(2,2) and (0,4) supersymmetric boundary conditions in 3d $\mathcal{N}=4$ theories and type IIB branes

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The half-BPS boundary conditions preserving $\mathcal{N}=(2,2)$ and $\mathcal{N}=(0,4)$ supersymmetry in 3d $\mathcal{N}=4$ supersymmetric gauge theories are examined. The BPS equations admit decomposition of the bulk supermultiplets into specific boundary supermultiplets of preserved supersymmetry. Nahm-like equations arise in the vector multiplet BPS boundary condition preserving $\mathcal{N}=(0,4)$ supersymmetry, and Robin-type boundary conditions appear for the hypermultiplet coupled to the vector multiplet when $\mathcal{N}=(2,2)$ supersymmetry is preserved. The half-BPS boundary conditions are realized in the brane configurations of type IIB string theory.

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I. INTRODUCTION

The boundary conditions for the supersymmetric field theory preserving a part of supersymmetries of the original bulk theory provide important new ingredients and insights to the original system, for example, the description of branes in string or M theory and in target space of the field theories, dualities or holography in the presence boundary conditions, mirror symmetry, and also the geometric Langlands program. The supersymmetric (SUSY) boundary conditions have been studied in a number of contexts such as 2d $\mathcal{N}=(2,2)$ theories [1–3], 4d $\mathcal{N}=4$ theories [4–7], 3d $\mathcal{N}=2$ theories [8,9], and 3d $\mathcal{N}=4$ theories [10] and Bagger-Lambert-Gustavsson (BLG) and Aharony-Bergman-Jafferis-Maldacena (ABJM) theories [11–13].

In this paper, we study the half-BPS boundary conditions of 3d $\mathcal{N}=4$ supersymmetric theories preserving $\mathcal{N}=(2,2)$ and $\mathcal{N}=(0,4)$ supersymmetry at the boundary, which we call A type and B type, respectively. We explicitly calculate the boundary BPS equations for the 3d $\mathcal{N}=4$ pure vector multiplet, pure hypermultiplet, hypermultiplet coupled to the vector multiplet such as supersymmetric quantum chromodynamics (SQCD), and also its supersymmetric deformations by Fayet-Iliopoulos (FI) parameters and mass parameters. For each A- and B-type boundary condition for the vector multiplet, we have two sets of boundary conditions, which we call “electriclike” and “magneticlike”. Interestingly, the half-BPS boundary conditions preserving $\mathcal{N}=(0,4)$ for the vector multiplet include a Nahm-like equation. For the hypermultiplet coupled to a vector multiplet, we see that certain types of Robin boundary conditions arise. By studying the BPS equations, we read off the boundary degrees of freedom arising from the bulk 3d $\mathcal{N}=4$ vector multiplet and hypermultiplet. These are discussed in Sec. II.

In Sec. III, we propose the brane configurations corresponding to the boundary conditions of 3d $\mathcal{N}=4$ theories preserving $\mathcal{N}=(2,2)$ and $\mathcal{N}=(0,4)$ supersymmetry by introducing additional branes to the brane configuration of Hanany and Witten in the type IIB string theory [14] realizing 3d $\mathcal{N}=4$ theories. We give a remark on the map of the boundary degrees of freedom from the bulk supermultiplet under S duality of the type IIB string theory.

In Sec. IV, we summarize our results and discuss future directions.

II. HALF-BPS BOUNDARY CONDITIONS IN 3D $\mathcal{N}=4$ THEORIES

In this section, we consider the (2,2)- or (0,4)-preserving boundary conditions for a pure vector multiplet, pure hypermultiplets, and hypermultiplets coupled to a vector multiplet with FI and mass deformations. We also see the decomposition of the bulk supermultiplet at the boundary as supermultiplets of preserved supersymmetries.

A. Vector multiplet

In this subsection, we study the half-BPS boundary conditions for the 3d $\mathcal{N}=4$ vector multiplet. The 3d $\mathcal{N}=4$ vector multiplet contains a three-dimensional gauge field A_μ, $\mu=0,1,2$, three real scalar fields ϕ^i, $i=3,4,5$, an auxiliary field F, and a fermionic field λ. They are in the adjoint representation of the gauge group G and transform as

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$$\begin{aligned} A_\mu &: (\mathbf{3},\mathbf{1},\mathbf{1}), \\ \phi^i &: (\mathbf{1},\mathbf{3},\mathbf{1}), \\ F &: (\mathbf{1},\mathbf{1},\mathbf{3}), \\ \lambda &: (\mathbf{2},\mathbf{2},\mathbf{2}) \end{aligned} \tag{2.1}$$

under the $SO(1,2)\times SO(3)_C\times SO(3)_H$. The 3d $\mathcal{N}=4$ supersymmetric field theories have the R-symmetry group $SO(4)_R\cong SU(2)_C\times SU(2)_H$, where the $SU(2)_C$ [respectively, $SU(2)_H$] is the double cover of the $SO(3)_C$ [respectively, $SO(3)_H$].

The 3d $\mathcal{N}=4$ vector multiplet can be expressed as 3d $\mathcal{N}=2$ vector multiplet $V(A_\mu,\sigma,\lambda,D)$ and adjoint chiral multiplet $\Phi(\phi,\psi_\phi,F_\phi)$. Our notations for the 3d $\mathcal{N}=2$ superspace and supermultiplet are summarized in the Appendix. The action of the 3d $\mathcal{N}=4$ vector multiplet in terms of 3d $\mathcal{N}=2$ supermultiplets is given by

$$S_V^{\mathcal{N}=4}=S_V^{\mathcal{N}=2}+S_\Phi^{\mathcal{N}=2} \tag{2.2}$$

with

$$S_V^{\mathcal{N}=2}=\frac{1}{g_{3\mathrm{d}}^2}\int d^3x d^4\theta \mathrm{Tr}(\Sigma^2), \tag{2.3}$$

$$S_\Phi^{\mathcal{N}=2}=-\frac{1}{g_{3\mathrm{d}}^2}\int d^3x d^4\theta \mathrm{Tr}(\overline{\Phi}e^{-2V}\Phi e^{2V}), \tag{2.4}$$

where Σ is a linear multiplet. In components, they are

$$S_V^{\mathcal{N}=2}=\frac{1}{g_{3\mathrm{d}}^2}\int d^3x\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}F^{\mu\nu}-\frac{1}{2}D^\mu\sigma D_\mu\sigma +\frac{1}{2}D^2-i\bar{\lambda}\sigma^\mu D_\mu\lambda+i\lambda[\sigma,\bar{\lambda}]\right], \tag{2.5}$$

$$S_\Phi^{\mathcal{N}=2}=\frac{1}{g_{3\mathrm{d}}^2}\int d^3x\mathrm{Tr}[\quad -D_\mu\overline{\phi}D^\mu\phi-i\overline{\psi}\sigma^\mu D_\mu\psi+\overline{F}_\phi F_\phi +\overline{\phi}[\phi,D]-\sqrt{2}i\psi[\overline{\phi},\lambda]+\sqrt{2}i\overline{\psi}[\phi,\bar{\lambda}] +i\overline{\psi}[\psi,\sigma]-\overline{\phi}[\sigma,[\sigma,\phi]]], \tag{2.6}$$

respectively. The actions are invariant under the supersymmetry transformations

$$\delta A_\mu=i\bar{\xi}\sigma_\mu\lambda+i\xi\sigma_\mu\bar{\lambda}, \tag{2.7}$$

$$\delta\sigma=\xi\bar{\lambda}-\bar{\xi}\lambda, \tag{2.8}$$

$$\delta\lambda=i\xi D-\frac{1}{2}\gamma^{\mu\nu}\xi F_{\mu\nu}-i\gamma^\mu\xi D_\mu\sigma, \tag{2.9}$$

$$\delta\bar{\lambda}=-i\bar{\xi}D-\frac{1}{2}\gamma^{\mu\nu}\bar{\xi}F_{\mu\nu}+i\gamma^\mu\bar{\xi}D_\mu\sigma, \tag{2.10}$$

$$\delta D=-\xi\sigma^\mu D_\mu\bar{\lambda}+\bar{\xi}\sigma^\mu D_\mu\lambda+\xi[\sigma,\bar{\lambda}]+\bar{\xi}[\sigma,\lambda], \tag{2.11}$$

for 3d $\mathcal{N}=2$ vector multiplet V with the Wess-Zumino gauge, and

$$\delta\phi=\sqrt{2}\xi\psi_\phi, \tag{2.12}$$

$$\delta\psi_\phi=\sqrt{2}i\gamma^\mu\bar{\xi}D_\mu\phi+\sqrt{2}\xi F-\sqrt{2}i\bar{\xi}[\sigma,\phi], \tag{2.13}$$

$$\delta F=\sqrt{2}i\bar{\xi}\sigma^\mu D_\mu\psi_\phi+2i\bar{\xi}[\bar{\lambda},\phi]-\sqrt{2}i\bar{\xi}[\psi_\phi,\sigma], \tag{2.14}$$

for the 3d $\mathcal{N}=2$ adjoint chiral multiplets Φ.

Suppose we have a boundary in the x^2 direction, say, at $x^2=0$. Employing the Noether method, we find the normal component J^2 of the supercurrent of the 3d $\mathcal{N}=4$ vector multiplet in terms of 3d $\mathcal{N}=2$ language from the action (2.2) and the supersymmetric transformations (2.7)–(2.14):

$$\begin{aligned} J^2&=J^2_{\mathrm{vec}}+J^2_{\mathrm{adj}}\\ &=-\frac{1}{4}i\xi^{2mn}F_{mn}\lambda+\frac{1}{2}iF^{m2}\sigma_m\lambda-\frac{1}{2}D_m\sigma\sigma^{m2}\lambda+\frac{1}{2}D^2\sigma\lambda\\ &\quad+\frac{1}{\sqrt{2}}D^2\phi\overline{\psi}_\phi-\frac{1}{\sqrt{2}}D_m\phi\sigma^{m2}\overline{\psi}_\phi\\ &\quad+\frac{1}{2}[\overline{\phi},\phi]\sigma^2\lambda-\frac{1}{\sqrt{2}}[\sigma,\phi]\sigma^2\overline{\psi}_\phi \end{aligned} \tag{2.15}$$

in the on shell, where $m,n,\ldots=0,1$ are space-time indices of the two-dimensional boundary.[1]

In the presence of a boundary, the translation invariance is broken, so the supersymmetry is broken, in general. However, some of the supersymmetry can be preserved at the boundary by imposing specific boundary conditions, i.e., the supersymmetric or BPS boundary conditions. The BPS boundary conditions can be found by demanding that the normal component of the supercurrent at the boundary vanishes:

[1]We can put the supercurrent of the 3d $\mathcal{N}=2$ vector and adjoint chiral multiplet in an $SU(2)_C\times SU(2)_H$ manifest expression, which leads to the 3d $\mathcal{N}=4$ manifest supercurrent for the 3d $\mathcal{N}=4$ vector multiplet. Denoting 3d $\mathcal{N}=4$ fermions and scalars by

$$\lambda^{\alpha A\dot{A}}=\begin{pmatrix}\lambda^\alpha & -\psi^\alpha_\phi\\ \overline{\psi}^\alpha_\phi & \bar{\lambda}^\alpha\end{pmatrix},\qquad \phi^A_B=\begin{pmatrix}\sigma & \sqrt{2}\phi\\ \sqrt{2}\overline{\phi} & -\sigma\end{pmatrix}, \tag{2.16}$$

respectively, the normal component of the supercurrent can be written as

$$\begin{aligned} J^2_{\mathrm{vec}}+J^2_{\mathrm{adj}}=J^2=(J^2)^{\alpha A\dot{A}}\\ &=\frac{1}{2}iF_{01}\lambda^{\alpha A\dot{A}}-\frac{1}{2}D^2\phi^A_B\lambda^{\alpha B\dot{A}}+\frac{1}{2}iF^2_m(\gamma^m)^\alpha_\beta\lambda^{\beta A\dot{A}}\\ &\quad+\frac{1}{4}[\phi^A_C,\phi^C_B](\gamma^m)^\alpha_\beta\lambda^{\beta B\dot{A}}+\frac{1}{2}D_m\phi^A_B(\gamma^{m2})^\alpha_\beta\lambda^{\beta B\dot{A}}. \end{aligned} \tag{2.17}$$

Here, A and $\dot{A}$ are indices for $SU(2)_C$ and $SU(2)_H$, respectively.

$$
\begin{aligned}
0 &= \bar{\xi}J^2 - \xi\bar{J}^2 \\
&= -\frac{1}{4}i\xi^{2mn}F_{mn}(\bar{\xi}\lambda) + \frac{1}{2}iF^{m2}(\bar{\xi}\sigma_m\lambda) - \frac{1}{2}D_m\sigma(\bar{\xi}\sigma^{m2}\lambda) + \frac{1}{2}D^2\sigma(\bar{\xi}\lambda) \\
&\quad - \frac{1}{4}i\xi^{2mn}F_{mn}(\xi\bar{\lambda}) + \frac{1}{2}iF^{m2}(\xi\sigma_m\bar{\lambda}) + \frac{1}{2}D_m\sigma(\xi\sigma^{m2}\bar{\lambda}) - \frac{1}{2}D^2\sigma(\xi\bar{\lambda}) \\
&\quad + \frac{1}{\sqrt{2}}D^2\phi(\bar{\xi}\bar{\psi}_\phi) - \frac{1}{\sqrt{2}}D_m\phi(\bar{\xi}\sigma^{m2}\bar{\psi}_\phi) + \frac{1}{2}[\bar{\phi},\phi](\bar{\xi}\sigma^2\lambda) - \frac{1}{\sqrt{2}}[\sigma,\phi](\bar{\xi}\sigma^2\bar{\psi}_\phi) \\
&\quad - \frac{1}{\sqrt{2}}D^2\bar{\phi}(\xi\psi_\phi) + \frac{1}{\sqrt{2}}D_m\bar{\phi}(\xi\sigma^{m2}\psi_\phi) - \frac{1}{2}[\bar{\phi},\phi](\xi\sigma^2\bar{\lambda}) - \frac{1}{\sqrt{2}}[\sigma,\bar{\phi}](\xi\sigma^2\psi_\phi),
\end{aligned}
\tag{2.18}
$$

where we impose the boundary condition on fermions such that the bulk equations of motion are still satisfied.

Although there are various solutions to the supersymmetric boundary conditions (2.18), in this paper we will focus on the half-BPS boundary conditions preserving $\mathcal{N}=(2,2)$ and $\mathcal{N}=(0,4)$ supersymmetry at the boundary, which we call A- and B-type boundary conditions, respectively [1,8].

1. A-type boundary conditions

For the A-type boundary conditions, the supersymmetric parameter ξ satisfies the projection condition[2]

$$\gamma^2\xi = \bar{\xi}. \tag{2.19}$$

To find the bosonic boundary conditions from (2.18), we choose the fermionic boundary conditions

$$\gamma^2\lambda = e^{2i\theta}\bar{\lambda}, \qquad \gamma^2\psi_\phi = e^{2i\theta}\bar{\psi}_\phi, \tag{2.20}$$

where $\theta \in \mathbb{R}$ is a constant parameter. Note that this form of fermionic boundary condition is compatible with the bulk equations of motion for fermions λ and ψ_ϕ. From (2.19) and (2.20), we obtain

$$
\begin{aligned}
&\bar{\xi}\lambda = -e^{2i\theta}\xi\bar{\lambda}, \qquad \bar{\xi}\sigma_m\lambda = e^{2i\theta}\xi\sigma_m\bar{\lambda}, \\
&\bar{\xi}\sigma^{m2}\lambda = e^{2i\theta}\xi\sigma^{m2}\bar{\lambda}, \qquad \bar{\xi}\sigma^2\lambda = -e^{2i\theta}\xi\sigma^2\bar{\lambda};
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
&\bar{\xi}\bar{\psi}_\phi = -e^{-2i\theta}\xi\psi_\phi, \qquad \bar{\xi}\sigma_m\bar{\psi}_\phi = e^{-2i\theta}\xi\sigma_m\psi_\phi, \\
&\bar{\xi}\sigma^{m2}\bar{\psi}_\phi = e^{-2i\theta}\xi\sigma^{m2}\psi_\phi, \qquad \bar{\xi}\sigma^2\bar{\psi}_\phi = -e^{-2i\theta}\xi\sigma^2\psi_\phi.
\end{aligned}
\tag{2.22}
$$

With the above fermionic boundary conditions and the above formulas (2.21) and (2.22), one can rewrite the general supersymmetric boundary conditions (2.18) as

$$
\begin{aligned}
0 &= \frac{1}{4}i\xi^{2mn}F_{mn}(e^{2i\theta}-1)(\xi\bar{\lambda}) - \frac{1}{2}iF^{2m}(e^{2i\theta}+1)(\xi\sigma_m\bar{\lambda}) \\
&\quad - \frac{1}{2}D_m\sigma(e^{2i\theta}-1)(\xi\sigma^{m2}\bar{\lambda}) - \frac{1}{2}D^2\sigma(e^{2i\theta}+1)(\xi\bar{\lambda}) \\
&\quad - \frac{1}{\sqrt{2}}(e^{-2i\theta}D^2\phi + D^2\bar{\phi})(\xi\psi_\phi) \\
&\quad - \frac{1}{\sqrt{2}}(e^{-2i\theta}D_m\phi - D_m\bar{\phi})(\xi\sigma^{m2}\psi_\phi) \\
&\quad - \frac{1}{2}[\bar{\phi},\phi](e^{2i\theta}+1)(\xi\sigma^2\bar{\lambda}) \\
&\quad + \frac{1}{\sqrt{2}}([\sigma,\phi]e^{-2i\theta} - [\sigma,\bar{\phi}])(\xi\sigma^2\psi_\phi).
\end{aligned}
\tag{2.23}
$$

Without further projection conditions, we can find supersymmetric bosonic configurations as the nontrivial solutions to (2.23) when $\theta = 0$ and $\frac{\pi}{2}$.

