Entropy of a box of gas in an external gravitational field revisited

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Earlier it was shown that the entropy of an ideal gas, contained in a box and moving in a gravitational field, develops an area dependence when it approaches the horizon of a static, spherically symmetric spacetime. Here we extend the above result in two directions; viz., to (a) the stationary axisymmteric spacetimes and (b) time-dependent cosmological spacetimes evolving asymptotically to the de Sitter or the Schwarzschild–de Sitter spacetimes. While our calculations are exact for the stationary axisymmetric spacetimes, for the cosmological case we present an analytical expression of the entropy when the spacetime is close to the de Sitter or the Schwarzschild–de Sitter spacetime. Unlike the static spacetimes, there is no hypersurface orthogonal timelike Killing vector field in these cases. Nevertheless, the results hold, and the entropy develops an area dependence in the appropriate limit.

I. INTRODUCTION

Killing horizons are null surfaces which act like one-way membranes, hiding information from observers who do not cross them. It is well known that such observers associate thermodynamic characteristics like entropy and temperature with these horizons (see, e.g., Chap. 8 of Ref. [1]). The well-known examples include the Schwarzschild and the Rindler spacetimes. In the Schwarzschild spacetime, an observer located at any constant radial coordinate r such that $r > 2M$ associates an entropy $S = A/4$ and a temperature $T = (1/8\pi M)$ with the black hole horizon located at $r = 2M$ [2–5], where A and M are the area of the event horizon and the black hole's mass, respectively. Similarly, an observer at constant spatial coordinate x (with $x > 0$) in a Rindler spacetime (with the metric $ds^2 = -g^2 x^2 dt^2 + d\mathbf{x}^2$) will associate a temperature $T = (g/2\pi)$ and entropy density $1/4$ with the Rindler horizon [6,7], located at $x = 0$ (see also Ref. [8] for a review). On the other hand, the freely falling observers in either of these spacetimes will not associate any thermodynamic features with the horizon, because they have access to regions in both the interior and exterior of the event horizon (or, the Rindler horizon), unlike the case of static observers (see Refs. [9–11] for details and also references therein) located outside the horizon.

The fact that we may erect locally a flat coordinate chart around a given event of any nonsingular spacetime, and that the flat chart could be mapped onto a Rindler frame, leads one to associate a local temperature and hence a local

thermodynamic description of spacetimes [9,10,12]. This has paved the way towards an emergent perspective for gravity, since one can associate thermodynamic quantities locally at any spacetime event, pointing towards an underlying microscopic statistical description for gravity [13–16]. This view has also received significant support from various other investigations—notably, one can introduce a thermodynamic perspective to the gravitational field equations near any null surface. In particular, various projections of the gravitational field equations can be written as the Navier-Stokes equation of fluid dynamics, as a thermodynamic identity, or as the heating and cooling of the spacetime [17–20]. Remarkably enough, most of these thermodynamic properties carry over quite naturally (albeit nontrivially) to a more general class of gravity theories [21,22]. These results also have implications for classical gravity: for example, until recently, it was not very clear how to formulate the variational principle for gravity when part of the boundary is null [23–26]. A natural question that emerges in this context is the following: Does the interpretation of (Area/4) as the entropy still apply to an arbitrary null surface which is not a Killing horizon? Surprisingly, it turns out that the existence of such thermodynamic properties is not necessarily true only for Killing horizons—in fact, it was shown recently in Ref. [27] using the Gaussian null coordinates that for a large class of generic null surfaces, which are *not* necessarily Killing horizons, the observers who do not cross them would still associate entropy densities (1/4) with the relevant areas.

Given that the black holes have thermodynamic properties, it is natural to ask whether such notions could be associated with cosmological or Hubble horizons as well. Among them, perhaps the most interesting one is that of the

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de Sitter space, which has a Killing horizon. The de Sitter horizon could also be associated with thermodynamic properties such as the entropy and temperature, qualitatively similar to that of the black hole [28–32]. For cosmological spacetimes other than the de Sitter, the Hubble horizon, $r(t) = H^{-1}(t)$ [where t is the comoving time, $r(t)$ is the proper radius, and $H(t)$ is the Hubble rate], is *not* a Killing horizon. For a spatially flat cosmology, the Hubble horizon also coincides with the apparent horizon, where the expansion corresponding to one of the two principle null congruences vanishes while the other is positive. The notion of apparent horizon is dependent upon the spacetime foliation. Attempts to build thermodynamics by associating entropy and temperature with such horizons can be seen in e.g. Refs. [33–35] and in references therein. However, we note that unlike the Killing horizons, the Hubble or apparent horizon is *not* a null surface in general [36], as the normal to such surfaces, $\nabla_a(r(t)H(t))$, is not in general a null vector.

In the usual statistical mechanics of normal matter, the entropy depends upon the volume of the system, instead of its area. Thus, the horizons are indeed some special objects—qualitatively or quantitatively, as far as the thermodynamic properties are concerned. It was shown in Refs. [37,38] that in order for the entropy-area relation to hold, the density of states near the horizon must have an exponential "pileup" behavior, and any effective theory describing a quantum field near the horizon must be nonlocal over the Planck-length scale near the horizon. In Ref. [39], the dynamics of a box containing an ideal gas moving in a static and spherically symmetric black hole spacetime was considered, and the area dependence of the entropy of the gas was demonstrated near the horizon by introducing the Planck length to be the cutoff near the horizon. Such a cutoff could be thought of as an ultraviolet cutoff, making some near-horizon divergent integrals finite. A similar result was obtained in a quasistatic gravitational collapse scenario in Ref. [40], for both Einstein and Lanczos-Lovelock gravity. We further refer our reader to Refs. [41–43] and references therein for different perspectives including the entanglement of black hole entropy using matter fields as probes.

As emphasized earlier, the entropy-area relation in general relativity transcends the horizons which arise in static spacetimes and holds true in stationary spacetimes and cosmological spacetimes. We already know from previous work that the entropy for a *normal* thermodynamic system also scales as area for static spacetimes when it is close to the horizon. These two facts raise the following

interesting question: viz., whether the entropy of a normal thermodynamic system also scales as area if it is located near the stationary or cosmological horizon. It is also of interest to understand what happens to the entropy of a box of ideal gas when it is near the apparent horizon in a cosmological spacetime which is *not* a null surface. Given this motivational backdrop we will, in this work, extend the formalism developed in Ref. [39] to spacetimes with a positive cosmological constant admitting a cosmological horizon or the Hubble horizon, as well to rotating solutions in general relativity. First, we shall consider a box of ideal gas moving in a general stationary axisymmetric spacetime endowed with a positive cosmological constant (e.g., the Kerr–Newman–de Sitter or the Plebanski–Demianski–de Sitter). We shall assume that the spacetime admits Killing horizons—both cosmological and black hole horizons. We shall demonstrate the area scaling of the gas's entropy for both cases. Next, we shall consider a box moving close to the Hubble horizon of a cosmological spacetime with the scale factor being appropriate for the Λ cold dark matter (ΛCDM for short) or the Λ radiation. We will show that an area scaling of the entropy of the gas always exists, and as the spacetime asymptotically evolves to the de Sitter, the entropy of the gas smoothly merges with the de Sitter Killing horizon behavior. We also will present a generalization of this result with a central mass/black hole.

The rest of the paper is organized as follows: In the next section, we describe the necessary geometric framework and derive the entropy in stationary axisymmetric spacetimes. Section III describes the calculation of the entropy and the emergence of the area law for cosmological spacetimes. The conclusions are summarized in Sec. IV. We shall assume mostly positive signature for the metric $(-, +, +, +)$ and will set $c = G = \kappa_B = 1$ throughout. We shall also assume spatially flat geometry ($k = 0$), as the other plausible choice, $k = +1$, seems to be relevant only for describing structures at scales much smaller than the Hubble horizon [44].

II. STATIONARY AXISYMMETRIC SPACETIMES

A. The metric and the coordinate systems

We consider a box of ideal gas moving in a stationary axisymmetric spacetime, endowed with a positive cosmological constant Λ. In order to model such a generic spacetime, let us first consider the example of the Kerr–Newman–de Sitter spacetime, representing a charged and rotating black hole sitting in the de Sitter universe (see e.g. Ref. [45] and also references therein), whose spacetime metric reads

$$ds^2 = -\frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\rho^2} dt^2 - \frac{2a\sin^2\theta}{\rho^2\Xi}((r^2 + a^2)\Delta_\theta - \Delta_r)dt d\phi$$
$$+ \frac{\sin^2\theta}{\rho^2\Xi^2}((r^2 + a^2)^2\Delta_\theta - \Delta_r a^2 \sin^2\theta)d\phi^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2, \tag{1}$$

where we have defined

$$\Delta_r = (r^2 + a^2)(1 - H_0^2 r^2) - 2Mr + Q^2,$$
$$\Delta_\theta = 1 + H_0^2 a^2 \cos^2\theta, \qquad \Xi = 1 + H_0^2 a^2,$$
$$\rho^2 = r^2 + a^2 \cos^2\theta, \qquad\qquad (2)$$

where $H_0^2 = \Lambda/3$, and M, Q, and a are the parameters specifying the mass, charge, and angular momentum, respectively, of the black hole. Setting $a = 0$ recovers the static Reissner–Nördstrom–de Sitter spacetime, whereas further setting $M = 0 = Q$ recovers the de Sitter spacetime written in the static coordinate system. The cosmological and the black hole event horizons are given by the largest (say, r_C) and the next-to-largest (say, r_H) roots of $\Delta_r = 0$, respectively.

