General relativistic polytropes in anisotropic stars

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Spherically symmetric relativistic stars with a polytropic equation of state (EoS), which possess local pressure anisotropy, are considered within the framework of general relativity. The generalized Lane-Emden equations are derived for the arbitrary anisotropy parameter $\Delta = p_t - p_r$ (p_t and p_r being the transverse and radial pressure, respectively). They are then applied to some special ansatz for the anisotropy parameter in the form of a differential relation between the anisotropy parameter Δ and the metric function ν. The analytical solutions of the obtained equations are found for incompressible fluid stars and then used for getting their mass-radius relation, gravitational energy, and binding energy. Also, following the Chandrasekhar variational approach, the dynamical stability of incompressible anisotropic fluid stars with a polytropic EoS against radial oscillations is studied. It is shown that the local pressure anisotropy with $p_t > p_r$ can make the incompressible fluid stars unstable with respect to radial oscillations, in contrast to incompressible isotropic fluid stars with a polytropic EoS which are dynamically stable.

I. INTRODUCTION

It was argued in Ref. [1] that, despite the spherically symmetric distribution of matter inside a compact stellar object, it can be characterized by local pressure anisotropy. An analysis of the generalized equations of hydrostatic equilibrium, allowing for pressure anisotropy, shows that anisotropy may have a substantial effect on the maximum equilibrium mass and gravitational surface redshift [1–3]. At not too high densities, the impact of anisotropy can be studied within Newtonian gravity [4,5]. At higher densities ($\varrho \gtrsim 10^{15}$ g/cm^3), both the relativistic effects and the effects of general relativity become important [6–12]. In order to study the influence of the local pressure anisotropy on the specific basis, it is necessary to know the concrete physical reasons responsible for its appearance, such as, e.g., the existence of a solid core [6,7], occurrence of spontaneous deformation of Fermi surfaces [13,14], availability of super-fluid states with the finite orbital momentum of Cooper pairs [15–19] or finite superfluid momentum [20,21], or the presence of strong magnetic fields inside a star [22–34].

With account of the pressure anisotropy, the equation of state (EoS) of the system will also be necessarily aniso-tropic. The EoS is the essential ingredient in solving the equations of the hydrostatic equilibrium, and its importance was underlined in Ref. [35]. In the given work, we choose a polytropic EoS, which is widely used in many astrophysical applications [36–42]. For the polytropic index $n = 2$, the exact solutions of Einstein equations with pressure anisotropy were obtained in Ref. [43]. Anisotropic spheres with uniform energy density in general relativity were studied in Refs. [12,44], and those with variable energy density were studied in Ref. [45]. Note also that the exact

solutions of Einstein equations for spherical anisotropic stars with a linear EoS were obtained in Ref. [46] for some particular mass distribution.

In the present work, we will study spherically symmetric relativistic anisotropic stars with a polytropic EoS, aiming to obtain the generalized Lane-Emden equations for the special ansatz for the anisotropy parameter in the form of a differ-ential relation between the anisotropy parameter and the metric function ν. This approach is different from that suggested in Refs. [1,43] which consists in setting the anisotropy parameter, or some metric function, in a specific functional form. In the general case, the obtained Lane-Emden equations can be integrated only numerically, but the analytic solutions can be found for incompressible fluid stars. These solutions can then be used to get the stars' mass-radius relation, gravitational energy, and binding energy. Also, we apply the Chandrasekhar variational procedure [47] to study the dynamical stability of incompressible anisotropic fluid stars with respect to radial oscillations.

II. BASIC EQUATIONS

For spherically symmetric stars, the line element is written in the form

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where we use the system of units with $c = 1$. While the matter distribution inside a star is spherically symmetric, we allow the existence of the local pressure anisotropy in its interior with the different values of the radial p_r and transverse $p_\theta = p_\varphi \equiv p_t$ pressures. The anisotropic energy-momentum tensor for the static configuration reads

$$T_i^k = \text{diag}(\varepsilon, -p_r, -p_t, -p_t), \quad (2)$$

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where ε is the energy density of the system. The space-time geometry and matter distribution are related by Einstein equations:

