Holographic cosmology and phase transitions of SYM theory

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We study the time development of strongly coupled $\mathcal{N} = 4$ supersymmetric Yang Mills (SYM) theory on cosmological Friedmann-Robertson-Walker (FRW) backgrounds via the AdS/CFT correspondence. We implement the cosmological background as a boundary metric fulfilling the Friedmann equation with a four-dimensional cosmological constant and a dark radiation term. We analyze the dual bulk solution of the type IIB supergravity and find that the time dependence of the FRW background strongly influences the dynamical properties of the SYM theory. We in particular find a phase transition between a confined and a deconfined phase. We also argue that some cosmological solutions could be related to the inflationary scenario.

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I. INTRODUCTION

The holographic approach [1–3] to supersymmetric Yang Mills (SYM) theory can be extended from the theory on flat Minkowski space-time to the one on curved boundary background space-times [4–7]. Such an extension is useful to understand for example the dynamical properties of the SYM theory on cosmologically developing four-dimensional (4D) Friedmann-Robertson-Walker (FRW) type space-times. Indeed, the influence of the cosmological evolution on several dynamical properties of the SYM theory such as e.g. the quark-antiquark potential have been clarified in [4–7]. Since the deconfined phase of QCD behaves in many ways similar to SYM theory at finite temperature, these studies yield interesting information about the behavior of the quark-gluon plasma in the early universe before big bang nucleosynthesis.

In the holographic approach, the scale factor $a_0(t)$ is undetermined by the bulk equations of motion due to gravity decoupling from the boundary and the boundary metric being nondynamical. For the same reason, the boundary cosmological constant can *a priori* be any function of time, $\lambda(t)$. In [5,6], the vacuum state of the SYM theory has been examined by setting $\lambda = (\dot{a}_0/a_0)^2 + k/a_0^2$ to a constant parameter. This is equivalent to setting $a_0(t)$ to be a solution of the 4D Friedmann equation with a 4D cosmological constant $\Lambda_4 = 3\lambda$. It has been found in particular [5,6] that the ground state of the SYM theory is in the deconfinement and confinement phases for the positive and the negative values of λ, respectively.

In [7], this analysis has been extended by including a new parameter C, the dark radiation constant, into the bulk solution. This parameter has first been introduced in [8,9] in the context of brane world models, and its origin has been discussed in [10,11]. From the holographic viewpoint, as shown in [7], the bulk solution is related by a large bulk diffeomorphism to the AdS_5-Schwarzschild metric with Hawking temperature set by C. The bulk diffeomorphism in particular acts on the boundary metric as a conformal transformation relating the FRW boundary metric to flat Minkowski space-time. Its field theory dual therefore represents a finite temperature SYM state in a FRW universe. In general, the dynamical properties of the dual theory are now controlled by the two parameters, λ and C. In [12], the analysis was extended to time-dependent $\lambda(t)$ by assuming the existence of the matter other than the 4D cosmological constant Λ_4 in the boundary space-time; however, the time-dependent properties of the SYM state were not discussed there. It was found [12] that the system is in the deconfinement phase when C is large enough even if the boundary cosmological constant λ is negative. Hence the two parameters C and λ can provide opposite dynamical effects in the theory, as shown in [12,13].

In this work we extend the analysis to the case where we can see how the state of the SYM system varies with the cosmological development of our Universe. Our purpose is to propose a self-consistent procedure how to do it and to find the time dependence of the state of the SYM theory. The procedure is as follows: First, we obtain the energy momentum tensor of the SYM theory, $\langle T_{\mu\nu}^{\mathrm{YM}} \rangle$, in the FRW boundary metric from holographic renormalization [14–16]. Then by using this tensor, the 4D Einstein equations [i.e. the Friedmann equation for the at-that-point arbitrary scale factor $a_0(t)$] with Λ_4 and SYM as matter are solved to obtain the boundary scale factor $a_0(t)$. After that, the time development of the SYM state is then analyzed by substituting this $a_0(t)$ back into the bulk solution. We notice that the 4D Einstein equations used here were

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obtained in [17] from a different holographic regularization procedure by adding a 4D boundary gravitational action. This formulation may be related also to the designer gravity approach of [18] to couple two AdS space-times together such that at the boundary a gravitational theory is induced.

The solutions of $a_0(t)$ obtained in this way provide a "sudden" singularity [19] in the 4D boundary space-time at the minimum value of the scale factor $a_0^{\rm min}$. The boundary cosmological constant λ is found as a two valued function of a_0 for $a_0 > a_0^{\rm min}$. The two branches merge at the minimal value. We then study the solutions for the two cases $\Lambda_4 < 0$ and $\Lambda_4 > 0$ separately.

The region of negative λ[1] is realized in the case of $\Lambda_4 < 0$, where we observe the expected phase transition between Wilson loop confinement and deconfinement. The equation of state $w = p/\rho$, where p and ρ denote the pressure and energy densities of the SYM fields, is found to be a useful parameter for this transition. The parameter w, which varies with time, characterizes the state of the SYM theory. We found that the critical point of the confinement-deconfinement transition is at exactly $w = -1/3$. We speculate on why this point is critical in terms of the virial theorem. For $\Lambda_4 > 0$, on the other hand, inflation behavior will be found through this analysis, and issues related to their cosmology are discussed.

As for the second branch of the solutions, which cover a larger positive region of λ, a simple behavior of the solution $a_0(t)$ is obtained, and the relation to the cosmology is discussed. In both cases, the value of $a_0(t)(> a_0^{\rm min})$ is bounded from below when the dark radiation C is present. This implies that we need to take into account quantum gravitational effects to approach the dynamics in the region of $a_0 < a_0^{\rm min}$ since the curvature diverges at $a_0^{\rm min}$. In this region, an approach from quantum cosmology in the mini-superspace would be available to resolve the dynamical properties of the system.

The outline of this paper is as follows: In the next section, a bulk $M_5 \times S^5$ space-time is given as a solution of ten-dimensional (10D) type IIB supergravity, and previous results for the dual of SYM theory in the FRW space-time are reviewed. In Sec. III, the procedure to obtain the cosmological time development of the SYM theory by self-consistently calculating $a_0(t)$ is laid out in detail, and the solutions which cross the (de)confinement transition point are studied. Other kinds of solutions and their relation to the inflationary scenario are discussed in Sec. IV. The solutions, which cover a large λ region, are given in Sec. V, and the relation to the cosmology is discussed. The summary and discussions are given in Sec. VI.

II. HOLOGRAPHY OF SYM THEORY IN A FRW METRIC

We start with ten-dimensional type IIB supergravity retaining the dilaton Φ, axion χ, as well as the self-dual five form field strength $F_{(5)}$,

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(R_{10} - \frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}e^{2\Phi}(\partial\chi)^2 - \frac{1}{4\times 5!}F_{(5)}^2\right). \tag{2.1}$$

All other fields are set to zero, and χ is Wick rotated to the Euclidean domain [20]. We are going to use the Freund-Rubin ansatz for $F_{(5)}$, $F_{\mu_1\cdots\mu_5} = -\sqrt{\Lambda_5}/2\epsilon_{\mu_1\cdots\mu_5}$ [21,22] to reduce to the five noncompact dimensions. The equations of motion of the above theory are then solved by an Ansatz for the 10D metric as $M_5 \times S^5$,

$$ds_{10}^2 = g_{MN}dx^M dx^N + g_{ij}dx^i dx^j = g_{MN}dx^M dx^N + R^2 d\Omega_5^2.$$

The five sphere radius R is defined via $1/R = \sqrt{\Lambda_5}/2$.