The Kerr–Newman–de Sitter spacetime is stationary and axisymmetric—it is endowed with two Killing vector fields $\xi^a \equiv (\partial_t)^a$ and $\phi^a \equiv (\partial_\phi)^a$, generating the stationarity and axisymmetry, respectively. These two vector fields, being coordinate fields, commute:

$$[\xi, \phi]^a = \pounds_\xi \phi^a = 0. \qquad\qquad (3)$$

Since the 2-planes orthogonal to these two Killing vector fields are spanned by coordinate vector fields, $(\partial_r)^a$ and $(\partial_\theta)^a$, they also commute with each other. This implies that any two vector fields spanning these 2-planes form a Lie algebra, and hence the $r - \theta$ planes are integral submanifolds of the spacetime [46]. In other words, when these planes are Lie-dragged along ξ^a and/or ϕ^a, they remain intact.

We note that, unlike in the case of the static spacetimes, the timelike Killing vector field is now not hypersurface orthogonal, as $\xi \cdot \phi = g_{t\phi} \neq 0$. The surface $\xi \cdot \xi = g_{tt} = 0$ defines the ergosphere. Since both the black hole and the cosmological horizons correspond to $\Delta_r = 0$, it is clear that unlike the static spacetimes, ξ^a is spacelike on the horizons. This implies that unlike the $\Lambda \leq 0$ stationary axisymmetric spacetimes, we have two ergospheres, instead of one. From Eq. (2), one can determine the locations of these ergospheres: one surrounding the black hole, and the other one appearing before the cosmological event horizon. These ergospheres intersect the respective horizons at the axial points $\theta = 0$, π, where the effect of rotation vanishes.

The angular velocities $\alpha = -(\xi \cdot \phi)/(\phi \cdot \phi) = -g_{t\phi}/g_{\phi\phi}$ at the two horizons are given by

$$\alpha_H = -\left.\frac{\xi \cdot \phi}{\phi \cdot \phi}\right|_{r=r_H} = \frac{a\Xi}{r_H^2 + a^2},$$
$$\alpha_C = -\left.\frac{\xi \cdot \phi}{\phi \cdot \phi}\right|_{r=r_C} = \frac{a\Xi}{r_C^2 + a^2}. \qquad (4)$$

The Killing vector fields that become null on the black hole and the cosmological horizons are given, respectively, by

$$\chi_H^a = \xi^a + \alpha_H \phi^a, \qquad \chi_C^a = \xi^a + \alpha_C \phi^a. \qquad (5)$$

It is easy to see using the metric functions that $\chi_H \cdot \chi_H (r \to r_H)$ and $\chi_C \cdot \chi_C (r \to r_C)$ vanish as $\mathcal{O}(\Delta_r)$. This completes the basic geometric description of the Kerr–Newman–de Sitter spacetime, given by Eq. (1).

Keeping this explicit example in mind, we shall now build a model for addressing general stationary axisymmetric spacetimes. Although we are primarily interested in $(3 + 1)$ dimensions, as we shall see at the end of this section, the results easily generalize to arbitrary higher dimensions as well. More details of the following discussions can be seen in Ref. [32] and references therein. We assume that the two Killing vector fields of the spacetime still commute [see, for example, Eq. (3)], and the 2-planes orthogonal to them are integral submanifolds. Note that this does not necessarily require that these vector fields be coordinate fields—in fact, it only means that we may locally define coordinates along those Killing vector fields.

We shall next construct a suitable $(3 + 1)$-foliation of the spacetime in between the two Killing horizons as follows: We define a vector field $\chi^a = \xi^a - (\xi \cdot \phi)/(\phi \cdot \phi)\phi^a$ such that $\chi^a \phi_a$ identically vanishes everywhere. Also, $\chi \cdot \chi = \xi \cdot \xi - (\xi \cdot \phi)^2/(\phi \cdot \phi) = -\tilde{\beta}^2$ (say), so that χ^a is timelike whenever $\tilde{\beta}^2 > 0$. Since the term $(\xi \cdot \phi)^2/(\phi \cdot \phi)$ is positive definite, it is clear that χ^a is manifestly timelike outside the ergospheres. Its behavior inside the ergosphere will be discussed in due course.

The fact that the 2-planes orthogonal to the Killing fields are integral submanifolds, and hence any two vector fields tangent to them form a Lie algebra, implies the following Frobenius-like conditions [46]:

$$\xi_{[a}\phi_b\nabla_c\phi_{d]} = 0 = \phi_{[a}\xi_b\nabla_c\xi_{d]}, \qquad (6)$$

which can be rewritten in terms of our vector field χ^a:

$$\chi_{[a}\phi_b\nabla_c\phi_{d]} = 0 = \phi_{[a}\chi_b\nabla_c\chi_{d]}. \qquad (7)$$

Denoting the function $-(\xi \cdot \phi)/(\phi \cdot \phi)$ by $\alpha(x)$, we have

$$\nabla_a\chi_b + \nabla_b\chi_a = \phi_a\nabla_b\alpha + \phi_b\nabla_a\alpha. \qquad (8)$$

We also get, from the commutativity of the two Killing vector fields,

$$\pounds_\phi\chi^a = 0 \quad \text{and} \quad \pounds_\chi\alpha = 0 = \pounds_\phi\alpha. \qquad (9)$$

Setting $\alpha = 0$, one immediately recovers the static spacetime. Using Eqs. (8) and (9), we can derive from the second equality of Eq. (7)

$$\nabla_{[a}\chi_{b]} = 2\tilde{\beta}^{-2}(\chi_b\nabla_a\tilde{\beta}^2 - \chi_a\nabla_b\tilde{\beta}^2), \qquad (10)$$

which implies that χ_a satisfies the Frobenius condition of hypersurface orthogonality; i.e., $\chi_{[a}\nabla_b\chi_{c]} = 0$. In other

words, the vector field χ^a is orthogonal to the family of spacelike hypersurfaces spanned by ϕ^a and the aforementioned 2-submanifolds. We also note that the price we have paid in making this foliation is that χ^a is not a Killing vector field now [see Eq. (8)], unlike the static spacetimes. The metric now reads

$$g_{ab} = -\tilde{\beta}^{-2}\chi_a\chi_b + h_{ab}. \qquad (11)$$

Note that since the foliation field χ^a can be thought of as orthogonal to $t = $ constant hypersurfaces, h_{ab} is the induced 3-metric over the spatial hypersurfaces orthogonal to χ^a, spanned, e.g. in the Boyer-Lindquist coordinates, by (r, θ, ϕ).

As we have seen earlier, χ^a is manifestly timelike outside the ergospheres. Since within the ergospheres, ξ^a is spacelike ($\xi \cdot \xi > 0$), we might expect that $\chi_a\chi^a = -\tilde{\beta}^2$ would be vanishing at some point. Equation (10) shows that *on such a surface* using the torsion-free condition, we must have $\chi_{[b}\nabla_{a]}\tilde{\beta}^2 = \tilde{\beta}^2\partial_{[a}\chi_{b]} = 0$. In other words, the normal $\nabla_a\tilde{\beta}^2$ to any $\tilde{\beta}^2 = 0$ surface becomes parallel to the foliation field, χ_a:

$$\nabla_a\tilde{\beta}^2 = -2\kappa(x)\chi_a, \qquad (12)$$

where $\kappa(x)$ is a function, assumed to be smooth. It turns out from the above equation that $\pounds_\chi\kappa(x) = 0$. Then, if s is a parameter along χ^a (i.e., $\chi^a\nabla_a s := 1$), it is easy to see that the vector field $k^a := e^{-\kappa(x)s}\chi^a$ is a null geodesic. By considering the Raychaudhuri equation for the null congruence $\{k^a\}$, one readily arrives at (see Ref. [32] and references therein)

$$T_{ab}k^ak^b + \frac{f^2e^{-2\kappa s}}{2}(D_a\alpha)(D^a\alpha) = 0, \qquad (13)$$

where we have denoted $\phi_a\phi^a = f^2$, and D_a is the derivative operator tangent to the $\tilde{\beta}^2 = 0$ hypersurface. Since $\pounds_\chi\alpha = 0$ always, the inner product $(D_a\alpha)(D^a\alpha)$ is positive definite. It is reasonable to assume that the matter energy-momentum tensor satisfies the weak and null energy condition, implying $T_{ab}k^ak^b \geq 0$. Then the left-hand side of Eq. (13) consists of positive-definite quantities, and the vanishing of their sum implies that each of them is vanishing. This means that the function $\alpha(x)$ is a constant over any $\tilde{\beta}^2 = 0$ hypersurface. In other words, the hypersurface orthogonal vector field $\chi^a = \xi^a + \alpha(x)\phi^a$ becomes Killing whenever it is null and that null hypersurface is a Killing horizon. This result is analogous to the explicit result of the Kerr–Newman–de Sitter spacetime discussed above. However, we note that the above general result has been proven without assuming *any* particular functional form of the norm of the various basis vector fields, nor assuming any particular matter field other than imposing the generic

energy conditions. In other words, we have found a well-behaved timelike vector field in a general stationary axisymmetric spacetime, foliating the region between the two Killing horizons and becoming smoothly the Killing fields on the horizons. This will be useful for our future purposes.

