# Discretization of 3d gravity in different polarizations

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We study the discretization of three-dimensional gravity with $\Lambda = 0$ following the loop quantum gravity framework. In the process, we realize that different choices of polarization are possible. This allows us to introduce a new discretization based on the triad as opposed to the connection as in the standard loop quantum gravity framework. We also identify the classical nontrivial symmetries of discrete gravity, namely the Drinfeld double, given in terms of momentum maps. Another choice of polarization is given by the Chern-Simons formulation of gravity. Our framework also provides a new discretization scheme of Chern-Simons, which keeps track of the link between the continuum variables and the discrete ones. We show how the Poisson bracket we recover between the Chern-Simons holonomies allows us to recover the Goldman bracket. There is also a transparent link between the discrete Chern-Simons formulation and the discretization of gravity based on the connection (loop gravity) or triad variables (dual loop gravity).

## I. INTRODUCTION

Three-dimensional (3d) gravity is a topological theory that can be exactly quantized [1]. Seen as a toy model for testing fundamental questions pertinent to four-dimensional quantum gravity, three-dimensional quantum gravity is also compelling for finding new approaches to the physically relevant four-dimensional theory. Three-dimensional gravity is appeals as well due to several existing quantum models [1]. For example, loop quantum gravity (LQG) and the combinatorial quantization formalism come from two different quantization and discretization processes.

LQG is a canonical quantization approach of which the starting point is the first order gravity action, the so-called Palatini action [2]. In the quantization process, two different procedures are achieved: the discretization and the quantization itself. These two strokes are usually done in one, but let us briefly describe them separately here. Discretization is done using a graph, embedded in the spatial surface, which in three dimensions can be chosen dual to a triangulation. The continuous phase space variables, the connection and the triad, are respectively smeared along the edges of the graphs and along the corresponding dual edges. As a result, the discrete phase space is described in terms of the so-called holonomy-flux variables, and their Poisson algebra is the holonomy-flux algebra. The result of the second stroke is the definition of the quantum states of space, the so-called spin networks. A key outcome of LQG is the discreteness of the geometric spatial operators. LQG is nevertheless not a discrete approximation of gravity. Indeed, one recovers

the continuum picture by taking a projective limit on the Hilbert spaces [3]. One might wonder how much quantization there is in the discretization process. The passage from the continuum to the discrete allows one to identify the discrete realization of the continuum symmetries (gauge symmetry and translational symmetry) with Poisson-Lie group symmetries given in terms of the classical Drinfeld double. Upon quantization, this will give rise to quantum group symmetries given by the Drinfeld double of $SU(2)$. In this sense, the discretization process identifies the origin of the quantum group symmetries.

The combinatorial quantization formalism is also a Hamiltonian quantization approach, which is based on the Chern-Simons action. The Chern-Simons action describes three-dimensional gravity but allows degenerate metrics as additional solutions. As previously mentioned, the discretization procedure is based on graphs embedded in the spatial surface. The discretization of the classical Chern-Simons theory, or the Atiyah-Bott symplectic form [4], was first identified for (closed) holonomies, giving rise to the Goldman bracket [5]. The Poisson algebra is not always well defined if one deals with open holonomies, but this problem was addressed by Fock and Rosly [6]. They postulated some Poisson structures for the (open) holonomies, which allow one to recover the moduli space with the right Poisson structure. They did not consider the link between the continuum variables Poisson bracket and their Poisson bracket between the holonomies. Later, Alekseev and Malkin proposed a change of coordinates in the phase space variables which made apparent nice structures such as the Drinfeld and Heinsenberg doubles [7]. Then, Alekseev *et al.* proposed a direct quantization procedure of the Fock and Rosly phase space [8,9].

Although both continuum theories are describing gravity, the loop gravity and Chern-Simons theories are difficult

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to compare (nevertheless, see Ref. [10]) at the discrete level and are written in terms of different mathematical structures at the quantum level.

In this paper, we want to focus on the discretization step of both theories and clarify the link between the two approaches. In the loop gravity context, the discrete theory is interesting *per se* as it provides a way to truncate the theory into some approximation, getting in particular finite-dimensional Hilbert spaces when the quantization is done. As in the quantum case, this truncation can be removed by considering the classical analog of the projective limit. One might wonder how much quantization there is in the discretization process. The passage from the continuum to the discrete allows one to identify the discrete realization of the continuum symmetries (gauge symmetry and translational symmetry) with Poisson-Lie group symmetries given in terms of the classical Drinfeld double. Upon quantization, this will give rise to quantum group symmetries given by the Drinfeld double of $SU(2)$. In this sense, the discretization process identifies the origin of the quantum group symmetries.

Starting from the Palatini action for three-dimensional gravity with a zero cosmological constant, we reproduce the analysis of Ref. [11], which focused on the four-dimensional case. Identifying precisely the passage from the continuum to the discrete reveals that some *choice of polarization* is actually made in the standard LQG approach. We discuss then the consequence of making *other choices of polarization*. We obtain three main results:

(i) We show that a different choice of polarization allows us to define the corresponding phase space based on the metric (triad) picture. This new discrete framework where the vanishing of the curvature around each face of the triangulation is automatically implemented could be of interest to write a more geometrical formulation of gravity.

(ii) Chern-Simons theory can be viewed as 3d gravity where no specific choice of polarization is made. We can also apply our "loopy" discretization scheme to Chern-Simons theory. The Poisson structure is well defined for any holonomy, meaning that we have derived an alternative to the usual Fock and Rosly regularization, while keeping a clear link with the continuous Chern-Simons variables.

(iii) This metric kinematical phase space can be shown to be the dual of the usual LQG holonomy kinematical phase space through the notion of symplectic dual pairs. These two dual representations are unified within the Chern-Simons framework. We illustrate in particular how different choices of polarization in the discretized Chern-Simons theory lead to the two dual discretized gravity pictures.

The scheme of the paper goes as follows. In Sec. II, we describe the discretization procedure of the gravity phase space variables and the associated symplectic form

following Ref. [11] for 3d gravity when $\Lambda = 0$. We identify the dual representation of the LQG phase space. In Sec. III, we propose a new discretization of Chern-Simons theory following the loop gravity discretization, and we show that the Goldman brackets are recovered without introducing any *ad hoc* regularization of the Poisson brackets. Finally, in Sec. IV, we show the link between the three different discrete pictures coming from either the Palatini action or the Chern-Simons action.

## II. 3D LQG AND ITS DUAL COUNTERPART

### A. Continuous phase space and constraints

We consider a principal G-bundle over $M$, a 3d manifold (with no boundary). We will consider in the following $G = SU(2)$ or $SU(1, 1)$. We note $\omega^A = \frac{1}{2}\epsilon^{ABC}\omega_{BC}$ the connection and $e_A$ the triad, which are both $\mathfrak{g}$ valued 1-form, with $\mathfrak{g} = \mathfrak{su}(2)$ or $\mathfrak{su}(1, 1)$. The transformation properties are as follows:

$$\omega \to \omega + d\zeta + [\omega, \zeta] = \omega + d_\omega\zeta,$$
$$e \to e + [e, \zeta] \quad \text{with} \quad \zeta \text{ a } \mathfrak{g}\text{-valued scalar.} \quad (1)$$

The curvature of the connection is the $\mathfrak{g}$-valued 2-form $F = d\omega + \omega \wedge \omega$. Given the Lie algebra $\mathfrak{g}$, with generators $\sigma_A$, we write its Killing form $\langle,\rangle$ as a normalized trace,[1]

$$\langle\sigma_A, \sigma_B\rangle = \text{Tr}(\sigma_A\sigma_B) = \eta_{AB}. \quad (2)$$

The 3d gravity action with zero cosmological constant is given by the $BF$ action

$$S_{\text{grav}}(e, \omega) = -\int_M \langle e \wedge F \rangle = -\int_M e^I \wedge F_I. \quad (3)$$

Capital indices are internal space indices, $I, J = 1, 2, 3$. The choice of gauge group G determines the signature of the spacetime under consideration. The equations of motion imply that the connection should be torsionless and flat,

$$d_\omega e = de + \omega \wedge e = 0, \qquad d\omega + \omega \wedge \omega = 0. \quad (4)$$

The $BF$ action is invariant under the gauge transformations (1) but also the translation

$$\omega \to \omega, \quad e \to e + d_\omega\phi, \quad \text{with} \quad \phi \text{ a } \mathfrak{g}\text{-valued scalar}, \quad (5)$$

thanks to the Bianchi identity $d_\omega F = 0$.

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[1] In the case of $\mathfrak{su}(2)$, or $\mathfrak{su}(1, 1)$, we can choose the generators to be anti-Hermitian and to satisfy the algebra $[\sigma_A, \sigma_B] = \epsilon_{ABC}\sigma^C$ and the normalized trace to be related to the two-dimensional trace by $\text{Tr}(A) = -2\text{tr}(A)$.

We assume that $M \sim \mathbb{R} \times \Sigma$ [with $\Sigma$ a smooth two-dimensional (2d) manifold with no boundary] and use the coordinates $(t, x_1, x_2)$ for a point in $M$. We can then proceed to the Hamiltonian formulation and identify the momentum variable which is of density weight 1. The natural choice is given by the dyad of density weight 1,

$$\frac{\delta S_{\text{grav}}}{\delta \dot{\omega}_{aJ}} \equiv \tilde{\mathbf{e}}^a = \tilde{\epsilon}^{ab} e_b. \tag{6}$$

Lowercase indices are space indices, $a$, $b = 1$, 2, and $\tilde{\epsilon}^{ab}$ is the antisymmetric tensor of density weight 1 such that $\tilde{\epsilon}^{12} = 1$. We introduced $\delta$, a variational differential [12] acting on fields which squares to zero $\delta^2 = 0$ and should not be confused with the space differential d. As such, the product $\delta A \delta B$ means the antisymmetric combination $\delta_{[1} A \delta_{2]} B$. We do not introduce a wedge notation for this skew symmetric product, but we have to remember that the product of two variational forms is anticommuting. We can then identify the symplectic potential (Liouville form) $\Theta_{\text{grav}}^{\text{LQG}}$,

$$\Theta_{\text{grav}}^{\text{LQG}} = \langle \tilde{\mathbf{e}}^a \delta \omega_a \rangle = \langle \tilde{\mathbf{e}} \cdot \delta \omega \rangle = \langle e \wedge \delta \omega \rangle. \tag{7}$$

Dynamics is given in terms of a pair of constraints implementing that the spatial parts of the curvature or the torsion are zero,

$$F^I = \tilde{\epsilon}^{ab} F_{ab}^I = 0, \quad T_J = (\partial_a \tilde{\mathbf{e}}_J^a + \epsilon_J{}^{IK} \omega_{Ia} \tilde{\mathbf{e}}_K^a) = 0. \tag{8}$$

Since a lot of attention will be given to the symplectic potential in the following, let us make some preliminary comments. The Liouville form we have obtained allows us to identify the phase space variables and provides the symplectic form, which in turns provides the Poisson bracket. However, there are in fact different possible choices of the Liouville form, equivalent up to boundary terms. Let us recall the case of the particle. In this case, the standard Liouville term is $p\,dq$ with $p$ the momentum variable and $q$ the configuration variable. This Liouville term is equivalent up to the boundary term to

$$-q\,dp, \quad \text{or also} \quad \frac{1}{2}(p\,dq - q\,dp). \tag{9}$$

Any of these Liouville forms leads to the same symplectic form (up to boundary terms). A similar ambiguity occurs in the gravity case. However, since some of the phase space variables should be of density weight 1, we have different options. The LQG case would consist in considering as the configuration variable the connection $\omega_a$, hence the Liouville term we introduced earlier in (7). The associated (nonzero) Poisson brackets would then read

$$\{\omega_a^I(x), \tilde{\mathbf{e}}_J^b(y)\} = \delta_b^a \delta_J^I \delta^2(x - y), \quad x, y \in \Sigma. \tag{10}$$

Instead, we could consider as a configuration variable the dyad $e_a$, which would be more in the philosophy of the ADM formalism. This is in essence the dual picture to the LQG framework. In this case, the symplectic 2-form would then read[2] (up to boundary terms)

$$\Omega_{grav}^{\text{LQG}^*} = \langle \delta \tilde{\omega}^a \delta e_a \rangle = \langle \delta \tilde{\omega} \cdot \delta e \rangle = \langle \delta \omega \wedge \delta e \rangle. \tag{11}$$

In this case, the momentum variables are given by $\tilde{\omega}^{Ia}(x) = \tilde{\epsilon}^{ab} \omega_b^I(x)$. The Poisson bracket is then

$$\{e_{Jb}(x), \tilde{\omega}^{Ia}(y)\} = \delta_J^I \delta_b^a \delta^2(x - y). \tag{12}$$

Finally, as we shall argue later, the symplectic 2-form

$$\Omega_{\text{grav}}^{\text{CS}} = \frac{1}{2}(\langle \delta \tilde{\mathbf{e}} \cdot \delta \omega \rangle + \langle \delta \tilde{\omega} \cdot \delta e \rangle)$$
$$= \frac{1}{2}(\langle \delta e \wedge \delta \omega \rangle + \langle \delta \omega \wedge \delta e \rangle) \tag{13}$$

will be related to the symplectic 2-form of Chern-Simons theory. The next sections will consist in finding a consistent discretization of these different symplectic 2-forms.

