Unified approach to the entropy of an extremal rotating BTZ black hole: Thin shells and horizon limits

José P. S. Lemos,[1,*] Masato Minamitsuji,[2,†] and Oleg B. Zaslavskii[3,‡]

[1]*Centro de Astrofisica e Gravitação–CENTRA, Departamento de Física, Instituto Superior Técnico—IST, Universidade de Lisboa—UL, Avenida Rovisco Pais 1, 1049-001, Portugal, and Gravitational Physics Group, Faculty of Physics, University of Vienna, Boltzmanngasse 5, A1090 Wien, Austria*

[2]*Centro de Astrofisica e Gravitação—CENTRA, Departamento de Física, Instituto Superior Técnico—IST, Universidade de Lisboa—UL, Avenida Rovisco Pais 1, 1049-001, Portugal*

[3]*Department of Physics and Technology, Kharkov V. N. Karazin National University, 4 Svoboda Square, Kharkov 61022, Ukraine, and Institute of Mathematics and Mechanics, Kazan Federal University, 18 Kremlyovskaya Street, Kazan 420008, Russia*

(Received 29 August 2017; published 31 October 2017)

Using a thin shell, the first law of thermodynamics, and a unified approach, we study the thermodymanics and find the entropy of a $(2+1)$-dimensional extremal rotating Bañados-Teitelbom-Zanelli (BTZ) black hole. The shell in $(2+1)$ dimensions, i.e., a ring, is taken to be circularly symmetric and rotating, with the inner region being a ground state of the anti-de Sitter spacetime and the outer region being the rotating BTZ spacetime. The extremal BTZ rotating black hole can be obtained in three different ways depending on the way the shell approaches its own gravitational or horizon radius. These ways are explicitly worked out. The resulting three cases give that the BTZ black hole entropy is either the Bekenstein-Hawking entropy, $S=\frac{A_+}{4G}$, or an arbitrary function of A_+, $S=S(A_+)$, where $A_+=2\pi r_+$ is the area, i.e., the perimeter, of the event horizon in $(2+1)$ dimensions. We speculate that the entropy of an extremal black hole should obey $0\le S(A_+)\le\frac{A_+}{4G}$. We also show that the contributions from the various thermodynamic quantities, namely, the mass, the circular velocity, and the temperature, for the entropy in all three cases are distinct. This study complements the previous studies in thin shell thermodynamics and entropy for BTZ black holes. It also corroborates the results found for a $(3+1)$-dimensional extremal electrically charged Reissner-Nordström black hole.

DOI: 10.1103/PhysRevD.96.084068

I. INTRODUCTION

One can argue that thin matter shells in general relativity provide the simplest class of spacetimes after vacuum spacetimes. Indeed, thin shells, besides giving instances of static and dynamic spacetimes, allow themselves to be scrutinized in relation to their entropic and thermodynamic matter and gravitational properties, and even from those properties to pick up the corresponding black hole properties. For static and rotating circularly symmetric thin shells, i.e., thin rings, in $(2+1)$-dimensional Bañados-Teitelbom-Zanelli (BTZ) spacetimes, their entropic and thermodynamic properties have been worked out in general and in the limit where the ring is taken to its own gravitational, or horizon, radius, i.e., in the black hole limit [1–4]. For static electric charged spherically symmetric thin shells in $(3+1)$-dimensional Reissner-Nordström spacetimes, these properties have also been worked out in detail in general and in the black hole limit [5–7]; see also Ref. [8] for neutral thin shells in Schwarzschild spacetimes. Related studies, where the entropy of black holes can be studied through systems with matter, involve quasiblack holes for which matter is spread over a 3-dimensional spatial region rather than on a 2-dimensional thin shell [9,10], and of quasistatic collapse of matter [11]. These works [1–11] stem from the fact that the concept of entropy is originally based on quantum properties of matter, and so it is very important to study whether and how black hole thermodynamics could emerge from thermodynamics of collapsing matter, when matter is compressed within its own gravitational radius. Conversely, it is through black hole entropy that we can grasp the microscopic aspects of a spacetime and hence of quantum gravity, and the fact that thermodynamics of a thin shell reflects thermodynamic properties of a black hole formed after quasistatic collapse of the shell indicates some connection between matter and gravitational degrees of freedom.

In this thin shell approach to black hole entropy, a clear cut distinction exists between nonextremal black holes and extremal black holes.