It then follows that the function κ, known as the surface gravity, is a constant on the Killing horizons [46]. We shall assume that $\kappa \neq 0$. We have explicitly specified two basis vectors so far: χ^a and ϕ^a. For our purpose, we shall now specify another one, which would in particular be useful while dealing with the near horizons' geometries. Let us consider the vector field $R_a = \frac{1}{\kappa}\nabla_a\tilde{\beta}$. Using the commutativity of the two Killing vector fields, it is easy to see that $\chi^a\nabla_a\tilde{\beta} = 0 = \phi^a\nabla_a\tilde{\beta}$. By Eq. (12), we have

$$\kappa^2 = \lim_{\tilde{\beta}^2\to 0} \frac{(\nabla_a\tilde{\beta}^2)(\nabla^a\tilde{\beta}^2)}{4\tilde{\beta}^2} \equiv \lim_{\tilde{\beta}^2\to 0}(\nabla_a\tilde{\beta})(\nabla^a\tilde{\beta}). \quad (14)$$

Thus, if R is a parameter along R^a [such that $R^a\nabla_aR := 1$ and if $\lambda(x)$ is any function, $R^a\nabla_a\lambda = \frac{d\lambda}{dR}$], then

$$R^aR_a = \frac{1}{\kappa}R^a\nabla_a\tilde{\beta} = \frac{1}{\kappa}\frac{d\tilde{\beta}}{dR} = 1, \qquad (15)$$

which means, since κ is a constant,

$$R = \frac{\tilde{\beta}}{\kappa}. \qquad (16)$$

On the other hand, if μ^a is the fourth basis tangent to the horizon, we must have $\mu^aR_a \sim \mathcal{O}(\tilde{\beta})$ infinitesimally close to the Killing horizon, as $\tilde{\beta} = 0$ on the horizon. Thus, with our specified basis, we have $\tilde{\beta}^2 = \kappa^2R^2$, and the metric in Eq. (11) near any of the Killing horizons could be written as

$$g_{ab} = -(\kappa R)^{-2}\chi_a\chi_b + R_aR_b + f^{-2}\phi_a\phi_b + \mu^{-2}\mu_a\mu_b, \quad (17)$$

where $\mu^2 = \mu_a\mu^a$. The χ-R part of the metric could now be identified with the Rindler space. Note that the square root of the norm $\tilde{\beta}$ has been used here as a coordinate along R^a, to express the notion of the "off the horizon" direction. This would be useful in defining the ultraviolet cutoff while probing the near-horizon geometry of a stationary axisymmetric spacetime.

Even though we have emphasized $\Lambda > 0$, the above analysis holds for anti–de Sitter or asymptotically flat spacetimes, for which the cosmological event horizon would be absent. We once again emphasize here that we have arrived at Eq. (17) using purely generic arguments and assumptions, incorporating the effect of any matter field satisfying the weak/null energy conditions.

As an explicit example, let us go back to the Kerr–Newman–de Sitter spacetime (1), for which we have for the Killing horizons

$$-\tilde{\beta}^2 = g_{tt} - \frac{(g_{t\phi})^2}{g_{\phi\phi}} = -\frac{\Delta_r \rho^2}{(r^2 + a^2)^2} + \mathcal{O}(\Delta_r^2),$$

$$\kappa_{H,C} = \left.\frac{\Delta_r'}{2(r^2 + a^2)}\right|_{r_H, r_C}$$

$$= \frac{r_{H,C}(1 - H_0^2 r_{H,C}^2) - H_0^2 r_{H,C}(r_{H,C}^2 + a^2) - M}{r_{H,C}^2 + a^2},$$

(18)

where the subscripts H and C denote the black hole and the cosmological event horizons, respectively, and the prime denotes differentiation with respect to the radial coordinate. The aforementioned basis vector μ^a equals $(\partial_\theta)^a$ here, and $(\partial_\theta)^a R_a = \mathcal{O}(\sqrt{\Delta_r})$. In other words, near the horizons, the four vector fields χ^a, R^a, ϕ^a, $(\partial_\theta)^a$ furnish a well-behaved basis for the near-horizon geometry.

Although we built our general formalism starting with the Kerr–Newman–de Sitter spacetime, it applies well to more general de Sitter black hole spacetimes as well. Such generalizations become necessary when we recall the potential non-uniqueness properties of de Sitter black holes [47]. For example, the general family of the Plebanski–Demianski–de Sitter–class spacetimes has the metric [48]

$$ds^2 = \frac{1}{\Omega^2}\left[-\frac{\Delta_r}{\rho^2}\left(dt - \left(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\right)d\phi\right)^2 \right.$$
$$+ \frac{\rho^2}{\Delta_r}dr^2 + \frac{P}{\rho^2}(adt - (r^2 + (a+l)^2)d\phi)^2$$
$$\left. + \frac{\rho^2}{P}\sin^2\theta d\theta^2 \right],$$

(19)

where

$$\Omega = 1 - \frac{\tilde{\alpha}}{\omega}(l + a\cos\theta)r, \qquad \rho^2 = r^2 + (l + a\cos\theta)^2,$$
$$P = \sin^2\theta(1 - a_3\cos\theta - a_4\cos^2\theta),$$
$$\Delta_r = (\omega^2 k + q^2 + q_m^2) - 2Mr + \epsilon r^2 - \frac{2\tilde{\alpha}n}{\omega}r^3$$
$$- (\tilde{\alpha}^2 k + H_0^2)r^4.$$

(20)

The parameters $\tilde{\alpha}$, ω, q, q_m, ϵ, and k are independent, and a_3 and a_4 are determined from them via some constraints. Physical meaning to these parameters could be asserted for only certain special subclasses of Eq. (19). In particular, for $\tilde{\alpha} = 0$, the above metric reduces to the Kerr–Newman–NUT–de Sitter solution [48],

$$ds^2 = -\frac{\Delta_r}{\rho^2}[dt - (a\sin^2\theta + 4l\sin^2\theta/2)d\phi]^2$$
$$+ \frac{\rho^2}{\Delta_r}dr^2 + \frac{P}{\rho^2}[adt - (r^2 + (a+l)^2)d\phi]^2$$
$$+ \frac{\rho^2}{P}\sin^2\theta d\theta^2,$$

(21)

where

$$\rho^2 = r^2 + (l + a\cos\theta)^2,$$
$$P = \sin^2\theta(1 + 4alH_0^2\cos\theta + H_0^2 a^2\cos^2\theta),$$
$$\Delta_r = (a^2 - l^2 + q^2 + q_m^2) - 2Mr + r^2$$
$$- 3H_0^2((a^2 - l^2)l^2 + (a^2/3 + 2l^2)r^2 + r^4/3),$$

(22)

where q and q_m are the electric and magnetic charges, respectively, and l is the Newman-Unti-Tamburino (NUT) parameter. Note that, in order to have a well-behaved black hole solution with $H_0 = 0$, one must have both the acceleration parameter and the NUT charge be vanishing. Nevertheless, in what follows, we shall formally consider the most general Plebanski–Demianski–de Sitter class given by Eq. (19), assuming *implicitly* that it indeed represents de Sitter black holes subject to suitable parameter values, which, for our current purpose, are not an explicit concern.

The black hole and the cosmological horizons are, as shown earlier, given by the two largest roots of $\Delta_r = 0$. The integrability of the 2-submanifolds orthogonal to the two commuting Killing vector fields follows trivially. The hypersurface orthogonal vector field χ^a behaves near the horizons as

$$\chi_{H,C} = \xi^a + \frac{a}{r_{H,C}^2 + (a+l)^2}\phi^a,$$

$$\chi^a\chi_a = -\tilde{\beta}^2 = -\frac{\Delta_r\rho^2}{\Omega^2[r^2 + (a+l)^2]^2} + \mathcal{O}(\Delta_r^2).$$

(23)

The above geometrical setup can also be generalized to higher-dimensional spacetimes as well, permitting larger numbers of commuting Killing vector fields whose orthogonal space is spanned by integral submanifolds of appropriate dimensions. In that case, the hypersurface orthogonal timelike vector field χ^a is given by $\chi^a = \xi^a + \alpha_{(i)}\phi^a_{(i)}$, where i denotes the number of axisymmetric Killing vector fields, and the $\alpha_{(i)}$ functions are determined by solving the algebraic equations, $\chi_a\phi^a_{(i)} = 0$ for each i. In fact, all existing stationary and axisymmetric solutions fall within the scope of this geometric setup. We shall not go into further details here, and refer the interested reader to Ref. [32] and references therein.

Previous work [39] shows that it is necessary to introduce a cutoff near the Killing horizons to regularize some divergent integrals in calculating the entropy. Such a cutoff could also be physically justified from the fact that for any finite Killing-coordinate time interval, no particle can reach the horizon. The natural cutoff in probing the near-horizon geometry is the Planck length. Specifically, for a static and spherically symmetric nonextremal black hole spacetime,

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2,$$

we introduce the Planck-length cutoff as the minimum proper radial distance of the approaching side of the box

$$L_P := \int_{r=r_H}^{r=L_a+r_H} \frac{dr}{\sqrt{f(r)}} \approx \sqrt{\frac{2L_a}{\kappa_H}}, \qquad (24)$$

where it is assumed that $L_a \ll r_H$, so that we could expand in the integrand, $f(r \to r_H) \approx 2\kappa_H(r - r_H)$. If we try to define a similar cutoff for the stationary axisymmetric spacetimes, we face an immediate difficulty, because the pure radial path near the horizon of a stationary axisymmetric spacetime makes no sense. In particular, within the ergosphere, all particles will rotate due to the frame-dragging effect [46]. Also, since the metric functions now depend upon both the radial and polar coordinates, defining proper length as above seems to be inconsistent. For example, in the case of Eq. (1), the proper radial distance should be $\int dr \sqrt{\frac{r^2 + a^2 \cos^2\theta}{\Delta_r}}$, which is θ dependent.

To handle this difficulty, we would instead use the coordinate R defined in Eq. (16) for our purpose. (Note that since $\tilde{\beta}$ is dimensionless and the surface gravity κ has inverse length dimension, R has the dimension of length.) Since $\tilde{\beta} = 0$ on the horizons, $R = 0$ there too, and we simply have, in this new coordinate,

$$L_P := \int_0^{L_a} dR. \qquad (25)$$

In other words, since $R \sim \tilde{\beta}$ and sufficiently close to the horizon $\tilde{\beta}$ must be monotonic, we are using the value of $\tilde{\beta}$ to indicate a measure of how far off we are from the horizon. This does not depend upon any specific path. Thus, the above generalization provides a natural way to impose a cutoff in probing the near-horizon geometry for stationary axisymmetric spacetimes.