The holographic dual to the large N SYM gauge theory embedded in 4D space-time with dark energy (i.e. boundary cosmological constant) and dark radiation is then given by the following form of the metric:

$$ds_{10}^2 = \frac{r^2}{R^2}(-\bar{n}^2 dt^2 + \bar{A}^2 a_0^2(t)\gamma_{ij}(x)dx^i dx^j) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2, \tag{2.2}$$

where

$$\gamma_{ij}(x) = \delta_{ij}\gamma^2(x), \qquad \gamma(x) = 1\Big/\left(1 + k\frac{\bar{r}^2}{4\bar{r}_0^2}\right), \qquad \bar{r}^2 = \sum_{i=1}^{3}(x^i)^2, \tag{2.3}$$

and $k = \pm 1$, or 0. The arbitrary scale parameter $\bar{r}_0$ of three space is set hereafter as $\bar{r}_0 = 1$. The solution obtained from 10D supergravity of type IIB theory [7,12,23] is

[1]The reader should not confuse λ with Λ_4. Their exact difference and relation is discussed in Sec. III.

$$\bar{A} = \left(\left(1 - \frac{\lambda}{4\mu^2}\left(\frac{R}{r}\right)^2\right)^2 + \tilde{c}_0\left(\frac{R}{r}\right)^4\right)^{1/2}, \quad (2.4)$$

$$\bar{n} = \frac{(1 - \frac{\lambda}{4\mu^2}(\frac{R}{r})^2)(1 - \frac{\lambda + \frac{a_0}{\dot{a}_0}\dot{\lambda}}{4\mu^2}(\frac{R}{r})^2) - \tilde{c}_0(\frac{R}{r})^4}{\sqrt{(1 - \frac{\lambda}{4\mu^2}(\frac{R}{r})^2)^2 + \tilde{c}_0(\frac{R}{r})^4}}, \quad (2.5)$$

where

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = \lambda(t), \quad (2.6)$$

and

$$\tilde{c}_0 = C/(4\mu^2 a_0^4). \quad (2.7)$$

Some comments are in order:

(i) The parameter C was first introduced as the "dark radiation" term in the context of the brane world model [8,9]. It has been reinterpreted as the projection of the five-dimensional (5D) Weyl term in [10,11].

(ii) From the holographic viewpoint, it has been made clear in [7] that C parametrizes the radiation energy and pressure (the energy-momentum tensor) of the SYM fields induced from the holographic bulk geometry. In the following, we will use this fact.

(iii) $\lambda(t)$ is a function of $a_0(t)$ defined via the Friedmann equation (2.6). Since $a_0(t)$ cannot be determined by the equations of motion of the 10D bulk gravity, $\lambda(t)$ can be set to an arbitrary function of t. For any $\lambda(t)$ and C, the above solution satisfies the 10D gravity equations of motion given by (2.1).

In general, we can analyze how the physical quantities of the SYM theory varies with time only after the t dependence of $a_0(t)$ has been specified. Before attempting to do so, we briefly review the results of the previous analyses where the time dependence of $a_0(t)$ was assumed to be adiabatically slow.

A. Phase diagram for $\lambda =$ const

The dynamical properties of the theory can be studied under the assumption that $a_0(t)$ varies very slowly compared to the time scale of the dynamics of the SYM theory [7].[2] Since there is no equation to determine $a_0(t)$ in solving the 5D Einstein equations, there is no constraint on the time dependence of $a_0(t)$. The parameters appearing in the bulk configuration (λ and b_0) can be chosen arbitrarily.

In the case of time-independent $a_0(t)$, the factors $\bar{A}$ and $\bar{n}$ of the above solution can be written as

$$\bar{A} = \left(\left(1 + \left(\frac{r_0}{r}\right)^2\right)^2 + \left(\frac{b_0}{r}\right)^4\right)^{1/2}, \quad (2.8)$$

$$\bar{n} = \frac{(1 + (\frac{r_0}{r})^2)^2 - (\frac{b_0}{r})^4}{\bar{A}}, \quad (2.9)$$

$$r_0 = \sqrt{|\lambda|}R^2/2, \qquad b_0 = R\tilde{c}_0^{1/4}. \quad (2.10)$$

In this case, the dynamical properties of the dual 4D SYM theory on the boundary are controlled by the parameters λ and C, or equivalently by r_0 and b_0. In the following we summarize the results:

(i) $\mathbf{C = 0}$ *and finite* λ: In this case, it has been found that the SYM theory is in the confined (deconfined) phase for negative (positive) λ [5,6].

(ii) *Finite* $\mathbf{C}$ *and* $\lambda = 0$: In this case, the 5D metric is reduced to the AdS_5-Schwarzschild form, in which $C(>0)$ represents the black hole mass. Then, from the holographic viewpoint, C corresponds to the thermal radiation of the SYM fields at a finite temperature [7]. So the system is in the deconfined phase.

(iii) From the above two cases, we can suppose that the two parameters, C and $\lambda(<0)$, compete with each other; namely $C > 0$ favors deconfinement and $\lambda < 0$ favors confinement. In fact, we found the phase transition at a point where these two opposite effects are balanced [7,12,23,24]. The critical line is given by $b_0 = r_0$, where b_0 and r_0 are defined in terms of C and λ by (2.10) above. The phase diagram of the SYM theory in the FRW space-time is given in Fig. 1.

(iv) In the deconfinement phase or in the region $b_0 \geq r_0$, the temperature T is given as the Hawking temperature of the 5D metric. Then T decreases with

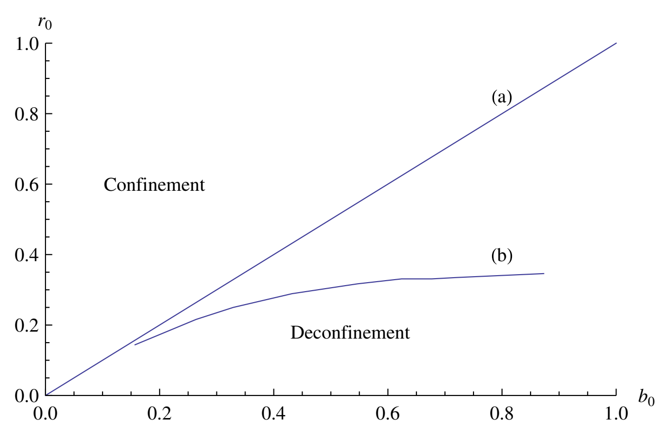

FIG. 1. Line (a) shows the critical line $r_0 = b_0$ separating the quark confinement phase from the deconfinement phase for the case of constant $a_0(t)$. Curve (b) represents the running $a_0(t)$ case obtained in this work. It obeys the formula $r_0 = (1 + \frac{\dot{\lambda} a_0}{\lambda \dot{a}_0})^{-1/4} b_0$ which is explained below.

