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Optical properties of a braneworld black hole: Gravitational lensing and retrolensing

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In this paper, we study optical properties in terms of weak and strong (retrolensing) gravitational lensing around a black hole on the brane. The black hole is described by the Reissener-Nordström metric, where the electric charge is replaced by the Weyl tidal charge which acts in consonance with mass. It is, therefore, expected that gravitational lensing effects would be enhanced by the tidal charge, and that is what we verify. We also study the shadow of a braneworld black hole in plasma and show that the tidal charge increases the size of the black hole shadow. We finally consider energy emission from the hole by thermal radiation which is also increased due to tidal charge.

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I. INTRODUCTION

The braneworld gravity model [1,2] offers an interesting and attractive device of studying higher dimensional gravitational effects on the 3-brane—our four dimensional spacetime. In this framework, the vacuum equation for gravitational field on the right has a term coming from projection of the bulk Weyl curvature. This was solved for a static spherically symmetric spacetime [3] to describe a black hole on the brane. It is obviously the Reissner-Nordström metric, where the charge is not electric but due to projection of the bulk Weyl curvature. It is called the tidal charge, W, and has the opposite sign in the metric $W^2 = -Q^2$. It was the first and simplest black hole on the brane solution. However, it is not a complete solution of bulk-brane system because it solves only the equation on the brane. There are also some other solutions for black holes on the brane [4–8]. However, there exists no complete solution in the literature. We shall, therefore, take the simplest solution for the brane black hole for studying the effect of tidal charge on gravitational lensing and emission of thermal radiation. Since the tidal charge now acts in consonance with the mass of the hole, it is therefore expected to contribute in line. This is what we wish to investigate in this paper.

The strong gravitational field surrounding a compact object causes deflection of light, giving rise to a gravitational lens. The gravitational lens differs from the optical one in the sense that deflection is maximum when light is passing closest to the central object, and it decreases as the distance from the center increases. Studying the gravitational lens systems gives us an opportunity to probe the image of the source which will be produced by the gravitational lensing object and the parameters of the compact object which is playing the role of the gravitational lens. The deflection of light decreases when it approaches the light sphere of the compact object, and this causes the so-called retrolensing.

In Ref. [9], the strong-field limit approach has been used to investigate the gravitational lensing properties of braneworld black holes. The bending angle of a light ray near the horizon of a braneworld black hole in the weak-field limit has been calculated using the variational principle in [10]. The gravitational lensing near the braneworld black holes have been studied in the various papers [4,5,11–27]. Different properties of the braneworld gravitational objects have been studied using solar system tests [28], quasiperiodic oscillations [29], and Jacobi stability analysis [30]. In [31], it has been shown that the shadow of rotating braneworld black holes depends on brane tension.

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The retrolensing by a charged rotating black hole and wormhole has been studied in [32,33], respectively. The different aspects of the weak and strong lensing, as well as photon motion in the vacuum and plasma environment, have been studied in Refs. [34–44]. There exists a large body of literature on the study of microlensing—beginning with the pioneering work on microlensing in Refs. [45–49], and proceeding to the strong gravitational lensing in the Schwarzschild metric studied in [43]. The review in [44] summarizes the gravitational lensing effect around compact gravitating objects in general relativity. The angular sizes and magnification factors for relativistic rings formed by photons which undergo one or several turns around the black hole have been explored in [43]. The influence of plasma on gravitational lensing has been investigated in [34,38–42]. In [50], the general-relativistic radiative transfer theory in refractive and dispersive media has been investigated. In general, optical properties including gravitational lensing and ghost images of regular as well as different kinds of black holes in different settings have been studied by various authors [51–70]. It turns out, in the weak-field approximation, that the expressions for deflection of light have the same character for both the regular black hole and no-horizon spacetimes which were extensively studied in the strong-field situations in [51,52].

The most interesting optical effect of the compact objects attracting the most attention during recent years is the black hole shadow. The accuracy of the Event Horizon Telescope is approaching that which is enough for direct observation of the shadow of the supermassive black holes SgrA* at the center of our Galaxy and M87. The observation and analysis of the black hole shadow, in principle, can help to get an estimation for the different parameters of the central black hole and surrounded spacetime and plasma [31,34,36,55,60,64,67–77]. Here we plan to study how the plasma and brane tidal charge together can change the form and the size of the shadow of the rotating black hole in the braneworld.

In this work, our main purpose is to study the weak and strong gravitational lensing as retrolensing for the black hole on the brane in the braneworld gravity model. In particular, we would verify that the Weyl tidal charge works in unison with the mass to produce enhanced light deflection as well as thermal radiation emission. The paper is organized as follows: Sec. II is devoted to the gravitational lensing in the weak-field approximation, while Sec. III focuses on the study of the photon deflection angle and magnification of the source image due to weak lensing around the black hole. The strong lensing as retrolensing, as well as magnification of the source image, has been investigated in Sec. IV. We study the shadow of the braneworld black hole in plasma in Sec. V. In Sec. VI, we conclude. Throughout the paper, we use the $G=c=1$ and $(-+++)$ signature. Greek indices run from 0 to 3 and latin from 1 to 3.

II. LIGHT DEFLECTION IN THE PRESENCE OF PLASMA

Consider the spacetime metric around the static and spherically symmetric black hole on the brane given by [3]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where the function $f(r)$ has the following form,

$$f(r) = 1 - \frac{2M}{r} - \frac{W^2}{r^2}, \quad (2)$$

and W is the Weyl tidal charge. The spacetime metric (1) mathematically has the same form as the Reissner-Nordström solution for a charged black hole but for the sign of W^2 which is opposite of Q^2. When the tidal charge $W=0$, it reduces to the Schwarzschild metric.