Before we end this discussion, let us illustrate how this works out explicitly for the Kerr–Newman–de Sitter spacetime, presented in Eq. (1). We consider the volume integral over the (r, θ, ϕ)-hypersurface,

$$\int \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi \frac{\rho}{\sqrt{\Delta_r}} dr, \qquad (26)$$

and take the near-horizon limit, $\Delta_r \to 0$. Taking the differential of Eq. (16), we get

$$\begin{aligned}
dR &= \frac{\rho \Delta_r' dr}{2(r^2 + a^2)\sqrt{\Delta_r}\kappa} - \frac{\sqrt{\Delta_r}}{\kappa(r^2 + a^2)} \\
&\quad \times \left(\frac{2\rho r}{r^2 + a^2} dr + \frac{a^2 \sin 2\theta}{2\rho} d\theta \right) \\
&\approx \frac{\rho \Delta_r' dr}{2(r^2 + a^2)\sqrt{\Delta_r}\kappa} = \frac{\rho dr}{\sqrt{\Delta_r}}, \qquad (27)
\end{aligned}$$

where in the last equality we have used the second equation of Eq. (18) and have also used the fact that since θ is tangent to the horizon, $d\theta$ really remains infinitesimal there. In other words, in the near-horizon limit we have

$$\int \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi \frac{\rho}{\sqrt{\Delta_r}} dr \to \int \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi dR, \qquad (28)$$

which is exactly analogous to the static horizons written in Rindler coordinates, thereby establishing the universality of the horizon properties of these two kind of spacetimes. With these necessary ingredients, we are now ready to calculate the entropy of a box of gas.

B. Entropy calculation

Let us consider a box of ideal gas containing N particles in thermal equilibrium with its surroundings at temperature β^{-1}. For a static spacetime, one can use the canonical ensemble [39]. Let us denote the density of states and the phase space volume (respectively) by $g(E)$ and $P(E)$. Then we have

$$g(E) = \frac{dP(E)}{dE},$$

$$Q(\beta) = \int e^{-\beta E} g(E) dE = \int e^{-\beta E} dP(E), \qquad (29)$$

where $Q(\beta)$ is the canonical one-particle partition function, related to the density of states via a Laplace transform. For general static spacetimes, the phase space volume can be written as [39,49]

$$P(E) = \int d^3x d^3p \, \Theta(E - |\xi^a p_a|), \qquad (30)$$

where the integration is over any spacelike hypersurface orthogonal to the timelike Killing vector field, ξ^a. The 4-momentum of a timelike or null geodesic is denoted by p^a, and $E = -\xi^a p_a$ is the conserved energy along the geodesic. By the definition of step function, it is clear that the above integral essentially computes the total phase space volume *within* the surface $E = |\xi^a p_a|$. Further, the integration measure can be seen to be invariant under local Lorentz transformations—the spatial volume gets contracted whereas the spatial momentum gets dilated—effectively leaving the phase space measure invariant (a proof of the same can be found in standard textbooks like Refs. [50–52]). A quick proof can be given using the invariance of d^3p/E_p and d^4x and using the fact that $dt/ds \sim E$, where the symbol $\sim$ stands for "transforms as." Then $(d^3p/E_p)(d^4x)(ds) \sim (d^3p/E_p)(d^3x)(dt/ds) \sim d^3p d^3x$. Since $(d^3p/E_p)(d^4x)(ds)$ is manifestly invariant, so is $d^3p d^3x$.

In a nonstatic spacetime, the definition of energy as a conserved quantity, with respect to a hypersurface

orthogonal timelike Killing vector field, does not exist. Nevertheless, the quantity $\Theta(E + \xi^a p_a)$ could still be used anyway as follows. Let us replace E with some scalar function $E(x)$ (x can stand for all spatial and the temporal coordinates), and we take $E(x) = -\chi^a p_a$, where χ^a is some timelike vector field orthogonal to a family of spatial hypersurfaces used in the integration measure. If we take both χ^a and p^a to be future directed (i.e., both χ^0, $p^0 > 0$ [46]), it is obvious that $E(x)$ would be positive. This is a formal argument in favor of using Eq. (30) in general nonstatic spacetimes. Thus, the above definition of the probability distribution for general spacetimes seems to be dependent on the foliation. Then, it is natural to ask what is the appropriate foliation to work with. The answer to this question lies in the notion of local temperature. Given a spacetime, only a particular foliation of it would lead to a correct value for the surface gravity in the Rindler limit, and one must use this foliation for getting meaningful results pertaining to spacetime thermodynamics. While for static spacetimes with Killing horizons there is no such ambiguity—one must always use the Killing time to define the foliation—for nonstatic spacetimes, we shall use only those which ensure the existence of the proper Rindler limit of the spacetime.

Let us evaluate $P(E)$ for the stationary axisymmetric spacetimes, taking χ^a to be the foliation vector field, as discussed in the previous section. Since the gas molecules individually move along geodesics, we note that the quantities $\epsilon = -\xi_a p^a$ and $\lambda = \phi_a p^a$ are conserved, being the energy and the orbital angular momentum, respectively. This means that $E(x) = -p_a \chi^a = (\epsilon - \lambda \alpha(x))$. Since $\alpha(x)$ is constant only on horizons, E is not conserved everywhere. Setting $\alpha = 0$ recovers the static spacetime. We have from Eq. (11), for a timelike geodesic,

$$-\frac{(\chi \cdot p)^2}{\tilde{\beta}^2} + h_{ab} p^a p^b = -m^2. \qquad (31)$$

But for the spatial momentum, we have clearly $p^2 \equiv h_{ab} p^a p^b$, where p denotes the magnitude square of the spatial momentum, giving

$$p^2 = \frac{(\epsilon - \lambda \alpha(x))^2}{\tilde{\beta}^2} - m^2. \qquad (32)$$

Thus, we have

$$P(\epsilon; \lambda) = \frac{4\pi}{3} \int \sqrt{h} d^3 x \left[\frac{(\epsilon - \lambda \alpha(x))^2}{\tilde{\beta}^2} - m^2 \right]^{\frac{3}{2}}. \qquad (33)$$

To arrive at the above integral, we have used the fact that the phase space volume includes a step function, and hence the only contribution to the phase space volume will come from the interior of the surface $E = |\xi^a p_a|$,

which corresponds to a sphere in the momentum space with the radius given by Eq. (32). Thus, in addition to a $\sqrt{h}$ factor, the momentum integral over the phase space simply yields $(4\pi/3)p^3$, where $|p| = \sqrt{h_{ab} p^a p^b}$. The above integral cannot be evaluated in general, even in the Schwarzschild spacetime. Nevertheless, apparently there is no reason for the above integral to lead to any area scaling of entropy, in general.

Let us evaluate the above integral, infinitesimally close to any of the horizons of a stationary axisymmetric spacetime. We use now Eq. (17), $\alpha = $ const.; the mass term goes away, giving

$$P(\epsilon; \lambda) = \frac{4\pi(\epsilon - \lambda \alpha)^3}{3} \int [d^2 X] \frac{dR}{\kappa^3 R^3}, \qquad (34)$$

where $[d^2 X]$ is the invariant "volume" measure of the spatial 2-surface on the horizon (spanned by μ^a and ϕ^a in Eq. (17); for general compact horizons such as that of the Kerr–Newman–de Sitter spacetime, $\mu^a = (\partial_\theta)^a$. Using the Planck length as the ultraviolet cutoff, we can perform the above integration. The side of the box away from the horizon gives very little contribution to the integral (this is in accordance with the exponential "pileup" of the accessible microstates near a Killing horizon [37,38]), because the size of the box must be much greater than the Planck length. We get the leading contribution, coming from near the horizon:

$$P(\epsilon; \lambda) \approx \frac{2\pi A_\perp (\epsilon - \lambda \alpha)^3}{3\kappa^3 L_P^2}, \qquad (35)$$

where $A_\perp = \int [d^2 X]$ is the transverse area of the side of the box close to and tangent to the horizon. Evidently, the above result holds irrespective of whether we are dealing with the black hole or the cosmological event horizon.

Considering the quantity $(\epsilon - \lambda \alpha)$ as the total or effective conserved energy E, and taking it to be positive definite (i.e., we are not considering any negative energy modes or the superradiant instability [46]), the second equation of Eq. (29) now gives

$$Q(\beta) = \frac{4\pi A_\perp}{\beta^3 \kappa^3 L_P^2}. \qquad (36)$$

We now expand the metric function $-g_{00} \equiv -g_{ab} \chi^a \chi^b = \kappa^2 R^2$ appearing in Eq. (17) around $R = 0$ in the radius L_P to get $-g_{00} = \kappa^2 L_P^2$. We can then rewrite $Q(\beta)$ as

$$Q(\beta) = \frac{4\pi A_\perp L_P}{(\beta \sqrt{-g_{00}})^3} = \frac{4\pi L_P A_\perp}{\beta_{\text{loc}}^3}, \qquad (37)$$

where $\beta_{\text{loc}} = \beta \sqrt{-g_{00}}$ is the inverse local Tolman temperature at a Planck distance away from any of the Killing

horizons. The N-particle partition function is given by $Q_N = (Q)^N$, and the entropy of the system is given by $S = (1 - \beta_{\text{loc}} \partial_{\beta_{\text{loc}}}) \ln Q_N$,

$$S = N \left[\ln \left(\frac{4\pi L_P A_\perp}{N \beta_{\text{loc}}^3} \right) + 3 \right], \qquad (38)$$

which is exactly similar to the static and spherically symmetric spacetimes [39]. This completes our main computational part pertaining to the stationary axisymmetric or rotating black holes.