From now on, we often identify the scalars σ and $\phi, \bar{\phi}$ of the 3d $\mathcal{N}=2$ vector and adjoint chiral multiplet with the scalars ϕ^i, $i = 3, 4, 5$, of the 3d $\mathcal{N}=4$ vector multiplet as

$$\sigma = \phi^3, \mathrm{Re}\phi = \phi^4, \mathrm{Im}\phi = \phi^5. \tag{2.24}$$

(i) $\gamma^2\lambda = \bar{\lambda}$ *and* $\gamma^2\psi_\phi = \bar{\psi}_\phi$ $(\theta = 0)$.—From (2.23) with $\theta = 0$, we find the boundary conditions

$$F_{2m} = 0, \tag{2.25}$$

$$D_2\phi^a = 0, \tag{2.26}$$

$$D_m\phi^5 = 0, \tag{2.27}$$

$$[\phi^a, \phi^5] = 0, \tag{2.28}$$

where $a = 3, 4$. The two-dimensional gauge field A_m and the two scalar fields ϕ^a satisfy Neumann-like boundary conditions (2.25) and (2.26) and so can fluctuate at the boundary. The condition (2.25) can be thought as the Dirichlet-like condition for the scalar field A_2. The scalar field ϕ^5 satisfies the Dirichlet-like condition (2.27). In particular, (2.27)

[2]Since the projection condition (2.19) is written in terms of 3d $\mathcal{N}=2$ SUSY parameters, it leads to $\mathcal{N}=(1,1)$ SUSY parameters at the boundary. But with the supersymmetry enhancement to 3d $\mathcal{N}=4$ in mind, as far as bosonic boundary BPS equations are concerned, it is okay to work with (2.19) for convenience.

and (2.28) can be solved by setting $\phi^5 = 0$. We call the above set of boundary conditions (2.25)–(2.28) the *electriclike* A-type boundary conditions, where the electriclike field $E_m = F_{2m}$ generated by scalar potential A_2 is required to be constant, while the magneticlike field $B = F_{01}$ can fluctuate at the boundary.

(ii) $\gamma^2\lambda = -\bar{\lambda}$ *and* $\gamma^2\psi_\phi = -\bar{\psi}_\phi$ $(\theta = \frac{\pi}{2})$.—In this case, the boundary conditions read

$$F_{01} = 0, \tag{2.29}$$

$$D_2\phi^5 = 0, \tag{2.30}$$

$$D_m\phi^a = 0, \tag{2.31}$$

$$[\phi^a, \phi^b] = 0, \tag{2.32}$$

where $a, b, \ldots = 3, 4$. We obtain the Dirichlet-like boundary condition for the two-dimensional gauge field A_m and the Neumann-like boundary condition for the scalar field ϕ^5. The third equation (2.31) is the Dirichlet-like condition on two scalar fields ϕ^a. The last constraint (2.32) implies that the two scalar fields satisfying the Dirichlet-like condition commute with each other. So one possible solution is to set them zero at the boundary. Meanwhile, as there is no constraint on F_{m2}, the scalar field A_2 is unconstrained and so can fluctuate at the boundary. We will call the set of boundary conditions (2.29)–(2.32) *magneticlike* A-type boundary conditions.

Boundary degree of freedom for the A type from the bulk vector multiplet

We see that two sets of the A-type boundary conditions, (2.25)–(2.28) and (2.29)–(2.31), provide decomposition of the 3d $\mathcal{N} = 4$ vector multiplet $V_{\mathcal{N}=4}$ under $\mathcal{N} = (2,2)$ supersymmetry at the boundary. The two-dimensional gauge field A_m and the two real scalar fields ϕ^a, $a = 3, 4$, which form a complex scalar field, are naturally packaged into a 2d $\mathcal{N} = (2,2)$ vector multiplet $V^{(2,2)}$ or field strength multiplet $\Sigma^{(2,2)}$. Meanwhile, from two real scalar fields A_2 and ϕ^5, one can form a 2d $\mathcal{N} = (2,2)$ twisted chiral multiplet $\widetilde{\Sigma}^{(2,2)}$, which is charged under the axial $U(1)_C$ R-symmetry group. We let ρ be the dual photons defined by $\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} = \partial_\mu\rho$ for each of the Abelian factors of the gauge group where A_2, which is a surviving degree of freedom when considering the Dirichlet-like boundary condition, appears in the left-hand side. Then $\widetilde{\rho} = \phi^5 + i\rho$ is charged under $U(1)_C$ and becomes a scalar component of the twisted chiral multiplet. Therefore, 3d $\mathcal{N} = 4$ vector multiplet $V_{\mathcal{N}=4}$ can be decomposed into 2d $\mathcal{N} = (2,2)$ vector multiplet $V^{(2,2)}$, or a field strength multiplet, which is a twisted chiral multiplet $\Sigma^{(2,2)}$, and 2d $\mathcal{N} = (2,2)$ twisted chiral multiplet $\widetilde{\Sigma}^{(2,2)}$;

$$V_{\mathcal{N}=4} \to (V^{(2,2)}, \widetilde{\Sigma}^{(2,2)}). \tag{2.33}$$

The 3d $\mathcal{N} = 4$ supersymmetric parameters consist of two copies of the 3d $\mathcal{N} = 2$ supersymmetric parameters ξ_1 and ξ_2. The projection (2.19) admits two right-moving supersymmetric parameters and two left-moving supersymmetric parameters.[3] Denoting the complex supersymmetric parameters of 2d $\mathcal{N} = (2,2)$ supersymmetry as

$$\begin{aligned} \xi^+ &\coloneqq \frac{1}{2}(\xi_1^+ + \bar{\xi}_1^+) + \frac{i}{2}(\xi_2^+ + \bar{\xi}_2^+), \\ \xi^- &\coloneqq \frac{1}{2i}(\xi_1^- - \bar{\xi}_1^-) + \frac{1}{2}(\xi_2^- - \bar{\xi}_2^-), \end{aligned} \tag{2.35}$$

with $\bar{\xi}^+ = (\xi^+)^*$ and $\bar{\xi}^- = (\xi^-)^*$ at the boundary, the axial $U(1)_A$ and the vector $U(1)_V$ of them may take

	$SO(1,1)$	$U(1)_A$	$U(1)_V$
ξ^+	+	−	+
ξ^-	−	+	+
$\bar{\xi}^+$	+	+	−
$\bar{\xi}^-$	−	−	−

(2.36)

For the electriclike A-type boundary conditions, which allow both left-moving and right-moving fermions, we similarly denote the two-dimensional fermionic fields by

$$\begin{aligned} \lambda^+ &\coloneqq \frac{1}{2}(\lambda + \bar{\lambda}) + \frac{i}{2}(\psi_\phi + \bar{\psi}_\phi), \\ \lambda^- &\coloneqq \frac{1}{2i}(\lambda - \bar{\lambda}) + \frac{1}{2}(\psi_\phi - \bar{\psi}_\phi), \end{aligned} \tag{2.37}$$

with $\bar{\lambda}^+ = (\lambda^+)^\dagger$ and $\bar{\lambda}^- = (\lambda^-)^\dagger$ at the boundary. Their R charges would be

	$SO(1,1)$	$U(1)_A$	$U(1)_V$
λ^+	+	+	+
λ^-	−	−	+
$\bar{\lambda}^+$	+	−	−
$\bar{\lambda}^-$	−	+	−

(2.38)

We write a complex scalar field as $\phi \coloneqq \phi^1 + i\phi^2$. Given the notation above, the supersymmetric transformation laws of the component fields $(A_m, \phi, \lambda^+, \lambda^-, \bar{\lambda}^+, \bar{\lambda}^-)$, which form an $\mathcal{N} = (2,2)$ vector multiplet, would be [15,16]

[3]We denote the right-moving fermion by Ψ^+ and the left-moving fermion by Ψ^-:

$$\gamma^2\Psi^+ = \Psi^+, \qquad \gamma^2\Psi^- = -\Psi^-. \tag{2.34}$$

One raises and lowers the spinor indices by the antisymmetric tensor $\Psi_\alpha = \epsilon_{\alpha\beta}\Psi^\beta$ with $\epsilon_{+-} = 1$ so that $\Psi^- = \Psi_+$, $\Psi^+ = -\Psi_-$.

$$\delta A_\pm = i\bar{\xi}_\pm \lambda_\pm + i\xi_\pm \bar{\lambda}_\pm, \tag{2.39}$$

$$\delta\phi = -i\bar{\xi}_+ \lambda_- - i\xi_- \bar{\lambda}_+, \tag{2.40}$$

$$\delta\lambda_+ = 2\partial_+ \bar{\phi}\xi_- + (iD - v_{01})\xi_+, \tag{2.41}$$

$$\delta\lambda_- = 2\partial_- \bar{\phi}\xi_+ + (iD + v_{01})\xi_-, \tag{2.42}$$

$$\delta D = -\bar{\xi}_+ D_- \lambda_+ - \bar{\xi}_- D_+ \lambda_- + \xi_+ D_- \bar{\lambda}_+ + \xi_- D_+ \bar{\lambda}_-, \tag{2.43}$$

where D is an auxiliary field, which is expressed as some function of ϕ where the detail form of it can be determined once the detail of coupling to the boundary fields is given.

The magneticlike A-type boundary conditions $\gamma^2\lambda = -\bar{\lambda}$ and $\gamma^2\psi_\phi = -\bar{\psi}_\phi$ also yield both left-moving and right-moving fermions. We similarly denote the two-dimensional fermions by

$$\chi^+ \coloneqq \frac{1}{2i}(\lambda - \bar{\lambda}) + \frac{1}{2}(\psi_\phi - \bar{\phi}_\phi),$$
$$\chi^- \coloneqq \frac{1}{2}(\lambda + \bar{\lambda}) + \frac{i}{2}(\psi_\phi + \bar{\phi}_\phi), \tag{2.44}$$

and their complex conjugate $\bar{\chi}^+ = (\chi^+)^\dagger$, $\bar{\chi}^- = (\chi^-)^\dagger$.

They would carry the R charges as

	$SO(1,1)$	$U(1)_A$	$U(1)_V$
χ^+	+	+	+
χ^-	−	−	+
$\bar{\chi}^+$	+	−	−
$\bar{\chi}^-$	−	+	−

(2.45)

The twisted chiral multiplet $\widetilde{\Sigma}^{(2,2)}$ has the component fields $(\tilde{\rho}, \chi^+, \chi^-, \bar{\chi}^+, \bar{\chi}^-)$. The supersymmetry transformation laws would take the form [15,16]

$$\delta\tilde{\rho} = \bar{\xi}_+ \chi_- - \xi_- \bar{\chi}_+, \tag{2.46}$$

$$\delta\bar{\chi}_+ = 2i\partial_+ \tilde{\rho}\bar{\xi}_- + G\bar{\xi}_+, \tag{2.47}$$

$$\delta\chi_- = -2i\partial_- \tilde{\rho}\bar{\xi}_+ + G\bar{\xi}_-, \tag{2.48}$$

$$\delta G = -2i\xi_+ \partial_- \bar{\chi}_+ - 2i\bar{\xi}_- \partial_+ \chi_-, \tag{2.49}$$

where G is some function of $\tilde{\rho}$.

2. B-type boundary conditions

Next we consider the B-type conditions where the projection condition on the supersymmetric parameter ξ is

$$\gamma^2\xi = -\xi. \tag{2.50}$$

Here and in the following, we choose a convention that this gives the right-moving supercharges, which leads to the chiral $\mathcal{N} = (0,4)$ supersymmetry at the two-dimensional boundary.[4]

Applying the ansatz

$$\gamma^2\lambda = e^{2i\theta}\lambda, \qquad \gamma^2\psi_\phi = e^{2i\theta}\psi_\phi \tag{2.51}$$

for the fermionic boundary conditions, which does not change the equations of motion when $\theta = 0$ or $\frac{\pi}{2}$, we can find the bosonic boundary conditions. These two boundary conditions for the fermionic fields determine their chiralities at the boundary. When $\theta = 0$ (respectively, $\theta = \frac{\pi}{2}$), the associated two-dimensional fermions are right moving (respectively, left moving).

(i) $\gamma^2\lambda = -\lambda$ *and* $\gamma^2\psi_\phi = -\psi_\phi$ $(\theta = \frac{\pi}{2})$.—With this choice of the fermionic boundary condition, it follows that

$$\bar{\xi}\lambda = 0, \qquad \bar{\xi}\sigma^2\lambda = 0, \qquad \bar{\xi}\bar{\psi}_\phi = 0, \qquad \bar{\xi}\sigma^2\bar{\psi}_\phi = 0, \tag{2.52}$$

so the general boundary conditions (2.18) turn into

$$0 = \frac{1}{2}(iF_{m2} - D_m\sigma)(\bar{\xi}\sigma^m\lambda) + \frac{1}{2}(iF^{m2} + D_m\sigma)(\xi\sigma^m\bar{\lambda}) - \frac{1}{\sqrt{2}}D_m\phi(\bar{\xi}\sigma^{m2}\bar{\psi}_\phi) + \frac{1}{\sqrt{2}}D_m\bar{\phi}(\xi\sigma^{m2}\psi_\phi). \tag{2.53}$$

Therefore, with identification (2.24), we find

$$F_{2m} = 0, \tag{2.54}$$

$$D_m\phi^i = 0. \tag{2.55}$$

The first condition (2.54) is the Neumann-like boundary condition for the two-dimensional gauge field A_m, while the second condition (2.55) is the Dirichlet-like boundary condition for the three scalar fields ϕ^i. The condition (2.54) can be rephrased as the Dirichlet-like boundary conditions for the scalar field A_2. We call this set of boundary conditions (2.54) and (2.55) the electriclike B-type boundary conditions.

(ii) $\gamma^2\lambda = \lambda$ *and* $\gamma^2\psi_\phi = \psi_\phi$ $(\theta = 0)$.—Choosing $\theta = 0$ for the fermionic boundary conditions in (2.51), we get

[4]In this convention, the projection condition for the supersymmetric parameter $\gamma^2\xi = \xi$ preserves $\mathcal{N} = (4,0)$ supersymmetry at the boundary.

$$\bar{\xi}\sigma_m\lambda = 0, \qquad \bar{\xi}\sigma^{m2}\lambda = 0, \qquad \bar{\xi}\sigma^{m2}\overline{\psi}_\phi = 0. \tag{2.56}$$

The generic boundary conditions (2.18) then reduce to

$$\begin{aligned} 0 = &\frac{1}{2}(-iF_{01} + D^2\sigma + [\phi, \overline{\phi}])(\bar{\xi}\lambda) \\ &+ \frac{1}{2}(-iF_{01} - D^2\sigma - [\phi, \overline{\phi}])(\xi\bar{\lambda}) \\ &+ \frac{1}{\sqrt{2}}(D^2\phi + [\sigma, \phi])(\bar{\xi}\overline{\psi}_\phi) \\ &- \frac{1}{\sqrt{2}}(D^2\overline{\phi} - [\sigma, \overline{\phi}])(\xi\psi_\phi). \end{aligned} \tag{2.57}$$

Thus, with identification (2.24), one finds

$$F_{01} = 0, \tag{2.58}$$

$$D_2\phi^i - \frac{1}{2}i\epsilon^{ijk}[\phi^j, \phi^k] = 0, \tag{2.59}$$

where ϵ^{ijk} is the Levi-Civita symbol with $\epsilon^{345} = 1$. The first condition (2.58) is the Dirichlet-like condition for the two-dimensional gauge field A_m. The scalar field A_2 is unconstrained and can fluctuate at the boundary. Note that the boundary conditions for three scalar fields ϕ^i is not Neumann-like, but rather they satisfy Nahm-like equations. They originate from the existence of fluctuating A_2 at the boundary [4]. We will call this set of boundary conditions (2.58) and (2.59) the magneticlike B-type boundary conditions.