The weak-field approximation is a useful tool to probe gravitational properties, and for that we write

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad (3)$$

with the notations

$$\begin{aligned} &\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1), \\ &h_{\alpha\beta} \ll 1, \qquad h_{\alpha\beta} \to 0 \quad \text{under } x^i \to \infty, \\ &g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}, \qquad h^{\alpha\beta} = h_{\alpha\beta}. \end{aligned} \quad (4)$$

We shall employ the Hamiltonian approach to study photon trajectories in a medium and write

$$\delta\left(\int p_\alpha dx^\alpha\right) = 0, \quad (5)$$

from which it easily follows that

$$\frac{dx^\alpha}{d\lambda} = \frac{\partial H}{\partial p_\alpha}, \qquad \frac{dp_\alpha}{d\lambda} = -\frac{\partial H}{\partial x^\alpha}, \quad (6)$$

where we have used the condition [78]

$$H(x^\alpha, p^\alpha) = \frac{1}{2}[g^{\alpha\beta}p_\alpha p_\beta - (n^2-1)(p_\alpha V^\alpha)^2] = 0. \quad (7)$$

Here p^α is the photon momentum, V^α is the four velocity of the observer at the plasma, n is the refractive index of the medium surrounding the central object, and λ is the affine parameter along the trajectory of the photon.

We will rewrite Eq. (7) in the form

$$H = \frac{1}{2}\left[g^{00}p_0p_0 + g^{ik}p_ip_k - (n^2-1)\frac{\hbar^2\omega^2(x^i)}{c^2}\right], \quad (8)$$

where $\hbar$ is the Planck constant, $\omega(x^i)$ is the photon frequency, which depends on coordinates x^i indicating its dependence on gravitational potential, c is the speed of light propagation in vacuum, and

$$p_\alpha V^\alpha = -\frac{\hbar\omega(x^i)}{c}. \tag{9}$$

The refractive index of plasma depends on the frequency $\omega(x^i)$ in the following way,

$$n^2 = 1 - \frac{\omega_0^2}{\omega^2(x^i)}, \qquad \omega_0^2 = \frac{4\pi e^2 N}{m} = K_e N, \tag{10}$$

where $N = N(x^i)$ is the electron density in plasma, e and m are electric charge and mass of electron, respectively. For simplicity, we adopt the following notations:

$$\omega = \omega(x^i \to \infty),$$
$$n_0 = n(x^i \to \infty). \tag{11}$$

Equations (8) and (10) allow us to find the equation of motion as

$$\frac{dx^i}{d\lambda} = g^{ik} p_k = p^i, \tag{12}$$

$$\frac{dp_i}{d\lambda} = -\frac{1}{2} g^{kl}_{,i} p_k p_l - \frac{1}{2} g^{00}_{,i} p_0^2 - \frac{1}{2}\frac{\hbar^2 K_e N_{,i}}{c^2}. \tag{13}$$

Taking into account the weak-field approximation and weak plasma strength, for photon propagation along the z direction, we write

$$p^\alpha = \left(\frac{\hbar\omega}{c}, 0, 0, \frac{n_0\hbar\omega}{c}\right), \qquad p_\alpha = \left(-\frac{\hbar\omega}{c}, 0, 0, \frac{n_0\hbar\omega}{c}\right). \tag{14}$$

The deflection angle in the plane orthogonal to the z axis is given in the following form:

$$\hat{\alpha}_k = [p_k(\infty) - p_k(-\infty)]/p,$$
$$p = \sqrt{p_1^2 + p_2^2 + p_3^2} = |p_3| = \frac{n_0\hbar\omega}{c}, \qquad k = 1, 2. \tag{15}$$

Using the equation of motion obtained earlier, one can easily obtain

$$\hat{\alpha}_k = \frac{1}{2}\int_{-\infty}^{\infty} \left(h_{33} + \frac{h_{00}\omega^2 - K_e N}{\omega^2 - \omega_0^2}\right)_{,k} dz. \tag{16}$$

Note that the negative and positive sign for $\hat{\alpha}_b$ indicate, respectively, deflection towards and away from the central object. At large r, the black hole metric could be approximated to [79]

$$ds^2 = ds_0^2 + \left(\frac{2M}{r} - \frac{W^2}{r^2}\right)dt^2 + \left(\frac{2M}{r} - \frac{W^2}{r^2}\right)dr^2, \tag{17}$$

where

$$ds_0^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

In the Cartesian coordinates, the components $h_{\alpha\beta}$ can be written as

$$h_{00} = \left(\frac{R_s}{r} - \frac{W^2}{r^2}\right),$$
$$h_{ik} = \left(\frac{R_s}{r} - \frac{W^2}{r^2}\right) n_i n_k,$$
$$h_{33} = \left(\frac{R_s}{r} - \frac{W^2}{r^2}\right)\cos^2\chi, \tag{18}$$

where $R_s = 2M$.