The symmetries of the action given by the gauge transformations (1) or the translations (5) can also be realized in terms of the Poisson brackets. The momentum maps (that is, the phase space functions that implement these symmetry transformations; see Appendix) are precisely these constraints. The curvature constraint $F$ implements the infinitesimal translation, whereas the torsion constraint $T$ implements the infinitesimal gauge transformation. By considering the smearing of the torsion and curvature $\mathcal{T} = \int \zeta^K T_K$, $\mathcal{F} = \int \phi_K F^K$ over the fields $N_K$, $\Lambda^K$, using the LQG phase space variables, we have that

$$\{\tilde{\mathbf{e}}, \mathcal{T}\} = -[\tilde{\mathbf{e}}, \zeta], \quad \{\omega, \mathcal{T}\} = -d_\omega \zeta, \tag{14}$$

$$\{\tilde{\mathbf{e}}, \mathcal{F}\} = \widetilde{d_\omega \phi}, \quad \{\omega, \mathcal{F}\} = 0. \tag{15}$$

The discretization scheme we will use should implement that the discretized constraints are the momentum maps implementing the discretized symmetries.

### B. Toward the discretization of the gravity phase space

We intend to construct a discretization of the gravity symplectic form inspired by previous works on discretizing gravity in order to get LQG [11]. One first chooses a triangulation[3] $\Gamma^*$ of $\Sigma$, and we denote by $\Gamma$ the graph given by its one skeleton. In the following, we will denote the vertices of the triangulation $\Gamma^*$ by $v$, $v'$ and the oriented

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[2]Recall that $\omega$ and $e$ are 1-forms and that we do an integration by parts.

[3]We could more generally choose any cellular decomposition. We restrict to triangulations for clarity of exposition only.
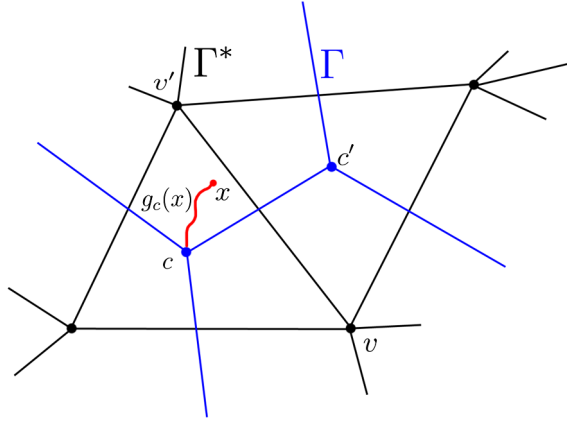
FIG. 1. Some components of the graphs $\Gamma$, $\Gamma^*$ and the holonomy $g_c(x)$. Curvature and torsion sit at the vertices of $\Gamma^*$. The face $c^*$ and the face $c'^*$ share the edge $\ell^* = [vv']$, dual to the link $\ell = [cc']$.

*edges* of $\Gamma^*$ by $\tilde{\ell} = [vv']$. We assume that in each face $[v_1 v_2 v_3]$ of the triangulation a center point $c$ has been chosen, and we denote the duality between centers and triangles by *: $c^* = [v_1 v_2 v_3]$. We connect the centers by *links* $\ell = [cc']$, and the graph made out of the centers and the links is denoted $\Gamma$ (see Fig. 1). This graph is dual to the triangulation graph $\Gamma^*$, and the duality between links and edges is written as

$$\ell^* = [cc']^* = [vv'] = \tilde{\ell} \quad \text{if } [vv'] = c^* \cap c'^*. \quad (16)$$

The duality is between oriented links and oriented edges. The orientation of the edge is chosen to be obtained from the orientation of the link by a counterclockwise rotation (see Fig. 1).

We intend to discretize the phase space variables $(\omega, \tilde{e})$ in which $\omega$ is the configuration variable or $(\tilde{\omega}, e)$ in which $e$ is the configuration variable. There is *a priori* no recipe on how the density weight 1 variables should be discretized. So, instead, we are going to discretize the 1-forms $(\omega, e)$ and see at the end that our discretization naturally provides a discretization of these density weight 1 variables.

We discretize the 1-form variables $(\omega, e)$ by assuming that the curvature $F(\omega)$ and torsion $T$, if any,[4] are concentrated on the vertices of $\Gamma^*$. This means that in the interior of each triangle $c^*$ we have $d_\omega e = 0 = d_\omega \omega$; that is, the connection $\omega$ is flat and torsionless inside the cell $c^*$. These equations can be solved easily inside each triangle $c^*$ in terms of a group element $g_c(x)$, normalized to $g_c(c) = 1$. This group element represents the *holonomy* of the flat connection from the center $c$ to the point $x \in c^*$ as illustrated in Fig. 1. The solution for the connection simply reads for $x \in c^*$,

$$\omega(x) \equiv (g_c^{-1} \mathrm{d} g_c)(x). \quad (17)$$

Given this parametrization of $\omega$, the zero torsion condition implies that the combination $(g_c e g_c^{-1})(x)$ is closed, hence exact on $c^*$. Therefore, we introduce the Lie algebra valued function $\mathbf{y}_c(x)$ on $c^*$, which solves the torsion condition as

$$e(x) \equiv (g_c^{-1} \mathrm{d} \mathbf{y}_c g_c)(x), \qquad x \in c^*. \quad (18)$$

The next step is to determine the discretization of the symplectic form behind the Poisson bracket (10). Let us consider the variations of $\omega$ and $e$, in each triangle $c^*$, from their definitions in (17). A key identity that we will repeatedly use [5] is that

$$\delta(g^{-1} \mathrm{d} g) = g^{-1} \mathrm{d}(\delta g g^{-1}) g. \quad (19)$$

We also need to extract the variation of the electric field $e = (g^{-1} \mathrm{d} y g)$. In order to do so, we similarly establish that

$$\delta(g^{-1} \mathrm{d} \mathbf{y} g) = g^{-1}(\mathrm{d} \delta \mathbf{y} + [\mathrm{d} \mathbf{y}, \delta g g^{-1}]) g. \quad (20)$$

Now that we have a simple expression of the fields inside each triangle $c^*$, we can decompose the full discretized symplectic structure as a sum $\Omega = \sum_c \Omega_c$, and $\Omega_c$ is defined by

$$\Omega_c = \int_{c^*} \Omega_{\text{grav}} = \int_{c^*} \langle \delta \omega \wedge \delta e \rangle. \quad (21)$$

From the values of $\delta \omega$ and $\delta e$ in $c^*$, we have

$$\Omega_c = \int_{c^*} \langle \mathrm{d}(\delta g_c g_c^{-1}) \wedge (\delta \mathrm{d} \mathbf{y}_c + [\mathrm{d} \mathbf{y}_c, \delta g_c g_c^{-1}]) \rangle$$

$$= \int_{c^*} \delta \langle \mathrm{d}(\delta g_c g_c^{-1}) \wedge \mathrm{d} \mathbf{y}_c \rangle. \quad (22)$$

The main point is that the integrand is an exact 2-form; it can therefore be entirely evaluated in terms of its boundary contribution. Also, we remark that, since $\mathbf{y}$ and $g$ enter asymmetrically, there are two different ways to integrate this form. In other words, we have

$$\Omega_c = \int_{\partial c^*} \delta \langle (\delta g_c g_c^{-1}) \mathrm{d} \mathbf{y}_c \rangle \quad (23)$$

$$= - \int_{\partial c^*} \delta \langle \mathrm{d}(\delta g_c g_c^{-1}) \mathbf{y}_c \rangle. \quad (24)$$

These two choices can be seen as the two natural choices of phase space variables $(\omega, \tilde{\mathbf{e}})$ or $(\tilde{\omega}, e)$ respectively. Indeed, the tilde variable in which the $\epsilon$ of weight density 1 sits

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[4]We leave open the possibility that there is some by considering the dynamics later or if we are introducing particles.

[5]An explicit derivation reads $\delta(h^{-1} \mathrm{d} h) = h^{-1} \mathrm{d} \delta h + \delta h^{-1} \mathrm{d} h = h^{-1}((\mathrm{d} \delta h) h^{-1} - (\delta h h^{-1})(\mathrm{d} h h^{-1})) h = h^{-1} \mathrm{d}(\delta h h^{-1}) h$.

indicates what variable will be discretized on the dual of $\Gamma$. It can be the flux $\tilde{\mathbf{e}}$, as in the loop polarization, or the holonomy in what we will call the dual loop polarization.

## C. Loop gravity phase space

We now work with the loop polarization $(\omega, \tilde{\mathbf{e}})$ represented by the choice (23). Since we have localized the symplectic structure on the boundary of the triangles, we can equivalently write the total symplectic structure as a sum of contribution associated with each link $[cc']$ of $\Gamma$ as

$$\Omega^{\text{LQG}} = \sum_{[cc'] \in \Gamma} \Omega_{cc'}, \qquad (25)$$

where the contribution from each link $\ell = [cc']$ is given by contributions from $c^*$ and from[6] $c'^*$. The edge shared by the faces $c^*$ and $c'^*$ is $\tilde{\ell} = [vv']$ (see Fig 1),

$$\Omega_{cc'} = \delta \int_{\tilde{\ell}} (\langle (\delta g_c g_c^{-1}) \mathrm{d}\mathbf{y}_c \rangle - \langle (\delta g_{c'} g_{c'}^{-1}) \mathrm{d}\mathbf{y}_{c'} \rangle). \qquad (26)$$

We now look at the matching condition across the edge $[vv'] = \tilde{\ell}$. Demanding the continuity of the connection across the edge implies that

$$g_c^{-1} \mathrm{d} g_c(x) = \omega(x) = g_{c'}^{-1} \mathrm{d} g_{c'}(x), \qquad x \in [vv']. \quad (27)$$

This condition means that there is a *constant* group element $h_{cc'}$, see Fig. 2, that relates both frames:

$$g_{c'}(x) = h_{c'c} g_c(x), \qquad x \in [vv']. \qquad (28)$$

$h_{cc'}$ represents the holonomy of the flat connection along the edge $[cc']$ of $\Gamma$. In the following, we use that $h_{cc'}^{-1} = h_{c'c}$. $h_{cc'}$ represents the usual group variables of loop gravity, in which $\Gamma$ is the support of the spin network.

The frame field $e(x)$ is also continuous, and hence we can relate the frames from different faces. Consider $x \in \tilde{\ell}$; then, we have

$$g_c^{-1} \mathrm{d}\mathbf{y}_c g_c = e = g_{c'}^{-1} \mathrm{d}\mathbf{y}_{c'} g_{c'}. \qquad (29)$$

Hence, we deduce that for $x \in \tilde{\ell}$ we have $\mathrm{d}\mathbf{y}_{c'} = h_{cc'}^{-1} \mathrm{d}\mathbf{y}_c h_{cc'}$. This can be integrated out, and we conclude that there are elements $x_{cc'} \in \mathfrak{g}$ such that

$$\mathbf{y}_{c'} = h_{cc'}^{-1}(\mathbf{y}_c + x_{cc'}) h_{cc'}. \qquad (30)$$

$x_{cc'}$ represents the translational holonomy. In other words, when going from $c$ to $c'$, the frame is rotated by $h_{cc'}$ but also translated by $x_{cc'}$. The combination $(h_{cc'}, x_{cc'})$ represents a

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[6]Note that the face $c'^*$ contributes with a minus sign due to the opposite orientation.