[2]We should notice that the solution, $a_0 = 1/\sqrt{|\lambda|} =$ const is actually found for negative constant λ and $k = -1$. This is considered as an extremal case of the slowly varying $a_0(t)$.

increasing r_0 up to the transition temperature (T_c) to the confinement phase. It is given as $T_c = 0$, which corresponds to the critical line $r_0 = b_0$ in Fig. 1.

(v) For positive λ, the theory is always in the deconfined and chiral symmetric phase.

B. The case of time-dependent λ

The case of a general time independent $a_0(t)$ and $\lambda(t)$ is treated in the following. As shown below, we solve the 4D Einstein equations to obtain the time dependence of $a_0(t)$ and $\lambda(t)$ by using the 4D energy-momentum tensor of the SYM theory which is obtained by holographic renormalization. Then the phase transition line given by (a) in Fig. 1 is replaced by (b), which obeys the following formula:

$$r_0 = \left(1 + \frac{\dot{\lambda} a_0}{\lambda \dot{a}_0}\right)^{-1/4} b_0. \tag{2.11}$$

The derivation and the details of this result are given below.

III. COSMOLOGICAL TIME DEVELOPMENT OF SYM SYSTEM

A. Holographic cosmology

As explained above, the scale factor $a_0(t)$ cannot be determined by the bulk equations of motion. It is obtained instead as a solution of 4D cosmological equations, where the 4D gravity couples to the various kinds of matter present. Here we obtain the scale factor by solving the 4D Einstein equations which couple to the holographic SYM theory. In the equations, we use the energy-momentum tensor of the SYM theory, $\langle T^{\rm SYM}_{\mu\nu}\rangle$, which is obtained holographically. Using the $a_0(t)$ obtained in this way, we can analyze the time development of the state of the SYM theory.

In our model, the matter part of the 4D cosmology is dominated by the SYM fields. The action to be solved is given as

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa_4^2}(R_4 - 2\Lambda_4)\right\} + S^{\rm eff}_{\rm SYM}, \tag{3.1}$$

where $\kappa_4^2 = 8\pi G_4$ and Λ_4 denote the 4D gravitational constant and cosmological constant, respectively. Furthermore, the matter part $S^{\rm eff}_{\rm SYM}$ is to be understood as an effective action which is obtained by integrating out all quantum fluctuations of the SYM fields under the FRW metric,

$$ds^2_{(4)} = -dt^2 + a_0(t)^2\gamma_{ij}dx^i dx^j. \tag{3.2}$$

The factor $a_0(t)$ is obtained by solving the following Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_4 g_{\mu\nu} = \kappa_4^2\langle T^{\rm YM}_{\mu\nu}\rangle, \tag{3.3}$$

where

$$\langle T^{\rm YM}_{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta S^{\rm eff}_{\rm SYM}}{\delta g^{\mu\nu}} \tag{3.4}$$

represents the energy-momentum tensor of SYM fields. It includes all quantum corrections of the interacting SYM fields in the background (3.2). As we mentioned in the introduction, Eq. (3.3) has been obtained from the holographic regularization procedure with a boundary condition different from the usual Dirichlet condition by adding an appropriate boundary 4D gravitational action in [17].

The strategy of our holographic approach is given as follows:

1. First, $\langle T^{\rm YM}_{\mu\nu}\rangle$ is calculated by holographic renormalization.
2. Then $a_0(t)$ is obtained by solving Eq. (3.3) for the metric (3.2).
3. Then we examine the time development of the state of the SYM theory based on the holographic principle by using (2.2)–(2.7).

B. Boundary cosmology

The independent equations of (3.3) are

$$\lambda \equiv \left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = \frac{\Lambda_4}{3} + \frac{\kappa_4^2}{3}\langle T^{\rm YM}_{00}\rangle, \tag{3.5}$$

$$2\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = \Lambda_4 - \kappa_4^2\langle T^{\rm YM}_{ii}\rangle. \tag{3.6}$$

The first Eq. (3.5) represents the tt component of (3.3), and it is called the Friedmann equation. From the above two equations, we obtain the following continuity equation:

$$\dot{\rho} + 3H(\rho + p) = 0, \qquad H = \dot{a}_0/a_0, \tag{3.7}$$

where $T^{\rm YM}_{\mu\nu}$ is supposed to be written by ρ and p as

$$\langle T^{\rm YM}_{\mu\nu}\rangle = {\rm diag}(\rho, p g^0_{ij}). \tag{3.8}$$

The form of $T^{\rm YM}_{\mu\nu}$ is given as follows [12,14–16]:

$$\rho = 3\alpha\left(\frac{\tilde{c}_0}{R^4} + \frac{\lambda^2}{16}\right), \tag{3.9}$$

$$p = \alpha\left\{\frac{\tilde{c}_0}{R^4} - 3\frac{\lambda^2}{16}\left(1 + \frac{2\dot{\lambda}}{3\lambda}\frac{a_0}{\dot{a}_0}\right)\right\}, \qquad \alpha = \frac{4R^3}{16\pi G_N^{(5)}}. \tag{3.10}$$

It is easy to see that Eq. (3.7) is satisfied by the above ρ and p.

We notice that solving Eqs. (3.5) and (3.6) is equivalent to solving Eqs. (3.5) and (3.7). On the other hand, Eq. (3.7) is satisfied for the above $\langle T_{\mu\nu}^{\rm YM}\rangle$ as mentioned above. So, our task to do here is to solve the Friedman equation (3.5) to obtain $a_0(t)$, and the equation to be solved is written as

$$\lambda = \frac{\Lambda_4}{3} + \frac{\kappa_4^2}{3}\langle T_{00}^{\rm SYM}\rangle = \frac{\Lambda_4}{3} + \tilde{\alpha}^2(\tilde{b_0}^4 + \lambda^2), \qquad (3.11)$$

where

$$\tilde{\alpha}^2 = \frac{\kappa_4^2}{16}\alpha = \frac{\kappa_4^2 N^2}{32\pi^2}, \qquad \tilde{b_0} = \frac{2}{R^2} b_0. \qquad (3.12)$$

Equation (3.11) is then solved first with respect to λ as follows[3]:

$$\lambda = \lambda_\pm = \frac{1 \pm \sqrt{1 - 4\tilde{\alpha}^2(\Lambda_4/3 + \tilde{\alpha}^2\tilde{b_0}^4)}}{2\tilde{\alpha}^2}. \qquad (3.13)$$

In the following, we solve the above equations for $\lambda = \lambda_-$ and $\lambda = \lambda_+$ separately. The solutions $a_0(t)$ of the two equations can be connected at a singular point $a_0 = a_0^{\rm min}$, which is given below. We now discuss the two solution branches separately.

IV. THE $\lambda = \lambda_-$ SOLUTION

A. The case of $\Lambda_4 < 0$

In this section we study the state of the SYM theory in the region of $\lambda < 0$, which is realized for $\Lambda_4 < 0$. This parameter region is of particular interest since a phase transition is expected there. The case of $\Lambda_4 > 0$ is studied in the next section.