Using the above expressions in the formula (16), one can calculate the light deflection angle for a black hole sitting in plasma [39,43],

$$\hat{\alpha}_b = \int_0^\infty \frac{\partial}{\partial b}\left[\left(\frac{R_s}{\sqrt{b^2+z^2}} - \frac{W^2}{r^2}\right)\frac{z^2}{b^2+z^2}, + \frac{1}{1-\omega_0^2/\omega^2}\left(\frac{R_s}{\sqrt{b^2+z^2}} - \frac{W^2}{r^2}\right)\right]dz, \tag{19}$$

where $b^2 = x_1^2 + x_2^2$ is the impact parameter, and x_1 and x_2 are the coordinates on the plane orthogonal to the z axis, and the plasma parameters are given by

$$\omega_0^2 = \frac{4\pi e^2 N(r)}{m}, \qquad \omega^2 = \frac{\omega_\infty^2}{f(r)}. \tag{20}$$

Here ω_0 and ω are, respectively, plasma and photon frequency, and ω_∞ is the asymptotic value of the photon frequency. Considering the power-law plasma density function as

$$N(r) = N_0 \frac{r_0}{r},$$

with the density number N_0 at the radial position of the inner edge of plasma environment r_0. In the approximation of the small tidal charge and large distance, the expression (20), after expanding in series on the powers of $1/r$, can be approximated to

$$n^2 = \left(1 - \frac{\omega_0^2}{\omega^2}\right)^{-1} \simeq 1 - \frac{4\pi e^2 N_0 r_0}{m\omega_\infty^2 r} + \frac{4\pi e^2 N_0 r_0 R_s}{m\omega_\infty^2 r^2}. \tag{21}$$

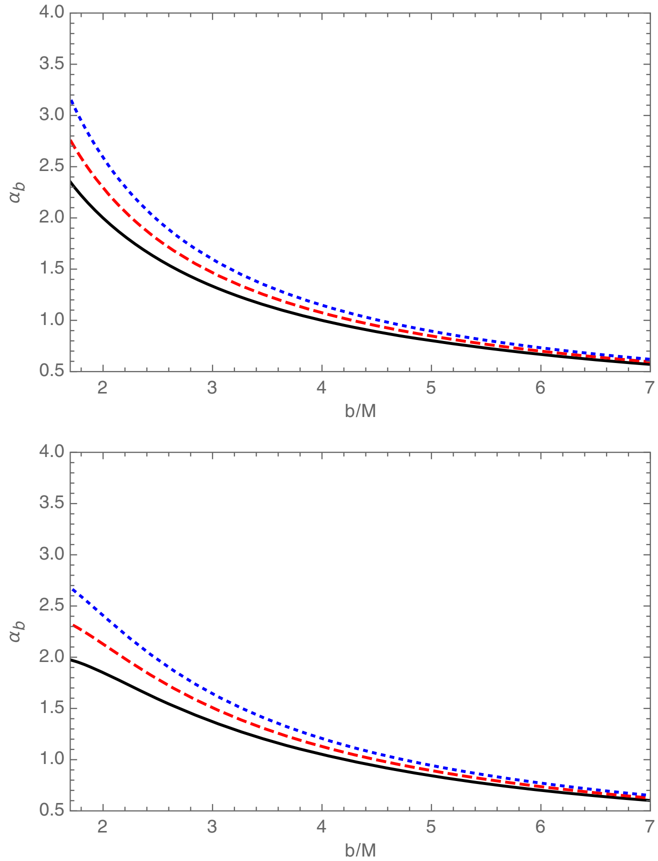

FIG. 1. Deflection angle α_b as a function of the impact parameter b for different tidal charge values: $W^2 = 0$, 0.5, 1.0, shown, respectively, as black-solid, red-dashed and blue-dotted. Left panel is for vacuum and right is for plasma ($4\pi e^2 N_0 r_0 / m\omega_\infty^2 M = 0.7$).

Using this approximation, one can easily find the deflection angle $\hat{\alpha}_b$ of the light around a black hole in presence of plasma:

$$\hat{\alpha}_b = \frac{2R_s}{b}\left(1 + \frac{\pi^2 e^2 N_0 r_0}{m\omega_\infty^2 b} - \frac{4\pi e^2 N_0 r_0 R_s}{m\omega_\infty^2 b^2}\right) - \frac{W^2}{4b^2}\left(3\pi + \frac{4\pi e^2 N_0 r_0}{m\omega_\infty^2 b}\left(8 - \frac{3\pi R_s}{b}\right)\right). \quad (22)$$

This reduces to $\hat{\alpha}_b = 2R_s/b$ for the Schwarzschild black hole when $e = W = 0$, indicating the absence of plasma and tidal charge [45,80]. Using the expression (22), we plot $\hat{\alpha}_b$ against the impact parameter b in Fig 1. It shows that the deflection angle increases with the value of the tidal charge, while the presence of plasma has the opposite effect.

III. THE IMAGE BRIGHTNESS

In this section, we will study the image source magnification using the expression for the deflection angle in the presence of plasma. Consider the gravitational lens equation in the following form [80,81],

$$\theta D_s = \beta D_s + \alpha D_{ls}, \quad (23)$$

where β is the angle of the real source from the observer-lens axis, θ is the angle of the apparent image of the source due to lensing with the deflection angle α, and D_s and D_{ls} are the distances from the observer to the lens and from the lens to the source, respectively. Since the impact parameter is $b = D_l\theta$, where D_l is the distance from the observer to the lens, we obtain the relation in the form [39,40]

$$\beta = \theta - \frac{D_{ls}}{D_s}\frac{F(\theta)}{D_l}\frac{1}{\theta}, \quad (24)$$

with the newly introduced quantity $F(\theta) = |\alpha_b| b = |\alpha_b(\theta)| D_l \theta$. Solutions of Eq. (24) give us the positions θ_k of the images of the object due to the lensing.

The special solution of Eq. (24) is called the Einstein angle θ_0, and it corresponds to the case where the object, lens, and observer are on a straight line. The corresponding radius of the Einstein ring is $R_0 = D_l\theta_0$, where θ_0 is the solution of Eq. (24) when $\beta = 0$. Usually the Einstein angle is very small in order to be resolved by modern telescopes. However, the lensing by some astrophysical objects like a star or stellar black hole can be detectable because it changes the apparent brightness of the source (magnification of the image brightness). The magnification of image brightness can be calculated using the formula

$$\mu_\Sigma = \frac{I_{\rm tot}}{I_*} = \sum_k \left|\left(\frac{\theta_k}{\beta}\right)\left(\frac{d\theta_k}{d\beta}\right)\right|, \qquad k = 1, 2, \ldots, s, \quad (25)$$

where s is the total number of images, $I_{\rm tot}$ is the total brightness of the images, I_* is the unlensed brightness of the source, and k refers to the number of images.