Poincaré transformation that maps the flat chart around $c$ to the flat chart around $c'$.

In order to evaluate the symplectic structure $\Omega_{cc'}$, we consider the variation of the frame relation (28) and get that

$$\delta g_{c'} g_{c'}^{-1} = h_{cc'}^{-1}(\delta g_c g_c^{-1} - \delta h_{cc'} h_{cc'}^{-1}) h_{cc'}. \qquad (31)$$

This together with (30) leads us to the simple expression

$$\Omega_{cc'} = \delta \langle (\delta h_{cc'} h_{cc'}^{-1}) | \tilde{X}_{\tilde{\ell}}^c \rangle, \qquad (32)$$

where we have introduce the *flux* vector $\tilde{X}_{\tilde{\ell}} \in \mathfrak{g}$ based at $c$ and defined as

$$\tilde{X}_{\tilde{\ell}}^c = \int_{\tilde{\ell}} \mathrm{d}\mathbf{y}_c = \int_{\tilde{\ell}} (g_c e g_c^{-1}) \equiv \int_{\ell} (g_c \tilde{\mathbf{e}} g_c^{-1}). \quad (33)$$

Such a flux is the natural candidate to encode the discretization of the density weight 1 vector $\tilde{\mathbf{e}}^a$. The presence of the 2d Levi-Cività tensor can be traced back to the fact that $\tilde{\mathbf{e}}$ is discretized on the dual of $\ell$, on which the connection is discretized through the holonomy $h_{\ell}$.

The discretized variables are therefore

$$(\tilde{X}_{\tilde{\ell}}^c, h_{\ell}^v = g_{cv} g_{c'v}^{-1}). \qquad (34)$$

We picked the point $v$ in $\tilde{\ell}$ to define $h_{\ell}$ to emphasize the symmetry between these variables and the dual LQG variables (54). In the following, when dealing with the usual LQG variables, we will drop the upper indices $c$ and $v$ to avoid cluttered notations.

The symplectic form associated to the link $\ell = [cc']$ then takes the following shape:

$$\Omega_{cc'}^{LQG} = \text{Tr}((\delta h_{\ell} h_{\ell}^{-1}) \delta \tilde{X}_{\tilde{\ell}} + (\delta h_{\ell} h_{\ell}^{-1})(\delta h_{\ell} h_{\ell}^{-1}) \tilde{X}_{\tilde{\ell}}). \quad (35)$$

We have recovered the standard symplectic form associated to $T^*G$. Extending the construction to all of the edges of $\Gamma$, we can decorate these edges with the phase space $T^*G$. The fluxes $\tilde{X}_{\tilde{\ell}}$ sit at the nodes of $\Gamma$ but depend on the edges $\tilde{\ell}$ of the graph $\Gamma^*$, whereas the holonomies $h_{\ell}$ decorate the links $\ell$ of $\Gamma$. This is the definition of the building blocks of the LQG phase space.

Note that we can invert the symplectic form to recover the standard Poisson bracket on $T^*G$ (see for example Ref. [13] for the details of the calculations),

$$\{\tilde{X}_{\tilde{\ell}}^A, \tilde{X}_{\tilde{\ell}}^B\} = \epsilon_C^{AB} \tilde{X}_{\tilde{\ell}}^C, \qquad \{\tilde{X}_{\tilde{\ell}}^A, h_{\ell}\} = \sigma^A h_{\ell},$$

$$\{h_{\ell}, h_{\ell}\} = 0. \qquad (36)$$

It is important to recognize that the data $(\tilde{X}_{\tilde{\ell}} = \tilde{X}_{[vv']}, h_{\ell})$ are not free. They do satisfy a discrete version of the Gauss law. This constraint follows from the continuous Gauss identity
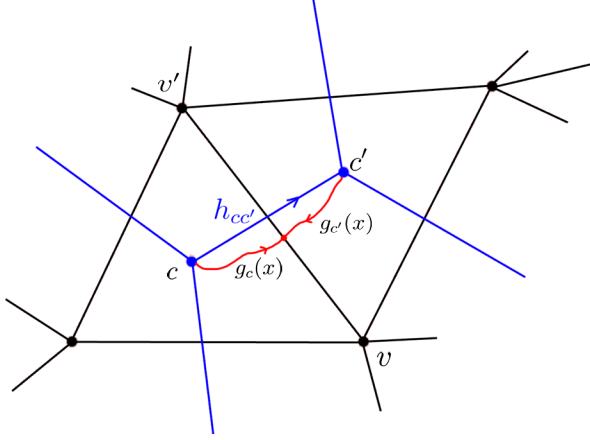
FIG. 2. The constant holonomy $h_{cc'}$ connects the frames of the two different faces. It is the standard LQG holonomy decorating the spin network $\Gamma$.



FIG. 3. The geometry of the triangle $c^*$ can be recovered from the vectors $\tilde{X}^c_{\tilde{\ell}}$.

$$\mathcal{J}_c \equiv \tilde{X}_{[v_1 v_2]} + \tilde{X}_{[v_2 v_3]} + \tilde{X}_{[v_3 v_1]} = \int_{\partial c^*} \mathrm{d}\mathbf{y}_c$$

$$= \int_{c^*} \mathrm{d}(\mathrm{d}\mathbf{y}_c) = \int_{c^*} g_c^{-1}(\mathrm{d}_\omega e) g_c = 0, \qquad (37)$$

which is essentially the sum over the three boundary edges of the triangle $c^*$ of $\Gamma^*$.

We see from this formula that $\mathcal{J}_c$ computes the violation of the torsion condition in the triangle $c^*$. Since we have assumed that it vanishes inside each triangle, the sum vanishes, and we recover the standard discretized Gauss law.

Since torsion is the momentum map at the infinitesimal level implementing the infinitesimal gauge transformation (14), and $\mathcal{J}_c$ can be seen as the discretization of torsion on $c^*$, it is natural to expect that $\mathcal{J}_c$ encodes a local G transformation at the trivalent vertex $c$ of $\Gamma$. As discussed in Appendix, $\mathcal{J}_c$ is the momentum map for the three copies of $T^*G$. Denoting $\mathcal{J}_c(\alpha) = \alpha^c_A \mathcal{J}^A_c$, we have

$$\delta^c_\alpha \tilde{X}_{\tilde{\ell}} = \{\tilde{X}_{\tilde{\ell}}, \mathcal{J}_c(\alpha)\}, \qquad \delta^c_\alpha h_\ell = \{h_\ell, \mathcal{J}_c(\alpha)\}. \quad (38)$$

As such, using (36), it implements the transformation $\delta^c_\alpha$ with $\alpha^c \in \mathfrak{g}$ at the nodes $c$ of $\Gamma$,

$$\delta^c_\alpha \tilde{X}_{\tilde{\ell}} = [\alpha_c, \tilde{X}_{\tilde{\ell}}], \qquad \delta^c_\alpha h_\ell = -\alpha_c h_\ell. \quad (39)$$

Following the Marsden-Weinstein theorem [14,15], we recover the usual kinematical LQG phase space as the double quotient,

$$\mathcal{P}^{\mathrm{kin}}_{\mathrm{LQG}} = (\times_{\ell \in \Gamma} T^*_\ell G) // (\times_{c \in \Gamma} \mathcal{J}_c), \qquad (40)$$

where the double quotient denotes the symplectic reduction by the constraint (37). The decorated graph $\Gamma$ is the classical analog of the spin network.

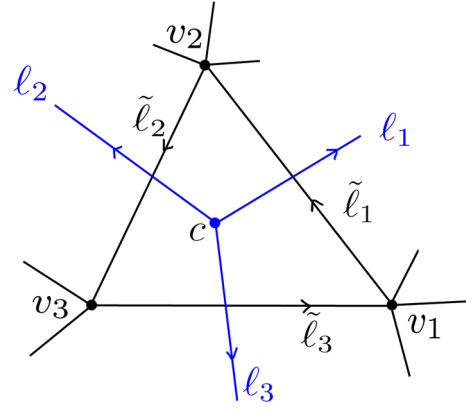The kinematical observables are functions built from the fluxes $\tilde{X}_{\tilde{\ell}}$ and holonomies $h_\ell$, such that they are invariant

under the transformations (39). Typically, such observables are then the Wilson loops $\mathrm{Tr}\prod_{\ell \in \lambda} h_\ell$, where $\lambda$ is a loop in $\Gamma$ or the scalar quantities built out from the fluxes. For example, the (kinematical) observables associated to the (triangle edge) length, (triangle) angle, (triangle) area, respectively $L_{v_1 v_2}$, $\theta_{v_1}$, $A_{v_1 v_2 v_3}$, are given by

$$L^2_{v_1 v_2} = |\tilde{X}_{[v_1 v_2]}|, \qquad \cos\theta_{v_1} = \frac{\tilde{X}_{[v_1 v_2]} \cdot \tilde{X}_{[v_1 v_3]}}{L_{v_1 v_2} L_{v_1 v_3}},$$

$$A_{v_1 v_2 v_3} = \frac{1}{2} |\tilde{X}_{[v_1 v_2]} \wedge \tilde{X}_{[v_1 v_3]}|. \qquad (41)$$

These observables allow us to reconstruct the triangle geometry dual to the vertex $c$; see Fig. 3.

To have the full description of 3d gravity at the discrete level, we just need to introduce the discretized version of the flatness constraint (considering the pure gravity case with no particles). This constraint is discretized by requiring that all the holonomies along the loops $\lambda$ in $\Gamma$ are flat,

$$\mathcal{G}_\lambda = \prod_{\ell \in \lambda} h_\ell = 1. \qquad (42)$$

This set of constraints together with the set of constraints generated by $\mathcal{J}_c$ form a first class system of constraints, just like in the continuum case.

The set of constraints $\{\mathcal{G}_\lambda\}_{\lambda \text{loops in}\Gamma}$ also implements some symmetry transformations. Note, however, that since we are dealing with elements in a non-Abelian group, this constraint can be seen as a *non-Abelian* momentum map. We consider therefore the symmetry action, labeled by the vertex $v$ and $\beta_v = \beta^A_v \sigma_A \in \mathfrak{su}(2)$, given by

$$\delta^v_\beta \tilde{X}^B_{\tilde{\ell}} \equiv \langle \mathcal{G}_\lambda^{-1} \{\tilde{X}^B_{\tilde{\ell}}, \mathcal{G}_\lambda\}, \beta^A_v \sigma_A \rangle,$$
$$\delta^v_\beta h_\ell \equiv \langle \mathcal{G}_\lambda^{-1} \{h_\ell, \mathcal{G}_\lambda\}, \beta^A_v \sigma_A \rangle. \qquad (43)$$

Let us consider the loop $\lambda = (c_1 c_2 \ldots c_i \ldots c_n c_1)$ surrounding a vertex $v \in \Gamma^*$, with initial point $c_1$ and for $i = 1, \ldots n$.

We define $\ell_i = [c_i c_{i+1}]$ with $[c_n c_{n+1}] \equiv [c_n c_1]$. The infinitesimal action $\delta_\beta^v$ is then explicitly choosing the ordering $\mathcal{G}_\lambda = h_{\ell_n} \ldots h_{\ell_1}$; we have

$$\delta_\beta^v \tilde{X}_{\tilde{\ell}_1} = h_{\ell_1}^{-1} \beta_v h_{\ell_1}, \qquad \delta_\beta^v \tilde{X}_{\tilde{\ell}_2} = h_{\ell_2}^{-1} h_{\ell_1}^{-1} \beta_v h_{\ell_1} h_{l_2}, \ldots,$$
$$\delta_\beta^v \tilde{X}_{\tilde{\ell}_i} = H_i^{-1} \beta_v H_i, \tag{44}$$

where we have defined the partial holonomies $H_i \equiv h_{\ell_1} h_{\ell_2} \ldots h_{\ell_i}$, while the holonomies remain invariant, $\delta_\beta^v h_{\ell_i} = 0$. We recognize the discretized action of the translations (5). This transformation seems to depend on the choice of initial point one starts with. However, the condition $\prod_{\ell \in \lambda} h_\ell = 1$ means that the different transformations generated by the different choices of initial point are all equivalent and related by a redefinition of the gauge parameter.