For nonextremal black holes, one finds that the entropy is

$$S=\frac{A_+}{4G}, \tag{1}$$

*joselemos@ist.utl.pt
†masato.minamitsuji@ist.utl.pt
‡zaslav@ukr.net

where A_+ is the area of the event horizon and G is the gravitational constant. Throughout the paper, we use units such that the velocity of the light, the Planck constant, and the Boltzmann constant are set to 1. This result has been found for static BTZ shells [1] and for rotating BTZ shells [2,3], as well as for Reissner-Nordström shells [5], all in the black hole limit. The result recovers the Bekenstein-Hawking entropy formula in $(2+1)$ dimensions [12,13] and in the original works in $(3+1)$ dimensions [14–16]. In $(2+1)$ dimensions, A_+ is a perimeter $A_+ = 2\pi r_+$, and in $(3+1)$ dimensions, $A_+ = 4\pi r_+^2$ is the usual area, with r_+ being the gravitational or horizon radius.

For extremal black holes, the ones which we will study in this paper, the situation is more subtle in the shell approach. Extremal black holes are those for which the angular momentum or electric charge is equal to the mass in some appropriate units. It has been found that the entropy of the extremal black hole can depend on the way the shell approaches its own gravitational radius. This results in three cases. On one hand, clearly, there is a case for an originally nonextremal shell, which we call *case 1*, in which after taking the black hole limit the shell turns into an extremal shell, where one finds $S = \frac{A_+}{4G}$ as in Eq. (1); see also Refs. [3,4] for BTZ and [5–7] for Reissner-Nordström. On the other hand, it was further found in the Reissner-Nordström situation that there is a new case [7], which we call *case 2*, in which the shell is turned extremal concomitantly with the spacetime being turned into a black hole. In this case, one finds also $S = \frac{A_+}{4G}$ as in Eq. (1). Finally, for an *ab initio* extremal shell that turns into an extremal black hole, one finds that the entropy is a generic function of A_+, i.e.,

$$S = S(A_+). \tag{2}$$

This result, which we call *case 3*, is found both in extremal rotating BTZ [4] and in extremal electric charged Reissner-Nordström [6].

Given the result (2) together with (1), one is led to speculate that the entropy of an extremal black hole should obey

$$0 \le S(A_+) \le \frac{A_+}{4G}. \tag{3}$$

The lower limit

$$S = 0 \tag{4}$$

is indeed found through a Euclidean path integral approach to extremal black hole entropy, both in BTZ black holes [17] and in Reissner-Nordström black holes [18], whereas in contradiction, the Bekenstein-Hawking upper limit of Eq. (3), $S = \frac{A_+}{4G}$, see also Eq. (1), is found through string theory techniques in extremal cases, namely, in $(2+1)$-dimensional extremal rotating BTZ black holes [19] and in $(3+1)$-dimensional extremal Reissner-Nordström black holes [20], following the breakthrough worked out in $(4+1)$ dimensions [21,22]; see also Refs. [23–32] for further studies on thermodynamics and entropy of extremal black holes. In a sense, Eq. (3) fills the gap between Euclidean path integral approaches and string theory techniques for the entropy of extremal black holes.

The aim of this paper is to complete the study on extremal rotating BTZ thin shell thermodynamics [1–4], in order to have a full understanding of the entropy of an extremal rotating BTZ black hole. We follow also the studies for electrically charged Reissner-Nordström shells [5,6], and in particular, we adopt the unified approach devised for an electrically charged Reissner-Nordström thin shell [7] and study the three different limits of a rotating thin shell in a $(2+1)$-dimensional rotating BTZ spacetime when it approaches both extremality and its own gravitational radius, i.e., in the extremal BTZ black hole limit. These three different limits yield the three cases, cases 1–3, already mentioned. Our analysis will point out the similarities between the rotating and the electric charged case and will show the contributions from the various thermodynamic quantities appearing in the first law to the entropy in all three cases. The approach developed in the present work can be of interest for the generic investigation of black hole entropy in the thin shell formalism, in particular, for the Kerr black hole, at least in the slow rotation approximation, or to other more complicated $(3+1)$- and $(n+1)$-dimensional black holes, with $n > 3$.

The paper is organized as follows. In Sec. II, we review the mechanics and thermodynamics of a rotating thin shell in $(2+1)$ dimensions with a negative cosmological constant, where the exterior spacetime is BTZ. In Sec. III, we introduce the three different limits, thus establishing three different cases, when the rotating thin shell is taken into its own gravitational radius and forms an extremal BTZ black hole. We define the good variables to study these limits and work out the geometry, the mass, and the angular momentum of the shell in the three different cases. In Sec. IV, we discuss the three different cases for the pressure, the circular velocity, and the local temperature of the shell. In Sec. V, we calculate the entropy of a rotating extremal BTZ black hole in the three different cases. In Sec. VI, we show, in the three different cases, which terms in the first law give the dominant contributions to the entropy. In Sec. VII, we conclude.