The case of $\lambda < 0$ is realized for the solution $\lambda = \lambda_-$ with negative $\Lambda_4 (= -|\Lambda_4|)$. In this case, the boundary Friedmann equation to be solved is

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = \lambda_- = \frac{1 - \sqrt{1 + 4\tilde{\alpha}^2|\Lambda_4|/3 - 4\tilde{\alpha}^4\tilde{b_0}^4}}{2\tilde{\alpha}^2}. \qquad (4.1)$$

Here λ_- is the corresponding solution of (3.13), and $\tilde{b_0}^4 = \frac{4C}{R^2 a_0^4}$. Here we specialized to the case of negative spatial curvature, $k = -1$, which is necessary to cover the region of negative λ. The solution for the scale factor $a_0(t)$ from (4.1) requires the value of λ to be in the range

$$\frac{1 - \sqrt{\tilde{\Lambda}_4}}{2\tilde{\alpha}^2} \le \lambda \le \frac{1}{2\tilde{\alpha}^2}, \qquad \tilde{\Lambda}_4 = 1 + 4\tilde{\alpha}^2|\Lambda_4|/3. \qquad (4.2)$$

Negative λ is hence found for

$$|\Lambda_4|/3 > \tilde{\alpha}^2\tilde{b_0}^4. \qquad (4.3)$$

On the other hand, in the range

$$|\Lambda_4|/3 < \tilde{\alpha}^2\tilde{b_0}^4 \le \frac{1}{4\tilde{\alpha}^2} + |\Lambda_4|/3, \qquad (4.4)$$

we find positive λ. The right-hand side of the inequality (4.4) denotes the condition for the reality of λ_-, and it provides the minimum of the allowed a_0. It is given as

$$a_0^{\rm min} = \tilde{\alpha}\left(\frac{16C}{R^2\tilde{\Lambda}_4}\right)^{1/4}. \qquad (4.5)$$

This point is called "sudden singularity" [19] since the scalar curvature diverges at this point due to the fact that $\ddot{a}_0 = \infty$ at this point as shown below.

Note that the left-hand inequality of (4.4) gives an upper constraint on a_0, but this is not the maximum value of a_0 possible by the Friedmann equation. Such a value is rather given by (4.8) below.

1. Solving the boundary Friedmann equation

First, we rewrite Eq. (4.1) in the following form:

$$\frac{1}{2}\dot{a}_0^2 + V_-(a_0) = 0, \qquad (4.6)$$

$$V_-(a_0) = \frac{k}{2} - \frac{1 - \sqrt{1 + 4\tilde{\alpha}^2|\Lambda_4|/3 - 4\tilde{\alpha}^4\tilde{b_0}^4}}{4\tilde{\alpha}^2} a_0^2. \qquad (4.7)$$

The value of the solution a_0 is restricted from below by $a_0^{\rm min}$ as given by (4.5). The upper bound, denoted $a_0^{\rm max}$, is given as the turning point, where $\dot{a}_0 = 0$ or $V_-(a_0^{\rm max}) = 0$. From (4.6) and (4.7), we obtain the upper bound to be

$$a_0^{\rm max} = \left(\frac{1 + \sqrt{1 + 4|\Lambda_4|\tilde{\alpha}^2(1 + 4C/R^2)/3}}{2|\Lambda_4|/3}\right)^{1/2}. \qquad (4.8)$$

We see that $V_-(a_0) > 0$ for $a_0 > a_0^{\rm max}$, and hence there is no real solution for $a_0 > a_0^{\rm max}$.

Here, starting from $a_0 = a_0^{\rm min}$, we solve (4.1) numerically for appropriate values of Λ_4 and C to see the characteristic properties of $a_0(t)$. A typical solution is shown in Fig. 2.

We should notice here that, at $t = 0$, $\dot{a}_0(t)$ is finite, but $\ddot{a}_0(t) = \infty$. This point is called a sudden singularity. At this point, the scalar curvature is divergent as seen from the curve (c), which represents $\ddot{a}_0(t)$, in Fig. 2. Our Friedmann

[3] We notice here

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = \lambda.$$

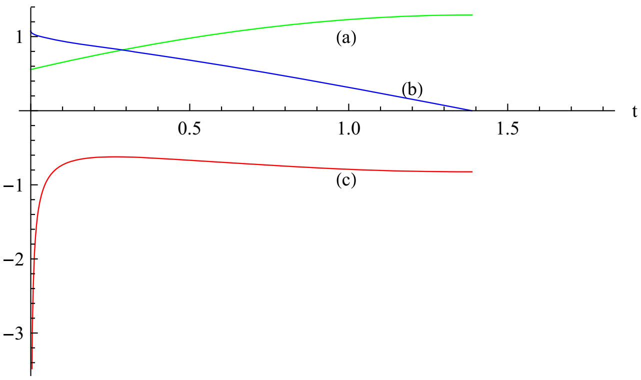

FIG. 2. Solution of (a) $a_0(t)$, (b) $\dot{a}_0(t)$, and (c) $\ddot{a}_0(t)$ for λ_-, where $\Lambda_4/3 = -1$, $C = 0.03$, and $R = 1$.

equation breaks down near this point, where higher order curvature terms and other quantum gravity effects would become important. We will further discuss this issue in future work.

Concentrating on the region $a_0 > a_0^{\min}$, the following points are observed from the solution of (4.1):

(i) It is well known that the solution $a_0(t)$ is given by a periodic function for constant, negative λ and $C = 0$. In the present case of $C \neq 0$, we find only one turning point at $a_0 = a_0^{\max}$ as seen from Fig. 2. Namely a_0 starts from $a_0^{\min}$ and arrives at this turning point, then it comes back to the starting point. After that, a_0 cannot return to the $a_0^{\max}$ since $\dot{a}_0 < 0$ there when it comes back.

(ii) In our analysis, a_0 is restricted in the range

$$a_0^{\min} < a_0(t) \leq a_0^{\max}. \tag{4.9}$$

However, the solution does not oscillate between $a_0^{\min}$ and $a_0^{\max}$ since the point of $a_0^{\min}$ is not a turning point but a singular point.

2. *Deconfinement transition in the SYM theory*

Using the solution of $a_0(t)$ given above, the time dependence of the parameters b_0 and r_0 is shown in the left of Fig. 3. We find their crossing point where the confinement-deconfinement phase transition occurs for the case of constant a_0. In the present case, as explained below, the transition point is given by the zero point of the curve (c) in the same figure, and it deviates slightly from the point $b_0 = r_0$ in the adiabatic approximation.

The transition point is determined here as follows.

1. For each solution $a_0(t)$, we search for the horizon as a zero of $\bar{n}(r,t)$, which is given by Eq. (2.5), as a function of r.
2. When a zero can (cannot) be found, we can say that the phase of the state of the SYM theory is in the deconfinement (confinement) phase.
3. Any point in time where the so-found horizon disappears is a transition point from deconfinement to the confinement phase, and vice versa.
4. Performing the above procedure for different parameters with the same initial condition, we find a critical curve in the $b_0 - r_0$ plane. It is shown in Fig. 1.