A momentum map with value in a non-Abelian group indicates the presence of a symmetry group (here, the translations $\mathbb{R}^3$) equipped with a nontrivial Poisson structure, hence a nontrivial Poisson-Lie group. This is reviewed in Appendix. At the quantum level, this leads to the notion of the quantum group (here, the Drinfeld double) as a symmetry group.

The Marsden-Weinstein theorem has been generalized to the case where the symmetries are Poisson-Lie group symmetries [15]. Hence, we can consider the symplectic reduction of the LQG kinematical phase with the symmetry action, acting at the vertices of the graph $\Gamma^*$ (vertices dual to the loops $\lambda \in \Gamma$).

$$\mathcal{P}_{\mathrm{LQG}}^{\mathrm{phys}} = \mathcal{P}_{\mathrm{LQG}} / / (\times_{\lambda \in \Gamma} \mathcal{G}_\lambda). \tag{45}$$

The physical observables, that is, the functions over copies of $T^* SU(2)$ invariant under the symmetries spanned by $\mathcal{J}_c$ and $\mathcal{G}_\ell$, have been discussed in Ref. [16].

### D. Dual representation of the loop gravity phase space

In the previous section, we have shown how the continuum symplectic structure reduces to the discrete loop gravity one. In order to do so, we have chosen the first decomposition in (23). This is the polarization were wave functions are functional of the connection. We now investigate what happens if we choose the second one, that is, the geometrical polarization where wave functions are functions of the frame $e$. In this case, we can write the edge symplectic structure as[7]

$$\tilde{\Omega}_{cc'} = \delta \int_{\tilde{\ell}} (\langle \mathrm{d}(\delta g_{c'} g_{c'}^{-1}) \mathbf{y}_{c'} \rangle - \langle \mathrm{d}(\delta g_c g_c^{-1}) \mathbf{y}_c \rangle). \tag{46}$$

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[7]This evaluation can be compared to (26). Once again, the face $c'^*$ has the opposite orientation, and we have implemented the minus sign coming from the integration by parts in (23).

This term differs from (26) by a boundary term, which disappears when we sum over all links. Indeed, we have that

$$\tilde{\Omega}_{cc'} = \Omega_{cc'} + \delta \int_{\tilde{\ell}} \mathrm{d}(\langle (\delta g_{c'} g_{c'}^{-1}) \mathbf{y}_{c'} \rangle - \langle (\delta g_c g_c^{-1}) \mathbf{y}_c \rangle). \tag{47}$$

Using (30) and (31), we can evaluate it as

$$\tilde{\Omega}_{cc'} = \delta \int_{\tilde{\ell}} \langle \mathrm{d}(\delta g_c g_c^{-1}) x_{cc'} \rangle. \tag{48}$$

Using the fact that the edge dual to the link $\ell$ is given by $\tilde{\ell} = [vv']$ and defining $g_{cv} := g_c(v)$, we get

$$\tilde{\Omega}_{cc'} = \delta \langle [(\delta g_{cv'} g_{cv'}^{-1}) - (\delta g_{cv} g_{cv}^{-1})] x_{cc'} \rangle. \tag{49}$$

If one introduces the holonomy from $v$ to $v'$ inside $c$,

$$\tilde{h}_{\tilde{\ell}}^c = \tilde{h}_{vv'}^c \equiv g_{cv}^{-1} g_{cv'}, \tag{50}$$

we get an equivalent but more familiar expression dual to (32)

$$\tilde{\Omega}_{cc'} = \delta \langle \delta \tilde{h}_{\tilde{\ell}}^c (\tilde{h}_{\tilde{\ell}}^c)^{-1} | X_\ell^v \rangle, \tag{51}$$

where we have defined

$$X_\ell^v \equiv (g_{cv}^{-1} x_{cc'} g_{cv}). \tag{52}$$

$x_{cc'}$ is a translational monodromy based at $c$, and the connectors $g_{cv}$ map it onto a field based at $v$. It is important to note that from the definition we have the relation

$$X_{-\ell}^v = X_{[c'c]}^v = -(\tilde{h}_{vv'}^c)^{-1} X_{[cc']}^v \tilde{h}_{vv'}^c = -(\tilde{h}_{\tilde{\ell}}^c)^{-1} X_\ell^v (\tilde{h}_{\tilde{\ell}}^c)^{-1}. \tag{53}$$

To summarize, the variables for the dual LQG formulation are given by

$$(X_\ell^v \equiv (g_{cv}^{-1} x_{cc'} g_{cv}), \tilde{h}_{\tilde{\ell}}^c = g_{cv}^{-1} g_{cv'}). \tag{54}$$

These variables are based at $v$ and can be seen as the dual picture to the standard LQG variables given in (34). The dual fluxes are depending on $\Gamma$, whereas the holonomies depend on the triangulation $\Gamma^*$.

Our construction provides a candidate for the discretization of the density weight 1 variable $\tilde{\omega}$,

$$\tilde{h}_{\tilde{\ell}}^c = P \exp \left( \int_{\tilde{\ell}} \omega \right) \equiv P \exp \left( \int_\ell \tilde{\omega} \right). \tag{55}$$

Once again the presence of the Levi-Cività tensor implies that the vector $\tilde{\omega}^a$ is discretized over the dual space of where the 1-form $e$ is discretized (into $x_\ell$).
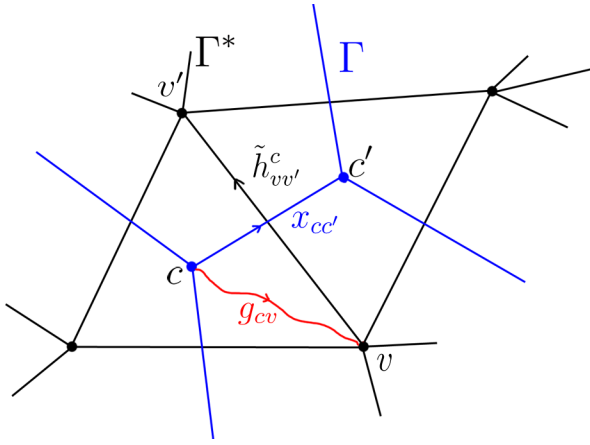
FIG. 4. The constant (Abelian) holonomy $x_{cc'}$ connects the frames of the two different faces as a translation. The holonomy $\tilde{h}^c_{vv'}$ lives on the face $c$ and connects the vertices of $\Gamma^*$. We have now a dual picture of the standard LQG picture, based on $\Gamma^*$.

The symplectic structure (51) is again the symplectic structure of $T^*G$ for the pair $(X_\ell, \tilde{h}^c_\ell)$ but based on the dual graph $\Gamma^*$ instead of $\Gamma$. The associated Poisson bracket is once again

$$\{X^A_\ell, X^B_\ell\} = \epsilon^{AB}_C X^C_\ell, \qquad \{X^A_\ell, \tilde{h}^c_{\tilde{\ell}}\} = \sigma^A \tilde{h}_{\tilde{\ell}},$$
$$\{\tilde{h}^c_{\tilde{\ell}}, \tilde{h}^c_{\tilde{\ell}}\} = 0. \tag{56}$$

We emphasize again that the holonomy is depending on the face $c$ from its definition (50); see Fig 4.

Just like in the standard LQG case, the variables we have integrated, namely here the holonomies, are not all independent. There is still a constraint naturally present due to the integration around $c^*$,

$$\mathcal{G}_c \equiv \tilde{h}^c_{v_1 v_2} \tilde{h}^c_{v_2 v_3} \tilde{h}^c_{v_3 v_1} = 1, \tag{57}$$

where $(v_1, v_2, v_3)$ are the three vertices of the triangle $c^*$. These constraints are the group analog of the Gauss constraints (37) but now they express the vanishing of the curvature inside $c^*$.

Once again, we can expect that the discretization of the curvature generates a momentum map $\mathcal{G}_c$, with value in a non-Abelian group, spanning the translations. We consider therefore the infinitesimal symmetry action, labeled by the node $c$ dual to the triangle $c^* = [v_1 v_2 v_3]$ and $\beta_c \in \mathbb{R}^3 \sim \mathfrak{su}(2)$, given by

$$\tilde{\delta}^c_\beta X^B_\ell \equiv \langle \mathcal{G}^{-1}_c \{X^B_\ell, \mathcal{G}_c\}, \beta^A_c \sigma_A \rangle,$$
$$\tilde{\delta}^c_\beta \tilde{h}^c_{\tilde{\ell}} \equiv \langle \mathcal{G}^{-1}_c \{\tilde{h}^c_{\tilde{\ell}}, \mathcal{G}_c\}, \beta^A_c \sigma_A \rangle. \tag{58}$$

This is to compare with the transformations generated by the LQG discrete flatness constraint (43). Explicitly, these transformations give, choosing the ordering $\mathcal{G}_c = \tilde{h}^c_{\tilde{\ell}_1} \dots \tilde{h}^c_{\tilde{\ell}_3}$ and the orientation as in Fig. 3,

$$\tilde{\delta}^c_\beta X^{v_3}_{\ell_3} = (\tilde{h}^c_{\tilde{\ell}_3})^{-1} \beta_c \tilde{h}^c_{\tilde{\ell}_3}, \qquad \tilde{\delta}^c_\beta X^{v_2}_{\ell_2} = (\tilde{h}^c_{\tilde{\ell}_2} \tilde{h}^c_{\tilde{\ell}_3})^{-1} \beta_c \tilde{h}^c_{\tilde{\ell}_2} \tilde{h}^c_{\tilde{\ell}_3},$$
$$\tilde{\delta}^c_\beta X^{v_1}_{\ell_1} = \mathcal{G}^{-1}_c \beta_c \mathcal{G}_c = \beta$$
$$\tilde{\delta}^c_\beta \tilde{h}^c_{\tilde{\ell}_i} = 0. \tag{59}$$

This transformation seems to depend on the choice of vertex one starts with, and accordingly it looks like one should have three transformations per center. However, the condition $\mathcal{G}_c = \tilde{h}^c_{\tilde{\ell}_1} \tilde{h}^c_{\tilde{\ell}_2} \tilde{h}^c_{\tilde{\ell}_3} = 1$ means that these three transformations are all equivalent and related by a redefinition of the gauge parameter $\beta_c \to \tilde{h}^c_{\tilde{\ell}_1} \beta_c (\tilde{h}^c_{\tilde{\ell}_1})^{-1} \to (\tilde{h}^c_{\tilde{\ell}_1} \tilde{h}^c_{\tilde{\ell}_2})^{-1} \beta_c \tilde{h}^c_{\tilde{\ell}_1} \tilde{h}^c_{\tilde{\ell}_2}$.

We define now the phase space of the dual LQG picture by considering the double quotient,

$$\mathcal{P}^{\text{kin}}_{\text{LQG}^*} = (\times_{\tilde{\ell} \in \Gamma^*} T^*_{\tilde{\ell}} G) // (\times_{c \in \Gamma} \mathcal{G}_c), \tag{60}$$

where the double quotient denotes the symplectic reduction by the constraint (57), thanks to the Marsden-Weinstein theorem generalized to the non-Abelian momentum map case [14]. Upon quantization, we expect to choose to construct our kinematical Hilbert space on the functions of $X_\ell$ together with the flatness constraint which implements a translation invariance at the centers of $\Gamma$. Hence, we expect to recover some spin networks based on the translational group. Our phase space is the classical analog of such a spin (or "momentum") network. We note that in the dual picture we implement at the kinematical level the nontrivial Poisson-Lie symmetry. Hence, at the quantum level, we shall expect to deal with representations of a quantum group (the Drinfeld double).

The kinematical observables invariant under the gauge transformations (59) are the group elements themselves. But there exist also some new observables located at the vertices $v$.

Let us consider a vertex $v_0$ surrounded by $n$ vertices $(v_1, \dots, v_n)$ ordered counterclockwise, and $n$ centers $(c_1, \dots c_n)$, as in Fig 5. As earlier, we will note $\lambda$ the loop around the vertex $v$. We chose the labels such that
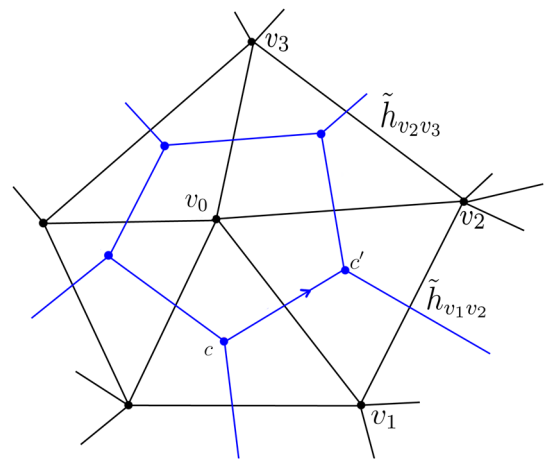


FIG. 5. We construct an observable associated to a vertex $v_0$.