$$R_i^k - \frac{1}{2} R\delta_i^k = 8\pi G T_i^k. \tag{3}$$

Given Eqs. (1) and (2), the Einstein equations read

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi G\varepsilon, \tag{4}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi G p_r, \tag{5}$$

$$\frac{1}{2} e^{-\lambda}\left(\nu'' - \frac{1}{2}\nu'\lambda' + \frac{1}{2}\nu'^2 + \frac{\nu' - \lambda'}{r}\right) = 8\pi G p_t, \tag{6}$$

where a prime denotes differentiation with respect to r. From Eqs. (4)–(6), or, equivalently, from the vanishing divergence of the energy-momentum tensor $T_{i;k}^k = 0$, one can get the equation for the hydrostatic equilibrium in the presence of the pressure anisotropy in the form

$$p_r' = -\frac{\nu'}{2}(\varepsilon + p_r) + \frac{2\Delta}{r}, \qquad \Delta \equiv p_t - p_r, \tag{7}$$

where Δ is the anisotropy parameter. The interior metric function λ can be found from Eq. (4):

$$e^{-\lambda(r)} = 1 - \frac{2G}{r} m(r), \qquad r < R, \tag{8}$$

where R is the radial coordinate at the surface of a star and $m(r)$ is the mass enclosed in the sphere of radius r:

$$m(r) = 4\pi \int_0^r \varepsilon r^2 dr. \tag{9}$$

From Eq. (5), given Eq. (8), one can find

$$\nu' = 2G\frac{m(r) + 4\pi p_r r^3}{r(r - 2Gm(r))}. \tag{10}$$

Hence, the equation of hydrostatic equilibrium for a spherically symmetric anisotropic star takes the form

$$p_r' = -G\frac{(\varepsilon + p_r)(m(r) + 4\pi p_r r^3)}{r(r - 2Gm(r))} + \frac{2\Delta}{r}, \tag{11}$$

which is the generalization of the Tolman-Oppenheimer-Volkov (TOV) equation taking into account the local pressure anisotropy. In order to solve Eq. (11), considered together with Eq. (9), it is necessary to set the EoS of the system, $\varepsilon = \varepsilon(p_r)$, and to use a specific model-dependent expression for Δ or some approximation for it. Further we will assume that any physical quantity or metric function in the interior of a star is free of singularities. As follows from Eq. (11), the gradient p_r' will be finite at $r = 0$, if, at least, $\Delta \propto r$ at $r \to 0$. As the boundary condition to Eq. (11), we will set the radial pressure at the center of a star

$p_r(0) = p_{r0}$, and we will determine the radial coordinate R at the surface from the condition $p_r(R) = 0$. The total mass then can be determined as $M = m(R)$, assuming that $m(0) = 0$. After finding the radial pressure distribution $p_r(r)$ [together with the mass distribution $m(r)$], the metric functions $\lambda(r)$ and $\nu(r)$ can be determined from Eqs. (8) and (10). At the boundary $r = R$, the metric functions are matchable to the exterior vacuum Schwarzschild metric:

$$\lambda(R) = -\nu(R) = -\ln\left(1 - \frac{2GM}{R}\right). \tag{12}$$

III. GENERALIZED LANE-EMDEN EQUATIONS

Further, as the EoS of the system, we choose the polytropic EoS in the form [37]

$$p_r = K\varrho^\gamma \equiv K\varrho^{1+\frac{1}{n}}, \tag{13}$$

where ϱ is the mass (baryon) density; K is some constant, which can be, in principle, temperature dependent; γ is the polytropic exponent and n is the polytropic index. Note that in some works [12,36,38,41] the polytropic EoS is set in the form $p_r = K\varepsilon^\gamma$, which will not be considered here. It is possible to show (see, e.g., Ref. [42]) that for the EoS (13) the energy density ε is related to the mass density ϱ and the radial pressure p_r by the equation

$$\varepsilon = \varrho + \frac{p_r}{\gamma - 1}. \tag{14}$$

It is convenient to introduce the auxiliary dimensionless Lane-Emden function θ according to the equations

$$p_r = p_{r0}\theta^{n+1}, \qquad \varrho = \varrho_0\theta^n, \tag{15}$$

where ϱ_0 is the central mass density. It follows from the boundary conditions for the radial pressure p_r that

