A + \frac{1}{2}B\mathcal{E}_3\left(\frac{\tau+k}{3}\right)\right) \qquad k \text{ runs from 0 to 2.} \tag{A33}$$

Finally the remaining sectors are given by

$$F^{(3r,5+3rk)} = \frac{1}{15}\left(\frac{A}{3} + B\left(-\frac{1}{4}\mathcal{E}_3(\tau) - \frac{1}{24}\mathcal{E}_5\left(\frac{\tau+k}{5}\right) + \frac{3}{8}\mathcal{E}_5\left(\frac{3\tau+3k}{5}\right) - \frac{3}{4}\eta(\tau+k)\eta(3\tau+3k)\eta\left(\frac{\tau+k}{5}\right)\eta\left(\frac{3\tau+3k}{5}\right)\right)\right),$$
$$F^{(3r,10+3rk)} = \frac{1}{15}\left(\frac{A}{3} + B\left(-\frac{1}{4}\mathcal{E}_3(\tau) - \frac{1}{24}\mathcal{E}_5\left(\frac{\tau+k}{5}\right) + \frac{3}{8}\mathcal{E}_5\left(\frac{3\tau+3k}{5}\right)\right.\right.$$
$$\left.\left. - \frac{3}{4}e^{-\frac{2\pi i}{5}}\eta(\tau+k)\eta(3\tau+3k)\eta\left(\frac{\tau+k}{5}\right)\eta\left(\frac{3\tau+3k}{5}\right)\right)\right); \tag{A34}$$

where k runs from 0 to 4 and $s = 1$ to 4.

$$F^{(5r,3s+5rk)} = \frac{1}{15}\left(\frac{A}{3} + B\left(\frac{5}{24}\mathcal{E}_5(\tau) + \frac{1}{12}\mathcal{E}_3\left(\frac{\tau+k}{3}\right) - \frac{5}{24}\mathcal{E}_5\left(\frac{\tau+k}{3}\right) + \frac{5}{4}\eta(\tau+k)\eta(5\tau+5k)\eta\left(\frac{\tau+k}{3}\right)\eta\left(\frac{5\tau+5k}{3}\right)\right)\right); \tag{A35}$$

where k runs from 0 to 2 and $s = 1$ to 2.

The low lying coefficients of the twisted elliptic genus in conjugacy classes $14A$ as well as $15A$ satisfy

$$c^{(0,s)}(\pm 1) = \frac{2}{N}, \qquad \sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2,$$
$$\sum_{s=0}^{N-1} c^{(0,)}(0) = 0, \qquad \sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z) \tag{A36}$$

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