Boundary degree of freedom for the B type from the bulk vector multiplet

We can also see the two sets of the B-type conditions (2.54) and (2.55) and (2.58) and (2.59) provide the decomposition of the 3d $\mathcal{N} = 4$ vector multiplet under the preserved $\mathcal{N} = (0,4)$ supersymmetry at the boundary. We observed that for the electriclike B-type boundary conditions the two-dimensional gauge field A_m can fluctuate and a pair of left-moving fermions transforming as $(\mathbf{2},\mathbf{2})_-$ survive at the boundary. They are part of the 2d $\mathcal{N} = (0,4)$ vector multiplet $V^{(0,4)}$, which also contains an auxiliary field transforming as $(\mathbf{1},\mathbf{3})_0$ that originates from the auxiliary field F in the 3d $\mathcal{N} = 4$ vector multiplet [see (2.1)]. On the other hand, for the magneticlike B-type boundary conditions, the scalar fields ϕ^i and A_2 can fluctuate at the boundary and can be combined into the two complex scalar fields transforming as $(\mathbf{2},\mathbf{1})_0$. Also, a pair of right-moving fermions $(\mathbf{1},\mathbf{2})_+$ survive at the boundary. Therefore, they form the $\mathcal{N} = (0,4)$ twisted hypermultiplets $\tilde{H}^{(0,4)}$. Hence, for the B-type conditions, the 3d $\mathcal{N} = 4$ vector multiplet $V_{\mathcal{N}=4}$ splits into two parts:

$$V_{\mathcal{N}=4} \to (V^{(0,4)}, \widetilde{H}^{(0,4)}). \tag{2.60}$$

The projection (2.50) reduces two copies ξ_1, ξ_2 of 3d $\mathcal{N} = 2$ supersymmetric parameters to four real left-moving supersymmetric parameters. We write them as $\xi^{A\dot{A}}$, where the indices $A, B, \ldots = 1, 2$ transform as a doublet under $SU(2)_C$ while the indices $\dot{A}, \dot{B}, \ldots = \dot{1}, \dot{2}$ transform as a doublet under $SU(2)_H$. We denote the four supersymmetric parameters of 2d $\mathcal{N} = (0,4)$ supersymmetry by

$$\xi^{-1\dot{1}} \coloneqq \xi_1^-, \qquad \xi^{-1\dot{2}} \coloneqq -\xi_2^-, \qquad \xi^{-2\dot{1}} \coloneqq \bar{\xi}_2^-, \qquad \xi^{-2\dot{2}} \coloneqq \bar{\xi}_1. \tag{2.61}$$

The electriclike B-type boundary conditions $\gamma^2\lambda = -\lambda$ and $\gamma^2\psi_\phi = -\psi_\phi$ lead to left-moving fermions. We take them as the doublets under the both $SU(2)_C$ and $SU(2)_H$ so that

$$\lambda^{-1\dot{1}} \coloneqq \lambda^-, \qquad \lambda^{-1\dot{2}} \coloneqq -\psi_\phi^-, \qquad \lambda^{-2\dot{1}} \coloneqq \overline{\psi}_\phi^-, \qquad \lambda^{-2\dot{2}} \coloneqq \bar{\lambda}^-. \tag{2.62}$$

Denoting the component fields for the vector multiplet $V^{(0,4)}$ by $(A_m, \lambda^{-A\dot{A}})$, the supersymmetry transformation[5] would be

$$\delta A_- = 2i\xi^-_{A\dot{A}}\lambda^{-A\dot{A}}, \tag{2.66}$$

$$\delta\lambda^{-A\dot{B}} = iD^A{}_C\xi^{-C\dot{B}} + F_{01}\xi^{-A\dot{B}}, \tag{2.67}$$

where $D^A{}_B$ would be some function of scalar field $\widetilde{X}^{AY'}$ in the twisted hypermultiplet for generic coupling to boundary fields. We use the antisymmetric tensor ϵ_{AB} and $\epsilon_{\dot{A}\dot{B}}$ with $\epsilon_{+-} = \epsilon_{\dot{+}\dot{-}} = 1$ to raise or lower indices $A, B, \ldots$ and $\dot{A}, \dot{B}, \ldots$, respectively.

The magneticlike B-type boundary conditions $\gamma^2\lambda = \lambda$ and $\gamma^2\psi_\phi = \psi_\phi$ lead to right-moving fermions. We can take them as a doublet under the $SU(2)_H$ and also a doublet under the additional global symmetry $SU(2)'_F$. We write them as $\widetilde{\Psi}^{+\dot{A}Y'}$:

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[5]From 3d $\mathcal{N} = 4$ supersymmetry transformation with projection and boundary conditions, we can see

$$\delta(A_0 - A_1) = 2i\xi^-_{A\dot{A}}\lambda^{-A\dot{A}}, \tag{2.63}$$

$$\delta(A_0 + A_1) = 0, \tag{2.64}$$

$$\delta\lambda^{-A\dot{B}} = iD^A_C\xi^{-C\dot{B}} + F_{01}\xi^{-A\dot{B}}, \tag{2.65}$$

at the boundary. Here, $D^A_B = \frac{1}{2}[\widetilde{X}^{AY'}, \widetilde{X}_{BY'}]$ with $\widetilde{X}^{AY'} = (\sigma - iA_2 - \sqrt{2}\phi\sqrt{2}\overline{\phi}\sigma + iA_2)$, which is a scalar component of a twisted hypermultiplet, where indices $Y' = 1', 2'$ denote the doublet under $SU(2)'_F$ global symmetry.

$$\widetilde{\Psi}^{+\dot{A}Y'} = \begin{pmatrix} \overline{\chi}^{+} & \lambda^{+} \\ \bar{\lambda}^{+} & -\chi^{+} \end{pmatrix}, \tag{2.68}$$

where the indices $Y' = 1', 2'$ represent the doublet under the $SU(2)'_F$. The supersymmetry transformations of the component fields $(\widetilde{X}^{AY'}, \widetilde{\Psi}^{+A'Y'})$, which form a twisted hypermultiplet, can also be obtained from 3d $\mathcal{N} = 4$ supersymmetry transformation with projection and boundary conditions:

$$\delta\widetilde{X}^{AY'} = -2\xi^{-A\dot{B}}\epsilon_{\dot{B}\dot{C}}\widetilde{\Psi}^{+\dot{C}Y'}, \tag{2.69}$$

$$\delta\widetilde{\Psi}^{+\dot{A}Y'} = -i\xi^{-B\dot{A}}\epsilon_{BC}(\partial_0 + \partial_1)\widetilde{X}^{CY'}, \tag{2.70}$$

which is the supersymmetry transformation of the $\mathcal{N} = (0,4)$ twisted hypermultiplet where Dirichlet-like condition $A_0 = A_1 = 0$ is incorporated. See also [17,18].

3. Reduction from the extended Bogomol'nyi equation

The BPS equations of the topologically twisted 4d $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theories on a 4-manifold M_4 have been studied in Ref. [19], which read

$$(F - \phi \wedge \phi + t d_A \phi)^+ = 0, \tag{2.71}$$

$$(F - \phi \wedge \phi - t^{-1} d_A \phi)^- = 0, \tag{2.72}$$

$$d_A \star \phi = 0, \tag{2.73}$$

where A is a four-dimensional anti-Hermitian gauge field that is a connection on a G bundle $E \to M_4$ and ϕ is a bosonic one-form field valued in an anti-Hermitian matrix given the adjoint representation of the Lie algebra of G. $d_A = d + [A, \cdot]$ is the covariant exterior derivative, $F = dA + A \wedge A$ is the field strength, $\star$ is the Hodge star operator, and t is a real constant parametrizing a family of topological twisted theories. Especially when $t = 1$, the set of equations (2.71)–(2.73) can be written as

$$F - \phi \wedge \phi + \star d_A \phi = 0, \tag{2.74}$$

$$d_A \star \phi = 0. \tag{2.75}$$

Equation (2.74) is called the extended Bogomol'nyi equation in Ref. [19]. It has been argued that the BPS equations (2.74) together with (2.75) provide a various family of the BPS equations in lower dimensions by performing the reduction on a given M_4, e.g., on $M_4 = C \times \Sigma$, where C and Σ are Riemann surfaces [19,20], and on $M_4 = M_3 \times \mathbb{R}_+$, where M_3 is a 3-manifold and $\mathbb{R}_+$ is a half line [21,22]. Here, we would like to see our BPS boundary conditions for the 3d $\mathcal{N} = 4$ vector multiplet in the reduction of the extended Bogomol'nyi equation.

We consider the equations on a 4-manifold $M_4 = \mathbb{R}_+ \times M_3$. We express the gauge field as $A = A_0 dx^0 + \widetilde{A}$, and the one-form as $\phi = \phi_0 dx^0 + \widetilde{\phi}$, where x^0 is the coordinate on the half line $\mathbb{R}_+$. Taking the x^0 independent parts from (2.74) and (2.75), one obtains the BPS equations on M_3:

$$\widetilde{F} - \widetilde{\phi} \wedge \widetilde{\phi} = \star(d_{\tilde{A}}\phi_0 - [A_0, \widetilde{\phi}]), \tag{2.76}$$

$$d_{\widetilde{A}} A_0 + [\phi_0, \widetilde{\phi}] = \star d_{\widetilde{A}} \widetilde{\phi}, \tag{2.77}$$

$$d^*_{\widetilde{A}} \widetilde{\phi} + [A_0, \phi_0] = 0, \tag{2.78}$$

where the exterior derivative $d_{\widetilde{A}}$, the Hodge operator $\star$, and $d^*_{\widetilde{A}} = \star d_{\widetilde{A}} \star$ are defined on the 3-manifold M_3. We further take $M_3 = \mathbb{R}_+ \times C$ and write $\widetilde{A} = A_2 dx^2 + A_z dz + A_{\bar{z}} d\bar{z}$, $\widetilde{\phi} = \phi_2 dx^2 + \phi_z dz + \phi_{\bar{z}} d\bar{z}$, where $\mathbb{R}_+$ is the half line $x^2 \geq 0$ and z and $\bar{z}$ are the local complex coordinates on the Riemann surface C. By squaring (2.76)–(2.78) and integrating by parts, one finds that $A_0 = \phi_2 = 0$. Let us denote the metric on 3-manifold M_3 by $ds^2 = (dx^2)^2 + 2|dz|^2$ and choose a gauge in which $A_2 = 0$. Then (2.76)–(2.78) are now simplified to [19]

$$F_{z\bar{z}} - [\phi_z, \phi_{\bar{z}}] = i\partial_2 \phi_0, \tag{2.79}$$

$$\partial_2 A_{\bar{z}} = -iD_{\bar{z}} \phi_0, \tag{2.80}$$

$$i[\phi_0, \phi_z] = \partial_2 \phi_z, \tag{2.81}$$

$$D_z \phi_{\bar{z}} = 0. \tag{2.82}$$

As a 4-manifold is now a product space $M_4 = \mathbb{R}_+ \times \mathbb{R}_+ \times C$, the topological twisting is not performed on the 4-manifold but on the two-dimensional surface C. When $C = \mathbb{R}^2$, which we will consider, the above configuration on a 3-manifold $M_3 = \mathbb{R}_+ \times C$ with a boundary at $x^2 = 0$ may admit maximally four supercharges. Regarding boundary conditions, given a field, it is reasonable to expect that there is either a normal derivative or tangential derivative of it but not both in the (BPS) boundary conditions or equations that the boundary fields should satisfy. So by picking sets of equations among (2.79)–(2.82)—more precisely, one in (2.79) or (2.80) and one in (2.81) or (2.82)—and by taking terms in equations to be separately zero, we may be able to find four consistent sets of BPS boundary conditions we are interested in. Meanwhile, we note that (2.79) and (2.80) have terms relevant to the Dirichlet-like and Neumann-like boundary conditions for the two-dimensional gauge fields A_z and $A_{\bar{z}}$, respectively, whereas (2.81) and (2.82) contain the Neumann-like and Dirichlet-like boundary conditions for the one-form fields ϕ_z and $\phi_{\bar{z}}$, respectively.

In order to see our boundary conditions in the reduced extended Bogomol'nyi equations (2.79)–(2.82), we take $C=\mathbb{R}^2$ where the one-form fields ϕ_z and $\phi_{\bar{z}}$ reduce to the scalar fields, and we set

$$\partial_z=\frac{1}{\sqrt{2}}(\partial_{\hat{0}}-i\partial_{\hat{1}}),\qquad \partial_{\bar{z}}=\frac{1}{\sqrt{2}}(\partial_{\hat{0}}+i\partial_{\hat{1}}),\tag{2.83}$$

$$A_z=\frac{1}{\sqrt{2}}(A_{\hat{0}}-iA_{\hat{1}}),\qquad A_{\bar{z}}=\frac{1}{\sqrt{2}}(A_{\hat{0}}+iA_{\hat{1}}),\tag{2.84}$$

$$\phi_0=\phi^5,\qquad \phi_z=\frac{1}{\sqrt{2}}(\phi^3-i\phi^4),\qquad \phi_{\bar{z}}=\frac{1}{\sqrt{2}}(\phi^3+i\phi^4),\tag{2.85}$$

where $m,n=\hat{0},\hat{1}$ are space-time indices on $\mathbb{R}^2$ while $i,j,\ldots=3,4,5$ and $a,b,\ldots=3,4$ label the scalar fields.
(Ai) *From (2.80) and (2.81)*.—By taking both of the lhs and the rhs of all the equations to be separately zero, we have

$$F_{2m}=0, D_m\phi^5=0,\qquad D_2\phi^a=0,\qquad [\phi^5,\phi^a]=0,\tag{2.86}$$

and one can identify them with the electriclike A-type conditions (2.25)–(2.28).
(Aii) *From (2.79) and (2.82)*.—We can obtain

$$F_{\hat{0}\,\hat{1}}=0,\qquad D_2\phi^5=0,\qquad [\phi^a,\phi^b]=0,\qquad D_m\phi^a=0\tag{2.87}$$

by setting every term in (2.79) to be zero. These are the magneticlike A-type conditions (2.29)–(2.32).
(Bi) *From (2.80) and (2.82)*.—Similarly, by taking both the lhs and the rhs in (2.80) to be separately zero, we get

$$F_{2m}=0,\qquad D_m\phi^i=0.\tag{2.88}$$

This set of equations are the electriclike B-type conditions (2.54) and (2.55).
(Bii) *From (2.79) and (2.81)*.—We can obtain

$$F_{mn}=0,\qquad D_2\phi^i+\epsilon^{ijk}[\phi^j,\phi^k]=0\tag{2.89}$$

by taking terms in (2.79) to be zero after arrangement, where we have restored the gauge fixed value $A_2=0$. Taking into account that ϕ^i's are anti-Hermitian here, we see that both equations are the magneticlike B-type conditions (2.58) and (2.59).[6]

[6]Equation (2.59) can be recovered from (2.89) via $A_2\to -iA_2$ and $\phi^j\to -\frac{1}{2}i\phi^j$.

B. Hypermultiplets

The 3d $\mathcal{N}=4$ hypermultiplets contain complex scalar fields $\boldsymbol{q}$ and fermionic fields $\boldsymbol{\psi}$ transforming as

$$\boldsymbol{q}:\ (\mathbf{1},\mathbf{1},\mathbf{2}),\qquad \boldsymbol{\psi}:\ (\mathbf{2},\mathbf{2},\mathbf{1})\tag{2.90}$$

under the $SO(1,2)\times SU(2)_C\times SU(2)_H$.

Also, the 3d $\mathcal{N}=4$ hypermultiplets in representation R of the gauge group can be expressed as a combination of the two 3d $\mathcal{N}=2$ chiral multiplets $Q(q,\psi,F_q)$ and $\widetilde{Q}(\widetilde{q},\widetilde{\psi},F_{\widetilde{q}})$ transforming in conjugate representations R and $\bar{R}$ of the gauge group. The action of the 3d $\mathcal{N}=4$ hypermultiplets coupled to 3d $\mathcal{N}=4$ vector multiplet is given by

$$S=S_K^{\mathcal{N}=2}+S_W^{\mathcal{N}=2},\tag{2.91}$$

where

$$S_K^{\mathcal{N}=2}=-\int d^3xd^4\theta(\overline{Q}e^{-2V}Q+\overline{\widetilde{Q}}e^{-2V}\widetilde{Q})\tag{2.92}$$

is the kinetic terms and

$$S_W^{\mathcal{N}=2}=-\sqrt{2}i\int d^3xd^2\theta(\widetilde{Q}\Phi Q)+\text{c.c.}\tag{2.93}$$

is the superpotential terms and c.c. stands for the complex conjugate.