The following point deserves mention. As is evident from Eq. (38), the entropy associated with normal matter (in this case, an ideal gas) behaves as ln(area) at the appropriate limit. On the other hand, we know that the black hole entropy scales as area. This shows that there is a large increase in the number of degrees of freedom associated with a gravitational system before and after the formation of the black hole. That is, as a star collapses to ultimately form a black hole, its entropy just before crossing the would-be event horizon scales as the logarithm of the area, while once the black hole has been formed, the entropy scales as area. This is possibly hinting towards the fact that the gravitational degrees of freedom associated with black holes have to be far greater in number compared to the degrees of freedom of a normal system. This may have interesting consequences, such as for one seeking a microscopic description of black hole entropy (and some possibilities have been previously discussed in Refs. [37,38]).

The result holds for higher-dimensional spacetimes as well, for which the near-Killing-horizon metric becomes, analogous to Eq. (17),

$$g_{ab} = -(\kappa R)^{-2} \chi_a \chi_b + R_a R_b + \gamma_{ab}^{(n-2)}, \qquad (39)$$

where $\gamma_{ab}^{(n-2)}$ is the $(n-2)$-dimensional analogue of the 2-metric $(\mu^{-2} \mu_a \mu_b + f^{-2} \phi_b \phi_b)$, appearing in Eq. (17). It is then straightforward to obtain after some algebra

$$Q(\beta) = \frac{(n-1)! \pi^{\frac{n-1}{2}} A_{(n-2),\perp} L_P}{\Gamma(\frac{n+1}{2})(n-2)\beta_{\text{loc}}^{n-1}}, \qquad (40)$$

and the entropy

$$S = N \left[\ln \left(\frac{(n-1)! \pi^{\frac{n-1}{2}} A_{(n-2),\perp} L_P}{\Gamma(\frac{n+1}{2})(n-2)\beta_{\text{loc}}^{n-1}} \right) + (n-1) \right]. \qquad (41)$$

For the sake of completeness, we will next present another derivation of the above result also, by explicitly using the lapse and shift functions. The chief goal is to explicitly demonstrate the existence of the coordinate R above, for a Killing horizon located at some constant radial coordinate—say, $r = r_H$.

C. An alternative derivation of the entropy

Any metric in $(3 + 1)$ dimensions could be expressed in terms of ten independent functions by choosing a particular time foliation, known as the Arnowitt-Deser-Misner decomposition [1,53]. However, the symmetries of the spacetime for the present case—i.e., stationary and axisymmetry—demand that five of the off-diagonal metric components vanish and that all the remaining components be independent of time as well as one of the angular coordinates. Thus, any stationary, axisymmetric black hole is characterized by five nonzero functions of (r, θ), which are denoted by $N(r, \theta)$, $N^\phi(r, \theta)$, $h_{rr}(r\theta)$, $h_{\theta\theta}(r, \theta)$, and $h_{\phi\phi}(r, \theta)$, so that the line element becomes

$$
\begin{aligned}
ds^2 &= -N^2 dt^2 + h_{rr} dr^2 + h_{\theta\theta} d\theta^2 + h_{\phi\phi}(d\phi + N^\phi dt)^2 \\
&= -\{N^2 - h_{\phi\phi}(N^\phi)^2\} dt^2 + 2h_{\phi\phi} N^\phi dt d\phi \\
&\quad + h_{rr} dr^2 + h_{\theta\theta} d\theta^2 + h_{\phi\phi} d\phi^2.
\end{aligned} \qquad (42)
$$

Among the above functions, the function $N(r, \theta)$ is known as the lapse, while $N^\phi(r, \theta)$ is known as the shift. The corresponding components for the inverse metric will be

$$g^{tt} = -\frac{1}{N^2}, \qquad g^{t\phi} = \frac{N^\phi}{N^2}, \qquad g^{rr} = h^{rr} = \frac{1}{h_{rr}},$$

$$g^{\theta\theta} = h^{\theta\theta} = \frac{1}{h_{\theta\theta}}, \qquad g^{\phi\phi} = \frac{1}{h_{\phi\phi}} - \left(\frac{N^\phi}{N} \right)^2, \qquad (43)$$

such that $h^{\phi\phi} = 1/h_{\phi\phi}$. Obviously, the above spacetime has two Killing vectors: $\xi^a = (\partial_t)^a$ corresponding to time translation, and $\phi^a = (\partial_\phi)^a$ corresponding to rotational invariance. The norms of these vectors are

$$\xi^a \xi_a = g_{tt} = -\{N^2 - h_{\phi\phi}(N^\phi)^2\}, \qquad (44)$$

$$\phi^a \phi_a = g_{\phi\phi} = h_{\phi\phi}. \qquad (45)$$

It is evident from the above equations that the vector ξ^a is not timelike everywhere—for the spacetime region in which $N^2 < h_{\phi\phi}(N^\phi)^2$, the norm of this vector is positive, indicating existence of the ergosphere. We introduce the following function:

$$\alpha(r, \theta) \equiv -\frac{\xi^a \phi_a}{\phi^a \phi_a} = -\frac{g_{t\phi}}{g_{\phi\phi}} = -N^\phi, \qquad (46)$$

which essentially coincides with the shift function, and another vector,

$$\chi^a = \xi^a - N^\phi \phi^a. \qquad (47)$$

The norm of this vector eventually evaluates to

$$\chi^a \chi_a = g_{ab} \chi^a \chi^b = g_{ab}(\xi^a - N^\phi \phi^a)(\xi^b - N^\phi \phi^b)$$
$$= g_{tt} - 2g_{t\phi}N^\phi + h_{\phi\phi}(N^\phi)^2 = -N^2. \quad (48)$$

To proceed further, we shall assume that there is some surface, say Φ, on which $N^2 = 0$, and on that surface α—i.e., N^ϕ—is a constant. (Note that this condition ensures that the geometry depicted above represents a black hole spacetime.) Thus, on that surface, χ^a is the Killing vector, and the Killing horizon of this spacetime corresponds to $\chi^2 = 0$—i.e., the surface Φ. We will also assume that the surface $N^2 = 0$ corresponds to some $r = r_H$, and thus one can write N^2 as

$$N^2 = \Delta(r)f(r,\theta), \quad (49)$$

such that $r = r_H$ is a solution of the equation $\Delta(r) = 0$. Since $r = r_H$ is a Killing horizon, on this surface the norm of the vector $\nabla_a r$ must vanish, suggesting $g^{rr} = 0$. Hence, we also have

$$g^{rr} = h^{rr} = \Delta(r)g(r,\theta). \quad (50)$$

It is useful to introduce yet another vector field,

$$R_a \equiv \nabla_a R \equiv \nabla_a \left(\frac{N}{\zeta}\right); \qquad \zeta = \text{constant}, \quad (51)$$

where ζ is a constant to be determined later. The components of this vector are

$$R_r = \frac{1}{\zeta}\partial_r(\sqrt{\Delta}\sqrt{f}) = \frac{\sqrt{f}\Delta'}{2\zeta\sqrt{\Delta}} + \frac{\sqrt{\Delta}f'}{2\zeta\sqrt{f}},$$

$$R^r = h^{rr}R_r = \frac{g\sqrt{f}\Delta'\sqrt{\Delta}}{2\zeta} + \frac{\Delta^{3/2}gf'}{2\zeta\sqrt{f}}, \quad (52)$$

$$R_\theta = \frac{\sqrt{\Delta}\partial_\theta f}{2\zeta\sqrt{f}}, \quad (53)$$

where a prime denotes a derivative with respect to the radial coordinate r. For stationary spacetime, one can prove that $(\nabla_a N)(\nabla^a N)$ is a constant in the limit $r \to r_H$, defined as the square of the surface gravity κ, leading to

$$\lim_{r \to r_H} R_a R^a = \frac{\kappa^2}{\zeta^2} = g^{rr}\left(\frac{\sqrt{f}\Delta'}{2\zeta\sqrt{\Delta}}\right)^2 = \frac{gf\Delta'^2}{4\zeta^2}. \quad (54)$$

Requiring $R_a R^a = 1$ on the surface $r = r_H$, we obtain

$$\zeta = \kappa = \sqrt{\frac{gf\Delta'^2}{4}}\Bigg|_{r=r_H}. \quad (55)$$

Since this has to be constant, one obtains that $f(r,\theta)g(r,\theta)$ should be independent of θ on the Killing horizon. Thus,

we get $N^2 = \kappa^2 R^2$. Using this expression for the lapse function N, the metric can be written as

$$g^{ab} = -\frac{1}{\kappa^2 R^2}\chi^a \chi^b + h^{ab}, \quad (56)$$

where, of course, h^{ab} is purely spatial. To rewrite it as fit for the near-horizon limit, we note that the spatial metric can be further expanded, leading to

$$h^{ab} = h^{\phi\phi}\phi^a\phi^a + h^{\theta\theta}(\partial_\theta)^a(\partial_\theta)^b + h^{rr}(\partial_r)^a(\partial_r)^b$$
$$\overset{r \simeq r_H}{=} h^{\phi\phi}\phi^a\phi^a + h^{\theta\theta}(\partial_\theta)^a(\partial_\theta)^b + R^a R^b + \mathcal{O}(\Delta). \quad (57)$$

The last line follows from the fact that $(R^r)^2 = \Delta(r)g(r,\theta) + \mathcal{O}(\Delta^2)$, coinciding with the correct leading-order behavior for h^{rr}, while both $R^r R^\theta$ and $R^\theta R^\theta$ are $\sim\mathcal{O}(\Delta)$ and do not contribute in the near-horizon limit. Thus, the full metric in the near-horizon regime becomes Rindler-like:

$$g^{ab} \overset{r \simeq r_H}{=} h^{\phi\phi}\phi^a\phi^a + h^{\theta\theta}(\partial_\theta)^a(\partial_\theta)^b + R^a R^b - \frac{1}{\kappa^2 R^2}\chi^a\chi^b. \quad (58)$$