Using the expression of the Einstein angle θ_0 for the Schwarzschild black hole,

$$\theta_0 = \sqrt{2R_s \frac{D_{ds}}{D_s D_d}}, \quad (26)$$

we generalize it for the Einstein ring $(\theta_0^{pl})_{\rm brane}$ in the case of a brane black hole in plasma by using Eqs. (22) and (24) to write

$$(\theta_0^{pl})_{\rm brane} = \theta_0\left[1 + \frac{\pi^2 e^2 N_0 r_0}{m\omega_\infty^2 b} - \frac{W^2}{8bR_s}\frac{4\pi e^2 N_0 r_0 R_s}{m\omega_\infty^2 b^2} \times \left(3\pi + \frac{4\pi e^2 N_0 r_0}{m\omega_\infty^2 b}\left(8 - \frac{3\pi R_s}{b}\right)\right)\right]^{1/2}. \quad (27)$$

Using Eq. (25), we obtain magnification of the image source as given by

$$\mu_{\rm tot}^{pl} = \mu_+^{pl} + \mu_-^{pl} = \frac{x^2 + 2}{x\sqrt{x^2 + 4}}, \quad (28)$$

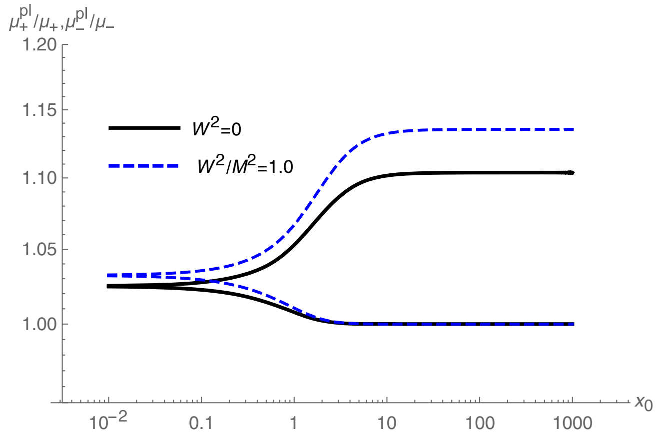

FIG. 2. μ_+^{pl}/μ_+ (lower curve) and μ_-^{pl}/μ_- (upper curve) of the image magnifications for the same valuye of tidal charge are shown for vacuum and plasma $4\pi e^2 N_0 r_0/m\omega_\infty^2 = 0.7$.

where the dimensionless parameter x is defined as

$$x = \frac{\beta}{(\theta_0^{pl})_{\rm brane}} = x_0\left[1 + \frac{\pi^2 e^2 N_0 r_0}{m\omega_\infty^2 b} - \frac{4\pi e^2 N_0 r_0 R_s}{m\omega_\infty^2 b^2} - \frac{W^2}{8bR_s}\left(3\pi + \frac{4\pi e^2 N_0 r_0}{m\omega_\infty^2 b}\left(8 - \frac{3\pi R_s}{b}\right)\right)\right]^{-1/2}. \quad (29)$$

Here, $x_0 = \beta/\theta_0$.

The magnification of the image source is expressed as

$$\mu_+^{pl} = \frac{1}{4}\left(\frac{x}{\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x} + 2\right), \quad (30)$$

$$\mu_-^{pl} = \frac{1}{4}\left(\frac{x}{\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x} - 2\right). \quad (31)$$

In Fig. 2, the image magnification parameter is plotted against x_0, showing the enhancing effect of tidal charge.

IV. RETROLENSING

In this section, we consider the deflection angle α_b in the strong gravity regime for the brane black hole in the absence of plasma. According to the work of [32], we look for the solution of the deflection angle in the form

$$\alpha_b = -\bar{a}\ln\left(\frac{b}{b_{cr}} - 1\right) + \bar{b} + \mathcal{O}((b - b_{cr})\log(b - b_{cr})), \quad (32)$$

where b_{cr} is the critical value of the impact parameter, and $\bar{a}$ and $\bar{b}$ are the functions of W^2/M^2. The first term on the right is due to the photon impact parameter, the second refers to the black hole geometry, and the third is the error term in the deflection angle in strong deflection limit [32].

The photon sphere radius $r_{\rm ph}$ for the brane black hole is given by

$$r_{\rm ph} = \frac{3M + \sqrt{9M^2 - 8W^2}}{2}. \quad (33)$$

The critical value of the impact parameter for the light ray can be found using the conditions [32,33]

$$b_{cr}(r_{\rm ph}) = \lim_{r_0\to r_{\rm ph}} b(r_0) = \lim_{r_0\to r_{\rm ph}} \frac{r_0^2}{\sqrt{\Delta(r_0)}}, \quad (34)$$

where $\Delta(r_0) = r_0^2 - 2Mr_0 + W^2$.