$$\theta(0) = 1, \qquad \theta(R) = 0. \tag{16}$$

Then Eq. (7) of hydrostatic equilibrium can be rewritten as

$$2q_0(n+1)d\theta - \frac{4\Delta dr}{\varrho_0 r\theta^n} + (1 + (n+1)q_0\theta)d\nu = 0, \tag{17}$$

where $q_0 \equiv \frac{p_{r0}}{\varrho_0}$. This equation can be integrated to give

$$\nu = \nu_0 - \ln\left(\frac{1 + (n+1)q_0\theta}{1 + (n+1)q_0}\right)^2$$
$$+ \frac{4}{\varrho_0} \int_0^r \frac{\Delta dr}{r\theta^n(1 + (n+1)q_0\theta)},$$

$\nu_0 \equiv \nu(0)$ being the integration constant. In order to find it, one can use the boundary condition (12). This gives

$$\nu_0 = \ln\frac{1 - \frac{2GM}{R}}{(1 + (n+1)q_0)^2} - \frac{4}{\varrho_0} \int_0^R \frac{\Delta dr}{r\theta^n(1 + (n+1)q_0\theta)}.$$

Therefore, the metric function $\nu(r)$ can be written as

$$\nu(r) = \ln \frac{1 - \frac{2GM}{R}}{(1 + (n+1)q_0\theta)^2} - \frac{4}{\varrho_0} \int_r^R \frac{\Delta dr}{r\theta^n(1 + (n+1)q_0\theta)}. \tag{18}$$

Let us define the auxiliary function

$$u(r) \equiv \frac{m(r)}{M} = \frac{r}{2GM}\left(1 - e^{-\lambda(r)}\right), \quad u(0) = 0, u(R) = 1, \tag{19}$$

which, according to Eq. (9), satisfies the differential equation

$$Mu' = 4\pi\varepsilon r^2. \tag{20}$$

Then, substituting $e^{-\lambda}$ from Eq. (19) and the derivative ν' from Eq. (17) into Eq. (5), and using Eq. (20), one gets

$$\frac{q_0(n+1)\theta' r}{1 + (n+1)q_0\theta}\left(1 - \frac{2GM}{r}u\right) + \frac{GMq_0\theta}{1 + nq_0\theta}u'$$
$$+ \frac{GM}{r}u - \frac{2\Delta}{\varrho_0\theta^n(1 + (n+1)q_0\theta)}\left(1 - \frac{2GM}{r}u\right) = 0. \tag{21}$$

Defining the dimensionless variable ξ and dimensionless function η by the equations

$$r = \alpha\xi, \qquad \eta = \frac{M}{4\pi\varrho_0\alpha^3}u, \tag{22}$$

where $\alpha^2 = \frac{q_0(n+1)}{4\pi G\varrho_0}$, Eqs. (20) and (21) can be rewritten as

$$\frac{\xi - 2(n+1)q_0\eta}{1 + (n+1)q_0\theta}\left\{\xi\frac{d\theta}{d\xi} - \frac{2\Delta}{\varrho_0 q_0(n+1)\theta^n(1 + (n+1)q_0\theta)}\right\}$$
$$+ \eta + q_0\xi^3\theta^{n+1} = 0, \tag{23}$$

$$\frac{d\eta}{d\xi} = \xi^2\theta^n(1 + nq_0\theta). \tag{24}$$

As follows from Eqs. (16) and (22), the boundary conditions for the functions $\theta(\xi)$ and $\eta(\xi)$ read

$$\theta(0) = 1, \qquad \theta(\xi_R) = 0, \tag{25}$$

$$\eta(0) = 0, \qquad \eta(\xi_R) = \frac{M}{4\pi\varrho_0\alpha^3}, \tag{26}$$

where $\xi_R = R/\alpha$. Equations (23) and (24) represent the generalized Lane-Emden equations for relativistic anisotropic polytropes with the EoS (13), after solving which one can find from Eqs. (15) and (22) the radial distribution

of the radial pressure and mass in the interior of a spherically symmetric relativistic anisotropic star. At $\Delta = 0$, Eqs. (23) and (24) go over to the equations for the relativistic isotropic polytropes [37].