In terms of the component fields, the action (2.92) can be expressed as

$$\begin{aligned}S_K^{\mathcal{N}=2}=\int d^3x[&-D_\mu\overline{q}D^\mu q-i\overline{\psi}\sigma^\mu D_\mu\psi+\overline{F}_qF_q-i\overline{\psi}\sigma\psi\\&-\sqrt{2}i\overline{\psi}\,\overline{\lambda}\,q-\sqrt{2}i\overline{q}\lambda\psi-\overline{q}Dq-\overline{q}\sigma^2q\\&-D^\mu\widetilde{q}D_\mu\overline{\widetilde{q}}-i\widetilde{\psi}\sigma^\mu D_\mu\overline{\widetilde{\psi}}+F_{\widetilde{q}}\overline{F}_{\widetilde{q}}+i\widetilde{\psi}\sigma\overline{\widetilde{\psi}}\\&+\sqrt{2}i\widetilde{q}\,\overline{\lambda}\,\overline{\widetilde{\psi}}+\sqrt{2}i\widetilde{\psi}\lambda\overline{\widetilde{q}}+\overline{\widetilde{q}}D\widetilde{q}-\widetilde{q}\sigma^2\overline{\widetilde{q}}],\end{aligned}\tag{2.94}$$

where $\sigma=\sigma^aT_R^a$, $D=D^aT_R^a$, and $\lambda=\lambda^aT_R^a, \overline{\lambda}=\overline{\lambda}^aT_R^a$, and the action (2.93) as

$$\begin{aligned}S_W^{\mathcal{N}=2}=\int d^3x[&-\sqrt{2}i(F_{\widetilde{q}}\phi q+\widetilde{q}F_\phi q+\widetilde{q}\phi F_q)\\&+i\sqrt{2}(\widetilde{\psi}\psi_\phi q+\widetilde{q}\psi_\phi\psi+\widetilde{\psi}\phi\psi)]+\text{c.c.},\end{aligned}\tag{2.95}$$

where the covariant derivatives are defined by

$$\begin{aligned}D_\mu q&=\partial_\mu q-iA_\mu q,\qquad D_\mu\overline{q}=\partial_\mu\overline{q}+i\overline{q}A_\mu,\\D_\mu\widetilde{q}&=\partial_\mu\widetilde{q}+i\widetilde{q}A_\mu,\qquad D_\mu\overline{\widetilde{q}}=\partial_\mu\overline{\widetilde{q}}-iA_\mu\overline{\widetilde{q}}.\end{aligned}\tag{2.96}$$

The action (2.91) is invariant under the supersymmetry transformations

$$\delta q = \sqrt{2}\xi\psi, \tag{2.97}$$

$$\delta\tilde{q} = \sqrt{2}\xi\tilde{\psi}, \tag{2.98}$$

$$\delta\psi = \sqrt{2}i\gamma^{\mu}\bar{\xi}D_{\mu}q + \sqrt{2}\xi F_{q} - \sqrt{2}i\bar{\xi}\sigma q, \tag{2.99}$$

$$\delta\tilde{\psi} = \sqrt{2}i\gamma^{\mu}\bar{\xi}D_{\mu}\tilde{q} + \sqrt{2}\xi F_{\tilde{q}} + \sqrt{2}i\bar{\xi}\sigma\tilde{q}, \tag{2.100}$$

$$\delta F_{q} = \sqrt{2}i\bar{\xi}\sigma^{\mu}D_{\mu}\psi + 2i(\bar{\xi}\,\bar{\lambda})q + \sqrt{2}i(\bar{\xi}\psi)\sigma, \tag{2.101}$$

$$\delta F_{\tilde{q}} = \sqrt{2}i\bar{\xi}\sigma^{\mu}D_{\mu}\tilde{\psi} - 2i(\bar{\xi}\,\bar{\lambda})\tilde{q} - \sqrt{2}i(\bar{\xi}\,\tilde{\psi})\sigma \tag{2.102}$$

for the 3d $\mathcal{N}=2$ chiral multiplets Q and $\tilde{Q}$ as well as the supersymmetry transformations (2.7)–(2.11) and (2.12)–(2.14), respectively, for the vector multiplet V and the adjoint chiral multiplet Φ. From the action (2.92) and the supersymmetric transformation laws, we obtain a supercurrent of the 3d $\mathcal{N}=4$ hypermultiplets:

$$\begin{aligned} J^{\mu} = &-\sqrt{2}D^{\mu}\bar{q}\psi - \bar{q}\gamma^{\mu}\bar{\lambda}q + \sqrt{2}D_{\nu}\bar{q}\gamma^{\mu\nu}\psi - \sqrt{2}\bar{q}\sigma\gamma^{\mu}\psi \\ &- \sqrt{2}\tilde{\psi}D^{\mu}\bar{\tilde{q}} + \tilde{q}\gamma^{\mu}\bar{\lambda}\,\bar{\tilde{q}} + \sqrt{2}\gamma^{\mu\nu}\tilde{\psi}D_{\nu}\bar{\tilde{q}} + \sqrt{2}\gamma^{\mu}\tilde{\psi}\sigma\bar{\tilde{q}} \\ &- 2\gamma^{\mu}(\bar{q}\,\bar{\phi}\,\bar{\tilde{\psi}} + \overline{q\psi_{\phi}}\bar{\tilde{q}} + \bar{\psi}\,\bar{\phi}\,\bar{\tilde{q}}). \end{aligned} \tag{2.103}$$

Using the 3d $\mathcal{N}=2$ notation, we get the supersymmetric boundary conditions for the 3d $\mathcal{N}=4$ hypermultiplets:

$$\begin{aligned} 0 = &-\sqrt{2}D_{2}\bar{q}(\xi\psi) - \sqrt{2}(\xi\tilde{\psi})D_{2}\bar{\tilde{q}} - \bar{q}(\xi\gamma^{2}\bar{\lambda})q + \tilde{q}(\xi\gamma^{2}\bar{\lambda})\bar{\tilde{q}} \\ &+ \sqrt{2}D_{\nu}\bar{q}(\xi\gamma^{2\nu}\psi) + \sqrt{2}(\xi\gamma^{2\nu}\tilde{\psi})D_{\nu}\bar{\tilde{q}} - \sqrt{2}\bar{q}\sigma(\xi\gamma^{2}\psi) + \sqrt{2}(\xi\gamma^{2}\tilde{\psi})\sigma\bar{\tilde{q}} \\ &+ 2\bar{q}\,\bar{\phi}(\xi\gamma^{2}\bar{\tilde{\psi}}) + 2\bar{q}(\xi\gamma^{2}\bar{\psi}_{\phi})\bar{\tilde{q}} + 2(\xi\gamma^{2}\bar{\psi})\bar{\phi}\,\bar{\tilde{q}} + \sqrt{2}(\bar{\xi}\,\bar{\psi})D_{2}q + \sqrt{2}D_{2}\tilde{q}(\bar{\xi}\,\bar{\tilde{\psi}}) + \bar{q}(\bar{\xi}\gamma^{2}\lambda)q - \tilde{q}(\bar{\xi}\gamma^{2}\lambda)\bar{\tilde{q}} \\ &- \sqrt{2}(\bar{\xi}\gamma^{2\nu}\bar{\psi})D_{\nu}q - \sqrt{2}D_{\nu}\tilde{q}(\bar{\xi}\gamma^{2\nu}\bar{\tilde{\psi}}) + \sqrt{2}(\bar{\xi}\gamma^{2}\bar{\psi})\sigma q - \sqrt{2}\tilde{q}\sigma(\bar{\xi}\gamma^{2}\bar{\tilde{\psi}}) - 2(\bar{\xi}\gamma^{2}\tilde{\psi})\phi q - 2\tilde{q}(\bar{\xi}\gamma^{2}\psi_{\phi})q - 2\tilde{q}\phi(\bar{\xi}\gamma^{2}\psi). \end{aligned} \tag{2.104}$$

One can generalize the boundary conditions and their solutions by introducing additional boundary degrees of freedom. Also, it would be intriguing to explore the space of the solutions for the given information. We defer these to later work. In this paper, we focus on the investigation of basic half-BPS boundary conditions for the hypermultiplets. As in the previous discussion on the vector multiplet, we examine the half-BPS boundary conditions of the A and B types for the pure hypermultiplets in this subsection and discuss the coupled hypermultiplets in next subsection.

1. A-type boundary condition

We are interested in A-type boundary conditions for bosonic fields given by $\gamma^{2}\xi = \bar{\xi}$ and fermionic boundary conditions

$$\gamma^{2}\psi = e^{2i\varphi}\bar{\psi}, \qquad \gamma^{2}\lambda = e^{2i\theta}\bar{\lambda}, \qquad \gamma^{2}\psi_{\phi} = e^{2i\theta}\bar{\psi}_{\phi}, \tag{2.105}$$

where $\varphi, \theta \in \mathbb{R}$ are constant phase parameters. Here and in the following, we consider the case $e^{2i\varphi} = -e^{2i\theta}$, i.e.,

$$\theta - \varphi = \frac{\pi}{2} + \pi\mathbb{Z}, \tag{2.106}$$

but for the A-type condition the case $e^{2i\varphi} = e^{2i\theta}$ provides equivalent results to the ones obtained from (2.106).

From (2.21) and (2.22), the generic boundary conditions (2.104) for the hypermultiplets become[7]

$$\begin{aligned} 0 = e^{-i\varphi}[&-\sqrt{2}(e^{i\varphi}D_{2}\cdot\bar{q} + e^{-i\varphi}D_{2}\cdot q)(\xi\psi) - \sqrt{2}(e^{i\varphi}D_{2}\cdot\bar{\tilde{q}} + e^{-i\varphi}D_{2}\cdot\tilde{q})(\xi\tilde{\psi}) \\ &+ \sqrt{2}(e^{i\varphi}D_{m}\cdot\bar{q} - e^{-i\varphi}D_{m}\cdot q)(\xi\gamma^{2m}\psi) + \sqrt{2}(e^{i\varphi}D_{m}\cdot\bar{\tilde{q}} - e^{-i\varphi}D_{m}\cdot\tilde{q})(\xi\gamma^{2m}\tilde{\psi}) \\ &- \sqrt{2}(e^{i\varphi}\sigma\cdot\bar{q} + e^{-i\varphi}\sigma\cdot q)(\xi\gamma^{2}\psi) + \sqrt{2}(e^{i\varphi}\sigma\cdot\bar{\tilde{q}} + e^{-i\varphi}\sigma\cdot\tilde{q})(\xi\gamma^{2}\tilde{\psi}) \\ &+ 2(e^{-i\varphi}\bar{\phi}\cdot\bar{q} + e^{-i\varphi}\phi\cdot q)(\xi\tilde{\psi}) + 2(e^{i\varphi}\bar{\phi}\cdot\bar{\tilde{q}} + e^{-i\varphi}\phi\cdot\tilde{q})(\xi\psi)] \\ &+ e^{-i\theta}[-(q\bar{q}e^{-i\theta} + q\bar{q}e^{i\theta})(\xi\lambda) + (\tilde{q}\,\bar{\tilde{q}}\,e^{-i\theta} + \tilde{q}\,\bar{\tilde{q}}\,e^{i\theta})(\xi\lambda) + 2(\bar{q}\,\bar{\tilde{q}}\,e^{i\theta} + q\tilde{q}e^{-i\theta})(\xi\bar{\psi}_{\phi})], \end{aligned} \tag{2.107}$$

[7]To see the general form of the supersymmetric boundary conditions of the coupled hypermultiplets, we obtained the condition (2.107) by using the componentwise projection condition (2.105) given fixed all the gauge and global symmetry indices. Given the data of preserved gauge and global symmetries at the boundary, a large family of the boundary conditions can be constructed from the results below by restoring the form of representation.

where the dot $\cdot$ indicates the gauge and global symmetry action on the hypermultiplets. Also, the generators for gauge group are implicit between the products of two scalars, e.g., $\overline{q}T_R^a q$.

We would like to find the solutions to the half-BPS boundary conditions of the pure hypermultiplet for $\varphi = 0$ and $\varphi = \frac{\pi}{2}$.[8]

(i) $\gamma^2\psi = \overline{\psi}$ *(when* $\varphi = 0$*).*—In the case with $\varphi = 0$, we find from (2.107) the following bosonic boundary conditions for the hypermultiplets:

$$\partial_2(\mathrm{Re}q) = 0, \qquad \partial_2(\mathrm{Re}\widetilde{q}) = 0, \tag{2.108}$$

$$\partial_m(\mathrm{Im}q) = 0, \qquad \partial_m(\mathrm{Im}\widetilde{q}) = 0. \tag{2.109}$$

(ii) $\gamma^2\psi = -\overline{\psi}$ *(when* $\varphi = \frac{\pi}{2}$*).*—For the other A-type boundary conditions with the fermionic boundary conditions $\varphi = \frac{\pi}{2}$, the bosonic boundary conditions for the hypermultiplets read

$$\partial_m(\mathrm{Re}q) = 0, \qquad \partial_m(\mathrm{Re}\widetilde{q}) = 0, \tag{2.110}$$

$$\partial_2(\mathrm{Im}q) = 0, \qquad \partial_2(\mathrm{Im}\widetilde{q}) = 0. \tag{2.111}$$

Boundary degree of freedom for the A type from the pure bulk hypermultiplet

The A-type conditions provide decomposition of the 3d $\mathcal{N} = 4$ hypermultiplets into the boundary supermultiplets in such a way that (Req, Re$\widetilde{q}$) fluctuate at the boundary and (Imq, Im$\widetilde{q}$) satisfy Dirichlet boundary conditions, or the other way around. Each of them forms the 2d $\mathcal{N} = (2,2)$ chiral multiplets $\Phi_I^{(2,2)}$ and $\Phi_{II}^{(2,2)}$ whose lowest components are the complex scalar fields, which consists of $(\mathrm{Re}q, \mathrm{Re}(\widetilde{q}))$ and $(\mathrm{Im}q, \mathrm{Im}(\widetilde{q}))$, respectively;

$$H_{\mathcal{N}=4} \to (\Phi_I^{(2,2)}, \Phi_{II}^{(2,2)}). \tag{2.112}$$

The A-type boundary conditions $\gamma^2\psi = \overline{\psi}$ give both left-moving and right-moving fermions. We may denote the two-dimensional fermions by

$$\psi^+ \coloneqq \frac{1}{2}(\psi + \overline{\psi}) + \frac{i}{2}(\widetilde{\psi} + \overline{\widetilde{\psi}}),$$
$$\psi^- \coloneqq \frac{1}{2i}(\psi - \overline{\psi}) + \frac{1}{2}(\widetilde{\psi} - \overline{\widetilde{\psi}}_\phi), \tag{2.113}$$

and their complex conjugate $\overline{\psi}^+ = (\psi^+)^\dagger$ and $\overline{\psi}^- = (\psi^-)^\dagger$. They would carry the R charges as

	$SO(1,1)$	$U(1)_A$	$U(1)_V$
ψ^+	$+$	$+$	$-$
ψ^-	$-$	$-$	$-$
$\overline{\psi}^+$	$+$	$-$	$+$
$\overline{\psi}^-$	$-$	$+$	$+$

(2.114)

We also denote a two-dimensional complex scalar field by $\varphi \coloneqq \mathrm{Re}q + i\mathrm{Re}\widetilde{q}$. The supersymmetry transformations of component fields $(\varphi, \psi^+, \psi^-, \overline{\psi}^+, \overline{\psi}^-)$, which form the chiral multiplet $\Phi_I^{(2,2)}$, would be given by

$$\delta\varphi = \xi_+\psi_- - \xi_-\psi_+, \tag{2.115}$$

$$\delta\psi_+ = 2i\partial_+\varphi\overline{\xi}_- + F\xi_+, \tag{2.116}$$

$$\delta\psi_- = -2i\partial_-\varphi\overline{\xi}_+ + F\xi_-, \tag{2.117}$$

$$\delta F = -2i\overline{\xi}_+\partial_-\psi_+ - 2i\overline{\xi}_-\partial_+\psi_-, \tag{2.118}$$

where F is an auxiliary field and (2.35) is used. One can similarly realize the supersymmetric transformation laws of the other chiral superfield $\Phi_{II}^{(2,2)}$.

2. B-type boundary conditions

The B-type conditions are characterized by the chiral projection (2.50) on the supersymmetric parameter. We can find the bosonic boundary conditions by considering the fermionic boundary conditions

$$\gamma^2\psi = e^{2i\varphi}\psi, \qquad \gamma^2\lambda = e^{2i\theta}\lambda, \qquad \gamma^2\psi_\phi = e^{2i\theta}\psi_\phi \tag{2.119}$$

with $(\varphi, \theta) = (0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, 0)$. When $(\varphi, \theta) = (0, \frac{\pi}{2})$, i.e., $\gamma^2\psi = \psi$, $\gamma^2\lambda = -\lambda$, and $\gamma^2\psi_\phi = -\psi_\phi$, by using the formulas (2.52) for λ and ψ_ϕ and (2.56) for ψ and $\widetilde{\psi}$, we obtain

$$0 = -\sqrt{2}(D_2 \cdot \overline{q} + \sigma \cdot \overline{q})(\xi\psi) - \sqrt{2}(D_2 \cdot \overline{\widetilde{q}} - \sigma \cdot \overline{\widetilde{q}})(\xi\widetilde{\psi})$$
$$+ 2\overline{\phi} \cdot \overline{q}(\xi\overline{\widetilde{\psi}}) + 2\overline{\widetilde{q}} \cdot \overline{\phi}(\xi\overline{\psi}). \tag{2.120}$$

Similarly, when $(\varphi, \theta) = (\frac{\pi}{2}, 0)$, i.e., $\gamma^2\psi = -\psi$, $\gamma^2\lambda = \lambda$, and $\gamma^2\psi_\phi = \psi_\phi$, the boundary condition becomes

$$0 = \sqrt{2}D_m \cdot \overline{q}(\xi\gamma^m\psi) + \sqrt{2}D_m \cdot \overline{\widetilde{q}}(\xi\gamma^m\widetilde{\psi})$$
$$- (|q|^2 - |\widetilde{q}|^2)(\xi\overline{\lambda}) + 2\overline{q}\,\overline{\widetilde{q}}(\xi\overline{\psi}_\phi). \tag{2.121}$$

The chiralities of the fermionic fields at the boundary from the bulk 3d $\mathcal{N} = 4$ hypermultiplet are determined by the phase factor $\varphi \in \mathbb{R}$. For $\varphi = 0$ (respectively, $\varphi = \frac{\pi}{2}$), the right-moving (respectively, left-moving) fermions survive at the two-dimensional boundary.