$c_a^* = [v_0 v_a v_{a+1}]$, with $v_{n+1} = v_1$. For simplicity, we denote $X_{[v_0 v_a]^*} \equiv X_a$, and we notice that all the $X_a$ sit at the vertex $v_0$. Let us consider the triangles sharing the edges $[v_0 v_1]$; see Fig 5. The flux $X_1$ gets gauge transformed from the face $c_0^*$ and $c_1^*$. Note, however, that from the perspective of the face $c_1^*$, $X_1$ has an opposite orientation. Hence, we should have in mind (53),

$$\tilde{\delta}_{\beta_{c_0}} X_1^{v_0} = \beta_{c_0}, \qquad \tilde{\delta}_{\beta_{c_1}} X_1^{v_0} = -\beta_{c_1}. \qquad (61)$$

Repeating the process for each edge $[v_0 v_a]$, we see that we can construct a kinematical observable (which commutes with $\mathcal{G}_c$),

$$\mathcal{J}_{v_0} = \sum_a X_a^{v_0}, \qquad (62)$$

which computes at the discrete level the integral of the torsion $\int_D \mathrm{d}_\omega e$ around a disk $D$ centered at $v_0$. This is the Abelian holonomy around $v_0$.

Let us note $\alpha_v = \alpha_{vA}\sigma^A$. $\mathcal{J}_v(\alpha_v) \equiv \alpha_{vA}\sum_{\ell \in \lambda} X_\ell^A$ is an Abelian momentum map which generates the following gauge transformations:

$$\tilde{\delta}_\alpha^v X_\ell^B = \{X_\ell^B, \mathcal{J}_v(\alpha_v)\} = [\alpha_v, X_\ell^B],$$
$$\tilde{\delta}_\alpha^v \tilde{h}_\ell^c = \{\tilde{h}_\ell^c, \mathcal{J}_v(\alpha_v)\} = -\alpha_v \tilde{h}_\ell^c. \qquad (63)$$

The dynamics of the dual formulation of LQG will be given by the constraint $\mathcal{J}_v = 0$. By construction, $\mathcal{G}_c$ and $\mathcal{J}_v$ form a first class system (in fact, they commute strongly, unlike the standard LQG case where they commute weakly).

The physical phase space of the dual LQG formulation is obtained by considering the symplectic reduction of the dual LQG kinematical phase by the symmetry action generated by $\mathcal{J}_v$, acting at the vertices of the graph $\Gamma^*$,

$$\mathcal{P}_{\mathrm{LQG}^*}^{\mathrm{phys}} = \mathcal{P}_{\mathrm{LQG}^*} // (\times_{v \in \Gamma^*} \mathcal{J}_v). \qquad (64)$$

## III. NEW DISCRETIZATION OF CHERN-SIMONS THEORY

To summarize the previous section, we have seen that we have two descriptions of the gravity phase space, one based on the connection picture and the other based on a triad (metric) picture. We would like to see if we can strengthen the relations between the two pictures, by *unifying* them. For this, we can expect that the Chern-Simons approach might provide the key to this unification. Indeed, in the Chern-Simons approach to gravity, the connection and the triad are put at the same level and unified through the Chern-Simons connection.

We have seen that the dual loop gravity picture can be seen as a different choice of polarization, having the triad as the configuration variable instead of the connection as in loop gravity. Another "choice" of polarization is to just remove the difference between configuration and momentum variables and work with these variables altogether. This is what the Chern-Simons connection does by unifying the triad and the connection. It provides a choice of coordinates on the gravity phase space agnostic in terms of what we call momentum or configuration.

We can therefore discretize the Chern-Simons theory in much the same way we proceeded for gravity. This will allow us to simplify the relation between the (dual) loopy picture and Chern-Simons. As such, we will provide a discretization of Chern-Simons which will keep clear the link between the continuum picture and the discrete picture (which is not the case for the Fock-Rosly formalism [6]). Our discretization scheme is also free of the issues that arise in a naive discretization (for example, the Poisson bracket between holonomies sharing an initial point would diverge).

To show that our discretization is consistent, we will prove that we can recover the Goldman bracket 6. The discretization will be done for any group, and in the next section, we will focus on the Poincaré/Euclidian group cases to connect with gravity with zero cosmological constant.

### A. Chern-Simons theory

We consider a principal $\mathcal{G}$-bundle over $M$, a 3d manifold (without any boundary). We note $\mathcal{A}$ its connection which is a *Lie* $\mathcal{G}$-valued 1-form. We note $\langle,\rangle$ or $\eta^{\mu\nu}$ the invariant form. The gauge transformation properties and the curvature tensor are the standard ones:

$$\mathcal{A} \to \mathcal{A} + \mathrm{d}_\mathcal{A}\xi,$$
$$\mathfrak{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}, \mathfrak{F} \to \mathfrak{F} + [\mathfrak{F}, \xi] \quad \text{with}$$
$$\xi \text{ a Lie } \mathcal{G}\text{-valued scalar}. \qquad (65)$$

The Chern-Simons action is given by

$$S(\mathcal{A}) = \int \frac{1}{2}\langle \mathcal{A} \wedge \mathrm{d}\mathcal{A}\rangle + \frac{1}{3}\langle \mathcal{A} \wedge [\mathcal{A} \wedge \mathcal{A}]\rangle. \qquad (66)$$

The equations of motion implement that the connection $\mathcal{A}$ is flat,

$$\mathfrak{F} = 0. \qquad (67)$$

Let us assume now that $M \sim \mathbb{R} \times \Sigma$ (with $\Sigma$ a smooth 2d manifold with no boundary) and use the coordinates $(t, x_1, x_2)$ for a point in $M$. We can then proceed to the Hamiltonian formulation. We identify the momentum variables to be the connection with density weight 1,

$$\frac{\delta S_{\mathrm{CS}}}{\delta \dot{\mathcal{A}}_{a\mu}} \equiv \tilde{\mathcal{A}}^{a\mu} = \frac{1}{2}\tilde{\epsilon}^{ab}\mathcal{A}_b^\mu. \qquad (68)$$

Lowercase indices of the beginning of the alphabet are space indices, $a, b = 1, 2$, while greek indices are internal indices. The symplectic 2-form is then

$$\Omega^{\text{CS}} = \langle \delta \tilde{\mathcal{A}} \cdot \delta \mathcal{A} \rangle = \frac{1}{2} \langle \delta \mathcal{A} \wedge \delta \mathcal{A} \rangle, \qquad (69)$$

while the dynamics is given in terms of the constraint

$$\tilde{\epsilon}^{ab} \mathfrak{F}_{ab}^{\mu} = 0. \qquad (70)$$

The canonical Poisson bracket is obtained by inverting the symplectic form given by $\Omega^{\text{CS}} = \delta \Theta^{\text{CS}}$,

$$\{ \mathcal{A}^{\mu a}(x), \tilde{\mathcal{A}}_{\nu b}(y) \} = \delta_b^a \delta_\nu^\mu \delta^2(x - y), \qquad x, y \in \Sigma. \qquad (71)$$

We note that $\mathcal{A}^{\mu a}(x)$ and $\tilde{\mathcal{A}}_{\nu b}(y)$ are dual to each other. Hence, upon discretization, we expect that the analog of (71) should be given in terms of holonomies living on dual edges.

Later on, we will also use the following notation for the Poisson bracket[8]:

$$\{ \mathcal{A}^{(1)}(x), \tilde{\mathcal{A}}^{(2)}(y) \} = \delta^2(x - y) T^\mu \otimes T_\mu, \qquad x, y \in \Sigma. \qquad (72)$$

A direct calculation shows that the smeared curvature is the momentum map implementing the infinitesimal gauge transformations (65). For example,

$$\left\{ \mathcal{A}, \int \xi^\nu \mathfrak{F}_\nu \right\} = \widetilde{\mathrm{d}_{\mathcal{A}} \xi}, \qquad (73)$$

where $\xi^\mu$ is the scalar field with a value in $Lie\mathcal{G}$. The physical phase space is therefore given by the quotient of the (infinite-dimensional) space of a flat connection on $\Sigma$ with the (infinite-dimensional) group of gauge transformations, the action of which is given by (65). This is the so-called *moduli space of flat connections*, which happens to be *finite* dimensional [4,6].

Since the flatness constraint is actually the same as the moment map, the Poisson structure on moduli space is induced by the quotient, through the Marsden-Weinstein theorem (adapted to deal with infinite-dimensional spaces).

It is actually more convenient to construct the moduli space of flat connections using discretized variables, i.e. holonomies. Fock and Rosly proposed such a construction [6]. Here, we want to propose a new approach which will make the link with gravity more transparent.

### B. Chern-Simons discretized 2-form

The new discretization of the Chern-Simons connection we want to propose follows the same line as the one we used

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[8]We use the standard notation $\mathcal{A}^{(1)} = \mathcal{A}^\mu T_\mu \otimes \mathbf{1}$ and $\mathcal{A}^{(2)} = \mathbf{1} \otimes \mathcal{A}^\mu T_\mu$ and similarly for $\tilde{\mathcal{A}}$. $T_\mu$ are the generators of $Lie\mathcal{G}$.

earlier for gravity. In particular, since we do not know how to discretize the density weight 1 variables, we discretize the variables $\mathcal{A}$ and expect the discretization to provide us the natural candidates for their discretization. The discussion in this section is done for any Lie algebra $Lie\mathcal{G}$.

We consider the triangulation $\Gamma^*$, on which vertices $v$ sit any curvature for the connections $\mathcal{A}$. Given a face $c^* \in \Gamma^*$, we consider the $\mathcal{G}$ holonomies $H_c$ from the center $c$ in $c^*$ to any other point $x$ in $c^*$. We have then

$$\mathcal{A}(x) = (H_c^{-1} \mathrm{d} H_c)(x). \qquad (74)$$

To build the discretized symplectic form, we will use that

$$\delta \mathcal{A} = \delta(H_c^{-1} \mathrm{d} H_c) = H_c^{-1} \mathrm{d}(\delta H_c H_c^{-1}) H_c. \qquad (75)$$

The smeared version of $\Omega^{\text{CS}}$ on the face $c^*$ is then

$$\Omega_c^{\text{CS}} = \frac{1}{2} \int_{c^*} \langle \mathrm{d}(\delta H_c H_c^{-1}) \wedge \mathrm{d}(\delta H_c H_c^{-1}) \rangle. \qquad (76)$$

Note that, unlike the gravity case (23), there is a symmetry between the two holonomy contributions.

Repeating the same steps as in Sec. II C, introducing $H_{cc'} = H_c(x) H_{c'}(x)^{-1}$ for $x \in c \cap c'$ and $H_{cv} := H_c(v)$ and $\tilde{H}_{vv'}^c = H_{cv}^{-1} H_{cv'}$ we obtain the discretized symplectic 2-form for the link $\ell = [cc']$,

$$\Omega_{cc'}^{\text{CS}} = \frac{1}{2} \langle H_{cv}(\delta \tilde{H}_{vv'}^c (\tilde{H}_{vv'}^c)^{-1}) H_{cv}^{-1}(\delta H_{cc'} H_{cc'}^{-1}) \rangle. \qquad (77)$$

The holonomies $H_{cv}$ are the connectors mapping the fields from $v$ to $c$. One can see the holonomies discretized on $\tilde{\ell}$ as the natural discretization of the density weight 1 variables $\tilde{\mathcal{A}}$. The Levi-Città tensor accounts for the discretization on the dual of $\ell$,

$$\tilde{H}_{\tilde{\ell}}^c = P \exp \left( \int_{\tilde{\ell}} \mathcal{A} \right) \equiv P \exp \left( \int_{\ell} \tilde{\mathcal{A}} \right). \qquad (78)$$

For a given link $\ell = [cc']$ and its dual $\tilde{\ell} = [vv']$, we build the phase space out of two copies of the group $\mathcal{G}$ as a manifold equipped with $\Omega_{cc'}^{\text{CS}}$ as a 2-form. To have a phase space, we need this 2-form to be closed, i.e. $\delta \Omega_{cc'}^{\text{CS}} = 0$, which is clearly not the case, due to the presence of the connectors $H_{cv}$. In Sec. IV C, we are going to see that if $\mathcal{G}$ is the Euclidian/Poincaré group then $\Omega_{cc'}^{\text{CS}}$ is indeed symplectic *up to boundary terms*, so that $\Omega_{\text{CS}}^{\text{discretized}} = \sum_\ell \Omega_\ell$ is symplectic. We leave the study of other groups such as $SL(2, \mathbb{C})$, relevant for gravity with a nonzero-cosmological constant, for later studies.