Near the photon sphere, the radial velocity vanishes, and we will have

$$\left(\frac{dr}{d\phi}\right) = r^4\left(\frac{1}{b^2} - \frac{\Delta}{r^4}\right), \quad (35)$$

where we have used the relation $b = L/E$. The deflection angle for light coming through the photon sphere can be found using the formula

$$\alpha_b = \lim_{r_0\to r_{\rm ph}} 2\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{\Delta}{r^4}}} - \pi, \quad (36)$$

and the technique described in [32], and we obtain

$$\bar{b} = \bar{a}\ln\left[\frac{8(3Mr_{\rm ph} - 4W^2)^3}{M^2 r_{\rm ph}^2 (Mr_{\rm ph} - W^2)^2} \times \left(2\sqrt{Mr_{\rm ph} - W^2} - \sqrt{3Mr_{\rm ph} - 4W^2}\right)^2\right] - \pi, \quad (37)$$

$$\bar{a} = \frac{r_{\rm ph}\sqrt{Mr_{\rm ph} - W^2}}{\sqrt{M(6M - r_{\rm ph})r_{\rm ph}^2 + 9r_{\rm ph}MW^2 - 4W^2}}. \quad (38)$$

The expressions for b_{cr}, b, $\bar{a}$, and $\bar{b}$ are obtained analytically. In the limiting case, when the brane tidal charge vanished, the values of $\bar{a}$ and $\bar{b}$ are the following: $\bar{a} = 1$ and $\bar{b} \simeq -2.92841$. Thus, the physical meanings of the quantities $\bar{a}$ and $\bar{b}$ can be interpreted as the deviation of the deflection angle due to modification of the spacetime metric. The dependences of these parameters are shown in Fig 3. In this plot, we presented b_{cr}, b, $\bar{a}$, and $\bar{b}$ depending on the brane parameter. It is clear that they have opposite properties of the braneworld with Reissner-Nordström gravity [32]. The horizon as well as the photon sphere radius and critical value of the impact parameter and the retrolensing parameter $\bar{b}$ increase with the increasing value

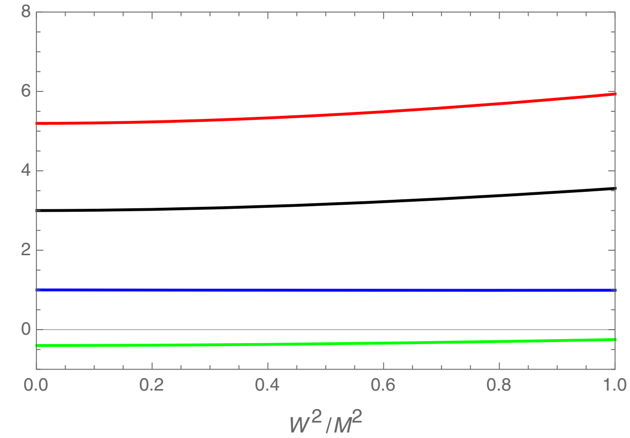

FIG. 3. b_c/M (red line), r_{pc}/M (black line), $\bar{a}$ (blue line), and $\bar{b}$ (green line) in the braneworld spacetime(solid line) as the functions of W^2/M^2.

of the tidal charge, while the parameter $\bar{a}$ has the opposite decreasing behavior.

The double image can be considered by the two image angles θ_+ and θ_-. In Fig. 4, the dependence of the double image on the tidal charge is exhibited, and it is clear that the image separation angle increases with the tidal charge.

In Fig. 5, the deflection angle of the light curve depending on the impact parameter of the photon motion has been shown for different value brane parameters.

V. SHADOW OF THE BRANEWORLD BLACK HOLE IN PLASMA

Using the Hamilton-Jacobi equation [69], one can obtain the equations of motion of the massless photon ($m = 0$) around the rotating braneworld black hole with the spacetime metric [3],

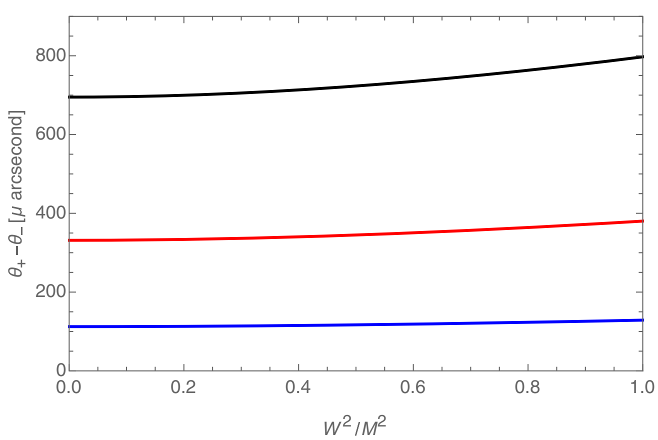

FIG. 4. Double image in the braneworld spacetime(solid line) as the functions of W^2/M^2 for the different masses of the black hole: $M = 10M_o$ (the black line, $M = 30M_o$ (the red line), $M = 60M_o$ (the blue line).

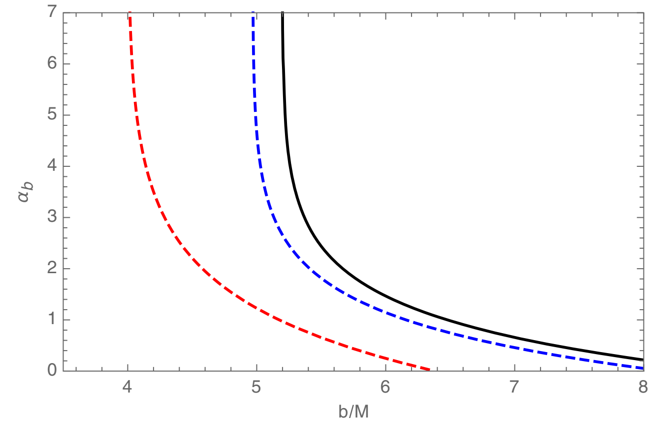

FIG. 5. Deflection angle of light curve in braneworld spacetime for different value of brane parameter W as $W^2 = 0$ (black, solid), $W^2 = 0.5$ (blue, dashed), $W^2 = 1.0$ (red, dashed).