Thus, given any stationary and axisymmetric black hole spacetime, the metric near the black hole horizon can always be written in a Rindler form. Since we are chiefly interested in the behavior of a box full of ideal gas in the near-horizon regime, the above Rindler form straightforwardly suggests that the entropy of the ideal gas should scale as area. To see this explicitly, we will first compute the probability of a particle in the ideal gas to have an energy E. Since the probability distribution involves an integral over phase space [see Eq. (30)] which extends up to the energy E in momentum space, one can integrate out the momentum degrees of freedom and finally obtain

$$P(E) = \frac{4\pi}{3}\int d^3x\sqrt{h}\left(\frac{E^2}{N^2} - m^2\right)^{3/2}, \quad (59)$$

where $E = -p_a\xi^a$, linearly dependent on the conserved energy and angular momentum of the particle, and we may identify $N \equiv \tilde{\beta}$ in our earlier analysis. Note that the above result holds for any stationary and axisymmetric spacetimes. Finally, specializing to the Kerr metric, the integral leading to the probability distribution near the horizon becomes

$$P(E) = \frac{8\pi^2 E^3}{3}\int dr d\theta \frac{\Sigma^4 \sin\theta}{\rho^2 \Delta^2}, \quad (60)$$

where $\Delta = r^2 + a^2 - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\theta$, and $\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$. In order to simplify the above result, we perform a change of coordinates, $(r,\theta) \to (R,\theta)$,

where $R = N/\kappa = \rho\sqrt{\Delta}/(\kappa\Sigma)$, with the last equality holding for the Kerr metric. The Jacobian associated with the above coordinate transformation becomes

$$
J = \det\begin{pmatrix} \frac{1}{\sqrt{g\Delta}} & \frac{\sqrt{\Delta}\partial_\theta f}{2\kappa\sqrt{f}} \\ -\frac{2\kappa\sqrt{f}}{\Delta\sqrt{g}\partial_\theta f} & 1 \end{pmatrix} = \frac{2}{\sqrt{g\Delta}} = \frac{2\rho}{\sqrt{\Delta}}. \quad (61)
$$

The Jacobian evaluated above is completely general and applies to any stationary and axisymmetric spacetimes; however, for the purpose of illustration, in the last expression we have used the results for the Kerr metric. Then the integral leading to probability distribution can be rewritten in the new coordinate system (R, θ) as

$$
\begin{aligned}
P(E) &= \frac{8\pi^2 E^3}{3}\int dRd\theta J^{-1}\frac{\Sigma^4\sin\theta}{\rho^2\Delta^2} \\
&= \frac{4\pi^2 E^3}{3}\int dRd\theta\frac{\Sigma^4\sin\theta}{\rho^3\Delta^{3/2}} \\
&= \frac{4\pi^2 E^3\Sigma}{3\kappa^3}\int dRd\theta\frac{\sin\theta}{R^3} = \frac{2\pi E^3 A_\perp}{3\kappa^3}\int\frac{dR}{R^3} \\
&= \frac{2\pi E^3 A_\perp}{3\kappa^3 L_P^2},
\end{aligned} \quad (62)
$$

where in the last line we have used the fact that the box is a Planck length away from the black hole. Given this probability distribution, one immediately observes that the distribution itself scales with area, and it is the source of the area dependence of entropy for the ideal gas. Since the distribution is known to us, one can compute the partition function using Eq. (29), where β will be the inverse temperature associated with the black hole. Thus, we arrive at the following expression for the partition function:

$$
Q(\beta) = \frac{2\pi A_\perp}{\kappa^3 L_P^2}\int dE E^2 e^{-\beta E} = \frac{2\pi A_\perp}{\kappa^3 L_P^2}\frac{\partial^2}{\partial\beta^2}\left(\frac{1}{\beta}\right) = \frac{4\pi A_\perp}{\beta^3\kappa^3 L_P^2}. \quad (63)
$$

Introducing a redshifted local temperature, $\beta_{\text{loc}} = N\beta = \beta(\kappa L_P)$, where $N = \kappa L_P$ is the redshift factor, we finally obtain

$$
Q(\beta_{\text{loc}}) = \frac{4\pi A_\perp}{\beta_{\text{loc}}^3}, \quad (64)
$$

which matches exactly with our earlier result. Thus, starting from the lapse and shift functions, and imposing the symmetries, one ends up with the Rindler form of the metric in the near-horizon region and an area scaling for entropy. This provides another way of looking into the problem by explicitly using coordinates and perhaps is interesting from the gravitational dynamics standpoint.

III. THE COSMOLOGICAL SPACETIMES

A. The metric and suitable coordinate choice

In this section, we shall further apply the formalism for cosmological spacetimes which are essentially nonstationary. Let us start with the Friedmann-Lemaitre-Robertson-Walker spacetime with flat spatial sections,

$$
ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]. \quad (65)
$$

The first of the Einstein equations reads

$$
H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = 8\pi\rho(t) + \Lambda. \quad (66)
$$

For any generic matter field with a linear equation of state, $p(t) = w\rho(t)$, where w is a constant, the energy-momentum conservation corresponding to Eq. (65) gives

$$
\rho(t) = \frac{\rho_0}{a^{3(1+w)}(t)},
$$

where ρ_0 is a constant. Plugging this into Eq. (66), we get

$$
a(t) = \left(\frac{8\pi\rho_0}{\Lambda}\right)^{\frac{1}{3(1+w)}}\left(\sinh\frac{3(1+w)H_0 t}{2}\right)^{\frac{2}{3(1+w)}}, \quad (67)
$$

where $H_0 = \sqrt{\Lambda/3}$, as earlier. We shall in particular be interested in cold dark matter ($w = 0$) and radiation ($w = 1/3$). For the first, the scale factor reads

$$
a(t) = \left(\frac{8\pi\rho_0}{\Lambda}\right)^{\frac{1}{3}}\sinh^{2/3}\frac{3H_0 t}{2}. \quad (68)
$$

For $t \to \infty$, the above scale factor evolves to that of the de Sitter, $a(t) \sim e^{H_0 t}$. Likewise, for the radiation we have

$$
a(t) = \left(\frac{8\pi\rho_0}{\Lambda}\right)^{\frac{1}{4}}\sinh^{1/2} 2H_0 t. \quad (69)
$$

Rewriting Eq. (65) in terms of spherical polar coordinates (R, θ, ϕ) and introducing the proper or the physical radius $r = Ra(t)$, we get

$$
ds^2 = -(1 - H^2(t)r^2)dt^2 - 2H(t)rdrdt + dr^2 + r^2 d\Omega^2. \quad (70)
$$

The Hubble horizon radius, $H^{-1}(t)$, is manifest in the above coordinate system. The coordinate singularity at the surface $r = H^{-1}(t)$ is not in general a null surface for any scale factor like $a(t) = a_0\sinh^p\frac{H_0 t}{p}$, as can be seen by computing the norm of the 1-form $\nabla_a(rH(t))$ normal to that surface,

$$g^{ab}\nabla_a(rH(t))\nabla_b(rH(t))$$

$$= -\frac{H_0^2/p}{\sinh^2(H_0 t/p)}\left[\frac{1}{p\cosh^2(H_0 t/p)} - 2\right] \neq 0, \qquad (71)$$

while the equality certainly holds in the de Sitter limit $H_0 t \to \infty$.

Let us suppose that we are dealing with the empty de Sitter space, so that $H(t)$ in Eq. (70) coincides with H_0. One can then introduce a new timelike coordinate, T (see, e.g., Ref. [54]),

$$t = T + \frac{1}{2H_0}\ln|1 - H_0^2 r^2|, \qquad (72)$$

to get

$$ds^2 = -(1 - H_0^2 r^2)dT^2 + (1 - H_0^2 r^2)^{-1}dr^2 + r^2 d\Omega^2, \qquad (73)$$

which is manifestly static inside the cosmological event horizon radius, H_0^{-1}. However, since we are interested in a time-dependent Hubble expansion rate, we cannot define a coordinate transformation like the above by just replacing H_0 with $H(t)$ [55].

On the other hand, using indeed the transformation of Eq. (72), the metric of Eq. (70) can be rewritten as

$$ds^2 = -(1 - r^2 H^2(T, r))dT^2$$
$$+ 2r\left(\frac{H_0(1 - H^2(T, r)r^2)}{1 - H_0^2 r^2} - H(T, r)\right)dTdr$$
$$+ \left(1 + \frac{2r^2 H(T, r)H_0}{1 - H_0^2 r^2} - \frac{H_0^2 r^2(1 - r^2 H^2(T, r))}{(1 - H_0^2 r^2)^2}\right)dr^2$$
$$+ r^2 d\Omega^2. \qquad (74)$$

Clearly the current Hubble radius, $H^{-1}(t)$, is smaller than the cosmological horizon radius, H_0^{-1}, of the de Sitter space. Accordingly, it is easy to see that the second term in g_{rr} is always greater than the third term, which is negative.

Also, in the limit $H(T, r) \to H_0$, we recover the static patch of the de Sitter space. Putting these all together, it is clear that the coordinate system used in Eq. (74) is well defined within the current Hubble radius. The above coordinate system would be very useful for our purpose of analyzing systems asymptotically evolving to the de Sitter spacetime.