[8]The half-BPS boundary conditions preserving $\mathcal{N} = (2,2)$ for hypermultiplet were also obtained in Ref. [23].

(i) $\gamma^2\psi = \psi$ *(when $\varphi = 0$).*—This boundary condition admits the right-moving fermions with $\varphi = 0$ in the hypermultiplets. For pure hypermultiplet, we turn off fields from the vector multiplet, so the condition (2.120) leads to the Neumann boundary conditions for the hypermultiplet scalars q and $\widetilde{q}$:

$$\partial_2 q = 0, \qquad \partial_2 \widetilde{q} = 0. \tag{2.122}$$

(ii) $\gamma^2\psi = -\psi$ *(when $\varphi = \frac{\pi}{2}$).*—In this case, the fermions in the hypermultiplet at the boundary are left moving. For the hypermultiplets without gauge coupling, (2.121) can be solved by requiring the Dirichlet boundary conditions for the hypermultiplet scalars q and $\widetilde{q}$:

$$\partial_m q = 0, \qquad \partial_m \widetilde{q} = 0. \tag{2.123}$$

Therefore, the bosonic degrees of freedom in the 3d $\mathcal{N} = 4$ hypermultiplets cannot survive at the boundary, while the left-moving fermions are free to fluctuate at the boundary.

Boundary degree of freedom for the B type from the pure bulk hypermultiplet

We saw that there are two types of B-type conditions for the 3d $\mathcal{N} = 4$ hypermultiplets. For boundary condition (i) with $\varphi = 0$, the full set of four bosonic fields as well as the right-moving fermions $(\mathbf{2}, \mathbf{1})_+$ can fluctuate at the boundary. They are packaged into the 2d $\mathcal{N} = (0, 4)$ hypermultiplets $H^{(0,4)}$. On the other hand, for the second condition (ii) with $\varphi = \frac{\pi}{2}$, the left-moving fermions $(\mathbf{1}, \mathbf{1})_-$ can fluctuate, but all the bosonic degrees of freedom satisfy the Dirichlet condition at the boundary. The fluctuating degrees of freedom can be packaged into the $\mathcal{N} = (0, 4)$ Fermi multiplets $\Lambda^{(0,4)}$. Therefore, we have the decomposition

$$H_{\mathcal{N}=4} \to (H^{(0,4)}, \Lambda^{(0,4)}). \tag{2.124}$$

The B-type boundary conditions $\gamma^2\psi = \psi$ lead to right-moving fermions. We write them as Ψ^{+AY}, where the indices $Y = 1, 2$ represent the doublet under the additional $SU(2)_F$ global symmetry:

$$\Psi^{+11} \coloneqq \overline{\widetilde{\psi}}, \qquad \Psi^{+12} \coloneqq \overline{\psi}, \qquad \Psi^{+21} \coloneqq \psi, \qquad \Psi^{+22} \coloneqq -\widetilde{\psi}. \tag{2.125}$$

Also, we denote the scalar component by

$$X^{\dot{1}1} \coloneqq q, \qquad X^{\dot{1}2} \coloneqq -\widetilde{q}, \qquad X^{\dot{2}1} \coloneqq \overline{\widetilde{q}}, \qquad X^{\dot{2}2} \coloneqq \overline{q}, \tag{2.126}$$

which transforms as a doublet under the $SU(2)_H$ and a doublet under the $SU(2)_F$. The supersymmetry transformation of component fields $(X^{\dot{A}Y}, \Psi^{+AY})$, which forms a hypermultiplet $H^{(0,4)}$, can be obtained from 3d $\mathcal{N} = 4$ supersymmetry transformation with projection and boundary conditions:

$$\delta X^{\dot{A}Y} = -\sqrt{2}\xi^{-B\dot{A}}\epsilon_{BC}\Psi^{+CY}, \tag{2.127}$$

$$\delta\Psi^{+AY} = \xi^{+A\dot{B}}\epsilon_{\dot{B}\dot{C}}(\partial_0 + \partial_1)X^{\dot{C}Y}, \tag{2.128}$$

which is a supersymmetry transformation of the $\mathcal{N} = (0, 4)$ hypermultiplet.

Another B-type boundary condition $\gamma^2\psi = -\psi$ leads to four real left-moving fermions, which are singlet under the R symmetry. These fermionic fields form a Fermi multiplet $\Lambda^{(0,4)}$. We can take them as two complex fermions, which we denote as

$$\zeta_1^- = \psi, \qquad \zeta_2^- = \overline{\widetilde{\psi}}, \tag{2.129}$$

where Hermitian conjugates are $\bar{\zeta}_1^- = \overline{\psi}$ and $\bar{\zeta}_2^- = \widetilde{\psi}$, respectively. Then, the supersymmetry transformation of these fields can be obtained, and they are

$$\delta\zeta_1^- = -\sqrt{2}i\xi^{-A\dot{A}}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\widetilde{X}^{B1'}X^{\dot{B}}, \tag{2.130}$$

$$\delta\zeta_2^- = -\sqrt{2}i\xi^{-A\dot{A}}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\widetilde{X}^{B2'}X^{\dot{B}}. \tag{2.131}$$

These can be organized into

$$\delta\Theta^{-Y'Y} = -\sqrt{2}i\xi^{-A\dot{A}}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\widetilde{X}^{BY'}X^{\dot{B}Y}, \tag{2.132}$$

where $\Theta^{Y'Y} = \begin{pmatrix} \zeta_1^- & \bar{\zeta}_2^- \\ \zeta_2^- & \bar{\zeta}_1^- \end{pmatrix}$, which is a supersymmetry transformation of the Fermi multiplet. When considering a generic interaction with boundary fields, the supersymmetry transformation would take a form

$$\delta\zeta_a^- = -\sqrt{2}i\xi^-_{A\dot{A}}C_a^{A\dot{A}}, \tag{2.133}$$

where ζ_a^-, $a = 1, 2, 3, 4$, denotes ζ_1^-, ζ_2^-, $\bar{\zeta}_1^-$, and $\bar{\zeta}_1^-$ and $C_a^{A\dot{A}}$ are some function of $X^{\dot{A}Y}$ and $\widetilde{X}^{AY'}$. See also [17,18].

C. Gauge coupling and SUSY deformations

We now discuss the half-BPS boundary conditions for the vector multiplets and the hypermultiplets in the 3d $\mathcal{N} = 4$ supersymmetric gauge theories with the supersymmetric deformation by FI parameters and mass parameters.

1. FI and mass deformations

The 3d $\mathcal{N} = 4$ gauge theories can be deformed by FI terms and mass terms while keeping supersymmetry.

We consider the effects of the supersymmetric deformations on the half-BPS boundary conditions.

If the 3d $\mathcal{N}=4$ supersymmetric gauge theories involve the $U(1)$ factors of the gauge group, they can be deformed in a supersymmetric way by introducing the baptized Fermi (BF) coupling of the topological currents for the $U(1)$ factors to a background Abelian $\mathcal{N}=4$ twisted vector multiplet (V_r, Φ_r) [24,25]:

$$S_{FI} = \int d^3x d^4\theta \mathrm{Tr}'(\Sigma V_r) + \frac{i}{2}\int d^3x d^2\theta \mathrm{Tr}'(\Phi\Phi_r) + \text{c.c.}, \qquad (2.134)$$

where $V_r = ir\bar{\theta}\theta$ and $\Phi_r = \phi_r$ with $r \in \mathbb{R}$, $\phi_r \in \mathbb{C}$. The trace Tr' takes only the $U(1)$ factors of the gauge group. Here $r^{\hat{i}} = (r, \mathrm{Re}(\phi_r), \mathrm{Im}(\phi_r))$, $\hat{i} = 7, 8, 9$, forms a triplet under the $SU(2)_H$. In terms of the component fields, we can express the action (2.134) as

$$S_{FI} = \int d^3x \left[-\frac{1}{2} rD + \frac{i}{2}\phi_r F_\phi - \frac{i}{2}\bar{\phi}_r \bar{F}_\phi\right], \qquad (2.135)$$

where r is a real FI parameter and ϕ_r a complex FI parameter. The conserved supercurrent is

$$J_r^\mu = \frac{1}{2} r(\gamma^\mu \bar{\lambda}) + \frac{\sqrt{2}}{2}\bar{\phi}_r(\gamma^\mu \bar{\psi}_\phi). \qquad (2.136)$$

One can also deform the 3d $\mathcal{N}=4$ supersymmetric gauge theories in a supersymmetric way by introducing mass terms for the hypermultiplets. It can be achieved by coupling Q and $\widetilde{Q}$ to a background Abelian $\mathcal{N}=4$ vector multiplet (V_M, Φ_M):

$$S_M = -\int d^3x d^4\theta (\bar{Q} e^{-2V_M} Q + \bar{\widetilde{Q}} e^{2V_M} \widetilde{Q}) + \sqrt{2} i \int d^3x d^2\theta (\widetilde{Q}\Phi_M Q) + \text{c.c.}, \qquad (2.137)$$

where $V_M = iM\bar{\theta}\theta$, $\Phi_M = \phi_M$, $M \in \mathbb{R}$ is real mass, and $\phi_M \in \mathbb{C}$ is complex mass parameters. Here $(M, \mathrm{Re}(\phi_M), \mathrm{Im}(\phi_M))$ forms a triplet under the $SU(2)_C$. In the component fields, the action (2.137) can be expressed as

$$\begin{aligned} S_M = \int d^3x [&-M^2(|q|^2 + |\widetilde{q}|^2) - iM(\bar{\psi}\psi - \bar{\widetilde{\psi}}\widetilde{\psi}) \\ &- (2F_q \bar{F}_q - 2F_{\widetilde{q}} \bar{F}_{\widetilde{q}}) + \sqrt{2} i \phi_M (F_{\widetilde{q}} q + F_q \widetilde{q}) \\ &- i\phi_M \widetilde{\psi}\psi + \sqrt{2} i \bar{\phi}_M (\bar{F}_{\widetilde{q}} \bar{q} + \bar{F}_q \bar{\widetilde{q}}) + i\bar{\phi}_M \bar{\widetilde{\psi}}\bar{\psi}]. \end{aligned} \qquad (2.138)$$

The conserved supercurrent is

$$\begin{aligned} J_M^\mu = &-\sqrt{2} M \bar{q}\gamma^\mu \psi + \sqrt{2} M \bar{\widetilde{q}}\gamma^\mu \widetilde{\psi} \\ &+ 2\bar{\phi}_M \bar{q}\gamma^\mu \bar{\widetilde{\psi}} + 2\bar{\phi}_M \bar{\widetilde{q}}\gamma^\mu \bar{\psi}. \end{aligned} \qquad (2.139)$$

The supercurrents (2.136) and (2.139) provide additional contributions to the supercurrents we obtained in previous sections and modify the supersymmetric boundary conditions.

2. *Coupled hypermultiplets*

We consider the half-BPS boundary conditions for the coupled hypermultiplet with FI and mass parameters turned on. Because of the coupling, the half-BPS boundary conditions for the hypermultiplets depends on the choice of the half-BPS boundary conditions for the vector multiplet discussed in Sec. II A with condition (2.106). This provides a large class of the half-BPS boundary conditions specified by the preserved gauge and flavor symmetries at the boundary. Here we want to find the general structure of deformed boundary conditions for the hypermultiplets due to gauge coupling, FI parameters, and mass parameters.

A-type boundary conditions.—

(i) $\gamma^2\psi = \bar{\psi}$, $\gamma^2\lambda = -\bar{\lambda}$, and $\gamma^2\psi_\phi = -\bar{\psi}_\phi$ (when $\varphi = 0$, $\theta = \frac{\pi}{2}$).—For the A-type conditions with $(\varphi, \theta) = (0, \frac{\pi}{2})$, we find from (2.107) the generic half of the supersymmetric boundary conditions for the hypermultiplets:

$$\begin{aligned} D_2 \cdot (\mathrm{Re} q) &= \sqrt{2}\mathrm{Re}[(\phi + \phi_M)\cdot \widetilde{q}], \\ D_2 \cdot (\mathrm{Re}\widetilde{q}) &= \sqrt{2}\mathrm{Re}[(\phi + \phi_M)\cdot q], \end{aligned} \qquad (2.140)$$

$$D_m \cdot (\mathrm{Im} q) = 0, \qquad D_m \cdot (\mathrm{Im}\widetilde{q}) = 0, \qquad (2.141)$$

$$(\sigma + M)\cdot(\mathrm{Re} q) = 0, \qquad (\sigma + M)\cdot(\mathrm{Re}\widetilde{q}) = 0, \qquad (2.142)$$

$$\mathrm{Im}(\widetilde{q} q) = \mathrm{Im}(\phi_r). \qquad (2.143)$$

The conditions (2.140) say that the real parts of the complex scalar fields q and $\widetilde{q}$ can fluctuate while satisfying the Robin-type boundary conditions, which specify a linear combination of the fields and the normal components of their derivatives at the boundary. The conditions (2.141) imply that the imaginary parts of q and $\widetilde{q}$ are subject to the Dirichlet-like boundary conditions.

The other set (2.142) and (2.143) are algebraic constraints which are responsible for the gauge coupling. The precise forms of the boundary conditions and the possible solutions depend on the detail of the 3d $\mathcal{N}=4$ vector multiplet and hypermultiplets, but these equations can be regarded as the basic building blocks of boundary conditions.

The real parts of q and $\tilde{q}$, which are fluctuating degrees of freedom at the boundary, satisfy conditions (2.142). As a coupled vector multiplet satisfies the magneticlike A-type boundary conditions when $\theta = \pi/2$, the two vector multiplet scalars $\mathrm{Re}\phi$ and σ obey the Dirichlet boundary conditions (2.31). Thus, the constraints (2.142) can be solved by setting σ to specific fixed values at the boundary.

The last condition (2.143) does not involve any bosonic fields in the vector multiplet, but it appears due to the gauge coupling and FI deformations as it is induced from the fermionic bilinear form involving ψ_ϕ. It is an imaginary part of the complex moment map $\mu_{\mathbb{C}}$ with fields restricted at the boundary.

(ii) $\gamma^2\psi = -\overline{\psi}$, $\gamma^2\lambda = \overline{\lambda}$, and $\gamma^2\psi_\phi = \overline{\psi}_\phi$ (when $\varphi = \frac{\pi}{2}$, $\theta = 0$).—The A-type conditions with $(\varphi, \theta) = (\frac{\pi}{2}, 0)$ are

$$D_2 \cdot (\mathrm{Im}q) = \sqrt{2}\mathrm{Im}[(\phi + \phi_M) \cdot \tilde{q}],$$
$$D_2 \cdot (\mathrm{Im}\tilde{q}) = \sqrt{2}\mathrm{Im}[(\phi + \phi_M) \cdot q], \qquad (2.144)$$

$$D_m \cdot (\mathrm{Re}q) = 0, \qquad D_m \cdot (\mathrm{Re}\tilde{q}) = 0, \qquad (2.145)$$

$$(\sigma + M) \cdot (\mathrm{Im}q) = 0, \qquad (\sigma + M) \cdot (\mathrm{Im}\tilde{q}) = 0, \qquad (2.146)$$

$$\mathrm{Re}(\tilde{q}q) = \mathrm{Re}(\phi_r), \qquad |q|^2 - |\tilde{q}|^2 = r. \qquad (2.147)$$

In this case, the real parts of q and $\tilde{q}$ satisfy Dirichlet-like boundary conditions (2.145), and the imaginary parts of q and $\tilde{q}$ can fluctuate while satisfying the Robin-type boundary conditions (2.144). Again, the remaining algebraic constraints (2.146) and (2.147) arise from the coupling of the 3d $\mathcal{N} = 4$ hypermultiplets to the vector multiplet.

For $\theta = 0$, the vector multiplet is subject to the electric A-type boundary conditions, where only the vector multiplet scalar $\mathrm{Im}\phi$ satisfies the Dirichlet-like boundary condition (2.72) and other scalars satisfy the Neumann-like boundary condition (2.26). As σ can fluctuate at the boundary, the constraint (2.146) are the conditions for the coupling of the hypermultiplet scalar fields q and $\tilde{q}$ and the vector multiplet scalar σ in a supersymmetric way when considering a boundary superpotential.