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &+ \frac{(\Sigma + a^2\sin^2\theta)^2 - \Delta a^2}{\Sigma}\sin^2\theta d\varphi^2 \\ &- 2\frac{\Sigma + a^2\sin^2\theta - \Delta}{\Sigma}a\sin^2\theta d\varphi dt, \end{aligned} \tag{39}$$

where $\Delta = r^2 + a^2 - 2Mr - W^2$, $\Sigma = r^2 + a^2\cos^2\theta$, and $a = J/M$ is the specific angular momentum of the rotating black hole with the total angular momentum J. The equation of motion for the massless particles will take the form [31,79]

$$\Sigma\frac{d\phi}{d\lambda} = \left(\frac{\mathcal{L}}{\sin^2\theta} - a\mathcal{E}\right) + \frac{a}{\Delta}[(r^2 + a^2)\mathcal{E} - a\mathcal{L}], \tag{40}$$

$$\Sigma\frac{dt}{d\lambda} = a(\mathcal{L} - n^2\mathcal{E}a\sin^2\theta) + \frac{r^2 + a^2}{\Delta}[(r^2 + a^2)n^2\mathcal{E} - a\mathcal{L}], \tag{41}$$

$$\Sigma\frac{dr}{d\lambda} = \sqrt{\mathcal{R}}, \tag{42}$$

$$\Sigma\frac{d\theta}{d\lambda} = \sqrt{\Theta}, \tag{43}$$

with $\mathcal{R}(r)$ and $\Theta(\theta)$ being functions of the radial and angle part of the photon motion as

$$\Theta = \mathcal{K} + \cos^2\theta\left(a^2\mathcal{E}^2 - \frac{\mathcal{L}^2}{\sin^2\theta}\right) - (n^2 - 1)a^2\mathcal{E}^2\sin^2\theta, \tag{44}$$

$$\begin{aligned} \mathcal{R} = &[(r^2 + a^2)\mathcal{E} - a\mathcal{L}]^2 + (r^2 + a^2)^2(n^2 - 1)\mathcal{E}^2 \\ &- \Delta[\mathcal{K} + (\mathcal{L} - a\mathcal{E})^2]. \end{aligned} \tag{45}$$