II. THIN SHELL THERMODYNAMICS IN A (2+1)-DIMENSIONAL BTZ SPACETIME

We consider general relativity in $(2+1)$ dimensions with a cosmological constant Λ, where we assume that $\Lambda < 0$, so that the spacetime is asymptotically anti-de Sitter (AdS), with curvature scale $\ell = \sqrt{-\frac{1}{\Lambda}}$. In an otherwise vacuum spacetime, we introduce a timelike rotating thin shell, i.e., a timelike thin ring in the $(2+1)$-dimensional spacetime, with radius R, that divides the spacetime into the inner and outer regions. We leave G explicitly in the formulas; the other physical constants are set to 1.

The spacetime inside the shell, $0 < r < R$, where r is a radial coordinate, is given by the zero mass $m = 0$ BTZ-AdS solution in $(2+1)$ dimensions.

The spacetime outside the shell, $r > R$, is generically described by the rotating BTZ solution with Arnowitt-Deser-Misner (ADM) mass m and angular momentum $\mathcal{J}$. Two important quantities of the outer spacetime, which are related to m and $\mathcal{J}$, are the gravitational radius r_+ and the Cauchy radius r_-. The relations between the quantities are [12]

$$8G\ell^2 m = r_+^2 + r_-^2, \tag{5}$$

$$4G\ell\mathcal{J} = r_+ r_-. \tag{6}$$

From Eqs. (5) and (6), one clearly sees that one can trade m and $\mathcal{J}$ for r_+ and r_- and vice versa. For a spacetime that is not over rotating, as will be the case considered here, one has that $m \geq \frac{\mathcal{J}}{\ell}$, which, in terms of r_+ and r_-, translates into $r_+ \geq r_-$. This inequality is saturated in the extremal case, $r_+ = r_-$, i.e., $m = \frac{\mathcal{J}}{\ell}$. The gravitational area A_+ defined as

$$A_+ = 2\pi r_+ \tag{7}$$

is actually a perimeter, since there are just 2 space dimensions.

The shell itself has radius R, and it is quasistatic in the sense that $\frac{dR}{d\tau} = \frac{d^2R}{d\tau^2} = 0$, where τ is the proper time on the shell. The area A of the shell defined as

$$A = 2\pi R \tag{8}$$

is also a perimeter, since there are 2 space dimensions. We assume that the shell is always located outside or at the gravitational radius,

$$R \geq r_+. \tag{9}$$

Note that the gravitational radius is not a horizon radius in this case, it is simply a feature of the spacetime. It would be a horizon radius only if $R \leq r_+$. Since from Eq. (9) one has $R \geq r_+$ there is a horizon in the limit $R = r_+$. In this limiting situation the shell is on the verge of becoming a black hole. Besides having a radius R, the shell has mass M and angular momentum J.

To find the properties of the shell and the connection to the inner and outer spacetime, one has to work out the junction conditions. The junction conditions determine the energy density of the shell σ and the angular momentum density of the shell j, or if one prefers, the rest mass of the shell $M \equiv 2\pi R\sigma$ and the angular momentum of the shell $J \equiv 2\pi R$. One finds that M and J are some specific functions of the ADM spacetime mass m, angular momentum $\mathcal{J}$, and the shell's radius R; see Ref. [3] for details (see also Ref. [2]). These relations can be inverted to give the ADM spacetime mass m as a function of M, J, and R, namely,

$$m(M, J, R) = \frac{RM}{\ell} - 2GM^2 + \frac{2G}{R^2}J^2, \tag{10}$$

and the ADM spacetime angular momentum $\mathcal{J}$ also as a function of M, J, and R, namely,

$$\mathcal{J}(M, J, R) = J. \tag{11}$$

In Eqs. (10) and (11), we have written m as $m(M, J, R)$ and $\mathcal{J}$ as $\mathcal{J}(M, J, R)$ in order to make manifest the explicit dependence of the ADM spacetime mass m and the ADM spacetime angular momentum $\mathcal{J}$ on the shell quantities, i.e., its rest mass M, its angular momentum J, and its radius R. This explicit dependence is also useful when we deal with the thermodynamics of the shell. The gravitational radius r_+ and the Cauchy radius r_- can be found inverting Eqs. (5) and (6) [12]. The gravitational radius is

$$r_+(M, J, R) = 2\ell\sqrt{Gm + \sqrt{(Gm)^2 - \frac{(G\mathcal{J})^2}{\ell^2}}}, \tag{12}$$

and the Cauchy radius is

$$r_-(M, J, R) = 2\ell\sqrt{Gm - \sqrt{(Gm)^2 - \frac{(G\mathcal{J})^2}{\ell^2}}}, \tag{13}$$

with m and $\mathcal{J}$ seen as functions of M, J, and R through Eqs. (10) and (11).