We note that as in the LQG or dual LQG case all the variables are not independent. Due to the integration around $c^*$, the Chern-Simons holonomy around the face $c^*$ is just the identity

$$\mathcal{H}_c = \tilde{H}^c_{v_1 v_2} \tilde{H}^c_{v_2 v_3} \tilde{H}^c_{v_3 v_1} = 1. \tag{79}$$

This is just the statement that there is no torsion or curvature on the face $c^*$; said otherwise, the Chern-Simons connection is flat on this face.

Given a graph $\Gamma$, with each edge, we associate two copies of $\mathcal{G}$, $\mathcal{P}_\ell = (\mathcal{G}_\ell, \mathcal{G}_{\tilde{\ell}})$. When $\Omega^{CS}_\ell$ is symplectic (up to boundary terms), we can then define the *kinematical* phase space of the discretized Chern-Simons connections on $\Gamma$ by

$$\mathcal{P}^{kin}_{CS} \equiv \times_{\ell \in \Gamma} \mathcal{P}_\ell // (\times_{c \in \Gamma} \mathcal{H}_c), \quad \text{with} \quad \Omega^{discrete}_{CS} = \sum_\ell \Omega^{CS}_\ell. \tag{80}$$

The physical phase space, i.e. the moduli space, is obtained by performing the further symplectic reduction with respect to the constraint $\mathcal{H}_v$ which encodes the flatness constraint (where there are no particles) around the vertices $v$ of $\Gamma^*$. We leave this for further studies.

### C. Recovering the Goldman bracket

We consider the $\mathcal{G}$ holonomies $H \equiv H_{cc'} H_{c'c}$, $\tilde{H} \equiv \tilde{H}_{v'v} \tilde{H}_{vv'}$ intersecting at $p$, as sketched in Fig. 6.

The Goldman bracket is a Poisson bracket between closed holonomies, the value of which is given in terms of the Casimir of $Lie\mathcal{G}$ 6,

$$\{H^{(1)}, \tilde{H}^{(2)}\} = \tilde{H}^{(2)}_{v'p} H^{(1)}_{cp} \mathcal{C} \tilde{H}^{(2)}_{pv'} H^{(1)}_{pc}, \tag{81}$$

where $\mathcal{C} = 2T^\mu \otimes T_\mu$ is the Casimir of $Lie\mathcal{G}$.

Explicitly, this Poisson bracket can be calculated from the Poisson brackets (72) by considering the contributions of the graph intersecting at $p$,

$$\{H^{(1)}, \tilde{H}^{(2)}\} = \tilde{H}^{(2)}_{v'v} \{H^{(1)}_{cc'}, \tilde{H}^{(2)}_{vv'}\} H^{(1)}_{c'c}, \tag{82}$$

and the use of (72) gives

$$\{H^{(1)}_{cc'}, \tilde{H}^{(2)}_{vv'}\} = H^{(1)}_{cp} \tilde{H}^{(2)}_{vp} \mathcal{C} H^{(1)}_{pc'} \tilde{H}^{(2)}_{pv'}, \tag{83}$$

to obtain (81).

The discretized symplectic 2-form we have obtained in (77) sits at $c$, so it is actually convenient to transport the bracket (83) from $p$ to $c$ to be able to compare them. This is done by taking advantage of the invariance of the Casimir under the adjoint action

$$\mathcal{C} = H_{pc} \otimes H_{pc} \mathcal{C} (H_{pc} \otimes H_{pc})^{-1}. \tag{84}$$

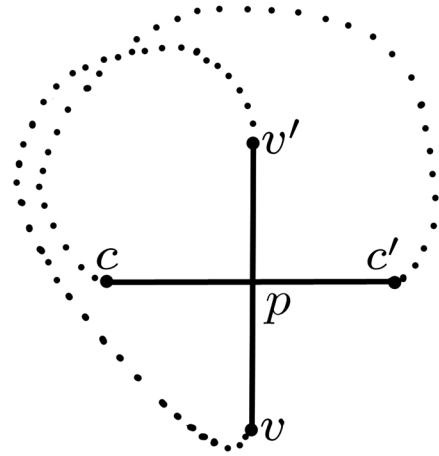These connectors $H_{pc}$ transport then the right-hand side of (83) to $c$,



FIG. 6. The holonomies $H$, $\tilde{H}$ intersect at $p$.

$$\{H^{(1)}_{cc'}, H^{(2)}_{vv'}\} = H^{(1)}_{cp} H^{(2)}_{vp} H^{(1)}_{pc} H^{(2)}_{pc} \mathcal{C} (H^{(1)}_{cp} H^{(2)}_{cp} H^{(1)}_{pc'} H^{(2)}_{pv'})$$
$$= H^{(2)}_{vc} \mathcal{C} H^{(1)}_{cc'} H^{(2)}_{cv'}. \tag{85}$$

Let us now reconsider the discretized Chern-Simons symplectic 2-form we have obtained,

$$\Omega^{CS}_{cc'} = \frac{1}{2} \langle H_{cv} (\delta \tilde{H}^c_{vv'} (\tilde{H}^c_{vv'})^{-1}) H^{-1}_{cv} (\delta H_{cc'} H^{-1}_{cc'}) \rangle, \tag{86}$$

which can be rewritten as

$$\Omega^{CS}_{cc'} = \frac{1}{2} \langle (H_{cv} \triangleright \tilde{\theta}^R) \wedge \theta^R \rangle. \tag{87}$$

$\tilde{\theta}^R$ and $\theta^R$ are the right-invariant 1-form on respectively $\mathcal{G}_{\tilde{\ell}}$ and $\mathcal{G}_\ell$. Their duals are the respective right-invariant vector fields $\tilde{\chi}^R$ and $\chi^R$. We have indeed with $H \in \mathcal{G}$

$$\chi \cdot f(H) = \frac{d}{dt} (f(e^{t \cdot T} H))|_{t=0}, \qquad \chi \cdot H = T^\mu H,$$
$$\langle \theta_\mu, \chi^\nu \rangle = \delta^\nu_\mu. \tag{88}$$

The dual of $H \triangleright \theta^R$ is $H^{-1} \triangleright \chi^R$. The 2-form $\Omega^{CS}_{cc'}$ is therefore readily invertible, and the associated Poisson bracket (which satisfies the Jacobi identity if and only if $\Omega^{CS}_{cc'}$ is closed) is given by

$$\{H^{(1)}_{cc'}, \tilde{H}^{(2)}_{vv'}\} = 2T^\mu H_{cc'} \otimes H^{-1}_{cv} T_\mu H_{cv} \tilde{H}_{vv'}$$
$$= H^{(2)}_{vc} \mathcal{C} H^{(1)}_{cc'} H^{(2)}_{cv'}, \tag{89}$$

which is exactly the bracket calculated in (85). Note that the Poisson bracket we have obtained in (89) does not obviously satisfy the Jacobi identity, due again to the presence of the connectors $H_{cv}$.

Our discretized symplectic 2-form allows one to therefore recover the Goldman bracket. Unlike the standard

calculation using (72), our Poisson bracket between holonomies is defined for any type of holonomies.[9]

In our formalism, we have recovered a Poisson bracket given in terms of the Casimir parallel transported which can be seen as a *symmetric* r-matrix [17]. Hence, we have recovered such an r-matrix by a discretization of the continuum theory. This is to be compared with the Fock-Rosly formalism [6] where such a structure is put by hand. Furthermore, in their framework, the r-matrix contains a nontrivial *antisymmetric* component which is essential for their construction.

Finally, we emphasize again that our derivation can be extended directly for any $Lie\mathcal{G}$.

## IV. CHERN-SIMONS THEORY AND GRAVITY

In Sec. II, we constructed two kinematical phase spaces. The first one is parametrized by $(h_\ell, \tilde{X}_{\tilde{\ell}}) \in T^*G$ and is associated with each *link* $\ell$ of $\Gamma$. The variables are subject to the Gauss constraints at each node $\mathcal{J}_c = 0$ where $\mathcal{J}_c \in \tilde{G} \sim \mathfrak{g}^*$ is in the *translational* component of $T^*G$ (i.e. the momentum component). Some gauge-invariant observables, the Wilson loops, are then generated by the *curvature* along $\Gamma$: $\prod_{\ell \in \Gamma} h_\ell \in G$.

The second one is parametrized by $(\tilde{h}_{\tilde{\ell}}, X_\ell) \in T^*G$ associated with each edge $\tilde{\ell}$ of $\Gamma^*$ and subject to the dual Gauss constraint (i.e. a flatness constraint) at each vertex $\mathcal{G}_c = 1$ where $\mathcal{G}_c \in G$ is in the *group* component of $T^*G$. Some gauge-invariant observables are then generated by the *torsion* along $\Gamma$: $\mathcal{J}_v \in \tilde{G}$. This apparent duality can be made more precise, through the notion of *symplectic dual pairs*.

**Definition 1.** A symplectic dual pair between two Poisson manifolds $(X_1, \{,\}_1)$ and $(X_2, \{,\}_2)$ is a correspondence symplectic manifold $(Y, \{,\}_Y)$ such that, given the Poisson maps $\iota_1$ and $\iota_2$,

$$
\begin{array}{ccc}
 & Y & \\
\iota_1 \swarrow & & \searrow \iota_2 \\
X_1 & & X_2,
\end{array} \tag{90}
$$

we have for any $f \in \mathcal{C}(X_1)$ and $h \in \mathcal{C}(X_2)$

$$\{\iota_1^* f, \iota_2^* h\}_Y = 0. \tag{91}$$

Such precise notion of duality will be useful when considering the quantum versions of our different formulations. The notion of duality then translates into the notion of Morita equivalence, which allows one to relate the different representations obtained upon quantization [18].

In our context, the correspondance symplectic manifold is the phase space generated by the discretized Poincaré

connections. To show the symplectic duality, we are going to show that the symplectic structure on the discretized Poincaré connections phase space can be expressed in terms of the each of the symplectic structures of the two LQG phase spaces,

$$\Omega_{\text{discrete}}^{\text{CS}} = \frac{1}{2}(\Omega^{\text{LQG}} + \Omega^{\text{LQG}^*}). \tag{92}$$

This is the discrete analog of the symmetric choice in (13).

### A. Gravity and Chern-Simons theory

The gravity action can be related to the Chern-Simons action by a symplectic transformation. Since we are interested in gravity with a zero cosmological constant, we shall consider a Chern-Simons theory built on the Euclidian group or the Poincaré group. The Poincaré (or Euclidian) Lie algebra $Lie\mathcal{G}$ generated by $T_\mu = (J_A, P_B)$, $A, B = 1, 2, 3$, with brackets,

$$[J_A, J_B] = \epsilon_{ABC}J^C, \qquad [J_A, P_B] = \epsilon_{ABC}P^C,$$
$$[P_A, P_B] = 0, \quad \text{and} \quad J_A^\dagger = -J_A, \qquad P_A^\dagger = P_A. \tag{93}$$

The indices are raised with the metric $\eta_{AB}$, the Minkowski or Euclidian metric, according to the choice of spacetime signature. A convenient parametrization of $Lie\mathcal{G}$ is given by setting [19]

$$P_A = \theta J_A, \quad \text{with} \quad \theta^2 = 0. \tag{94}$$

$\theta$ can be seen as a Grassmanian number, which plays a role similar to the imaginary number *i*. Following Ref. [19], given two real numbers *a* and *b*, we have

$$\overline{(a + \theta b)} = a - \theta b. \tag{95}$$

The pairing between the generators (see Ref. [20] for a discussion on the most general pairing one can consider) is the Killing form of $Lie\mathcal{G}$,

$$\langle P_A, J_B \rangle = \langle J_A, P_B \rangle = \eta_{AB} = -2 \int d\theta \text{tr}(J_A P_B), \tag{96}$$

where $\eta_{AB}$ is the Minkowski (respectively Euclidian) metric if we deal with a Lorentzian (respectively Euclidian) spacetime.