The metric in Eq. (74) can also be generalized easily to include a central mass/black hole. The version of Eq. (65) is now given by the McVittie spacetime written in the spherical polar coordinates [56,57]:

$$ds^2 = -\left(\frac{1 - \frac{M}{2Ra(t)}}{1 + \frac{M}{2Ra(t)}}\right)^2 dt^2 + a^2(t)\left(1 + \frac{M}{2Ra(t)}\right)^4$$
$$\times (dR^2 + R^2 d\Omega^2). \qquad (75)$$

Defining the proper radius $r = Ra(t)(1 + M/2Ra(t))^2$, the above metric can be rewritten as

$$ds^2 = -\left(1 - \frac{2M}{r} - H^2 r^2\right)dt^2 - 2Hr\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}drdt$$
$$+ \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2. \qquad (76)$$

For $H = H_0$, the above simply represents the Schwarzschild–de Sitter spacetime. In particular, defining

$$t = T - H_0\int\frac{rdr}{(1 - 2M/r)^{\frac{1}{2}}(1 - 2M/r - H_0^2 r^2)},$$

the above metric with $H = H_0$ takes the standard form of the Schwarzschild–de Sitter,

$$ds^2 = -\left(1 - \frac{2M}{r} - H_0^2 r^2\right)dT^2 + \left(1 - \frac{2M}{r} - H_0^2 r^2\right)^{-1}dr^2$$
$$+ r^2 d\Omega^2; \qquad (77)$$

whereas for a generic scale factor, we find, in place of Eqs. (74) or (76),

$$ds^2 = -\left(1 - \frac{2M}{r} - H^2(T, r)r^2\right)dT^2 + \frac{2rdTdr}{\sqrt{(1 - 2M/r)}}\left(\frac{H_0(1 - \frac{2M}{r} - H^2(T, r)r^2)}{(1 - \frac{2M}{r} - H_0^2 r^2)} - H(T, r)\right)$$
$$+ \frac{dr^2}{(1 - 2M/r)}\left(1 + \frac{2H(T, r)H_0 r^2}{(1 - \frac{2M}{r} - H_0^2 r^2)} - \frac{H_0^2 r^2(1 - \frac{2M}{r} - H^2(T, r)r^2)}{(1 - \frac{2M}{r} - H_0^2 r^2)^2}\right) + r^2 d\Omega^2, \qquad (78)$$

in which setting $H \to H_0$ recovers the Schwarzschild–de Sitter spacetime. Note that the above metric could also represent the dynamical horizon of a black hole. The dynamical cosmological and black hole apparent horizons are given by the larger and the smaller roots of $g_{TT} = 0$, respectively.

First, we shall be interested in the dynamics of spacetimes in Eqs. (74) or (78) when they are close to the de Sitter or Schwarzschild–de Sitter spacetimes. In that case, the Hubble rate, $H = \dot{a}/a$, corresponding to the scale factors of Eqs. (68) and (69), takes the form $H_0(1 + \delta H(T, r))$ (with $H_0 t \gg 1$), where

$$\delta H(T, r) \approx 6H_0 t(T, r)e^{-3H_0 t(T,r)} \quad \text{(for } \Lambda \text{CDM)},$$

$$\delta H(T, r) \approx 8H_0 t(T, r)e^{-4H_0 t(T,r)} \quad \text{(for } \Lambda \text{ radiation).}$$

$$(79)$$

The above relations show that the Λ-radiation universe evolves faster to the de Sitter spacetime than does ΛCDM, just because of the fact that the pressure of the radiation dilutes itself faster than the cold dark matter. Equation (74) takes the form up to $\mathcal{O}(\delta H)$:

$$ds^2 = -(1 - r^2 H_0^2 - 2r^2 H_0^2 \delta H(T, r))dT^2$$

$$- 2\delta H(T, r)H_0 r \frac{1 + H_0^2 r^2}{1 - H_0^2 r^2} dT dr$$

$$+ (1 - H_0^2 r^2)^{-1} \left(1 + \frac{2\delta H(T, r)r^2 H_0^2}{1 - H_0^2 r^2} \right) dr^2 + r^2 d\Omega^2.$$

$$(80)$$

We note that we must take the $\delta H \to 0$ limit before taking the limit $r \to H_0^{-1}$, as the de Sitter cosmological horizon radius is larger than the dynamical Hubble radius $H^{-1}(T, r)$, and our coordinate system is well behaved only up to the latter. Likewise, Eq. (78) becomes

$$ds^2 = -\left(1 - \frac{2M}{r} - r^2 H_0^2 - 2r^2 H_0^2 \delta H(T, r) \right) dT^2$$

$$- \frac{2H_0 \delta H(T, r)r \left(1 - \frac{2M}{r} + H_0^2 r^2 \right)}{\sqrt{(1 - 2M/r)} \left(1 - \frac{2M}{r} - H_0^2 r^2 \right)} dT dr$$

$$+ \frac{dr^2}{(1 - 2M/r - H_0^2 r^2)} \left(1 + \frac{2H_0^2 r^2 \delta H(T, r)}{(1 - 2M/r - H_0^2 r^2)} \right)$$

$$+ r^2 d\Omega^2.$$

$$(81)$$

The spacetimes in Eqs. (80) and (81), now being close to the de Sitter or the Schwarzschild–de Sitter (respectively) can be treated as quasistatic. We shall use these two forms of the actual metrics in order to explicitly evaluate the entropy of a box filled with ideal gas in the next section.

B. Calculation of the entropy

Let us consider Eq. (80), in which we need to first obtain a suitable spatial hypersurface and an orthogonal timelike vector field, χ^a. Certainly, the most convenient choice of the spatial hypersurface would be the (r, θ, ϕ) hypersurfaces. Since $(\partial_T)^a$ is not orthogonal to that hypersurface, we define a vector field

$$\chi^a = (\partial_T)^a - \frac{(\partial_T \cdot \partial_r)}{(\partial_r \cdot \partial_r)} (\partial_r)^a.$$

$$(82)$$

Clearly, the above construction is analogous to what we did in the stationary axisymmetric case, with the difference

that $(\partial_r)^a$ is not a Killing vector field here. We have the norm of χ^a,

$$\chi^a \chi_a = -\tilde{\beta}^2 = -(1 - r^2 H_0^2 - 2r^2 H_0^2 \delta H(T, r)) + \mathcal{O}(\delta H)^2,$$

$$(83)$$

guaranteeing that χ^a is indeed timelike within the current Hubble radius. Since by construction $\chi_a(\partial_r)^a = 0$, we shall use χ^a in Eq. (30). Putting these all together, and using the momentum dispersion relation $p_a p^a = -m^2$, the phase space volume, Eq. (30), becomes

$$P(E(T)) = \frac{4\pi}{3} \int dr d\theta \sin\theta d\phi \frac{r^2}{\sqrt{1 - H_0^2 r^2}}$$

$$\times \left(1 + \frac{r^2 H_0^2 \delta H(T, r)}{1 - H_0^2 r^2} \right)$$

$$\times \left(\frac{E^2(r, T)}{1 - H_0^2 r^2 - 2r^2 H_0^2 \delta H(T, r)} - m^2 \right)^{\frac{3}{2}}, \quad (84)$$

where we have defined $E(r, T) = -p_a \chi^a$; and since χ^a is not a Killing vector field here, this quantity is not a constant. Setting $H_0 = 0$ corresponds to the flat spacetime, whereas setting $\delta H(T, r) = 0$ corresponds to the de Sitter spacetime, which we shall briefly consider first, following Ref. [39]. Note that in this limit, the vector field χ^a is Killing, and E should be regarded as the conserved energy.

Once again, we need to impose the Planck-scale cutoff in probing the near-horizon geometry. But we can now use the radial coordinate itself for this purpose, unlike in the stationary axisymmetric spacetimes. Let us imagine that the box is close to the cosmological event horizon, with its sides located at r_a (the outer side, closer to the horizon) and r_b (the inner side). Let $r_a = H_0^{-1} - L_a$ and $r_b = H_0^{-1} - L_b$. We choose L_a to be such that it is related to the Planck length L_P as

$$L_P = \int_{H_0^{-1} - L_a}^{H_0^{-1}} \frac{dr}{\sqrt{1 - H_0^2 r^2}} = \sqrt{\frac{2L_a}{H_0}}. \quad (85)$$

We assume that $L_P \ll H_0^{-1}$, which is very natural, owing to the observed tiny value of the cosmological constant. The cosmological event horizon has a negative surface gravity $-\kappa_C = -H_0$, owing to the repulsive effect. The integral in Eq. (84) for the de Sitter space becomes, as $r \to H_0^{-1}$,

$$P(E) = \frac{4\pi E^3}{3} \int_{r_b}^{r_a} dr \sin\theta d\theta d\phi \frac{r^2}{(1 - H_0^2 r^2)^2}, \quad (86)$$

which yields, after using Eq. (85) at the leading order,

$$P(E) \approx \frac{2\pi E^3}{3} \frac{A_\perp}{L_P^2 \kappa_C^3}, \quad (87)$$

where as earlier, $A_\perp$ is the transverse area of the box, tangent to the cosmological event horizon. The partition function is obtained from the second relation in Eq. (29),

$$Q(\beta) = \frac{4\pi A_\perp}{\beta^3 \kappa_C^3 L_P^2}. \qquad (88)$$

Now, using Eq. (85), we find $(1 - H_0^2 r^2)|_{r=r_a} \approx L_P^2 H_0^2$. Putting this back into the above equation, we get

$$Q(\beta) = \frac{4\pi A_\perp L_P}{\beta_{\mathrm{loc}}^3(r_a)}, \qquad (89)$$

where $\beta_{\mathrm{loc}}(r_a) = \beta\sqrt{L_P^2 H_0^2}$ is the inverse of the blueshifted local temperature as earlier. For N noninteracting particles, we find the partition function to be $Q_N = Q^N(\beta)$ and the entropy

$$S = N\left[\ln\left(\frac{4\pi L_P A_\perp}{N\beta_{\mathrm{loc}}^3(r_a)}\right) + 3\right], \qquad (90)$$

which, as expected, is exactly formally similar to the result for stationary axisymmetric spacetimes, Eq. (38).