In step 1 of the above procedure, the zero of the numerator of $\bar{n}(r,t)$ in Eq. (2.5) is found by rewriting it as the quadratic equation in terms of $x \equiv R^2/r^2 > 0$, whose solutions are given by

$$x_\pm = \frac{n_a + n_b \pm \sqrt{(n_a + n_b)^2 - 4(n_a n_b - n_c)}}{2(n_a n_b - n_c)}. \tag{4.10}$$

Here

$$n_a = \frac{\lambda R^2}{4}, \qquad n_b = \frac{\lambda + \dot{\lambda}\frac{a_0}{\dot{a}_0}}{4} R^2, \qquad n_c = \tilde{c}_0. \tag{4.11}$$

Then we search for a positive solution of $x_\pm$, which we interpret as a horizon of the 5D metric (2.2).

At the beginning, or at small a_0, a horizon can be observed at large r (small x_-). In other words, the SYM system is in a very hot plasma phase. Then, we expect that the horizon approaches $r = 0$ with growing a_0 and disappears at the transition point. We find that this transition is realized at the point of

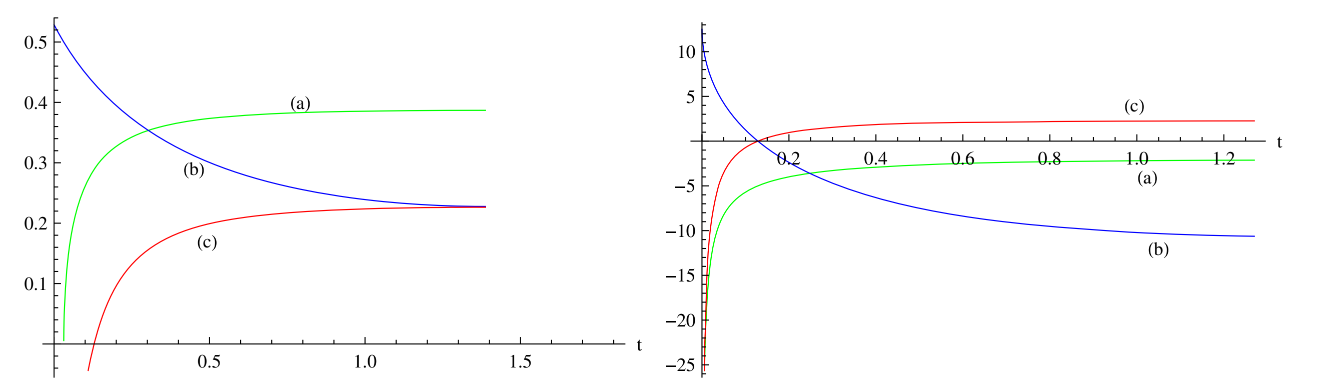

FIG. 3. Left: Time dependence of (a) r_0, (b) b_0, and (c) $n_a n_b - n_c$. Right: Time dependence of (a) $10/x_-$, (b) $10^2/x_+$, and (c) $10^2 \times (n_a n_b - n_c)$ for λ_-, where $\Lambda_4/3 = -1$, $C = 0.03$, and $R = 1$.

$$n_a n_b - n_c = 0 \Leftrightarrow r_0 = \left(1 + \frac{\dot{\lambda} a_0}{\lambda \dot{a}_0}\right)^{-1/4} b_0, \qquad (4.12)$$

where the second equation follows from (2.11). An example is shown in the right-hand side of Fig. 3, where the transition point is found at $t \sim 0.12$. Then the values of b_0 and r_0 are found from the solution shown on the left-hand side of that figure. We notice that this point shows a definite shift from the point of $b_0 = r_0$. In summary, we find the critical curve (b) in the $b_0 - r_0$ plane as shown in Fig. 1.

3. Tension of the $q\bar{q}$ potential

Note that the above transition point is also reproduced from the potential between the quark and antiquark. In the confinement phase, a finite tension ($\tau_{q\bar{q}}$) for the linear quark-antiquark potential appears in the confinement phase. The tension $\tau_{q\bar{q}}$ is given by [7]

$$\tau_{q\bar{q}} = \frac{n_s(r^*)}{2\pi\alpha'}, \qquad n_s(r) = \left(\frac{r}{R}\right)^2 \bar{n} \bar{A}, \qquad (4.13)$$

where r^* denotes the minimum point of $n_s(r)$. A typical result for n_s is shown in Fig. 4. From this figure, we can see that the transition point coincides with the one given above by observing the disappearance of the horizon.

Equation of state parameter w.—The two critical lines (a) and (b) in Fig. 1 are qualitatively different. This difference is reduced to the correction factor, $(1 + \frac{\dot{\lambda} a_0}{\lambda \dot{a}_0})^{-1/4}$, which appears in the case of time-dependent scale factor $a_0(t)$. On the other hand, if we find a quantity which characterizes the critical point by its special value even in the case of a time-dependent $a_0(t)$, we can use this quantity as an order parameter for the state of the system. Such a well-behaved quantity is the ratio $w(= p/\rho)$ of the pressure p and energy density ρ. This is called an equation of state parameter. At the critical point of the confinement-deconfinement transition, we find

$$w = -\frac{1}{3}. \qquad (4.14)$$

This is seen analytically by using (3.9), (3.10), and (4.12). Note that the relation (4.14) is satisfied at the critical point independently of the various parameters entering the solution. Furthermore, the relation (4.14) also holds for the case of constant a_0 at the critical point $r_0 = b_0$. The latter point has not been discussed in the literature on the topic before.

In the present model, after the phase transition to the confinement phase the system remains at $T = 0$. However, the state of the SYM theory continues to change with time since other quantities such as ρ and p and hence also w change. This fact implies that the dynamical properties of the system might be better characterized by w in the present case rather than by the temperature T. We should introduce a new form of thermodynamics with a newly defined temperature to understand the dynamics in this region. Such an idea would be realized by introducing the entanglement temperature which has been discussed in [25]. In this article, we do not discuss this point further. It is opened here.

Relating the critical point $w = -1/3$ to screening via the virial theorem.—When $a_0(t)$ varies with time, for small a_0 or for high density, we find $w > -1/3$ and a horizon is seen. As a_0 increases and the horizon approaches $r = 0$ and disappears for $w < -1/3$ in the large a_0 region, what has happened at the critical point, $w = -1/3$?

For $C = 0$, there is no radiation and only the negative boundary cosmological constant Λ_4 exists. This case has been examined in [6], where it was found that the SYM theory is in the confinement phase with the linear quark potential. In the case of $C > 0$, the above vacuum state is excited to a finite temperature state filled with SYM radiation, i.e. gluons.

These excited gluons can be observed as the thermal radiation in the deconfinement phase where the screening effect for the confinement force is overwhelming. Here we replace the strength of the screening effect by a force with the following form of potential:

$$V_{\text{screen}} = aL^{\alpha}, \qquad (4.15)$$

where a and α are some constants and L denotes the distance between the gluons (or between the quark and antiquark). This force would be repulsive.