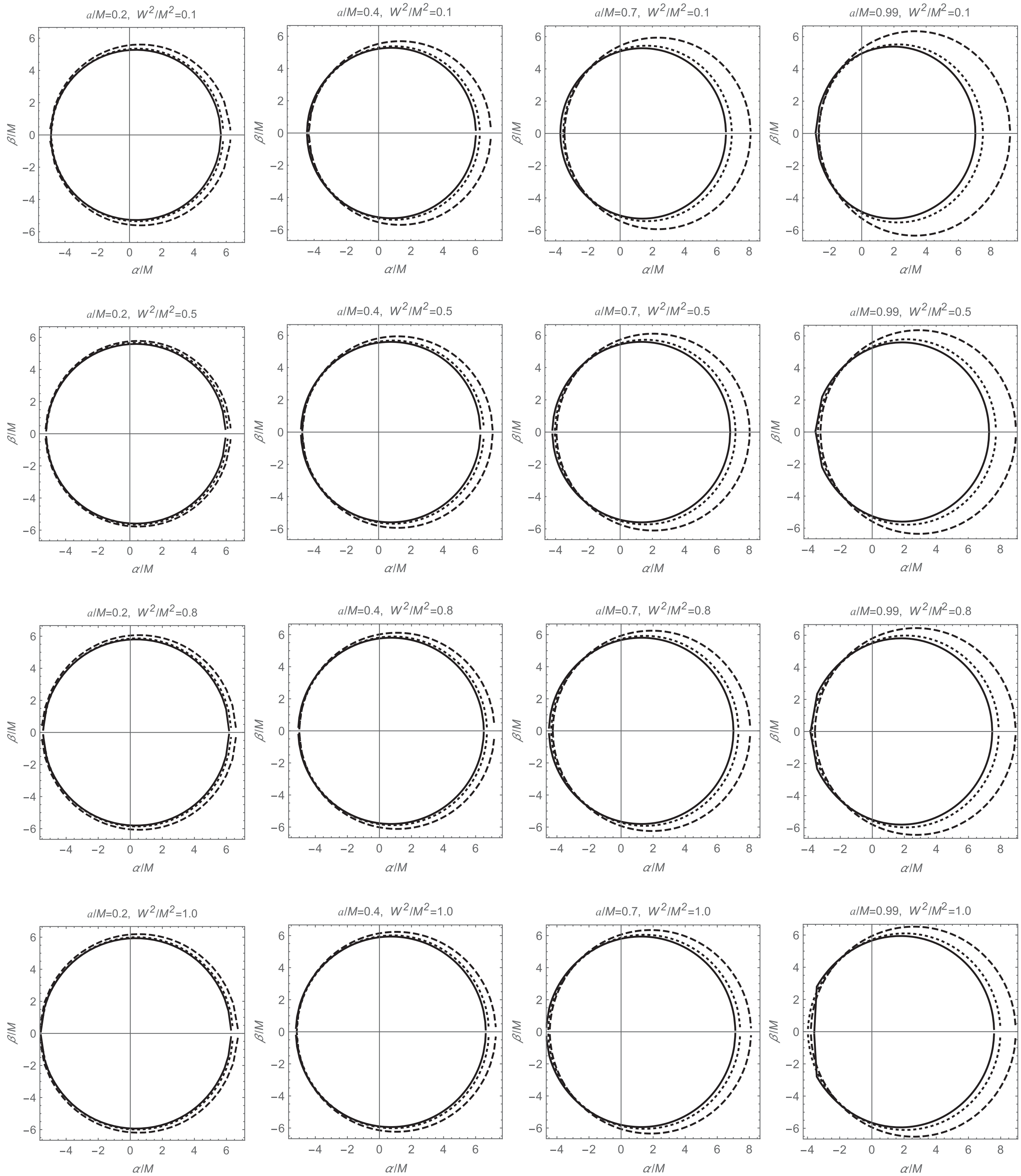

FIG. 6. The shadow of the rotating black hole in the braneworld surrounded by the plasma for the different values of the rotation parameter a, tidal charge W, and the refraction index. The solid lines in the plots correspond to the vacuum case, while for dotted and dashed lines we choose the plasma frequency $\omega_0/\omega_\infty = k/r$ and $(k/M)^2 = 0.5$ and $(k/M)^2 = 1$, respectively.

In Eqs. (40)–(43), $\mathcal{K}$ is the Carter constant, $\mathcal{E}$ and $\mathcal{L}$ are the energy and the angular momentum of the photon, respectively. It is very convenient to introduce new impact parameters $\xi = \mathcal{L}/\mathcal{E}$ and $\eta = \mathcal{K}/\mathcal{E}^2$ to describe the borders of the photon trajectory around the braneworld black hole.

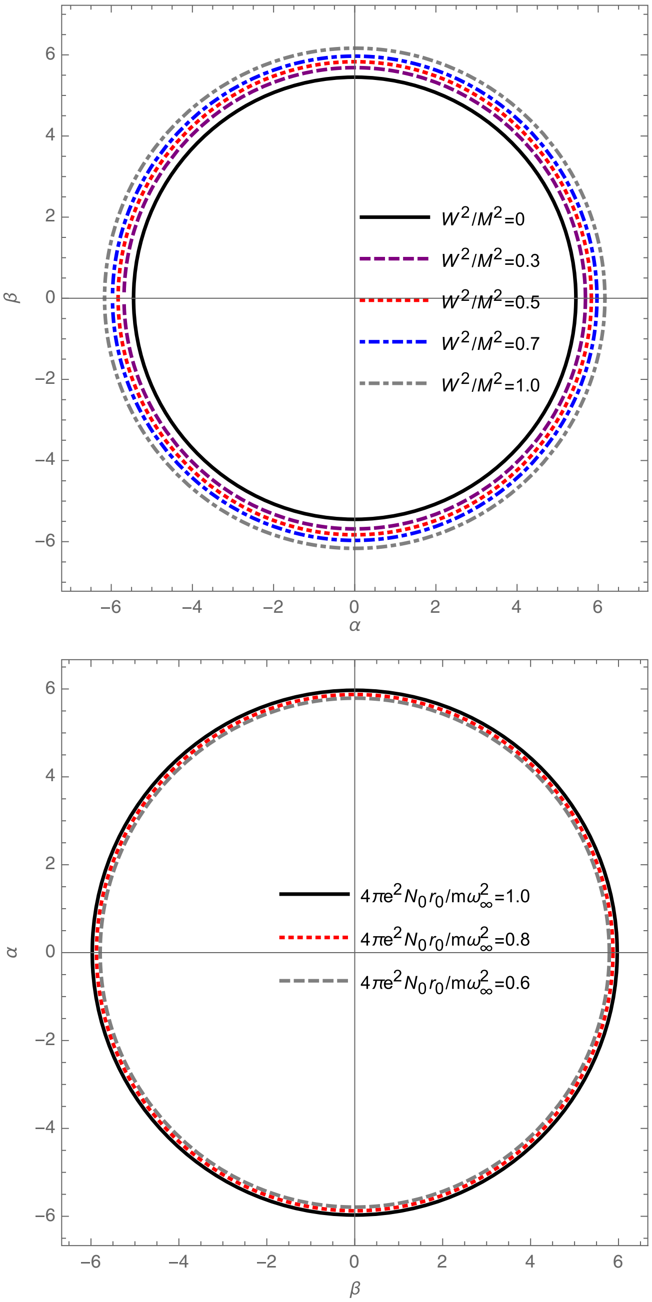

FIG. 7. Shadow of the black hole for the different values of the tidal charge W and plasma parameter. For the top plot, $4\pi e^2 N_0 r_0/(m\omega_\infty^2 M) = 0.7$; for the bottom plot, $W^2/M^2 = 0.5$.

In order to describe the shadow of the black hole, one can introduce celestial coordinates in the following way [69,82]:

$$\alpha = \lim_{r\to\infty}\left(-r^2 \sin\theta \frac{d\phi}{dr}\right), \tag{46}$$

$$\beta = \lim_{r_0\to\infty} r_0^2 \frac{d\theta}{dr}. \tag{47}$$

Using the equations of motion (40)–(43), one can find the form of α and β as

$$\alpha = -\frac{\xi}{n \sin\theta}, \tag{48}$$

$$\beta = \frac{\sqrt{\eta + a^2 - n^2 a^2 \sin^2\theta - \xi^2 \cot^2\theta}}{n}. \tag{49}$$

In Fig. 6, the dependence of the shape and size of the shadow of the rotating black hole in the braneworld in the plasma have been shown for the different values of the rotation parameter, brane tidal charge, and plasma refraction parameter. One can see that the increase of the brane tidal charge and plasma refraction index cause the increase of the shadow radius. Moreover, the plasma creates additional asymmetries in the shape of the shadow which could provide additional information from the analysis of the shadow [67].