Two conditions in (2.147) are the constraints on the bulk hypermultiplets due to the gauge coupling and FI deformations, which are a real part of the complex moment map $\mu_{\mathbb{C}}$ and the real moment map $\mu_{\mathbb{R}}$, respectively, with fields restricted at the boundary.

B-type boundary conditions.—

(i) $\gamma^2\psi = \psi$, $\gamma^2\lambda = -\lambda$, and $\gamma^2\psi_\phi = -\psi_\phi$ (when $\varphi = 0$ and $\theta = \frac{\pi}{2}$).—As the full R symmetry $SU(2)_C \times SU(2)_H$ is maintained for the B-type conditions, a pair of fermionic fields ψ and $\bar{\tilde{\psi}}$ may form the supermultiplet. For $(\varphi, \theta) = (0, \frac{\pi}{2})$, we obtain, from (2.120), (2.136), and (2.139),

$$D_2 \cdot q + (\phi^i + M^i) \cdot q = 0,$$
$$D_2 \cdot \tilde{q} - (\phi^i + M^i) \cdot \tilde{q} = 0, \qquad (2.148)$$

where we have defined the triplet $\phi^i = (\sigma, \mathrm{Re}\phi, \mathrm{Im}\phi)$ and $M^i = (M, \mathrm{Re}\phi_M, \mathrm{Im}\phi_M)$ of the $SU(2)_C$. The bosonic degrees of freedom q and $\tilde{q}$ can fluctuate at the boundary while satisfying Robin-type boundary conditions (2.148). In this case, the vector multiplet obeys the electric B-type conditions that admit the Dirichlet-like boundary condition (2.54) for all the vector multiplet scalars. Also, the detail forms of boundary conditions depend on the specific data of the theories; however, (2.148) can be viewed as the basic building blocks for the boundary conditions.

(ii) $\gamma^2\psi = -\psi$, $\gamma^2\lambda = \lambda$, and $\gamma^2\psi_\phi = \psi_\phi$ (when $\varphi = \frac{\pi}{2}$ and $\theta = 0$).—From (2.121), (2.136), and (2.139), the general B-type conditions with $(\varphi, \theta) = (\frac{\pi}{2}, 0)$ read

$$D_m \cdot q = 0, \qquad D_m \cdot \tilde{q} = 0, \qquad (2.149)$$

$$|q|^2 - |\tilde{q}|^2 = r, \qquad q\tilde{q} = \phi_r. \qquad (2.150)$$

Similarly as before, the algebraic conditions (2.150) come from the gauge coupling and FI deformation.

3. BPS boundary conditions and 3d $\mathcal{N} = 4$ vacua

The classical moduli space of the 3d $\mathcal{N} = 4$ supersymmetric gauge theory on $\mathbb{R}^{1,2}$ is determined by the set of equations

$$[\phi^i, \phi^j] = 0, \qquad (2.151)$$

$$(\phi^i + M^i) \cdot (q, \tilde{q}) = 0, \qquad (2.152)$$

$$\mu^{\hat{i}} + r^{\hat{i}} = 0, \qquad (2.153)$$

where the dot $\cdot$ implies the action of the gauge and flavor symmetry group on the hypermultiplet scalars $(q, \tilde{q})$. Here $\mu^{\hat{i}}$ are the three hyper-Kähler moment maps for the action of the gauge symmetry group on the hypermultiplets. They split into the real and complex moment maps $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}$ [10]. They are, respectively, associated to the Kähler form

$$\omega = \sum_I (dq^I \wedge d\bar{q}^I + d\tilde{q}^I \wedge d\bar{\tilde{q}}^I) \tag{2.154}$$

and the holomorphic symplectic form

$$\Omega = \sum_I (dq^I \wedge d\tilde{q}^I) \tag{2.155}$$

and given by

$$\mu_{\mathbb{R}} = |q|^2 - |\tilde{q}|^2, \tag{2.156}$$

$$\mu_{\mathbb{C}} = q\tilde{q}. \tag{2.157}$$

We remark that the half-BPS boundary conditions detect the set of the defining equations (2.151)–(2.153) of the vacua. We have encountered Eq. (2.151) in the vector multiplet boundary conditions (2.32) and (2.59) where fields are restricted at the boundary, which can be expected as it characterizes the Coulomb branch. The second set of equations (2.152) specifies the coupling between the vector multiplet scalars and the hypermultiplet scalars. We have met these equations with fields restricted on the boundary in the boundary conditions constraining the fluctuation of hypermultiplet scalars. As (2.152) suggests, these conditions can be shifted by turning on the mass parameters $\sigma \to \sigma + M, \phi \to \phi + \phi_M$. We also saw that the moment maps (2.153) with FI parameters appear as algebraic constraints for the scalar component of hypermultiplets at the boundary.

III. BRANE CONSTRUCTION

In this section, we propose the brane configurations in the type IIB string theory corresponding to the half-BPS boundary conditions of the 3d $\mathcal{N}=4$ supersymmetric theories discussed in Sec. II. We also study the map between boundary supermultiplets arising from 3d bulk supermultiplets for simplest examples by considering S duality of the type IIB theory.

A. Type IIB configuration

We consider the brane realization of 3d $\mathcal{N}=4$ theories in the type IIB string theory on $\mathbb{R}^{1,9}$ [14]. Let Q_L (respectively, Q_R) be the supercharge generated by the left- (respectively, right-) moving world-sheet degrees of freedom which satisfies the chirality condition of the type IIB string theory:

$$\Gamma_{0123456789} Q_L = Q_L, \qquad \Gamma_{0123456789} Q_R = Q_R. \tag{3.1}$$

We consider D3-branes supported on (x^0, x^1, x^2, x^6) and bounded along the x^6 direction by two NS5-branes supported on $(x^0, x^1, x^2, x^3, x^4, x^5)$ or by two D5-branes supported on $(x^0, x^1, x^2, x^7, x^8, x^9)$:

	0	1	2	3	4	5	6	7	8	9
D3	∘	∘	∘	−	−	−	∘	−	−	−
NS5	∘	∘	∘	∘	∘	∘	−	−	−	−
D5	∘	∘	∘	−	−	−	−	∘	∘	∘

(3.2)

where ∘ denotes the directions in which branes are supported whereas − stands for the directions at which branes are located. The brane configuration (3.2) preserves linear combination of supercharges $\epsilon_L Q_L + \epsilon_R Q_R$ with

$$\Gamma_{012345}\epsilon_L = \epsilon_L, \qquad \Gamma_{012345}\epsilon_R = -\epsilon_R \tag{3.3}$$

and

$$\Gamma_{012789}\epsilon_R = \epsilon_L, \tag{3.4}$$

$$\Gamma_{0126}\epsilon_R = \epsilon_L. \tag{3.5}$$

Here, the first condition (3.3) is the projection condition on spinors ϵ_L and ϵ_R imposed by the NS5-branes, while (3.4) and (3.5) are the conditions by the D5-branes and the D3-branes, respectively. From (3.3)–(3.5), we can find two nontrivial conditions on the spinors. So there remain eight supercharges.

As D3-branes are bounded in the x^6 direction, the low-energy effective theory of world volume of D3-branes is described by 3d $\mathcal{N}=4$ supersymmetric theories after decoupling the gravity. The above brane configuration breaks the Lorentz symmetry group $SO(1,9)$ into $SO(1,2)_{012} \times SO(3)_{345} \times SO(3)_{789}$, where $SO(1,2)$ is Lorentz symmetry and the double covers of $SO(3)_{345} \times SO(3)_{789}$ give $SU(2)_C \times SU(2)_H \cong SO(4)_R$ R symmetry of 3d $\mathcal{N}=4$ theories.

B. D3-NS5 branes

Let us first consider the case where the N coincident D3-branes are stretched between the two parallel NS5-branes. The low-energy effective theory is the 3d $\mathcal{N}=4$ $U(N)$ pure SYM theory [14]. The three-dimensional coupling constant g_{3d}^2 is classically given by $\frac{1}{g_{3d}^2} = \frac{\Delta x^6(\mathrm{NS5})}{g_{4d}^2}$, where $\Delta x^6(\mathrm{NS5})$ is the interval of the stretched D3-branes along x^6 and g_{4d}^2 is the gauge coupling of 4d $\mathcal{N}=4$ SYM theory. The bosonic massless modes of the world-volume theory of D3-branes are the fluctuations of the D3-branes in transverse directions x^3, x^4, and x^5 and three-dimensional gauge fields. The $U(N)$ gauge symmetry has a nontrivial center $U(1)$, which parametrizes the motion of the center of mass of the N D3-branes. The FI parameters $\{r, \mathrm{Re}(\phi_r), \mathrm{Im}(\phi_r)\}$ are described by the relative positions of two NS5-branes along x^7, x^8, and x^9.

1. A-type boundary conditions

The half-BPS boundary conditions for the pure 3d $\mathcal{N}=4$ vector multiplet discussed in Sec. II A can be realized in the D3-NS5 brane system by introducing additional branes. We call such additional branes the NS5′-brane and D5′-brane where they are supported on $(x^0, x^1, x^3, x^4, x^6, x^9)$ and $(x^0, x^1, x^5, x^6, x^7, x^8)$, respectively. They are located at $x^2=0$ and D3-branes are extended in the half space $x^2 \geq 0$:

	0	1	2	3	4	5	6	7	8	9
D3	○	○	○	—	—	—	○	—	—	—
NS5	○	○	○	○	○	○	—	—	—	—
NS5′	○	○	—	○	○	—	○	—	—	○
D5′	○	○	—	—	—	○	○	○	○	—

(3.6)

Therefore, the additional 5-branes provide the two-dimensional boundary at $x^2=0$ in the effective 3d $\mathcal{N}=4$ SYM theories (see Fig. 1). Also, the original $SO(1,2)\times SO(3)_{345}\times SO(3)_{789}$ symmetry is broken to $SO(1,1)\times SO(2)_{34}\times SO(2)_{78}$.

The NS5′-brane and D5′-brane provide additional projection conditions, respectively,

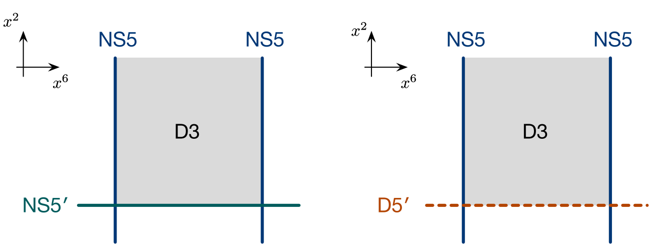
$$\Gamma_{013469}\epsilon_L = \epsilon_L, \qquad \Gamma_{013469}\epsilon_R = -\epsilon_R, \tag{3.7}$$

$$\Gamma_{015678}\epsilon_R = \epsilon_L. \tag{3.8}$$

From the conditions (3.3), (3.5), (3.8), and (3.7), there are three nontrivial projection conditions, so four supercharges are preserved in the brane configuration (3.6). In order to see the chirality of the two-dimensional supersymmetry, we note the conditions

$$\Gamma_{26}\epsilon_L = -\Gamma_{59}\epsilon_L, \qquad \Gamma_{26}\epsilon_R = -\Gamma_{59}\epsilon_R \tag{3.9}$$

from the above brane configurations. Since the four-dimensional world volume of the D3-branes is finite along x^6 and the effective field theory is three-dimensional,



(a)NS5′-brane (b)D5′-brane

FIG. 1. D3-NS5 system with a NS5′-brane or D5′-brane. The NS5′- (D5′-) brane provides the electriclike (magneticlike) A-type boundary conditions for the vector multiplet where the 2d $\mathcal{N}=(2,2)$ vector multiplet (twisted chiral multiplet) can fluctuate at the boundary.

we may treat Γ_6 essentially proportional to the identity matrix. Here Γ_2 plays the role of the two-dimensional chirality matrix for the two-dimensional boundary of the three-dimensional field theory, while Γ_5 (respectively, Γ_9) is the chirality matrix for the $SO(2)_{34}$ [respectively, $SO(2)_{78}$]. Let $(\pm,\pm,\pm)$ be the representation under the $SO(1,1)\times SO(2)_{34}\times SO(2)_{78}$, where $\pm$ denote the two-dimensional chiralities. Suppose that chiral supersymmetry is preserved at the two-dimensional boundary, say, the right-moving $(+,\cdot,\cdot)$ supersymmetry. As the $SO(2)_{34}$ charge and the $SO(2)_{78}$ charge are constrained via (3.9), we would have only two supercharges with $(+,+,-)$ and $(+,-,+)$ if we choose a positive multiplicative constant for Γ_6, which we treated as the identity matrix. However, since we have four supercharges in the brane setup (3.6), this implies that there should also be left-moving supersymmetry. Therefore, the additional NS5′- and D5′-branes preserve the nonchiral $\mathcal{N}=(2,2)$ supersymmetry where $SO(2)_{34}\times SO(2)_{78}\cong U(1)_{\rm axial}\times U(1)_{\rm vector}$ are axial and vector R symmetry of the 2d $\mathcal{N}=(2,2)$ supersymmetry.

(i) NS5′-*brane.*—The D3-branes ending on the NS5′-brane can fluctuate along x^3, x^4, and the two-dimensional gauge field A_m can fluctuate at the boundary. On the other hand, the NS5′-brane gives a Dirichlet boundary condition for A_2 and also for ϕ^5 as it is localized at x^5. These boundary conditions are consistent with the electriclike A-type boundary conditions (2.25)–(2.28):

$$\begin{aligned} F_{2m}&=0 \qquad \text{(Neumann-like)},\\ D_2\phi^a&=0 \qquad \text{(Neumann-like)},\\ D_m\phi^5&=0 \qquad \text{(Dirichlet-like)}. \end{aligned} \tag{3.10}$$

(ii) D5′-*brane.*—As the x^3 and x^4 position of the D3-branes are fixed by the D5′-brane but the motion of the D3-brane along x^5 is unconstrained, ϕ^3 and ϕ^4 satisfy the Dirichlet-like condition but ϕ^5 would satisfy the Neumann-like condition. The boundary condition for the two-dimensional gauge field A_m imposed by the D5′-brane is the Dirichlet-like boundary condition. Therefore, inserting D5′-brane would give

$$\begin{aligned} F_{01}&=0 \qquad \text{(Dirichlet-like)},\\ D_2\phi^5&=0 \qquad \text{(Neumann-like)},\\ D_m\phi^a&=0 \qquad \text{(Dirichlet-like)}. \end{aligned} \tag{3.11}$$

In addition, the attached D5′-brane can leave A_2 unconstrained. This is consistent with the field theoretic analysis in Sec. II A.

2. B-type boundary conditions

There are other additional 5-branes which can preserve $\mathcal{N}=(0,4)$ supersymmetry at the two-dimensional

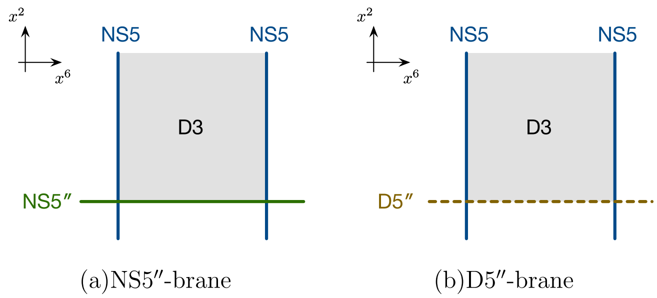


FIG. 2. D3-NS5 system with a NS5″-brane or D5″-brane. The NS5″- (D5″-)brane provides the electriclike (magneticlike) B-type boundary conditions for the vector multiplet where the 2d $\mathcal{N} = (0,4)$ vector multiplet (twisted hypermultiplet) can fluctuate at the boundary.

boundary of the 3d effective theories. We consider the NS5″-brane with world volume $(x^0, x^1, x^6, x^7, x^8, x^9)$ or the D5″-brane with world volume $(x^0, x^1, x^3, x^4, x^5, x^6)$ located at $x^2 = 0$, where D3-branes are extended on the half-space $x^2 \geq 0$ (see Fig. 2);

	0	1	2	3	4	5	6	7	8	9	
D3	○	○	○	—	—	—	○	—	—	—	
NS5	○	○	○	○	○	○	—	—	—	—	(3.12)
NS5″	○	○	—	—	—	—	○	○	○	○	
D5″	○	○	—	○	○	○	○	—	—	—	

These additional NS5″ and D5″ give constraints, respectively,

$$\Gamma_{016789}\epsilon_L = \epsilon_L, \qquad \Gamma_{016789}\epsilon_R = -\epsilon_R \tag{3.13}$$

$$\Gamma_{013456}\epsilon_R = \epsilon_L. \tag{3.14}$$

From the conditions (3.13), (3.14), (3.3), and (3.5), we have three nontrivial projection conditions, so there are four supercharges in the brane system (3.12). Also, the set of conditions leads to

$$\Gamma_{01}\epsilon_L = \epsilon_L, \qquad \Gamma_{01}\epsilon_R = \epsilon_R, \tag{3.15}$$

which implies that we have chiral $\mathcal{N} = (0,4)$ supersymmetry at the two-dimensional boundary.