To obtain the gravity variables from the Chern-Simons ones, we introduce a pair of connections $\mathcal{A}_\pm$,

$$\mathcal{A}_\pm \equiv J_I \omega^I \pm P_I e^I,$$
$$\tilde{\mathcal{A}}_\pm^a \equiv \tilde{\epsilon}^{ba}(J_I \omega_b^I \pm P_I e_b^I) = J_I \tilde{\omega}^{Ia} \pm P_I \tilde{\mathbf{e}}^{Ia}. \tag{97}$$

Note that these connections are not independent since we have that $\mathcal{A}_+^\dagger = -\mathcal{A}_-$ and similarly for $\tilde{\mathcal{A}}_\pm$.

---

[9]The Poisson bracket of holonomies sharing some initial/final point is not defined if using (72). Our discretization scheme regularized this issue.

We are now interested in recovering the symplectic 2-forms related to gravity. First, we note that

$$\Omega_{\pm}^{\mathrm{CS}} = \frac{1}{2}\langle \delta\mathcal{A}_{\pm} \wedge \delta\mathcal{A}_{\pm}\rangle = \pm\frac{1}{2}(\langle \delta e \wedge \delta\omega\rangle + \langle \delta\omega \wedge \delta e\rangle)$$
$$= \pm\langle \delta e \wedge \delta\omega\rangle, \tag{98}$$

so that the symmetric expression in terms of the gravity variables comes indeed from the Chern-Simons symplectic form as alluded to in (13). Conversely, we have also

$$\Omega_{\mathrm{grav}}^{\mathrm{LQG}} = \langle \delta e \wedge \delta\omega\rangle = \frac{1}{4}\langle \delta(\mathcal{A}_+ - \mathcal{A}_-) \wedge \delta(\mathcal{A}_+ + \mathcal{A}_-)\rangle$$
$$= \frac{1}{2}(\Omega_+^{\mathrm{CS}} - \Omega_-^{\mathrm{CS}}). \tag{99}$$

Finally, we can relate the gravity action to the Chern-simons action. Using the Killing form based on the Casimir $J^I \otimes P_I + P^I \otimes J_I$ for the Chern-Simons actions, we have

$$\mathcal{S}_{\mathrm{grav}} = \frac{1}{2}(\mathcal{S}_{\mathrm{CS}}(\mathcal{A}_+) - \mathcal{S}_{\mathrm{CS}}(\mathcal{A}_-)). \tag{100}$$

### B. Relating both Chern-Simons and gravity discretized variables

As mentioned earlier, we are interested in the specific case where $Lie\mathcal{G} = \mathfrak{iso}(3)$ or $\mathfrak{iso}(2,1)$, that is the gravity case with $\Lambda = 0$. First, we split the Poincaré holonomy $H_c = t_c h_c$ into the translational and rotational parts, respectively $t_c$ and $h_c$. The Chern-Simons connection can also be expressed in terms of the gravity variables,

$$H_c^{-1}\mathrm{d}H_c = h_c^{-1}(t_c^{-1}\mathrm{d}t_c)h_c + h_c^{-1}\mathrm{d}h_c$$
$$= h_c^{-1}(\bar{\mathbf{e}}^K P_K)h_c + \bar{\omega}^K J_K$$
$$= \mathcal{A} = e^K P_K + \omega^K J_K. \tag{101}$$

By construction, $t_c$ is an element of $\mathbb{R}^3$, so $\bar{\mathbf{e}}$ is a $\mathbb{R}^3$-connection. Furthermore, there is only an action of $\mathfrak{su}(2)$ on $\mathbb{R}^3$ and no back-action. Therefore, we can identify the connection components term by term,

$$\bar{\omega} = \omega, \qquad h_c^{-1}\bar{\mathbf{e}}h_c = e. \tag{102}$$

Bearing in mind the discretization of gravity of Sec. II B, it is natural to identify the discretized Chern-Simons components $(h_c, t_c)$ with the discretized gravity ones $(g_c, \mathbf{y}_c)$ in (18),

$$\bar{\omega} = \omega \Rightarrow h_c \leftrightarrow g_c. \tag{103}$$

Taking advantage of the Grassmannian parameter $\theta$ which gets the exponential linearized (since $\theta^2 = 0$), we have

$$H_c = t_c g_c = e^{\theta \int_c^x d\mathbf{y}_c} g_c = (1 + \theta\mathbf{y}_c)g_c, \quad \text{with}$$
$$\mathbf{y}_c = \int_c^x d\mathbf{y}_c. \tag{104}$$

We can recover the different fluxes we have introduced when considering the gravity discretization,

$$t_{cv}^{-1}t_{cv'} = 1 + \theta\left(\int_c^v d\mathbf{y}_c - \int_c^{v'} d\mathbf{y}_c\right) = 1 + \theta\tilde{X}_{vv'}$$

$$t_c(h_{cc'}t_{c'}^{-1}h_{cc'}^{-1}) = 1 + \theta\left(\int_c^x d\mathbf{y}_c - h_{cc'}\left(\int_{c'}^x d\mathbf{y}_c\right)h_{cc'}^{-1}\right)$$
$$= 1 + \theta(\mathbf{y}_c - h_{cc'}\mathbf{y}_{c'}h_{cc'}^{-1}) = 1 - \theta x_{cc'}, \tag{105}$$

where we used first (33) and then (30).

We can then go further and relate the holonomies entering into the Chern-Simons symplectic form to the (dual) LQG ones. Bearing in mind (50) and (104), we have

$$\tilde{H}_{vv'} = H_{cv}^{-1}H_{cv'} = g_{cv}^{-1}(t_{cv}^{-1}t_{cv'})g_{cv}g_{cv'}^{-1}g_{cv'}$$
$$= g_{cv}^{-1}(t_{vv'})g_{cv}\tilde{h}_{vv'} = g_{cv}^{-1}(1 + \theta\tilde{X}_{vv'}^c)g_{cv}\tilde{h}_{vv'}$$
$$\equiv \tilde{L}_{vv'}\tilde{h}_{vv'}. \tag{106}$$

In a similar manner, bearing in mind (28), (104) and (52),

$$H_{cc'} = H_c H_{c'}^{-1} = t_c g_c g_{c'}^{-1} t_{c'}^{-1} = (t_c(h_{cc'}t_{c'}^{-1}h_{cc'}^{-1}))h_{cc'}$$
$$= (1 + \theta x_{cc'})h_{cc'}$$
$$\equiv L_{cc'}h_{cc'}. \tag{107}$$

Having identified the relationship between the Chern-Simons and gravity discretized variables, it is interesting to see how the Chern-Simons kinematical flatness constraint (80) appears from the gravity perspective. A simple calculation shows that it is equivalent to the "kinematical" constraints we have obtained for the LQG and dual LQG variables,

$$\tilde{H}_{v_1v_2}^c\tilde{H}_{v_2v_3}^c\tilde{H}_{v_3v_1}^c = 1 \Leftrightarrow \tilde{X}_{v_1v_2} + \tilde{X}_{v_2v_3} + \tilde{X}_{v_3v_1} = 0,$$
$$\tilde{h}_{v_1v_2}\tilde{h}_{v_2v_3}\tilde{h}_{v_3v_1} = 1. \tag{108}$$

### C. Relating the discretized symplectic 2-forms

We would like to recover the discretized analog of (13). To this aim, it will be useful to redo the full analysis of Sec. III B, while keeping track of the link between the Chern-Simons and gravity discrete variables.

We start with the Chern-Simons discretized symplectic form on a face $c^*$,

$$\Omega_{c^*}^{CS} = \frac{1}{2} \int_{c^*} \langle d(\delta H_c H_c^{-1}) \wedge d(\delta H_c H_c^{-1}) \rangle, \qquad (109)$$

and perform part of the integration to consider only the integration on the boundary $\partial c^*$ of the face,

$$\Omega_{c^*}^{CS} = \frac{1}{2} \int_{\partial c^*} \langle (\delta H_c H_c^{-1}) d(\delta H_c H_c^{-1}) \rangle. \qquad (110)$$

We recall that the gravity symplectic form is related to the Chern-Simons symplectic form as given in (98). So, we can check now how the discretization affects this, by recovering (23) or (24) from (110). We have that

$$\delta H_c H_c^{-1} = \delta t_c t_c^{-1} + t_c (\delta g_c g_c^{-1}) t_c^{-1}$$
$$= \delta g_c g_c^{-1} + \theta(\delta \mathbf{y}_c + [\mathbf{y}_c, \delta g_c g_c^{-1}]). \qquad (111)$$

Plugging this back into (110), and restricting $\partial c^*$ to the edge $\tilde{\ell}$, we have

$$\begin{aligned}
\Omega_c^{CS} &= \frac{1}{2} \int_{\tilde{\ell}} \langle (\delta H_c H_c^{-1}) d(\delta H_c H_c^{-1}) \rangle \\
&= - \int_{\tilde{\ell}} \langle d(\delta g_c g_c^{-1})(\delta \mathbf{y}_c + [\mathbf{y}_c, \delta g_c g_c^{-1}]) \rangle \\
&\quad + \frac{1}{2} [\langle (\delta g_c g_c^{-1})(\delta \mathbf{y}_c + [\mathbf{y}_c, \delta g_c g_c^{-1}]) \rangle]_v^{v'} \\
&= - \int_{\tilde{\ell}} \delta \langle d(\delta g_c g_c^{-1}) \mathbf{y}_c \rangle \\
&\quad + \frac{1}{2} [\langle (\delta g_c g_c^{-1})(\delta \mathbf{y}_c + [\mathbf{y}_c, \delta g_c g_c^{-1}]) \rangle]_v^{v'} \\
&= \Omega_c + \text{boundary terms}. \qquad (112)
\end{aligned}$$

We have recognized in the first term the analog of the discretized symplectic form (24), now restricted to the edge $\tilde{\ell}$, whereas the second contribution is a boundary term. Summing up over all of $\partial c^*$, this boundary term gives 0.

Since we have managed to recover $\Omega_c$, we can recover either $\Omega_{cc'}^{\text{LQG}}$ or $\Omega_{cc'}^{\text{LQG}^*}$ when considering the contributions from the faces $c^*$ and $c'^*$, according to where we put the integration, either on $\delta g_c g_c^{-1}$ or $\mathbf{y}_c$. Hence, we have that for the pair of faces $c^*$ and $c'^*$

$$\Omega_{cc'}^{CS} = \frac{1}{2} (\Omega_{cc'}^{\text{LQG}} + \Omega_{cc'}^{\text{LQG}^*}) + \text{boundary terms}. \qquad (113)$$

When considering the full graph $\Gamma$, we have then

$$\Omega_{\text{discrete}}^{CS} = \frac{1}{2} (\Omega^{\text{LQG}} + \Omega^{\text{LQG}^*}). \qquad (114)$$

## V. OUTLOOK

Three-dimensional gravity is a nice laboratory to explore some aspects of quantum gravity that could be relevant for four-dimensional gravity. Its connection with Chern-Simons theory also offers rich links with other theories such as the Wess-Zumino model or conformal theory [1].

In this paper, we had a fresh look at the discretization of the 3d gravity with zero cosmological constant, as done in the standard LQG framework. Keeping track of how the continuum variables are related to the discrete variables allowed us to uncover a new discretization, based on the metric variable (triad). This new framework can be seen as a different choice of polarization, hence in a sense as a dual view of the standard LQG approach. Although the quantization of the model is still to be done, it could be related to the recent work of Dittrich and Geiller [21,22]. In the dual polarization, we see now the flatness constraint as the kinematical constraint, whereas the torsionless constraint will become the dynamical constraint. In this sense, this is a dual picture, where both constraints are exchanged. Hence, in terms of a spinfoam model, we expect to have a model dual to the Ponzano-Regge model. This is a work in progress.

This fresh look highlighted the fact that nontrivial symmetries are at play, even in the $\Lambda = 0$ case. These nontrivial Poisson-Lie group symmetries will lead to quantum group symmetries upon quantization of the theory. Such quantum group symmetries were already identified from different quantized approaches [23–25], but this time, we identified them at the classical level.