Considering the virial theorem for the thermal gluon system one obtains

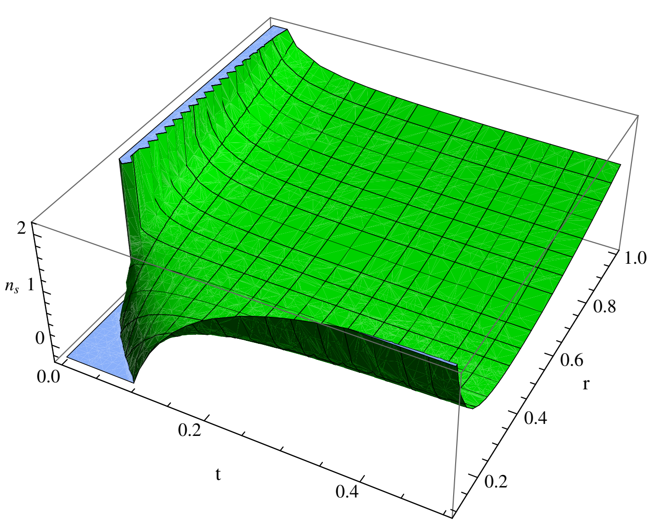

FIG. 4. Time dependence of $n_s(r, t)$ for $\Lambda_4/3 = -1$, $C = 0.03$, and $R = 1$.

$$\langle K \rangle = \frac{\alpha}{2}\langle V \rangle, \tag{4.16}$$

where K and V denote the kinetic and potential energy for the gluons, respectively. The vacuum expectation values are to be understood quantum mechanically. The equation of state is related to these vacuum expectation values,

$$w = \frac{\langle K \rangle - \langle V \rangle}{\langle K \rangle + \langle V \rangle} = \frac{\alpha - 2}{\alpha + 2}. \tag{4.17}$$

From this, we find $\alpha = 1$ for $w = -1/3$. The system is in the confinement phase for $\alpha < 1$. This implies that, in the region of $w < -1/3$, the confining potential ($V = \tau L$ [6]) overwhelms the screening (or thermal) effect at large L. As a result, the system should be in the confinement phase for $w < -1/3$, which is consistent with the disappearance of the horizon.

It should be noticed that this is true when the system is in the confinement phase for $C = 0$. When the system is in the deconfinement phase for $C = 0$, the system remains in a deconfinement phase even if $w < -1/3$.

Behavior of energy density ρ and pressure p.—It is necessary to investigate the time dependence of the physical quantities near the transition point. We show the energy density ρ, the pressure p, and the parameter w, in Fig. 5 in the case under consideration. Unexpectedly, we do not find any sign of a phase transition in these quantities themselves since they all change very smoothly with time.

B. The case of $\Lambda_4 > 0$

For $\Lambda_4 > 0$, λ_- is positive at any time in the region $a_0 > a_0^{\rm min}$. In this case, the Friedmann equation to be solved is

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = \frac{1 - \sqrt{1 - 4\tilde{\alpha}^2\Lambda_4/3 - 4\tilde{\alpha}^4\tilde{b_0}^4}}{2\tilde{\alpha}^2}. \tag{4.18}$$

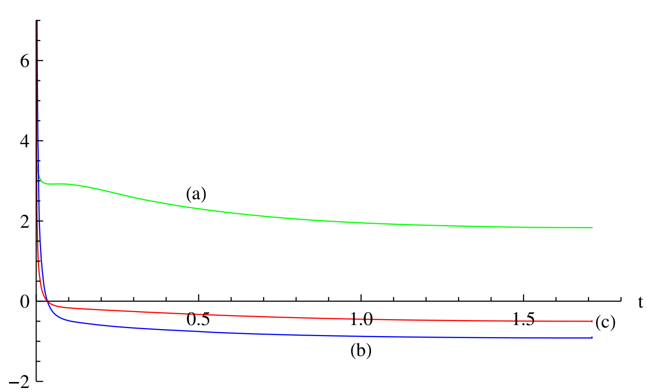

FIG. 5. Time dependence of (a) ρ, (b) p, and (c) $w = p/\rho$ for λ_-, where $\Lambda_4/3 = -1$, $C = 0.03$, and $R = 1$.

Although this equation has a solution for $k = -1, +1$, and 0, it is discussed here for the case of negative spatial curvature $k = -1$, in order to compare the case of negative Λ_4 in Sec. IV A.

The solution for the scale factor $a_0(t)$ of (4.18) requires the value of λ to vary in the range

$$\frac{1 - \sqrt{\tilde{\Lambda}_{4\_}}}{2\tilde{\alpha}^2} \le \lambda \le \frac{1}{2\tilde{\alpha}^2}, \qquad \tilde{\Lambda}_{4\_} = 1 - 4\tilde{\alpha}^2\Lambda_4/3. \tag{4.19}$$

In this way we ensure that λ is positive. The minimum value of a_0 is given by

$$a_0^{\rm min} = \tilde{\alpha}\left(\frac{16C}{R^2\tilde{\Lambda}_{4\_}}\right)^{1/4}. \tag{4.20}$$

1. Features of the typical solution of (4.18)

The characteristic features of the solution of (4.18) can be read from the typical numerical solution given in Fig. 6. The equation has been solved by the initial condition $a_0(0) = a_0^{\rm min}$, as in the case of negative Λ_4 discussed before.

(i) The scale a_0 increases monotonically with almost constant velocity $\dot{a}_0$. However, $\ddot{a}_0(t)$ changes its sign at $t = t_1$, so the expansion of the Universe changes from decelerating to accelerating at $t = t_1$. This implies that the dominant source of the expansion changes from the radiation to the positive dark energy at this time.

(ii) The value of w is interesting at the turning point. We find it as

$$w(t_1) = \frac{1}{3}. \tag{4.21}$$

This relation is again independent of the parameters of the theory as seen at the phase transition point above. The meaning of this fact will be discussed in future work.

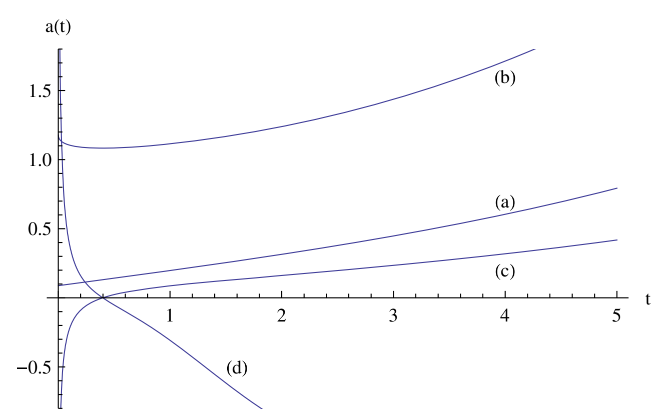

FIG. 6. Solution of (a) $a_0(t)/10$, (b) $\dot{a}_0(t)$, (c) $\ddot{a}_0(t)$, and (d) $(w - 1/3)$ for λ_-, where $\Lambda_4/3 = 5 \times 10^{-2}/(4\alpha^2)$, $C = 0.03$, and $R = 1$.