Let us consider the generic case now. We cannot evaluate Eq. (84) along the above lines, as we do not know the behavior of the function $E(r, T)$. However, we can still determine the scaling properties of the entropy when the spacetime is close to the de Sitter, as follows.

The Hubble horizon radius $r_C(T)$ is given in this case by $g_{TT} = -(1 - H_0^2 r^2 - 2r^2 H_0^2 \delta H) = 0$:

$$r_C(T) \approx \frac{1}{H_0}(1 - \delta H). \qquad (91)$$

When one side of the box is close to the Hubble horizon, the integral of Eq. (84) diverges. In order to regularize this integral, we need to define a time-dependent cutoff, $L_P(T)$, as

$$L_P(T) = \int_{r_C(T)-L_a}^{r_C(T)} \frac{dr}{\sqrt{1 - H_0^2 r^2 - 2r^2 H_0^2 \delta H}}, \qquad (92)$$

where, as in Eq. (85), $(r_C(T) - L_a)$ is the (time-dependent) radial coordinate of the side of the box which is close to the Hubble horizon. (If we set $\delta H = 0$ above, we recover the Planck length L_P as the cutoff.) Assuming that the length scale of the box is much smaller compared to the horizon length scale, we have

$$L_P(T) \approx \sqrt{\frac{2L_a}{H}}. \qquad (93)$$

We can also define the "surface gravity" of the Hubble horizon as $\kappa^2(T) := \frac{1}{4\tilde{\beta}^2} g^{ab}(\nabla_a \tilde{\beta}^2)(\nabla_b \tilde{\beta}^2)$, which becomes, to the leading order,

$$-\kappa(T) \approx -H(T) = -H_0(1 + \delta H) + \mathcal{O}(\delta H)^2. \qquad (94)$$

Now, by noting that in the leading order we have

$$\sqrt{1 - H_0^2 r^2} = \sqrt{1 - r^2 H^2(T, r)}\left[1 + \frac{H_0^2 r^2 \delta H}{1 - H_0^2 r^2}\right],$$

Eq. (84) becomes, when the box is close to the Hubble horizon,

$$P(E(T)) \approx \frac{4\pi}{3}\int_{r_b}^{r_a} dr \sin\theta d\theta d\phi \frac{r^2 E^3(r, T)}{(1 - r^2 H^2(T, r))^2}, \qquad (95)$$

which is formally similar to the Killing horizon case, certainly due to the quasi–de Sitter approximation. Since in this limit $\delta H \ll 1$ in Eq. (79), it follows that the function $E(r, T)$ varies slowly compared to the divergent denominator of the above integrand. Also, we can pull the $r^2 = r_C^2(T)$ term outside the integration. Putting all of these together and performing the angular integration, we obtain

$$P(E(T)) \approx \frac{4\pi E^3(T) A_\perp(T)}{3}\int_{r_b=r_C(T)-L_b}^{r_a=r_C(T)-L_a} \frac{dr}{(1 - r^2 H^2(T, r))^2}, \qquad (96)$$

where $A_\perp(t) = \Omega r_C^2(T)$ is the section of the transverse area of the Hubble horizon, parallel to the approaching side of the box, with Ω being the relevant solid angle. This expression could further be simplified as

$$P(E(T)) \approx \frac{\pi E^3(T) A_\perp(T)}{3}\int_{r_b=r_C(T)-L_b}^{r_a=r_C(T)-L_a} \frac{dr}{\kappa^2(T)(r_C(T) - r)^2} \qquad (97)$$

and could easily be evaluated by using Eqs. (91), (93), and (94), and once again the fact that δH is a slowly varying quantity:

$$P(E(T)) \approx \frac{2\pi E^3(T) A_\perp(T)}{3\kappa^3(T) L_P^2(T)}. \qquad (98)$$

The corresponding expression for the entropy becomes

$$S \approx N\left[\ln\left(\frac{4\pi L_P(T) A_\perp(T)}{N\beta_{\mathrm{loc}}^3(T)}\right) + 3\right], \qquad (99)$$

which is formally similar to the result in the case of the de Sitter spacetime. Since all the time-dependent quantities appearing above have a smooth de Sitter limit, the above expression for the entropy asymptotes to Eq. (90) at late times. (To the best of our knowledge, this is the first demonstration of the emergence of the entropy-area

relation in a physical scenario for a Hubble horizon.) Even though we have performed the above computations in the asymptotic de Sitter limit, we shall argue at the end of this section that such area scaling should occur in the case of a generic Hubble horizon as well. In other words, our analysis shows that the area scaling for the entropy for a Killing horizon, Eq. (90), should come from a smooth limit from a non-Killing or dynamical horizon.

There is, however, one caveat to the above derivation. The notion of the entropy of a given system is usually related to the lack of information to the observer, as happens in the case of a null surface, e.g. Ref. [27]. For such surfaces, we get an area scaling due to the piling up of accessible states near the horizon [37,38] for the observers who do not cross the surface. This also makes the Planck-scale cutoff imposed earlier physically meaningful. On the other hand, since the Hubble horizons are not null surfaces, the time-dependent cutoff $L_P(T)$ used in Eq. (98) should perhaps be regarded as merely a regulator to handle the divergence of the near-horizon integral.

The above analysis works for the McVittie spacetime of Eq. (81) as well, for which the dynamical black hole $(r_H(T))$ and the Hubble horizons $(r_C(T))$ are the two real positive roots of

$$(1 - 2M/r - r^2 H_0^2 - 2r^2 H_0^2 \delta H(T, r)) = 0.$$

The integral of Eq. (95) now modifies to, when close to either of these horizons,

$$P(E(T)) \approx \frac{4\pi}{3} \int_{r_b}^{r_a} dr \sin\theta d\theta d\phi$$
$$\times \frac{r^2 E^3(T)}{(1 - 2M/r - r^2 H_0^2 - 2r^2 H_0^2 \delta H)^2}. \quad (100)$$

By taking one side of the box close to either of the horizons, the above integral could be evaluated as earlier to obtain similar results, by expanding various terms to $\mathcal{O}(\delta H)$ from their stationary (i.e., Schwarzschild–de Sitter) values.

Let us finally consider the generic case of Eq. (74) and take the near-Hubble-horizon limit,

$$P(E(T))$$
$$\approx \frac{4\pi}{3} \int_{r_b}^{r_a} dr \sin\theta d\theta d\phi$$
$$\times \frac{r^2 E^3(r, T)}{\left(1 + \frac{2r^2 H(T,r)H_0}{1 - H_0^2 r^2} - \frac{H_0^2 r^2 (1 - r^2 H^2(T,r))}{(1 - H_0^2 r^2)^2}\right)^{1/2} (1 - r^2 H^2)^{3/2}}. \quad (101)$$

To the best of our knowledge, the above integral cannot be evaluated analytically. However, we may easily estimate it and argue in favor of the area scaling near the Hubble horizon as follows. Since the $(1 - H^2(T, r)r^2)$ term in the denominator is divergent near the Hubble horizon and the integration is defined on a constant time hypersurface, we may safely pull the energy function out of the integral as earlier. Also, we can define the time-dependent cutoff near the Hubble horizon:

$$L_P(T) = \int_{r_C(T) - L_a}^{r_C(T)} \frac{dr}{\sqrt{1 - r^2 H^2(T, r)}}.$$

Now, if we are far away from the de Sitter limit, $H_0 \ll H(T, r)$, we may pull the $\sqrt{g_{rr}}$ term out of the integral as well. Putting these all together, and performing the trivial angular integration, we obtain

$$P(E(T)) \approx \frac{4\pi A_\perp(T) E^3(T)/3}{\sqrt{1 + \frac{2H_0/H}{1 - H_0^2/H^2}}} \int_{r_b}^{r_a} \frac{dr}{(1 - r^2 H^2)^{3/2}}. \quad (102)$$

Even though it is not similar to what we obtain for the case of the Killing horizon, Eq. (86), an area scaling is manifest. In fact, we note that the quadratic divergence in the denominator for the Killing horizon is "tamed" here a little bit—indicating a lesser value of the probability function compared to a Killing horizon of the same area. Perhaps this should be attributed to the fact that a Hubble horizon cannot hide information completely—modes can reenter through them.

IV. SUMMARY AND DISCUSSIONS

It was shown a few years back in Ref. [39] that when a box of ideal gas approaches the Killing horizon of a static spherically symmetric spacetime, the entropy of the gas develops an area dependence. In this work, we have extended this result for nonstatic spacetimes with a positive cosmological constant, like the stationary axisymmetric and the cosmological spacetimes.

For the stationary axisymmetric spacetimes, the imposition of a Planck-length cutoff in the near-horizon divergent integral requires care, and we have accomplished this by using the norm of the timelike, hypersurface orthogonal vector field χ^a. The final result in Eq. (38) is exactly the same as that of the static and spherically symmetric spacetime. We also have provided an expression for generic spacetime dimensions in Eq. (41). These results hold on an equal footing for both the black hole and the cosmological event horizons. We note that the techniques developed in Ref. [32] (and references therein) were very essential in this computation, as the coordinate R defined in Eq. (16) was very crucial in defining the Planck-length cutoff near the horizon, irrespective of the path of the box.

In the case of cosmological spacetimes evolving to the (Schwarzschild-) de Sitter spacetime, we first built a suitable coordinate system to deal with the Hubble horizon, Eqs. (74) and (78). When the spacetime is close to the de Sitter model, we have explicitly evaluated the leading-order expression of the entropy of the gas when it is close to the

Hubble horizon and have shown that it is formally similar to that of the de Sitter. For a generic cosmological spacetime, also, we also have formally demonstrated the area scaling when the box is close to the Hubble horizon.

It would be highly interesting to apply our methods to deal with various nonstatic spacetimes in other approaches as well, such as the so-called brick wall formalism [58]. We hope to return to this issue in the near future.

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