Note that for a given q_0, the radius and mass can be found as functions of the constant K in EoS (13),

$$R = R^* q_0^{\frac{1-n}{2}}\xi_R, \qquad M = M^* q_0^{\frac{3-n}{2}}\eta(\xi_R), \tag{27}$$

where the dependence on K goes through the quantities R^* and M^*:

$$R^* = \sqrt{\frac{n+1}{4\pi G}}K^{\frac{n}{2}}, \qquad M^* = \frac{1}{\sqrt{4\pi}}\left(\frac{n+1}{G}\right)^{\frac{3}{2}}K^{\frac{n}{2}}. \tag{28}$$

The mass-radius relation has the form

$$\frac{GM}{R} = (n+1)q_0\frac{\eta(\xi_R)}{\xi_R}. \tag{29}$$

The total energy of a star is

$$E = 4\pi\int_0^R \varepsilon r^2 dr = M^* q_0^{\frac{3-n}{2}}\eta(\xi_R). \tag{30}$$

The proper energy E_0 is the integral of the energy density over the proper spatial volume [36]:

$$E_0 = 4\pi\int_0^R \varepsilon e^{\frac{\lambda}{2}}r^2 dr$$
$$= 4\pi\varrho_0\alpha^3\int_0^{\xi_R}(1 + nq_0\theta)\frac{\xi^2\theta^n}{\sqrt{1 - \frac{2q_0(n+1)}{\xi}\eta(\xi)}}d\xi. \tag{31}$$

The gravitational potential energy Ω is determined as

$$\Omega = E - E_0 = M^* q_0^{\frac{3-n}{2}}\eta(\xi_R)\left(1 - \frac{1}{\eta(\xi_R)}\int_0^{\xi_R}(1 + nq_0\theta)\right.$$
$$\left.\times\frac{\xi^2\theta^n}{\sqrt{1 - \frac{2q_0(n+1)}{\xi}\eta(\xi)}}d\xi\right). \tag{32}$$

The binding energy is the difference between the energy of the particles scattered to infinity and the total energy of the system [36]:

$$E_B = E_{0g} - E, \qquad E_{0g} = 4\pi\int_0^R \varrho e^{\frac{\lambda}{2}}r^2 dr, \tag{33}$$

which reads

$$E_B = M^* q_0^{\frac{3-n}{2}}\eta(\xi_R)\left(\frac{u_g(\xi_R)}{\eta(\xi_R)} - 1\right),$$
$$u_g(\xi) = \int_0^\xi \frac{\xi^2\theta^n}{\sqrt{1 - \frac{2q_0(n+1)}{\xi}\eta(\xi)}}d\xi. \tag{34}$$

IV. ANALYTICAL SOLUTIONS AND NUMERICAL RESULTS FOR INCOMPRESSIBLE ANISOTROPIC FLUID STARS

In order to solve the generalized Lane-Emden Eqs. (23), (24) for the functions θ, η with the boundary conditions (25), (26), one needs to specify the anisotropy parameter Δ. This can be done within the given model framework with the concrete physical mechanism responsible for the appearance of the local pressure anisotropy. The other approach to study the effects of the pressure anisotropy is to set the anisotropy parameter Δ in a phenomenological way, with the use of some phenomenological ansatz for the anisotropy parameter. This approach allows us to study the general properties of spherically symmetric relativistic anisotropic stars and will be followed in the given research. The conclusions obtained within this approach are rather of a general character and independent of the details of a specific physical mechanism.

In fact, we will follow the point of view, suggested in Refs. [5,48], which consists in setting some additional differential relation between the unknown functions. Namely, we will suppose that the presence of the anisotropy parameter Δ does not change the general form of Lane-Emden equations for relativistic isotropic stars [37], but can only change the coefficients in these equations. Specifically, we will assume that the parameter Δ and the metric function ν are related by the differential equation

$$-\frac{4\Delta dr}{\varrho_0 r\theta^n} + (1 + (n+1)q_0\theta)d\nu = (1 + \beta q_0\theta)d\nu, \quad (35)$$

where β is some real constant. Substituting Eq. (35) into Eq. (17) and integrating it, one can obtain the metric function $\nu(r)$ in the form

$$\nu(r) = \ln\frac{1 - \frac{2GM}{R}}{(1 + \beta q_0\theta)^{\frac{2(n+1)}{\beta}}}. \quad (36)$$