In order to extract the pure effect of the Weyl charge and plasma, we start with the nonrotating black hole ($a = 0$) with the Weyl tidal charge pierced in the plasma. First, we

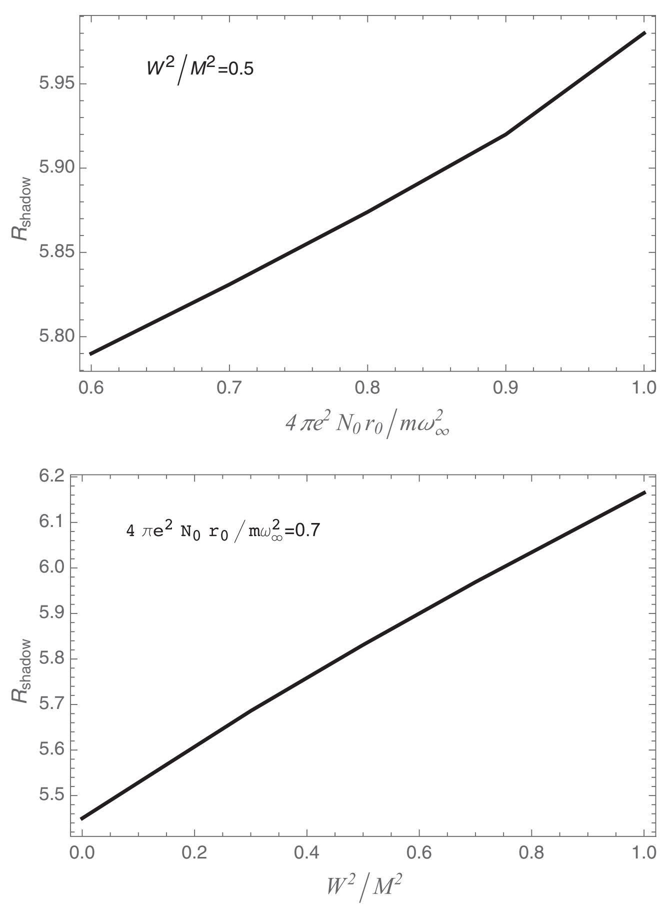

FIG. 8. Observational radius of the black hole shadow for the different values of the tidal charge W and plasma parameter.

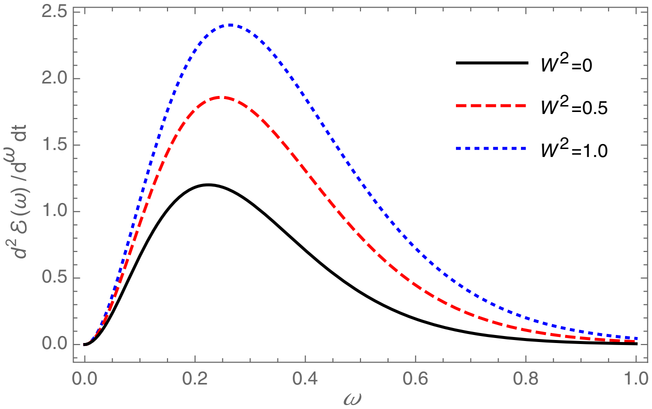

FIG. 9. Emission energy by the braneworld black hole for the different values of the tidal charge W.

introduce the radius of the black hole shadow $R_{\rm sh}$:

$$R_{\rm sh} = \alpha^2 + \beta^2 \tag{50}$$

as a new observable parameter.

One can get the radius of the shadow of the black hole with the Weyl charge in a plasma medium as

$$\begin{aligned}R_{\rm sh} = \frac{1}{2(r-1)^2 r^2}[&(3A-4)r^4 - (4+3A)r^5 + 4r^6 \\ &- 4r^3W^2 + 2\sqrt{2}r^3\{A(r-1)(4r-r^2+3W^2) \\ &+ r(2r-r^2+W^2)^2\}^{1/2}],\end{aligned} \tag{51}$$

where, for simplicity, we have set $M = 1$ and $A = 4\pi e^2 N_0 r_0/m\omega_\infty^2$.

In Figs. 7 and 8, the dependence of the black hole shadow size from the Weyl tidal charge and plasma parameters has been presented according to how the Weyl charge and plasma parameters increase the black hole shadow size.

A. Energy emission by braneworld black hole through thermal radiation

A black hole radiates energy thermally by the well-known Hawking process. Here we investigate the effect of the Weyl charge on this process and expect that it has to increase the rate of emission. The rate of energy emission is given by the formula [83]

$$\frac{d^2E(\omega)}{d\omega dt} = \frac{2\pi^2\sigma}{e^{\omega/T}-1}\omega^3, \tag{52}$$

where $\sigma \approx \pi R^2_{\rm shadow}$ is the cross section and $T = \frac{\sqrt{M^2-W^2}}{2\pi(M+\sqrt{M^2-W^2})}$ is the Hawking temperature. For the braneworld black hole spacetime given in Eqs. (1) and (2), we have plotted the energy emission rate in Fig. 9 for the different values of the tidal charge. It shows that the peak height increases with increasing tidal charge, and it occurs at increasing frequency.

VI. CONCLUSION

In this paper, we have studied optical properties that include gravitational lensing, weak/strong retrolensing, image magnification, and shadow and have computed the thermal energy radiation rate for a black hole on the brane. In the braneworld gravity model, the vacuum equation on the brane also has a contribution coming from the bulk spacetime as a projection of its Weyl curvature. This embodies the black hole with an additional charge, called the Weyl tidal charge, which appears in the metric exactly in the same way as the electric charge for the Reissner-Nordström metric but with the opposite sign. Thus, mass and tidal charge act in unison and, hence, it is expected that it should contribute positively to all gravitational properties. This is what we have set out to verify and have indeed established that the tidal charge enhances the effect in all the gravitational properties investigated.

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