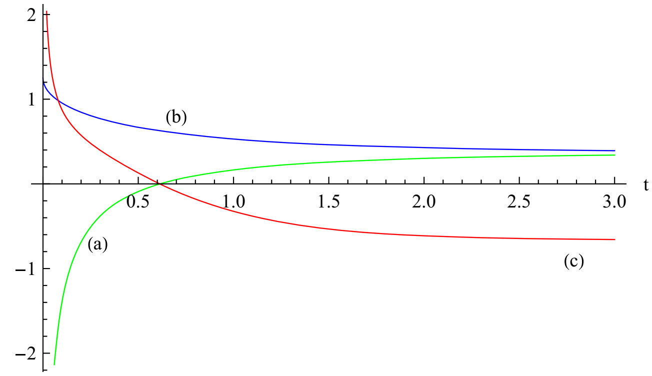

FIG. 7. Solution of (a) $10/x_+$, (b) $10/x_-$, and (c) $w+1/3$ for λ_-, where $\Lambda_4/3 = 0.5/(4\alpha^2)$, $C = 0.03$, and $R = 1$.

2. Emergence of a second horizon at $w = -1/3$

Next, we examine what happens at $w = -1/3$. In the present case of positive Λ_4, λ_- is positive at any point of $a_0(t)$. As a result, the horizon represented by x_- exists and does not disappear at any time, as seen by curve (b) in Fig. 7. In other words, the theory is in the deconfinement phase at any time although the temperature decreases with increasing a_0. In this sense, there is no phase transition to the confinement phase.

On the other hand, we observe that w decreases from a large positive value and crosses $-1/3$ at a given time, and then decreases further and approaches -1 for $t \to \infty$. It is interesting to see what happens in the present case after w crosses $w = -1/3$. Our observation is summarized as follows:

1. There is no confinement-deconfinement phase transition at this point as mentioned above. This is because of the fact that the theory for $C = 0$ and $w = -1$ is in the deconfinement phase, as shown for the $\lambda > 0$ case. So the system should be continuing in the deconfinement phase even if the state is in the region $-1/3 > w(> -1)$.
2. The horizon (x_-) observed for $w > -1/3$ smoothly varies as a horizon with positive value. On the other hand, when w crosses $w = -1/3$, the value of x_+ changes its sign to positive. This implies that the second horizon, x_+, appears for $w > -1/3$. Then we could say that in the holographic sense, two theories are explicitly described by the dual 5D geometry for $-1 < w < -1/3$. Two boundaries for each horizon exist in this case, so we will find two 4D field theories on those boundaries. A similar case has been discussed in [26]. However, the two CFTs would be noninteracting since they are not causal due to the two horizons between the boundaries [27]. A similar situation has been found and discussed in the case of negative λ [23]. We will further discuss this point in a future work.

3. Instanton solution for $k = +1$

Up to here, we have examined the case of Lorentzian time, and we find one turning point for the case of $\Lambda_4 < 0$. For $\Lambda_4 > 0$, we can find a Euclidean time solution with two turning points for $k = 1$. Barvinsky, Kamenshchik, and Nesterev [28] have used this solution to construct a new hilltop inflation scenario. In this sense, this type of solution is very interesting, and we will discuss this solution in the future.

V. THE SOLUTION BRANCH $\lambda = \lambda_+$

In this section we discuss the positive sign solution in (3.13). At the point of $a_0^{\min}$, λ_- takes its maximum value $\frac{1}{2\alpha^2}$. The value of λ larger than this maximum of λ_- is realized by λ_+. So the solution of the equation solved here covers the side of larger λ compared to the case of $\lambda = \lambda_-$.

In the case discussed in this section, λ_+ is always positive, so we can solve the Friedmann equation again for any value of the spatial curvature k, $k = 0, 1, -1$. In all cases, a_0 grows exponentially with the t at enough large t, where λ is positive and almost constant as shown in Fig. 8. The typical solution is shown in the left panel of Fig. 8. In this case, there is no turning point, so the Universe continues to expand exponentially forever.

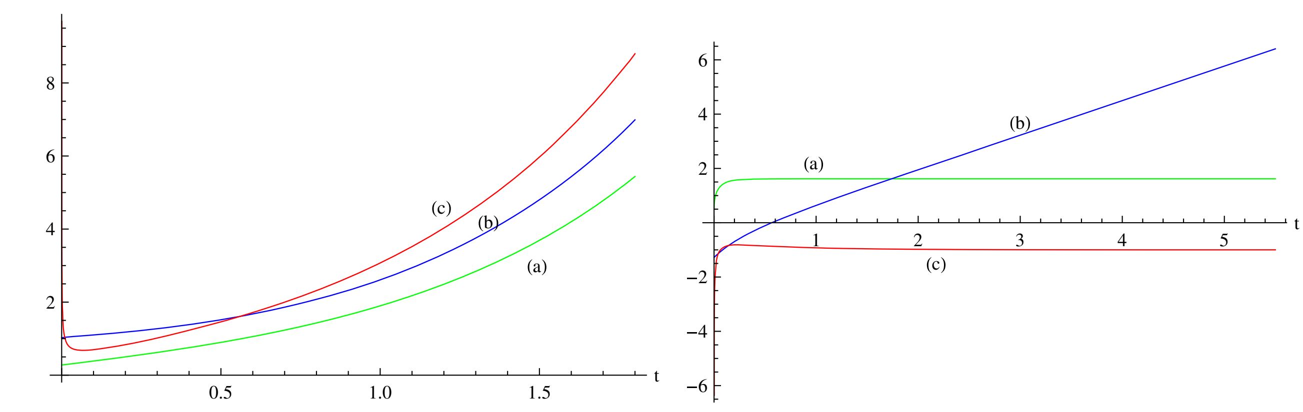

FIG. 8. Left: Time dependence of (a) $a_0(t)$, (b) $\dot{a}_0(t)$, and (c) $\ddot{a}_0(t)$ for λ_+, where $\Lambda_4/3 = -1$, $C = 0.03$, and $R = 1$. Right: Time dependence of (a) λ_+, (b) $\ln a_0(t)$, and (c) $w = p/\rho$.

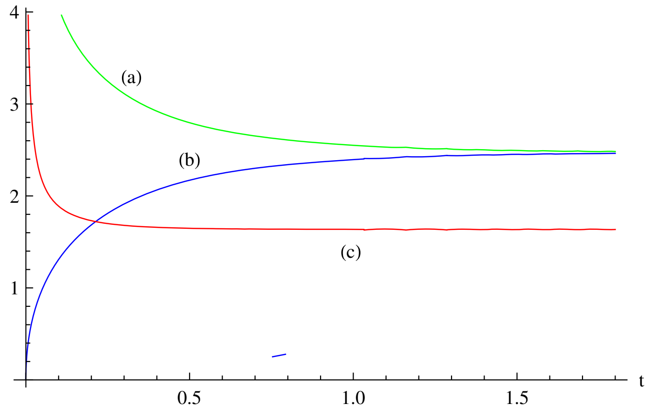

FIG. 9. Solution of (a) x_+, (b) x_-, and (c) $10(n_a n_b - n_c)$ for λ_+ and $k = -1$, where $\Lambda_4/3 = -1/(4\alpha^2)$, $C = 0.03$, and $R = 1$.

Some important observations in this case concerning the equation of state parameter w are summarized in the following:

(i) First, we observe that w increases from $-\infty$ to -1 with increasing time. The state is restricted to the region of $w < -1$, which is usually called the phantom state since the kinetic energy is negative. The physical interpretation of this situation is an open problem which we leave to future work.