As a thermodynamic system, the shell has a locally measured temperature T and an entropy S. We consider that the shell is adiabatic;, i.e., it does not radiate to the exterior. The entropy S of a system can be expressed as a function of the state independent variables which for the rotating shell can be chosen as the shell's locally measured proper mass M, the shell's angular momentum J, and the shell's area A. Thus, $S = S(M, J, A)$, and in these variables, the first law of thermodynamics reads

$$TdS = dM + pdA - \Omega dJ, \tag{14}$$

where p is the tangential pressure at the shell, Ω is the thermodynamic angular velocity of the shell, and T is the temperature of the shell. These quantities, p, Ω, and T are equations of state functions of (M, J, A), i.e., $p = p(M, J, A)$, $\Omega = \Omega(M, J, A)$, and $T = T(M, J, A)$.

In $(2+1)$ dimensions, the shell's area is a perimeter, namely, $A = 2\pi R$, and we can thus express S, p, Ω, and T, as functions of the shell radius R, instead of its area A. This simplifies the presentation. Thus, $S = S(M, J, R)$, $T = T(M, J, R)$, $p = p(M, J, R)$, and $\Omega = \Omega(M, J, R)$. In order to have a well-defined entropy S, there are integrability conditions for $T = T(M, J, R)$, $p = p(M, J, R)$, and $\Omega = \Omega(M, J, R)$; see Ref. [3].

The first law for the shell, Eq. (14), is clearly displayed and has a clear physical meaning in the variables M, J, and R. As it turns out and as we will see, it is much simpler mathematically to work instead in the variables r_+, r_-, and R. Indeed, from Eqs. (12) and (13) together with Eqs. (10) and (11), one can swap the variables M, R, and J, into r_+, r_-, and R. So, from now on, we express our quantities in terms of (r_+, r_-, R).

Inverting Eq. (10) and using Eqs. (12) and (13) together with Eq. (11), one finds

$$M(r_+, r_-, R) = \frac{R}{4G\ell}\left(1 - \frac{1}{R^2}\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}\right). \tag{15}$$

Inverting Eq. (11) and using Eqs. (12) and (13) [or more simply Eq. (5)], one finds

$$J(r_+, r_-, R) = \frac{r_+ r_-}{4G\ell}. \tag{16}$$

The tangential pressure p at the shell found through the junction conditions [3] (see also Ref. [2]) is

$$p(r_+, r_-, R) = \frac{1}{8\pi G\ell}\left(\frac{R^4 - r_+^2 r_-^2}{R^2\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}} - 1\right). \tag{17}$$

The angular velocity Ω and the corresponding linear or circular velocity $v = R\Omega$ can be found either by the junction conditions or from one integrability condition of the first law of thermodynamics Eq. (14) [3,4]. The integrability condition gives that the angular velocity defined thermodynamically can be expressed by $\Omega(r_+, r_-, R) = \frac{r_+ r_-}{R\sqrt{(1-\frac{r_+^2}{R^2})(1-\frac{r_-^2}{R^2})}}(c(r_+, r_-) - \frac{1}{R^2})$, where $c(r_+, r_-)$ is an integrating arbitrary function of r_+ and r_- (see Eq. (59) in Ref. [3] and Sec. VI in Ref. [4]). We choose $c(r_+, r_-) = \frac{1}{r_+^2}$, in order to have the well-defined black hole limit [3,4]. In this case, one sees that Ω vanishes when the shell approaches the gravitational radius, $R \to r_+$. Since the circular velocity of the shell is $v = R\Omega$, one has, with the choice $c(r_+, r_-) = \frac{1}{r_+^2}$ and after simplifications, that

$$v(r_+, r_-, R) = R\Omega(r_+, r_-, R) = \frac{r_-}{r_+}\sqrt{\frac{R^2 - r_+^2}{R^2 - r_-^2}}. \tag{18}$$