The inclusion of these additional 5-branes does not break the symmetry $SO(3)_{345} \times SO(3)_{789} \cong SU(2)_C \times SU(2)_H \cong SO(4)_R$, which is the R symmetry of 2d $\mathcal{N} = (0,4)$ supersymmetry. Under the $SO(1,1) \times SU(2)_C \times SU(2)_H$, the preserved right-moving supercharges transform as $(\mathbf{2},\mathbf{2})_+$.

(i) NS5″*-brane.*—The NS5″-brane fixes the motion of the D3-branes in x^3, x^4, and x^5, so three scalar fields ϕ^i obey the Dirichlet-like boundary conditions. On the other hand, the two-dimensional gauge field A_m can fluctuate at the boundary, and A_2 satisfy the Dirichlet-like condition. Therefore, the NS5″-brane imposes the boundary conditions

$$\begin{aligned} F_{2m} &= 0 \qquad \text{(Neumann-like)}, \\ D_m\phi^i &= 0 \qquad \text{(Dirichlet-like)}, \end{aligned} \tag{3.16}$$

which are consistent with NS5″-like B-type boundary conditions (2.54) and (2.55).

(ii) D5″*-brane.*—Since the D5″-brane is extended along x^3, x^4, and x^5, the three scalar fields ϕ^i are free to move at the boundary. They transform as $(\mathbf{3},\mathbf{1})$ under $SO(3)_{345} \times SO(3)_{789}$. Meanwhile, the two-dimensional gauge field A_m satisfies the Dirichlet condition, because it is tangent to the D5″-brane, but the scalar field A_2 can fluctuate at the boundary. Thus, for a single D3-brane, the D5″-brane would give the boundary conditions

$$\begin{aligned} F_{01} &= 0 \qquad \text{(Dirichlet-like)}, \\ D_2\phi^i &= 0 \qquad \text{(Neumann-like)}. \end{aligned} \tag{3.17}$$

However, considering the field theory result discussed in Sec. II A, we expect that the above boundary condition is generalized to

$$\begin{aligned} F_{01} &= 0 \qquad \text{(Dirichlet-like)}, \\ D_2\phi^i - \frac{1}{2} i\epsilon^{ijk}[\phi^j, \phi^k] &= 0 \qquad \text{(Nahm-like)}. \end{aligned} \tag{3.18}$$

That is, we expect that D3-NS5-D5″ realize the magneticlike B-type boundary conditions, which are described by (3.18) including the Nahm-like equation. This is reminiscent of the appearance of the Nahm equation in half-BPS boundary conditions of 4d $\mathcal{N} = 4$ theories discussed in Ref. [4], where the nontrivial boundary conditions for a multiple stack of D3-branes provided by a D5-brane are described by the Nahm equation due to the existence of the fluctuating scalar fields A_2.

C. D3-D5 branes

Next, we consider the N D3-branes suspended between the two parallel D5-branes. In the low-energy limit, the world-volume theory of the D3-branes is a theory of N massless 3d $\mathcal{N} = 4$ hypermultiplets [14]. The bosonic massless modes in the theories are the fluctuations of the D3-branes in transverse positions x^7, x^8, and x^9, which we will denote by $X^{\hat{7}}$, $X^{\hat{8}}$, and $X^{\hat{9}}$, respectively, and the scalar field A_6. They combine into two complex scalar fields transforming as (1,2) under $SU(2)_C \times SU(2)_H$. The mass parameters $\{M, \phi_M\}$ are given by the relative position of the D5-branes along $\{x^3, x^4, x^5\}$.

1. A-type boundary conditions

As discussed in Sec. III B, we can realize the two-dimensional nonchiral $\mathcal{N} = (2,2)$ supersymmetry by the introduction of the NS5′- or D5′-branes

	0	1	2	3	4	5	6	7	8	9
D3	○	○	○	—	—	—	○	—	—	—
D5	○	○	○	—	—	—	—	○	○	○
NS5′	○	○	—	○	○	—	○	—	—	○
D5′	○	○	—	—	—	○	○	○	○	—

(3.19)

as in the configuration (3.6) (see Fig. 3).

Under the space-time symmetry $SO(1,1) \times SO(2)_{34} \times SO(2)_{78} \cong SO(1,1) \times U(1)_C \times U(1)_H$, the three scalar fields $X^{\hat{i}}$, $\hat{i} = 7,\ 8,\ 9$, are divided into the two scalar fields $X^{\hat{a}}$, $\hat{a} = 7$, 8, and the scalar field $X^{\hat{9}}$. As $SU(2)_H$ is broken to $U(1)_H$, these scalar fields are charged under the vector R symmetry of 2d $\mathcal{N} = (2,2)$ theories.

(i) NS5′*-brane.*—As the D3-branes can move along x^9 in the presence of NS5′, the scalar field $X^{\hat{9}}$, which describes the position of the D3-branes along x^9, can fluctuate at the boundary. In addition, the massless modes of the scalar field A_6 can also fluctuate as the NS5′-brane is extended along x^6. Thus, the additional NS5′-brane keeps the half of the bosonic degrees of freedom of the 3d $\mathcal{N} = 4$ hypermultiplet at the boundary:

$$\partial_m X^{\hat{7}} = 0, \qquad \partial_m X^{\hat{8}} = 0 \qquad \text{(Dirichlet-like)},$$
$$\partial_2 X^{\hat{9}} = 0, \qquad \partial_2 A_6 = 0 \qquad \text{(Neumann-like)}. \tag{3.20}$$

Let $q = X^{\hat{7}} + iA_6$ and $\tilde{q} = X^{\hat{8}} + iX^{\hat{9}}$ be two complex scalar fields. Then we have

$$\partial_m(\mathrm{Re}q) = 0, \quad \partial_m(\mathrm{Re}\tilde{q}) = 0 \quad \text{(Dirichlet-like)},$$
$$\partial_2(\mathrm{Im}q) = 0, \quad \partial_2(\mathrm{Im}\tilde{q}) = 0 \quad \text{(Neumann-like)}. \tag{3.21}$$

(ii) D5′*-brane.*—As D3-branes can move along x^7 and x^8 directions, scalar fields $X^{\hat{7}}$ and $X^{\hat{8}}$ corresponding to directions x^7 and x^8 can fluctuate at the boundary. On the other hand, the scalar field $X^{\hat{9}}$ corresponding to x^9 cannot fluctuate at the boundary. Also, the massless modes associated to A_6 cannot fluctuate at the boundary, since D5′ is extended along x^6. Similarly to the case with the NS5′-brane, the half of the bosonic degrees of freedom of the 3d $\mathcal{N} = 4$ hypermultiplet can survive at the boundary. Therefore, we have

$$\partial_2 X^{\hat{7}} = 0, \qquad \partial_2 X^{\hat{8}} = 0 \qquad \text{(Neumann-like)},$$
$$\partial_m X^{\hat{9}} = 0, \qquad \partial_m A_6 = 0 \qquad \text{(Dirichlet-like)}. \tag{3.22}$$

Again, in terms of q and $\tilde{q}$ we have

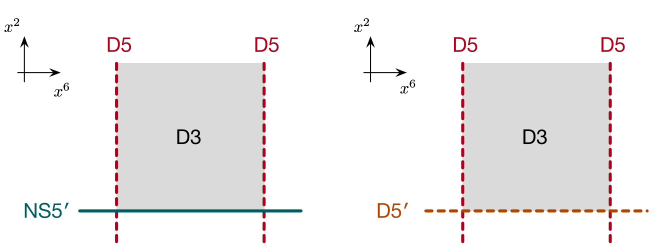
$$\partial_2(\mathrm{Re}q) = 0, \quad \partial_2(\mathrm{Re}\tilde{q}) = 0 \quad \text{(Neumann-like)},$$
$$\partial_m(\mathrm{Im}q) = 0, \quad \partial_m(\mathrm{Im}\tilde{q}) = 0 \quad \text{(Dirichlet-like)}. \tag{3.23}$$

2. B-type boundary conditions

Following the arguments for the D3-NS5 brane system, $\mathcal{N} = (0,4)$ supersymmetry can be preserved at the boundary by adding the NS5″- or D5″-branes at $x^2 = 0$ to the D3-D5 brane configuration where D3-branes are extended along $x^2 \geq 0$ (see Fig. 4) as

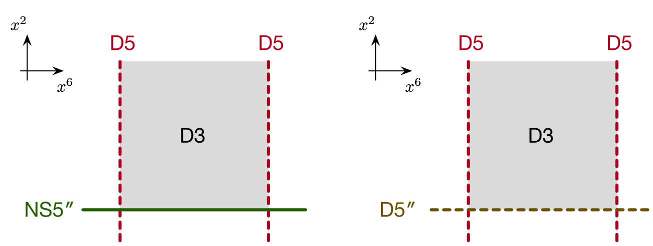
	0	1	2	3	4	5	6	7	8	9
D3	○	○	○	—	—	—	○	—	—	—
D5	○	○	○	—	—	—	—	○	○	○
NS5″	○	○	—	—	—	—	○	○	○	○
D5″	○	○	—	○	○	○	○	—	—	—

(3.24)



(a)NS5′-brane (b)D5′-brane

FIG. 3. D3-D5 system with a NS5′-brane or D5′-brane. The NS5′ and D5′-brane provide the A-type boundary conditions for the pure hypermultiplet where 2d $\mathcal{N} = (2,2)$ chiral multiplets can fluctuate at the boundary.



(a)NS5″-brane (b)D5″-brane

FIG. 4. D3-D5 system with a NS5″-brane or D5″-brane. The NS5″- (D5″-)brane provides the B-type boundary conditions for the pure hypermultiplet where the 2d $\mathcal{N} = (0,4)$ hypermultiplet [$\mathcal{N} = (0,4)$ Fermi multiplet] can fluctuate at the boundary.

The brane configuration (3.24) preserves R symmetry $SO(4)_R = SU(2)_C \times SU(2)_H \cong SO(3)_{345} \times SO(3)_{789}$ of 3d $\mathcal{N} = 4$ theories, and the three scalar fields $X^{\hat{i}}$ transform as a triplet under $SO(3)_{789}$.

(i) NS5″-*brane*.—Since the NS5″-brane is supported on the x^6, x^7, x^8, and x^9 directions, scalar field A_6 and three scalar fields $X^{\hat{i}}$ can fluctuate at the boundary. The NS5″-brane would lead to the Neumann conditions for these scalar fields:

$$\partial_2 X^{\hat{i}} = 0, \qquad \partial_2 A_6 = 0. \tag{3.25}$$

These conditions correspond to Neumann boundary conditions (2.122) for the pure hypermultiplets:

$$\partial_2 q = 0, \quad \partial_2 \tilde{q} = 0 \quad \text{(Neumann-like)}. \tag{3.26}$$

(ii) D5″-*brane*.—Since the D5″-brane is extended in x^6 and located at x^7, x^8, and x^9, the scalar field A_6 and the three scalar fields describing the position of the D3-branes all satisfy the Dirichlet condition at the boundary:

$$\partial_m X^{\hat{i}} = 0, \qquad \partial_m A_6 = 0. \tag{3.27}$$

We see that the above conditions (3.27) are equivalent to the conditions (2.123).

Hence, in terms of q and $\tilde{q}$, the conditions read

$$\partial_m q = 0, \quad \partial_m \tilde{q} = 0 \quad \text{(Dirichlet-like)}. \tag{3.28}$$

D. D3-NS5-D5 branes

We consider A- and B-type boundary conditions for SQCD in the context of brane configuration (3.2).

1. A-type boundary conditions

Similarly as before, we consider the extra NS5′- or D5′-branes at $x^2 = 0$ in the following brane configurations:

	0	1	2	3	4	5	6	7	8	9
D3	∘	∘	∘	–	–	–	∘	–	–	–
NS5	∘	∘	∘	∘	∘	∘	–	–	–	–
D5	∘	∘	∘	–	–	–	–	∘	∘	∘
NS5′	∘	∘	–	∘	∘	–	∘	–	–	∘
D5′	∘	∘	–	–	–	∘	∘	∘	∘	–

(3.29)

As usual, the 3d $\mathcal{N} = 4$ vector multiplet is realized in the world volume of D3-branes. Also, the hypermultiplet is realized as strings connecting D3-branes and D5-branes. We expect that, when NS5′-(D5′-)brane is added, it provides a Neumann- (Dirichlet-) like condition for $\{A_m, \phi^5\}$ and $\{\mathrm{Im}(q), \mathrm{Im}(\tilde{q})\}$, but a Dirichlet- (Neumann-) like condition for $\{A_2, \phi^a\}$ and $\{\mathrm{Re}(q), \mathrm{Re}(\tilde{q})\}$ where $m = 0, 1$, $a = 3, 4$.

Since the NS5′-brane is located at x^7 and x^8, two of the FI parameters of the 3d $\mathcal{N} = 4$ theory would arise in the boundary conditions as the relative positions of NS5-branes in the x^7 and x^8 directions. This brane picture is consistent with the result that the deformed A-type boundary conditions (2.147) for the coupled hypermultiplets involve the two FI parameters r and $\mathrm{Re}\phi_r$. In this brane configuration, mass parameters of the 3d $\mathcal{N} = 4$ theory are given by the relative distance between D3- and D5-branes in the x^3, x^4, and x^5 directions. The mass parameter $\mathrm{Im}\phi_M$, which generalizes the hypermultiplet Neumann boundary conditions (2.111) to the Robin-type boundary conditions (2.144), is given by the relative distance between D3-branes and D5-branes along the x^5 direction where the position of D3-branes along the x^5 direction is fixed by the NS5′-brane. Meanwhile, the mass parameter M, which is related to the vacuum expectation value of ϕ^3 of the background vector multiplet, has a different nature from $\mathrm{Im}\phi_M$ above. Given the position of the D5-brane at a fixed location of the x^3 and x^4 directions, since the NS5′-brane is supported on the x^3 and x^4 directions, the D3-brane can still move along those directions. This is compatible with the BPS equations (2.146) that mass parameter M appears in boundary coupling rather than boundary conditions.

For the D5′-brane, since it is located at the x^9 direction, the boundary conditions would be deformed by one of the FI parameters of the 3d $\mathcal{N} = 4$ theory as the relative position of NS5-branes in the x^9 direction. In a field theory analysis, we see that a single FI parameter $\mathrm{Im}\phi_r$ appears in the deformed A-type boundary conditions (2.143) for the coupled hypermultiplets. As the D5′-brane is located at the x^3 and x^4 directions, in a similar manner discussed above, two mass parameters would generalize the hypermultiplet boundary conditions. Those corresponding two mass parameters M and $\mathrm{Im}\phi_M$ appear in the deformed hypermultiplet boundary conditions (2.140) and (2.142).

2. B-type boundary conditions

Also, we consider the extra NS5″- or D5″-branes at $x^2 = 0$ to the following brane configurations:

	0	1	2	3	4	5	6	7	8	9
D3	∘	∘	∘	–	–	–	∘	–	–	–
NS5	∘	∘	∘	∘	∘	∘	–	–	–	–
D5	∘	∘	∘	–	–	–	–	∘	∘	∘
NS5″	∘	∘	–	–	–	–	∘	∘	∘	∘
D5″	∘	∘	–	∘	∘	∘	∘	–	–	–

(3.30)

Similarly as before, we expect that when a NS5″- (D5″-) brane is added, it provides a Neumann- (Dirichlet-) like condition for $\{A_m\}$ and $\{q, \tilde{q}\}$, but a Dirichlet- (Neumann-) like condition for $\{A_6, \phi^i\}$.

As the NS5″-brane is supported on the x^7, x^8, and x^9 directions, none of the FI parameters of the 3d $\mathcal{N}=4$ theory would deform the boundary conditions. This brane perspective is consistent with the result that the deformed B-type boundary conditions (2.148) for the coupled hypermultiplets involve no FI parameters in the boundary conditions. As the NS5″-brane is located at the x^3, x^4, and x^5 directions, in a similar manner discussed above, three mass parameters (M, ϕ_M) would appear in boundary conditions. In a field theory analysis, we see that the hypermultiplet Neumann boundary condition (2.122) is generalized to the Robin-type boundary condition (2.148) by all three mass parameters M^i.

In the case of the D5″-brane, which is located at the x^7, x^8, and x^9 directions, all three FI parameters $r^{\hat{i}}$ would deform the boundary conditions. This can be seen from the deformed B-type boundary conditions (2.150) for the coupled hypermultiplets. Since the D5″-brane is supported in the x^3, x^4, and x^5 directions, mass parameters would not appear in boundary conditions. In fact, the deformed B-type hypermultiplet boundary conditions (2.149) and (2.150) are not affected by mass parameters.

E. *S* duality

From the analysis on the half-BPS boundary condition of the 3d $\mathcal{N}=4$ theory, we saw which 2d supermultiplet of $\mathcal{N}=(2,2)$ and $\mathcal{N}=(0,4)$ from the bulk 3d $\mathcal{N}=4$ multiplet arises at the boundary. We also found that such a boundary condition can be consistently understood in terms of brane configurations of the type IIB string theory.