Since the phase space $T^*SU(2)$ is at the root of LQG both in three and four dimensions, it is likely that our construction, based on the *metric* formalism, should be generalizable to the four-dimensional (4d) picture. Note, however, that the 3d case is easier than the 4d case, since we essentially deal with graphs ($\Gamma$ and $\Gamma^*$) dual to each other. In four dimensions, we would have to deal with graphs dual to 2-complexes, which could make the discretization of the holonomy harder. We leave this interesting issue for later.

We have used our discretization approach to Chern-Simons theory as well. This provided a new scheme which allowed us to recover the Goldman bracket, while keeping a clear link with the continuum variables, unlike the Fock-Rosly formalism. Since we used the same discretizing scheme as for the gravity one, the link between the gravity and the Chern-Simons variables is very clear. Note that this link between the Fock-Rosly formalism and gravity was discussed in Refs. [10,26]. Our new approach to Chern-Simons should be studied

further. For example, one should check whether we recover the right spaces (Heisenberg or Drinfeld doubles) [7,8] when dealing with a punctured space (i.e. particles) or if handles are present. Due to the transparent link with gravity, it would also be interesting to explore how, when introducing boundaries, we can connect the Wess-Zumino model or the conformal theory field theory to loop (quantum) gravity.

Finally, we hope also that our approach should shed some light on why we deal with quantum group structures when the cosmological constant is not zero. As we mentioned already, quantum group structures already appear when $\Lambda = 0$, but from the LQG perspective, there have been many discussions on why we have to deal with a quantum group such as $SU_q(2)$ (in the Euclidian case) when $\Lambda < 0$ [27–29]. Our present work showed that the choice of polarization matters, and it is likely that such a quantum group will appear when a different polarization than the usual one is chosen. This is a work in progress.

## APPENDIX: MOMENTUM MAPS AND SYMMETRIES

In this Appendix, we review the key notions of symplectic geometry that are relevant for our results. There are two different ways to discuss the notion of symmetries when dealing with symplectic (or more generally Poisson) geometry. In this Appendix, we would like to review the different approaches.

The first one, more mathematically inclined, consists in defining the action $\rhd$ of a group $G$ on the symplectic manifold $M$. To this aim, it is necessary to equip $G$ with a Poisson structure (which cannot be symplectic by construction) compatible with the group product (so that the action is transitive). Hence, $G$ must be a *Poisson-Lie group* [15,17]. We demand then that

$$g \rhd \{f_1, f_2\}_M = \{g \rhd f_1, g \rhd f_2\}_{G \times M}$$
$$= \{g \rhd f_1, g \rhd f_2\}_G + \{g \rhd f_1, g \rhd f_2\}_M,$$
$$f_1, f_2 \in \mathcal{C}(M). \qquad (A1)$$

In physical language, this is stating that a symmetry action on the Poisson bracket of two observables must be the same

as the Poisson bracket of the symmetry transformed observables.

If the Poisson bracket on $G$ is trivial, i.e. 0, as it often is in standard physics, the only relevant contribution is coming from $M$. However, we can put nontrivial Poisson bracket structures on $G$ and get new realizations of symmetries which are not accessible if setting the Poisson bracket on $G$ to be zero.

For example, consider the phase space $M = \mathbb{C}^2 \ni z_i, i = 1, 2$, equipped with the following symplectic structure [30],

$$\{z_1, z_2\}_M = \frac{i}{\beta} z_1 z_2, \qquad \{z_1, \bar{z}_2\}_M = \frac{i}{\beta} z_1 \bar{z}_2, \qquad (A2)$$

$$\{z_1, \bar{z}_1\}_M = -i\left(1 - \frac{2}{\beta}(z_1 \bar{z}_1)\right),$$
$$\{z_2, \bar{z}_2\}_M = -i\left(1 - \frac{2}{\beta}(z_1 \bar{z}_1 + z_2 \bar{z}_2)\right), \qquad (A3)$$

where $\beta$ is a deformation parameter and all other brackets are zero. These Poisson brackets are not covariant under the transformations of $SU(2) \ni g = \begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$ with $\det g = 1$, unless there is a nontrivial Poisson structure on $SU(2)$,

$$\{\alpha, \bar{\alpha}\} = -\frac{2i}{\beta} \gamma \bar{\gamma}, \qquad \{\alpha, \gamma\} = \frac{i}{\beta} \alpha \gamma,$$

$$\{\alpha, \bar{\gamma}\} = \frac{i}{\beta} \alpha \bar{\gamma}, \qquad \{\gamma, \bar{\gamma}\} = 0. \qquad (A4)$$

As such, symmetries with nontrivial Poisson bracket can be seen as "hidden symmetries."

The second one, more commonly found in physics, consists in representing the infinitesimal symmetry transformations using a function on $M$ (with a value in a space we are going to determine shortly) and the Poisson bracket on $M$. This function will be called a *momentum map* since, for example, it is well known that when dealing with $M$ being a cotangent bundle the momentum coordinates generate the infinitesimal transformations on configuration space.

However, these are not the only symmetry transformations one might be interested in. For example, we could be interested in the infinitesimal transformations generated by the configuration space (provided it has a group structure) on momentum space, or it might happen that the phase space $M$ is not of the cotangent type, so that we need to have in hand a formalism that accounts for these general situations.

Let us note $\mathfrak{g}$ and $\mathfrak{g}^*$, the Lie algebra of $G$ and its dual, the Lie algebra of the group $G^*$. We have also $e_a$, $e^b$ their respective basis, and $\chi$ the vector field on $M$ implementing the infinitesimal transformation spanned by the Lie algebra element $x \in \mathfrak{g}$,

$$\chi \cdot f = \frac{d}{dt}(e^{tx} \rhd f)|_{t=0}. \tag{A5}$$

We define the momentum map $\mathcal{P}$ which will implement the infinitesimal transformation using the Poisson bracket on $M$ [14,31],

$$\mathcal{P} \colon M \to G^* \xi \to g_*(\xi) = e^{-Q(\xi)} \text{ such that } \chi \cdot f$$
$$= \langle g_*^{-1}\{f, g_*\}_M, x\rangle, f \in \mathcal{C}(M), \tag{A6}$$

where $g_*^{-1}$ is the inverse of the group element in $G^*$ and $\langle,\rangle$ is the bilinear form between $\mathfrak{g}$ and $\mathfrak{g}^*$. $Q(\xi)$ is a $\mathfrak{g}^*$-valued function which is called the *charge* generating the Poisson-Lie group action. Note that in this definition, $\chi$ is a right-invariant vector; a similar definition would hold for the left-invariant one.

We can connect this definition to the more abstract definition of the notion of symmetry action (A1). By considering the infinitesimal version of (A1), bearing in mind that the infinitesimal version of the Poisson bracket on $G$ is given by the structure constant $C_a^{bc}$ of $\mathfrak{g}^*$, we get that

$$e_a \cdot \{f_1, f_2\}_M = C_a^{bc}(e_b \cdot f_1)(e_c \cdot f_2) + \{e_a \cdot f_1, f_2\}_M$$
$$+ \{f_1, e_a \cdot f_2\}_M. \tag{A7}$$

On the other hand, using the definition of the symmetry action involving the momentum map, we recover the same expression,

$$\chi \cdot \{f_1, f_2\} - \{\chi \cdot f_1, f_2\} - \{f_1, \chi \cdot f_2\}$$
$$= \langle g_*^{-1}\{\{f_1, f_2\}, g_*\} - \{g_*^{-1}\{f_1, g_*\}, f_2\}$$
$$- \{f_1, g_*^{-1}\{f_2, g_*\}\}, X\rangle$$
$$= \langle [g_*^{-1}\{f_1, g_*\}, g_*^{-1}\{f_2, g_*\}], X\rangle, \tag{A8}$$

where we used the Jacobi identity and the last term provides the contribution $C_a^{bc}$ accounting for the nontrivial Poisson structure on $G^*$.

Let us illustrate this construction when we deal with $M = T^*G \sim G \rhd\!\!< \mathfrak{g}^*$ [or example $G = SU(2)$], with coordinates $(X, g)$ in the left trivialization. In this case, $G^* \sim \mathfrak{g}^* \sim \mathbb{R}^3$ is an Abelian group. We use the Poisson bracket

$$\{X_i, X_j\} = \epsilon_{ij}^k X_k, \quad \{X_i, g\} = J_i g, \quad \{g, g\} = 0. \tag{A9}$$

We note therefore that $G$ has a trivial Poisson structure, whereas $G^* \sim \mathfrak{g}^*$ is equipped with a nontrivial one, specified by the Lie algebra structure constants $\epsilon_{ij}^k$ of $\mathfrak{g}$.

Since we have two groups, $G$ and $G^*$, we can construct two types of momentum maps: the standard one $\mathcal{J}$, with

value in $G^* \sim \mathfrak{g}^*$ and a may be less usual one $\mathcal{G}$ with value in $G$. The former one will implement the infinitesimal rotations on configuration space, while the latter one will implement the infinitesimal translations (since $G$ is non-Abelian, there will be a difference between left and right) on $G^*$.

We define then (we recall we use the left trivialization of $T^*G$ and we note respectively $P_i$ and $J_j$ the generators of the Lie algebras $\mathbb{R}^3$ and $\mathfrak{g}$, such that $\langle P_i, J_j\rangle = \delta_{ij}$)

$$\mathcal{J} \colon T^*G \to \mathfrak{g}^* \sim G^* \sim \mathbb{R}^3 \quad \mathcal{G} \colon T^*G \to G$$
$$(X, g) \to g_*(X) = e^{-X^i P_i} \quad (X, g) \to g. \tag{A10}$$

The infinitesimal transformations they implement are respectively ($\beta^i \in \mathbb{R}^3$)

$$\chi^i_{\mathcal{J}} \cdot X^j = \langle g_*^{-1}\{X^j, g_*\}, J^i\rangle = \epsilon^{ijk} X_k,$$
$$\chi^i_{\mathcal{J}} \cdot g = \langle g_*^{-1}\{g, g_*\}, J^i\rangle = -J^i g \tag{A11}$$

$$\beta_i \chi^i_{\mathcal{G}} \cdot X^j = \langle g^{-1}\{X^j, g\}, \beta_i P^i\rangle = g \rhd \beta^i,$$
$$\beta_i \chi^i_{\mathcal{G}} \cdot g = \langle g^{-1}\{g, g\}, \beta_i P^i\rangle = 0, \tag{A12}$$

where $g \rhd \beta^i$ denotes the coadjoint action[10] of $g$ on the vector $\beta^i$.

A phase space based on the manifold $M = G^* \bowtie G$ is called the Heisenberg double [15,17,32]. The cotangent bundle $T^*SU(2) \sim SU(2) \rhd\!\!< \mathbb{R}^3$ is a simple example of such a structure. The general Poisson-Lie symmetry of the Heisenberg double is given by the Drinfeld double $D = G^* \bowtie G$, which acts by left or right translation on $M$. As we discussed in our simple example, due to the symmetry between $G$ and $G^*$, we can build momentum maps with value in $G$ or $G^*$. The general momentum map associated to the Drinfeld double has value in $D^* \sim D$.

In our construction, we are interested in putting together many copies of the same Heisenberg double $M$ and considering some global symmetry transformations. The momentum map is then extended using the group product [14]. First, let us recall that the Poisson structure on $\mathcal{C}(M^{\times n}) \sim \mathcal{C}(M)^{\otimes n}$ is given by the sum of the Poisson bracket on each of the individual components $\mathcal{C}(M)$. If $G$ acts on $M^{\times n}$, then we define the associated momentum map and the global infinitesimal (right) action by

$$\mathcal{P}_{\text{tot}} \colon M^{\times n} \to G^*$$
$$(\xi_1, \dots \xi_n) \to g_*(\xi_1) \dots g_*(\xi_n)$$
$$= g_{*\text{tot}} \text{ such that } \chi_{\text{tot}} \cdot f = \langle g_{*\text{tot}}^{-1}\{f, g_{*tot}\}_{M^{\times n}}, x\rangle, \tag{A13}$$

with $f \in \mathcal{C}(M^{\times n})$.

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[10]Hence if $G = SU(2)$, $\beta = \beta^i J_i$, then $g \rhd \beta = g^{-1}\beta g$.

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