The temperature T being a pure thermodynamic quantity is found from another integrability condition of the first law of thermodynamics Eq. (14) [3,4]. As found in Ref. [3], the temperature can be expressed as $T(r_+, r_-, R) = \frac{T_0(r_+, r_-)}{\frac{R}{\ell}\sqrt{(1-\frac{r_+^2}{R^2})(1-\frac{r_-^2}{R^2})}}$, where $T_0(r_+, r_-)$ is an arbitrary function of r_+ and r_- (see also Eqs. (C2) and (C3) from Ref. [4]). Now, $T_0(r_+, r_-)$ is chosen to be the Hawking temperature of the BTZ black hole, i.e., $T_0(r_+, r_-) = T_H(r_+, r_-) = \frac{1}{2\pi\ell^2}\frac{r_+^2 - r_-^2}{r_+}$ [12]. Thus, we have

$$T(r_+, r_-, R) = \frac{r_+^2 - r_-^2}{2\pi\ell R r_+}\frac{R^2}{\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}}. \tag{19}$$

For the outer spacetime, it is usually useful to define the redshift function k that appears naturally in several instances, namely, $k(r_+, r_-, R) = \frac{R}{\ell}\sqrt{(1 - \frac{r_+^2}{R^2})(1 - \frac{r_-^2}{R^2})}$. With this quantity, the temperature T assumes the familiar form $T(r_+, r_-, R) = \frac{T_H(r_+, r_-)}{k(r_+, r_-, R)}$, and so the function $T_H(r_+, r_-)$ can be interpreted as the temperature of the shell located at the radius where $k = 1$, the Hawking temperature. Seen in this fashion, the formula for T expresses then the gravitational redshift of the temperature of the shell; namely, it is an instance of the Tolman temperature formula.

Note that the choices, $c(r_+, r_-) = \frac{1}{r_+^2}$ for the velocity v and $T_0(r_+, r_-) = T_H(r_+, r_-) = \frac{1}{2\pi\ell^2}\frac{r_+^2 - r_-^2}{r_+}$ for the temperature T that lead to Eqs. (18) and (19), respectively, are essential if we want to take the black hole limit, i.e., when the shell is taken to its gravitational radius, $R \to r_+$ [3,4]. So, we stick to these choices.

III. THE THREE DIFFERENT APPROACHES AND THE THREE LIMITS TO THE BTZ EXTREMAL BLACK HOLE

A. The variables useful to define the three different approaches and limits to an extremal horizon

To study the entropy of the BTZ extremal black hole, we take a unified approach; see Ref. [7] for an extremal electric charged shell in $3+1$ dimensions. For that, we introduce the dimensionless parameters ε and δ through

$$\varepsilon^2 = 1 - \frac{r_+^2}{R^2}, \tag{20}$$

$$\delta^2 = 1 - \frac{r_-^2}{R^2}. \tag{21}$$

From Eqs. (20) and (21), we see that we can change the independent thermodynamics variables (r_+, r_-, R) into the new variables (ε, δ, R). In this set of variables, for example, the redshift function k defined above takes the simple form $k(\varepsilon, \delta, R) = \frac{R}{\ell}\varepsilon\delta$.

B. Geometry and the three horizon limits

The three relevant limits to an extremal black hole are:

(i) *Case 1.*—$r_+ \neq r_-$ and $R \to r_+$, i.e.,

$$\delta = O(1), \qquad \varepsilon \to 0. \tag{22}$$

In evaluating the entropy S, we then take $r_+ \to r_-$, i.e., the $\delta \to 0$ limit, to make the shell extremal at its own gravitational radius $R = r_+$.

(ii) *Case 2.*—$r_+ \to r_-$ and $R \to r_+$, i.e.,

$$\delta = \frac{\varepsilon}{\lambda}, \qquad \varepsilon \to 0, \tag{23}$$

where the constant λ is finite, not infinitesimal, and must satisfy $\lambda < 1$ due to $r_+ > r_-$. The limit $\varepsilon \to 0$ means here that simultaneously $R \to r_+$ and $r_+ \to r_-$ in such a way that $\delta \sim \varepsilon$. In other words, extremality and black holeness are approached concomitantly.

(iii) *Case 3.*—$r_+ = r_-$ and $R \to r_+$, i.e.,

$$\delta = \varepsilon, \qquad \varepsilon \to 0. \tag{24}$$

This is the case in which there is an extremal shell from the very beginning and then one pushes it to its own gravitational radius.