(ii) Since the state in this case is restricted to $w < -1$, the equation of state parameter does not arrive at $w = -1/3$. As a result, the number of the horizons does not change as time evolves. We observe two horizons at any time t, as shown in Fig. 9.

About the connectedness of λ_- and λ_+ at $a_0 = a_0^{\min}$

As pointed out above, the solutions $\lambda = \lambda_-$ and $\lambda = \lambda_+$ of (3.13) are connected at the sudden singularity, namely at the point of $a_0 = a_0^{\min}$. The scalar curvature is divergent at this point. So it is not easy to connect the two solutions due to the singularity. As for this problem, there is an interesting approach proposed by Awad [19] connected to an inflationary scenario; however, we postpone the discussion of this issue to a future work.

VI. SUMMARY AND DISCUSSION

In AdS/CFT, boundary gravity decouples from the bulk, and the boundary metric becomes nondynamical. Hence, the time-dependent scale factor $a_0(t)$ in a FRW space-time on the boundary cannot be determined self-consistently from the equations of motion of the bulk gravity theory, which in our case is a truncation of 10D supergravity. On the other hand, the bulk solution induces energy and momentum sourcing gravity on the boundary via the usual AdS/CFT dictionary for the metric. For example, a finite temperature state in the bulk (an AdS-Schwarzschild black hole), will induce a radiation energy density and a corresponding pressure on the boundary. Hence, in order to obtain a self-consistent holographic cosmological evolution, we need to impose the dynamics of the scale factor $a_0(t)$ by hand.

In this work we proposed a way to do so by imposing the Friedmann equations for the boundary metric which relate energy density and pressures of the dual field theory to the time development of the CFT by giving a solution for the scale factor $a_0(t)$. We hence solved the 4D boundary Einstein equations coupled to the SYM theory energy-momentum tensor $\langle T_{\mu\nu}\rangle$ and to a boundary cosmological constant Λ_4. The energy-momentum tensor $\langle T_{\mu\nu}\rangle$ itself is obtained from the holographic renormalization procedure in the 5D reduction of type IIB 10D supergravity. By using the $a_0(t)$ solved in this way we were able to find a time-dependent 5D gravitational background, which consistently describes the time dependence of the state of $\mathcal{N} = 4$ SYM theory in the FRW space-time via the AdS/CFT correspondence. The solution is characterized by two free parameters, Λ_4 and C, the 4D cosmological constant and the dark radiation constant of the SYM fields. These two parameters control the dynamical properties of the SYM theory.

We then proceeded to analyze the phase structure of the SYM theory for different cases of these parameters. Our general finding is that negative $\Lambda_4 < 0$ drives the theory to a Wilson loop confinement phase. On the other hand, the dark radiation C counteracts this tendency by the screening of the confinement force. Hence these competing effects lead to a confinement-deconfinement transition. The same phenomenon had been observed for the case of slowly varying a_0 already in [7–12,24].

We find that the solution for $a_0(t)$ has two branches, $\lambda = \lambda_-$ and $\lambda = \lambda_+$. These two branches arise from solving a quadratic equation for λ.[4] Both solutions have a minimum $a_0^{\min}$ of the scale factor $a_0(t)$, and both branches $\lambda_\pm$ meet at this point. This point turns out to be singular since the acceleration $\ddot{a} = \infty$ diverges. At this point, the classical Friedmann equation breaks down, and quantum gravitational effects have to be taken into account to resolve the singularity, which is beyond the scope of this work.

The first branch $\lambda = \lambda_-$ itself separates into two cases depending on the sign of Λ_4. For negative Λ_4, the region of negative values of λ_- is covered. The Friedmann equation is then solved for hyperbolic ($k = -1$) three-dimensional spatial topology, and we find that the solution $a_0(t)$ increases with time and arrives at a maximum turning point with $\dot{a}_0 = 0$ and $a_0^{\max}$. After that, a_0 turns back to the singular point $a_0^{\min}$. An important phenomenon in this case is that, before arriving at $a_0^{\max}$, we find a phase transition of the SYM theory from the (Wilson loop) deconfinement to the confinement phase at a critical time $t = t_c$. This transition happens exactly when the horizon zero in g_{tt},

[4] We should notice here that $\lambda_\pm$ is not Λ_4 but it is given by Λ_4 and the dark radiation as shown above.

which is present in the deconfinement phase at small $a_0(t < t_c)$, disappears. We have observed that at this transition point, the value of the ratio $w = p/\rho$ of the pressure to the energy density is exactly $-1/3$. This value is in particular independent of the two free parameters of the theory C and Λ_4. The meaning of this value $w = -1/3$ has been discussed from the viewpoint of the virial equilibrium. We would like to stress that the energy-momentum tensor components ρ and p (and hence also w) vary smoothly across the transition, which hence only shows in the Wilson loop potential.

In the case of $\Lambda_4 > 0$, λ is always positive, and we can solve the Friedmann equation for any value of the spatial curvature k, namely for $k = -1$, $+1$, and 0. In order to compare with the case of $\Lambda_4 < 0$, we have first examined the case of negative curvature, $k = -1$. In this case, the solution a_0 increases monotonically and has no turning point. It furthermore has the following properties: First, the acceleration $\ddot{a}(t)$ changes its sign from positive to negative at an appropriate time. While we find that $w = +1/3$ is realized at this changing point independently of the parameters, its physical interpretation is an open question here. On the other hand, we find no phase transition at the point of $w = -1/3$ since the horizon zero in g_{tt} remains present for all regions of $a_0(t)$. Third, we noticed that a second horizon appears for $w < -1/3$, an interesting observation whose meaning we plan to investigate in a future work. Finally, for $k = +1$ we found a Euclidean solution with two turning points. This solution has been used in [28] as an instanton solution to drive an inflation scenario. In our case such solutions exist for special values of the parameters, Λ_4 and C. We postpone the discussion about this kind of solution related to the inflation scenario and the quantum cosmology.

For the second branch, $\lambda = \lambda_+$, the solution a_0 increases monotonically with time in all cases and never has a turning point. At large enough a_0, it expands exponentially as expected since λ_+ is positive and grows $a_0(t)$. The value of w increases monotonically from $-\infty$ to -1, so $w < -1$ at any a_0. So the matter is in a phantom phase. We always find two horizons in this case.

Of course, the time dependence of the solutions a_0 obtained here depends on the 4D gravity model on the boundary as well as on the 5D bulk theory and the chosen solution therein. In the present case, it is constructed by the Einstein-Hilbert action with a cosmological constant Λ_4 and SYM theory. In general, other matter fields might be included, or the 4D Einstein equations could be modified. For example, higher order curvature terms may be needed near the sudden singularity to resolve it. In this sense, some parts of the above results will be model dependent, but the chosen model, an otherwise conformal ground state in the bulk with only the temperature turned on and the prudent choice of the standard two-derivative Einstein-Hilbert action coupled minimally to energy and momentum as well as a cosmological constant on the boundary is the simplest reasonable choice. Also, the results obtained here will remain as a useful clue when we perform the analysis of more complicated models.

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