Upon S duality of the type IIB string theory, the 3d $\mathcal{N}=4$ theory arising from a given brane configuration enjoys mirror symmetry [14]. With additional branes that provide the half-BPS boundary condition discussed in previous sections, it is interesting to see the relation between the boundary degrees of freedom arising from a particular brane configuration and those arising from an S-dual configuration of the original brane configuration. In general, this could be a nontrivial task, but here we just take the simplest cases, pure vector multiplet and pure hypermultiplet, discussed in the previous section, and would like to see how the boundary degrees of freedom from the bulk 3d $\mathcal{N}=4$ multiplet are mapped to each other.

1. *A type*

In this case, we have $U(1)_C \times U(1)_H$ R symmetry of the 2d $\mathcal{N}=(2,2)$ theory, which is the axial and vector R symmetry, from original $SU(2)_C \times SU(2)_H$ R symmetry of 3d $\mathcal{N}=4$. As $SU(2)_C$ and $SU(2)_H$ are exchanged under a RS map,[9] so $U(1)_C$ and $U(1)_H$ are exchanged. Hence, it is expected that 3d $\mathcal{N}=4$ mirror symmetry is closely related to 2d $\mathcal{N}=(2,2)$ mirror symmetry through S duality in the type IIB string theory. In fact, it has been argued that 3d mirror symmetry descends to 2d mirror symmetry via compactification [26] and also that the 2d $\mathcal{N}=(2,2)$ interface theory between 3d $\mathcal{N}=4$ mirror pairs produces a mirror map of 2d $\mathcal{N}=(2,2)$ chiral and twisted chiral operators [10]. We see for the following simplest example that the 2d mirror map is realized as S duality in the type IIB string theory.

(i) D3-NS5-NS5′ $\overset{S\text{-dual}}{\longleftrightarrow}$ D3-D5-D5′.—The boundary degree of freedom from the bulk 3d $\mathcal{N}=4$ vector multiplet arising in a D3-NS5-NS5′ system is the 2d $\mathcal{N}=(2,2)$ vector multiplet or field strength multiplet, which is a twisted chiral multiplet. On the other hand, the one from the bulk 3d $\mathcal{N}=4$ hypermultiplet arising in a D3-D5-D5′ system is the 2d $\mathcal{N}=(2,2)$ chiral multiplet in the adjoint representation. As two brane configurations are S dual, which gives rise to a mirror pair between the pure vector multiplet and the pure hypermultiplet in the bulk, we see that the twisted chiral multiplet and chiral multiplet at the boundary $x^2=0$ are exchanged under S duality of the type IIB string theory or 3d $\mathcal{N}=4$ mirror symmetry. This is consistent with 2d $\mathcal{N}=(2,2)$ mirror symmetry.

(ii) D3-NS5-D5′ $\overset{S\text{-dual}}{\longleftrightarrow}$ D3-D5-NS5′.—Similarly, in this case, the boundary degree of freedom from the bulk vector multiplet arising in a D3-NS5-D5′ system is the 2d $\mathcal{N}=(2,2)$ twisted chiral multiplet, and the one from the bulk hypermultiplet arising in D3-D5-NS5′ is the 2d $\mathcal{N}=(2,2)$ chiral multiplet. Under S duality of the brane configuration, those two 2d $\mathcal{N}=(2,2)$ supermultiplets are mapped to each other, which is consistent with 2d $\mathcal{N}=(2,2)$ mirror symmetry.

2. *B type*

The 2d $\mathcal{N}=(0,4)$ mirror symmetry has not been studied much in the literature.[10] We expect that the $\mathcal{N}=(0,4)$ theory arising from (a more general or complicated version of) our brane configuration and the theory arising from the corresponding S-dual configuration give rise to the $\mathcal{N}=(0,4)$ mirror pair. In the 2d $\mathcal{N}=(0,4)$ gauge theory may receive the anomaly from massless charged chiral fermions running in one loop [28], and we should take into account the cancellation of the gauge anomaly to obtain the effective theories. We hope to revisit this issue in the context of the brane configuration. Here, we consider only the map between the 2d $\mathcal{N}=(0,4)$ supermultiplets at the boundary arising from the 3d $\mathcal{N}=4$ pure vector multiplet

[9] In the brane configuration of the type IIB string theory, R of the RS map denotes the map x^i to x^{i+4} where $i=3,4,5$ and S denotes S duality [14]. In the following, we mean S duality by RS duality.

[10] The 2d $\mathcal{N}=(0,4)$ mirror symmetry could be understood as the special case of 2d $\mathcal{N}=(0,2)$ mirror symmetry [27].

and the pure hypermultiplet discussed in the previous section.

(i) D3-NS5-NS5$''\overset{S\text{-dual}}{\longleftrightarrow}$D3-D5-D5$''$.—The boundary degree of freedom from the 3d $\mathcal{N}=4$ vector multiplet arising in a D3-NS5-NS5$''$ system is the 2d $\mathcal{N}=(0,4)$ vector multiplet, and the one from the 3d $\mathcal{N}=4$ hypermultiplet arising in a D3-D5-D5$''$ system is the 2d $\mathcal{N}=(0,4)$ Fermi multiplet.

The 2d $\mathcal{N}=(0,4)$ vector multiplet is made of an $\mathcal{N}=(0,2)$ vector multiplet and Fermi multiplet in adjoint representation, where the $\mathcal{N}=(0,2)$ vector multiplet can be expressed as an $\mathcal{N}=(0,2)$ field strength multiplet, which is the $\mathcal{N}=(0,2)$ Fermi multiplet. The fermions in the $\mathcal{N}=(0,4)$ vector multiplet are charged under $SO(1,1)\times SU(2)_C\times SU(2)_H$ as $(\mathbf{2},\mathbf{2})_-$. Meanwhile, the $\mathcal{N}=(0,4)$ Fermi multiplet is made of two $\mathcal{N}=(0,2)$ Fermi multiplets in a conjugate representation of gauge group G, and it is charged under $SU(2)_C\times SU(2)_H$ as $(\mathbf{1},\mathbf{1})_-$. Since there are four real fermions in the vector multiplet, under the S duality of the IIB theory, the number of fermions is matched with the number of them in the Fermi multiplet, though it is not quite sure to explain the relation of their R charges in the scope of this paper. It seems that better understanding is needed for this case.

(ii) D3-NS5-D5$''\overset{S\text{-dual}}{\longleftrightarrow}$D3-D5-NS5$''$.—The boundary degree of freedom from the bulk 3d $\mathcal{N}=4$ vector multiplet arising in a D3-NS5-D5$''$ system is the 2d $\mathcal{N}=(0,4)$ twisted hypermultiplet, and the one from the 3d $\mathcal{N}=4$ hypermultiplet arising in a D3-D5-NS5$''$ system is the 2d $\mathcal{N}=(0,4)$ hypermultiplet. Upon S duality, $SU(2)_C$ and $SU(2)_H$ are exchanged, so twisted hypermultiplets are mapped hypermultiplets, and vice versa.

IV. CONCLUSION AND DISCUSSION

In this paper, we studied the half-BPS boundary conditions in 3d $\mathcal{N}=4$ gauge theories preserving $\mathcal{N}=(2,2)$ and (0,4) supersymmetries at the boundary, which we call A type and B type, respectively. We calculated the BPS boundary equations for a vector multiplet and hypermultiplet involving gauge coupling, FI, and mass deformations. We also saw that 3d bulk supermultiplets are decomposed to the boundary supermultiplet of preserved supersymmetry. We found that the boundary BPS equations for the vector multiplet, in particular, give rise to a Nahm-like equation in the magneticlike B-type boundary conditions. For the hypermultiplet, we saw that the Neumann-like boundary conditions for scalar components of the hypermultiplet are generalized to a Robin-type boundary condition upon turning on gauge coupling and mass deformation. We proposed brane configurations in the type IIB string theory realizing such $\mathcal{N}=(2,2)$ and (0,4) BPS boundary conditions in 3d $\mathcal{N}=4$ theories and checked that they are consistent with the analysis in the field theory. We also saw how the boundary supermultiplets from the bulk supermultiplets are mapped under S duality of the type IIB theory.

In order to study the supersymmetric vacua of 3d $\mathcal{N}=4$ gauge theory on a half-space, it is necessary to study the BPS boundary conditions in detail. A notable consequence is that we get a Nahm-like equation in vector multiplet boundary conditions of B type. It is interesting to analyze these BPS equations in a similar way as discussed in Ref. [4] for 4d $\mathcal{N}=4$ SYM theories.

Brane realization of 2d gauge theories with (2,2) and (0,4) supersymmetries is one of the interesting subjects.[11] In particular, there is an anomaly issue in 2d $\mathcal{N}=(0,4)$ theories, and it would be interesting to know how such an anomaly condition can arise in the type IIB string theory. Also, as we briefly discussed for the boundary degrees of freedom from the bulk supermultiplets, the realization of 2d $\mathcal{N}=(0,4)$ theories in the brane configuration will tell us, via S duality of the type IIB theory, a *mirror* dual theory of a given 2d $\mathcal{N}=(0,4)$ theory from the corresponding brane configurations. With the anomaly issue taken into account, the study of mirror symmetry of the 2d $\mathcal{N}=(0,4)$ theory via type IIB S duality would be one intriguing direction.

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APPENDIX: 3d $\mathcal{N}=2$ SUPERSPACE AND SUPERFIELDS

1. Spinors and superspace

We use the metric $\eta_{\mu\nu}=\eta^{\mu\nu}=\mathrm{diag}(-1,1,1)$ and 2×2 γ^μ matrices to satisfy

$$\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}. \tag{A1}$$

γ^0 is taken as anti-Hermitian and γ^1 and γ^2 as Hermitian. We introduce a three-dimensional charge conjugation matrix ϵ, which has the following properties:

[11] Other brane realizations for 2d $\mathcal{N}=(2,2)$ and (0,4) have been discussed in Refs. [29,30], respectively.

$$\epsilon^\dagger = \epsilon^{-1}, \qquad \epsilon^T = -\epsilon, \qquad (\epsilon\gamma^\mu)^T = \epsilon\gamma^\mu. \tag{A2}$$

Two-component spinors ψ^α with upper or lower indices transform as

$$\psi_\alpha := \epsilon_{\alpha\beta}\psi^\beta, \qquad \psi^\alpha = (\epsilon^{-1})^{\alpha\beta}\psi_\beta. \tag{A3}$$

We use the following summation convention:

$$(\chi\psi) := \chi^\alpha\psi_\alpha = \chi^\alpha\epsilon_{\alpha\beta}\psi^\beta, \qquad (\gamma^\mu\psi)^\alpha = \gamma^{\mu\alpha}{}_\beta\psi^\beta,$$
$$(\epsilon\gamma^\mu\psi)_\alpha = (\epsilon\gamma^\mu)_{\alpha\beta}\psi^\beta. \tag{A4}$$

We define σ matrices as

$$\sigma^\mu := \epsilon\gamma^\mu \tag{A5}$$

and use the summation expression $\xi\sigma^\mu\psi := \xi^\alpha(\epsilon\gamma^\mu)_{\alpha\beta}\psi^\beta$. We define the conjugation by

$$\overline{\psi}_\alpha := -\psi^\dagger_\beta(\gamma^0)^\beta{}_\alpha. \tag{A6}$$

Here are useful spinor formulas:

$$\xi\psi = \psi\xi, \qquad \xi\sigma^\mu\psi = -\psi\sigma^\mu\xi,$$
$$\psi\sigma^\mu\psi = 0, \qquad \psi\epsilon\gamma^{\mu\nu}\chi = -\chi\epsilon\gamma^{\mu\nu}\psi, \tag{A7}$$

$$(\xi\psi)^\dagger = -\overline{\psi}\,\overline{\xi}, \qquad (\xi\sigma^\mu\psi)^\dagger = \overline{\psi}\sigma^\mu\overline{\xi} = -\overline{\xi}\sigma^\mu\overline{\psi}, \tag{A8}$$

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \qquad \theta^\alpha\theta^\beta = -\frac{1}{2}(\epsilon^{-1})^{\alpha\beta}\theta\theta, \tag{A9}$$

$$(\theta\psi)(\theta\chi) = -\frac{1}{2}(\theta\theta)(\psi\chi), \tag{A10}$$

$$(\theta\sigma^\mu\chi)(\theta\psi) = -\frac{1}{2}\theta\theta\psi\sigma^\mu\chi, \tag{A11}$$

$$\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} = \frac{1}{2}\theta\theta\bar{\theta}\,\bar{\theta}\,\eta^{\mu\nu}, \tag{A12}$$

$$-\frac{1}{2}(\chi\lambda)(\psi\xi) - \frac{1}{2}(\chi\sigma^\mu\lambda)(\psi\sigma_\mu\xi) = (\chi\xi)(\psi\lambda), \tag{A13}$$

where ψ, ξ, θ, and λ are two-component spinors.

We consider the 3d $\mathcal{N}=2$ superspace coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^\alpha)$ which transform as $x^\mu \to x^\mu - i\epsilon\sigma^\mu\bar{\theta} - i\bar{\epsilon}\sigma^\mu\theta$, $\theta \to \theta + \epsilon$, and $\bar{\theta} \to \bar{\theta} + \bar{\epsilon}$ under the supersymmetry transformations. Let us define the following supersymmetric derivatives:

$$Q_\alpha := \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu, \qquad \overline{Q}_m := -\frac{\partial}{\partial\bar{\theta}^\alpha} + i(\sigma^\mu\theta)_\alpha\partial_\mu, \tag{A14}$$

$$D_\alpha := \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu, \qquad \bar{D}_\alpha := -\frac{\partial}{\partial\bar{\theta}^\alpha} - i(\sigma^\mu\theta)_\alpha\partial_\mu. \tag{A15}$$

They have the anticommutation relations

$$\{Q_\alpha, \overline{Q}_\beta\} = 2i\sigma^\mu_{\alpha\beta}\partial_\mu, \qquad \{D_\alpha, \bar{D}_\beta\} = -2i\sigma^\mu_{\alpha\beta}\partial_\mu, \tag{A16}$$

with all the other anticommutators vanishing. The supersymmetry transformation of a superfield $\Phi(x,\theta,\bar{\theta})$ is expressed as

$$\delta\Phi(x,\theta,\bar{\theta}) = (\xi Q - \bar{\xi}\,\bar{Q})\Phi. \tag{A17}$$

2. Supermultiplet

a. Chiral multiplet

Chiral superfield $\Phi(x,\theta,\bar{\theta})$ is defined by the constraint

$$\bar{D}_\alpha\Phi = 0. \tag{A18}$$

Using $y^\mu := x^\mu + i\theta\sigma^\mu\bar{\theta}$, one can obtain the component field representations:

$$\begin{aligned}\Phi &= \Phi(y,\theta)\\ &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\\ &= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\,\bar{\theta}\,\partial^2\phi(x)\\ &\quad + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}(\theta\theta)(\bar{\theta}\sigma^\mu\partial_\mu\psi(x)) + \theta\theta F(x).\end{aligned} \tag{A19}$$

Similarly, the antichiral superfield $\bar{\Phi}(x,\theta,\bar{\theta})$ obeying the constraint $D_m\bar{\Phi} = 0$ can be obtained from (A19) by conjugation:

$$\begin{aligned}\overline{\Phi} &= \overline{\phi}(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\overline{\phi}(x) - \frac{1}{4}\theta\theta\bar{\theta}\,\bar{\theta}\,\partial^2\overline{\phi}(x)\\ &\quad - \sqrt{2}\bar{\theta}\,\overline{\psi}(x) - \frac{i}{\sqrt{2}}(\bar{\theta}\,\bar{\theta})(\theta\sigma^\mu\partial_\mu\overline{\psi}(x)) - \bar{\theta}\,\bar{\theta}\,\overline{F}(x).\end{aligned} \tag{A20}$$

b. Vector multiplet

The vector superfield satisfies the relation

$$V = \bar{V}. \tag{A21}$$

Choosing the Wess-Zumino gauge, we obtain

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\bar{\theta}\sigma - i\theta\theta\bar{\theta}\,\bar{\lambda} + i\bar{\theta}\,\bar{\theta}\,\theta\lambda - \frac{1}{2}\theta\theta\bar{\theta}\,\bar{\theta}\,D(x). \tag{A22}$$

One can express a field strength as a linear multiplet:

$$\Sigma \coloneqq -\frac{i}{2}\bar{D}DV. \tag{A23}$$

In components, it is expressed as

$$\Sigma = \sigma + \theta\bar{\lambda} - \lambda\bar{\theta} - i(\bar{\theta}\theta)D + \frac{1}{2}(\bar{\theta}\epsilon\gamma^{\mu\nu}\theta)F_{\mu\nu} - \frac{i}{2}\theta\theta(\bar{\theta}\sigma^\mu\partial_\mu\bar{\lambda}) + \frac{i}{2}\bar{\theta}\,\bar{\theta}(\theta\sigma^\mu\partial_\mu\lambda) + \frac{1}{4}\theta\theta\bar{\theta}\,\bar{\theta}\,\partial^\mu\partial_\mu\sigma. \tag{A24}$$

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