C. Mass and angular momentum in the three horizon limits

In the variables ε and δ of Eqs. (20) and (21), the shell's rest mass M in Eq. (15) can be written as

$$M(\varepsilon, \delta, R) = \frac{R}{4G\ell}(1 - \varepsilon\delta). \tag{25}$$

In addition, from Eq. (16), the shell's angular momentum J is now

$$J(\varepsilon, \delta, R) = \frac{R^2}{4G\ell}\sqrt{(1-\varepsilon^2)(1-\delta^2)}. \tag{26}$$

In all cases 1–3, the limits defined in Eqs. (22)–(24) yield

$$M(\varepsilon, \delta, r_+) = \frac{r_+}{4G\ell} \tag{27}$$

and

$$J(\varepsilon, \delta, r_+) = \frac{r_+^2}{4G\ell} \tag{28}$$

for the shell's mass and angular momentum, respectively. Thus, the three limits, not surprisingly, yield the same extremal condition,

$$J = r_+ M. \tag{29}$$

IV. PRESSURE, CIRCULAR VELOCITY, AND LOCAL TEMPERATURE: THE THREE EXTREMAL BTZ BLACK HOLE LIMITS

A. Pressure in the three horizon limits

In the variables ε and δ of Eqs. (20) and (21), the shell's pressure p in Eq. (17) can be written as

$$p(\varepsilon, \delta, R) = \frac{1}{8\pi G\ell}\left(\frac{\delta}{\varepsilon} + \frac{\varepsilon}{\delta} - 1 - \varepsilon\delta\right). \tag{30}$$

For cases 1–3, the limits defined in Eqs. (22)–(24) yield from Eq. (30) the expressions for the pressure as below:

(i) *Case 1.*—For $\delta = O(1)$ and $\varepsilon \to 0$,

$$p(\varepsilon, \delta, r_+) = \frac{\delta}{8\pi G\ell\varepsilon}, \tag{31}$$

up to leading order. Equation (31) means that the pressure is divergent as $1/\varepsilon$.

(ii) *Case 2.*—For $\delta = \frac{\varepsilon}{\lambda}$ and $\varepsilon \to 0$,

$$p(\varepsilon, \delta, r_+) = \frac{1}{8\pi G\ell}\left(\frac{1}{\lambda} + \lambda - 1\right), \tag{32}$$

up to leading order. Equation (32) means that the pressure remains finite but nonzero, since λ is finite and fixed with $\lambda < 1$.

(iii) *Case 3.*—For $\delta = \varepsilon$ and $\varepsilon \to 0$,

$$p(\varepsilon, \delta, r_+) = \frac{1}{8\pi G\ell}, \tag{33}$$

up to leading order. Equation (33) means that the pressure remains finite and nonzero. Note the difference from the $(3+1)$-dimensional electric extreme shell in an asymptotically flat spacetime studied in Refs. [6,7], where in this same limit one found instead $p = 0$. This difference arises from the different asymptotic behaviors of the spacetime, namely, asymptotically flat spacetime in Refs. [6,7] and an asymptotically AdS spacetime here; see also Ref. [4].

B. Circular velocity in the three horizon limits

With the variables ε and δ defined in Eqs. (20) and (21), the shell's circular velocity v in Eq. (18) can be written as

$$v(R, \varepsilon, \delta) = \sqrt{\frac{1-\delta^2}{1-\varepsilon^2}\frac{\varepsilon}{\delta}}. \tag{34}$$

For cases 1–3, the limits defined in Eqs. (22)–(24) yield from Eq. (34) the expressions for the circular velocity as below:

(i) *Case 1*.—For $\delta = O(1)$ and $\varepsilon \to 0$,

$$v(\varepsilon, \delta, r_+) = 0 \tag{35}$$

up to leading order.

(ii) *Case 2*.—For $\delta = \frac{\varepsilon}{\lambda}$ and $\varepsilon \to 0$,

$$v(\varepsilon, \delta, r_+) = \lambda, \tag{36}$$

up to leading order. Equation (36) means that the circular velocity is nonzero since λ is finite and fixed with $\lambda < 1$.

(iii) *Case 3*.—For $\delta = \varepsilon$ and $\varepsilon \to 0$,

$$v(\varepsilon, \delta, r_+) \leq 1. \tag{37}$$

This result is not found directly from Eq. (34). Indeed, from Eq. (34), it follows that $v(r_+, \varepsilon, \delta) = 1$. However, in this case, the condition $c(r_+ r_-) = 1/r_+^2$ imposed to obtain Eq. (18) is no longer valid. An independent calculation is requested for an *ab initio* extremal shell as shown in Ref. [4]. In this case, there is also an interesting relationship between the impossibility for a material body to reach the velocity of light $v = 1$ and the unattainability of the absolute zero of temperature [4].

It is also worth remembering that the property of $v < 1$ was found for near-horizon particle orbits in the background of near-extremal black holes for the Kerr metric [33] and in Ref. [34] for a much more general case. Thus, we see an interesting analogy between limiting behaviors of self-gravitating shells in $(2+1)$-dimensional spacetimes and test particles in $(3+1)$-dimensional spacetimes.

C. Temperature in the three horizon limits

In the variables ε and δ of Eqs. (20) and (21), the shell's local temperature T in Eq. (19) can be written as

$$T(\varepsilon, \delta, R) = \frac{\delta^2 - \varepsilon^2}{2\pi\ell\delta\varepsilon\sqrt{1-\varepsilon^2}}. \tag{38}$$

For cases 1–3, the limits defined in Eqs. (22)–(24) yield from Eq. (38) the expressions for the local temperature as below:

(i) *Case 1*.—For $\delta = O(1)$ and $\varepsilon \to 0$,

$$T(\varepsilon, \delta, r_+) = \frac{\delta}{2\pi\ell\varepsilon}, \tag{39}$$

up to leading order. Equation (39) means that the temperature is divergent as $1/\varepsilon$.

(ii) *Case 2*.—For $\delta = \frac{\varepsilon}{\lambda}$ and $\varepsilon \to 0$,

$$T(\varepsilon, \delta, r_+) = \frac{1-\lambda^2}{2\pi\ell\lambda}, \tag{40}$$

up to leading order. Equation (40) means that the local temperature is nonzero since λ is finite and fixed with $\lambda < 1$. It is worth noting a simple formula that follows from (32) and (40) and relates the pressure and temperature in this horizon limit, namely, $\frac{p}{T} = \frac{1}{4G}\frac{1+\lambda^2-\lambda}{1-\lambda^2}$. Thus, if we believe that the horizon of a black hole probes quantum gravity physics, we find that in this case the quantum gravity regime obeys an ideal gas law.

(iii) *Case 3*.—For $\delta = \varepsilon$ and $\varepsilon \to 0$,

$$T(\varepsilon, \delta, r_+) = \text{finite}. \tag{41}$$

This was shown in Ref. [4]. Equation (41) does not follow directly from Eq. (38). The condition for T should be modified. It turns out that T_0 may depend not only on r_+ and r_- but also on R. As a result, it may happen that $T_0 \to 0$ but the local temperature on the shell T remains finite [4].

V. ENTROPY: THE THREE EXTREMAL BTZ BLACK HOLE LIMITS

Having carefully studied the equations of state for p, v, and T, we can now calculate the entropy by integrating the first law, see Eq. (14), in all three cases:

(i) *Case 1*.—$\delta = O(1)$ and $\varepsilon \to 0$. Here, we use first the expressions in terms of (ε, δ, R), i.e., Eqs. (25), (26), (30), (34), and (38). Then, one finds that the first law Eq. (14) can be expressed in terms of the differentials of $d\epsilon$, $d\delta$, and dR as $dS(\varepsilon, \delta, R) = \frac{\pi}{2G}(-\frac{R\varepsilon}{\sqrt{1-\varepsilon^2}} d\varepsilon + \sqrt{1-\varepsilon^2} dR)$. Then, taking $\varepsilon \to 0$, i.e., $R \to r_+$, we get $dS(\varepsilon, \delta, r_+) = \frac{\pi}{2G} dr_+$. Since it does not depend on δ, the expression is also valid in the $\delta \to 0$ case, i.e., in the extremal case $r_+ \to r_-$. Then, integrating with the condition $S \to 0$ as $r_+ \to 0$, we get in this extremal limit

$$S = \frac{A_+}{4G}, \tag{42}$$

where $A_+ = 2\pi r_+$ is the area, i.e., the perimeter, of the shell, i.e., the ring, see Eq. (7), when it is pushed to its gravitational radius. The entropy in Eq. (42) is nothing but the Bekenstein-Hawking entropy; see Eq. (1).

(ii) *Case 2*.—$\delta = \frac{\varepsilon}{\lambda}$ and $\varepsilon \to 0$. Here, we also have to use the expressions in terms of (ε, δ, R) i.e., Eqs. (25), (26), (30), (34), and (38), and then the first law Eq. (14) can be expressed in terms of the differentials of $d\epsilon$, $d\delta$, and dR, as $dS(\varepsilon, \delta, R) = \frac{\pi}{2G}(-\frac{R\varepsilon}{\sqrt{1-\varepsilon^2}} d\varepsilon + \sqrt{1-\varepsilon^2} dR)$, which is the same formula as in case 1. Then, taking $\varepsilon \to 0$, i.e., $R \to r_+ \to r_-$, we get $dS(\varepsilon, \delta, r_+) = \frac{\pi}{2G} dr_+$. This means that the entropy is independent of the

TABLE I. The contributions of the pressure p, angular velocity v, and temperature T to the entropy of the extremal black hole $S(A_+)$, according to the first law.

Case	Pressure p	Velocity v	Local temperature T	Entropy $S(A_+)$	Contribution
1	Infinite	1	Infinite	$\frac{A_+}{4G}$ Eq. (42)	Pressure
2	Finite nonzero	<1	Finite nonzero	$\frac{A_+}{4G}$ Eq. (43)	Mass, pressure and angular velocity
3	Finite nonzero	≤ 1	Finite zero and nonzero	$0 \leq S(A_+) \leq \frac{A_+}{4G}$ Eq. (44)	Mass, pressure and angular velocity

parameter λ. Then, integrating with the condition $S \to 0$ as $r_+ \to 0$, we get in this extremal limit

$$S = \frac{A_+}{4G}, \tag{43}$$

where again $A_+ = 2\pi r_+$; see Eq. (7). The entropy in Eq. (43) is again the Bekenstein-Hawking entropy; see Eq. (1). This result was not involved in former studies [3,4].

(iii) *Case 3.*—$\delta = \varepsilon$ and $\varepsilon \to 0$. This case is special. One takes the extremality condition $\delta = \varepsilon$ from the beginning, and thus another route to calculate the entropy has to be followed. This was performed in Ref. [4], and the result is

$$S = S(A_+), \tag{44}$$

where $S(A_+)$ is a well-behaved, but otherwise arbitrary, function of A_+; see also Eq. (2). One can argue, as was done in Ref. [4], that the lower and upper bounds for the entropy in this case are given by the zero entropy, Eq. (4), and the Bekenstein-Hawking entropy, Eq. (1), i.e., $0 \leq S(r_+) \leq \frac{A_+}{4G}$; see Eq. (3). In addition, Eq. (44) suggests that the entropy of an extremal black hole does not take a unique value, but instead it may depend on the preceding history that led to the formation of precisely that extremal black hole; see also Ref. [11].

VI. CONTRIBUTIONS TO THE ENTROPY IN THE THREE EXTREMAL HORIZON LIMITS

Finally, for all three different cases, we state which terms in the first law (14) give the dominant contributions to the entropy:

(i) *Case 1.*—$\delta = O(1)$ and $\varepsilon \to 0$. We have that the pressure term alone, see Eq. (31), contributes to the entropy. Taking then into account Eq. (39), we obtain the Bekenstein-Hawking entropy (42).

(ii) *Case 2.*—$\delta = \frac{\varepsilon}{\lambda}$ and $\varepsilon \to 0$. All three terms in the first law (14) equally contribute to the entropy. Thus, the mass, pressure, and circular velocity terms give contributions to the Bekenstein-Hawking entropy (43).

(iii) *Case 3.*—$\delta = \varepsilon$ and $\varepsilon \to 0$. All three terms in the first law (14) contribute to the entropy; see Ref. [4]. We note that in contrast to the electrically charged case [6] the pressure does not vanish in the extremal limit and contributes to the entropy in the first law as all other terms do; see Eq. (44).

We summarize these results in Table I.

VII. CONCLUSIONS

We have presented a unified framework to explain how the different entropies of an extremal BTZ black hole arise from an extremal shell. Cases 1 and 2 agree in the entropy but disagree in all other thermodynamic quantities. Cases 2 and 3 disagree in the entropy but agree in all other thermodynamic quantities. Therefore, in this sense, case 2 is intermediate between cases 1 and 3. These results complement the former studies in a $(2+1)$-dimensional BTZ spacetime [1–4] and have much in common with those in the $(3+1)$-dimensional electrically charged case [5,6], in particular, with Ref. [7].

Consideration of astrophysically relevant rotating black holes in $(3+1)$ dimensions is too complex. In this regard, using the $(2+1)$-dimensional rotating BTZ exact solution enables one to trace quite subtle details that are expected in the more realistic $(3+1)$ case. Therefore, we hope that the present work can shed light on the entropy issue for the $(3+1)$-dimensional black holes as well.

ACKNOWLEDGMENTS

We thank Fundação para a Ciência e Tecnologia (FCT), Portugal, for financial support through Grant No. UID/FIS/00099/2013. J. P. S. L. thanks Coordenação de Aperfeiçoamento do Pessoal de Nível Superior (CAPES), Brazil, for support within the Programa CSF-PVE, Grant No. 88887.068694/2014-00. J. P. S. L. is also thankful for FCT Grant No. SFRH/BSAB/128455/2017. M. M. thanks FCT for financial support through Grant No. SFRH/BPD/88299/2012. O. B. Z. is thankful for support from SFFR, Ukraine, Project No. 32367. O. B. Z. has also been partially supported by the Kazan Federal University through a state grant for scientific activities.

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