Using Eqs. (8) and (36), and introducing the same dimensionless variable ξ and dimensionless function η as in Eq. (22), we have the modified Lane-Emden equations, corresponding to the ansatz (35), in the form

$$\frac{\xi - 2(n+1)q_0\eta}{1 + \beta q_0\theta}\xi\frac{d\theta}{d\xi} + \eta + q_0\xi^3\theta^{n+1} = 0, \quad (37)$$

$$\frac{d\eta}{d\xi} = \xi^2\theta^n(1 + nq_0\theta), \quad (38)$$

with the same boundary conditions (25) and (26). One can see that the obtained Lane-Emden equations formally look as in the isotropic case [37], but with the difference that the impact of the anisotropy parameter is reflected in the coefficient β [substituting the multiplier $(n+1)$].

In the general case, the Lane-Emden Eqs. (37) and (38) can be integrated only numerically, but the analytical solutions can be found for incompressible anisotropic fluid stars, characterized by the constant density $\varrho = $ const. At $n = 0$, the function $\eta(\xi)$, with account of the boundary condition $\eta(0) = 0$, is given by

$$\eta(\xi) = \frac{\xi^3}{3}. \quad (39)$$

The solution for the function $\theta(\xi)$ reads

$$\frac{1 + 3q_0\theta}{1 + \beta q_0\theta} = \pm\frac{1 + 3q_0}{1 + \beta q_0}\left(1 - \frac{2q_0}{3}\xi^2\right)^{\frac{3-\beta}{4}}. \quad (40)$$

In the last equation, only the branch corresponding to the upper plus sign satisfies the boundary conditions (25), and the respective solution is given by

$$\theta(\xi) = \frac{1}{q_0}\frac{(1 + 3q_0)(1 - \frac{2q_0}{3}\xi^2)^{\frac{3-\beta}{4}} - (1 + \beta q_0)}{3(1 + \beta q_0) - \beta(1 + 3q_0)(1 - \frac{2q_0}{3}\xi^2)^{\frac{3-\beta}{4}}}. \quad (41)$$

The positive root of $\theta(\xi)$ reads

$$\xi_R = \sqrt{\frac{3}{2q_0}\left[1 - \left(\frac{1 + \beta q_0}{1 + 3q_0}\right)^{\frac{4}{3-\beta}}\right]}. \quad (42)$$

It is possible to check that the subradical function is non-negative at any β satisfying the inequality $1 + \beta q_0 > 0$.

Calculating the function $u_g(\xi)$ in Eq. (34) at $n = 0$, the binding energy of incompressible anisotropic fluid stars can be written as

$$E_B = M^*\left\{-\frac{3}{4}\xi_R\sqrt{q_0\left(1 - \frac{2q_0}{3}\xi_R^2\right)}\right.$$
$$\left. + \frac{1}{2}\sqrt{\left(\frac{3}{2}\right)^3}\arcsin\left(\sqrt{\frac{2q_0}{3}}\xi_R\right) - \frac{1}{3}(\sqrt{q_0}\xi_R)^3\right\}. \quad (43)$$

It can be verified that at $n = 0$ the gravitational potential energy is just $\Omega = -E_B$.

Using the obtained analytical results for incompressible anisotropic fluid stars, we can also represent the basic quantities of interest in a graphical form. From Eq. (35), one can find the anisotropy parameter Δ at $n = 0$ in terms of the dimensionless variable ξ:

$$\Delta(\xi) = \frac{p_{r0}\xi}{4}(1 - \beta)\theta(\xi)\frac{\partial\nu}{\partial\xi}. \quad (44)$$

The radial pressure p_r, in turn, is determined from Eq. (15). Figure 1 shows the ratios $\Delta(\xi)/p_{r0}$, $p_r(\xi)/p_{r0}$, and $p_t(\xi)/p_{r0}$ at some fixed values of the parameter $q_0 = \frac{p_{r0}}{\varrho_0}$
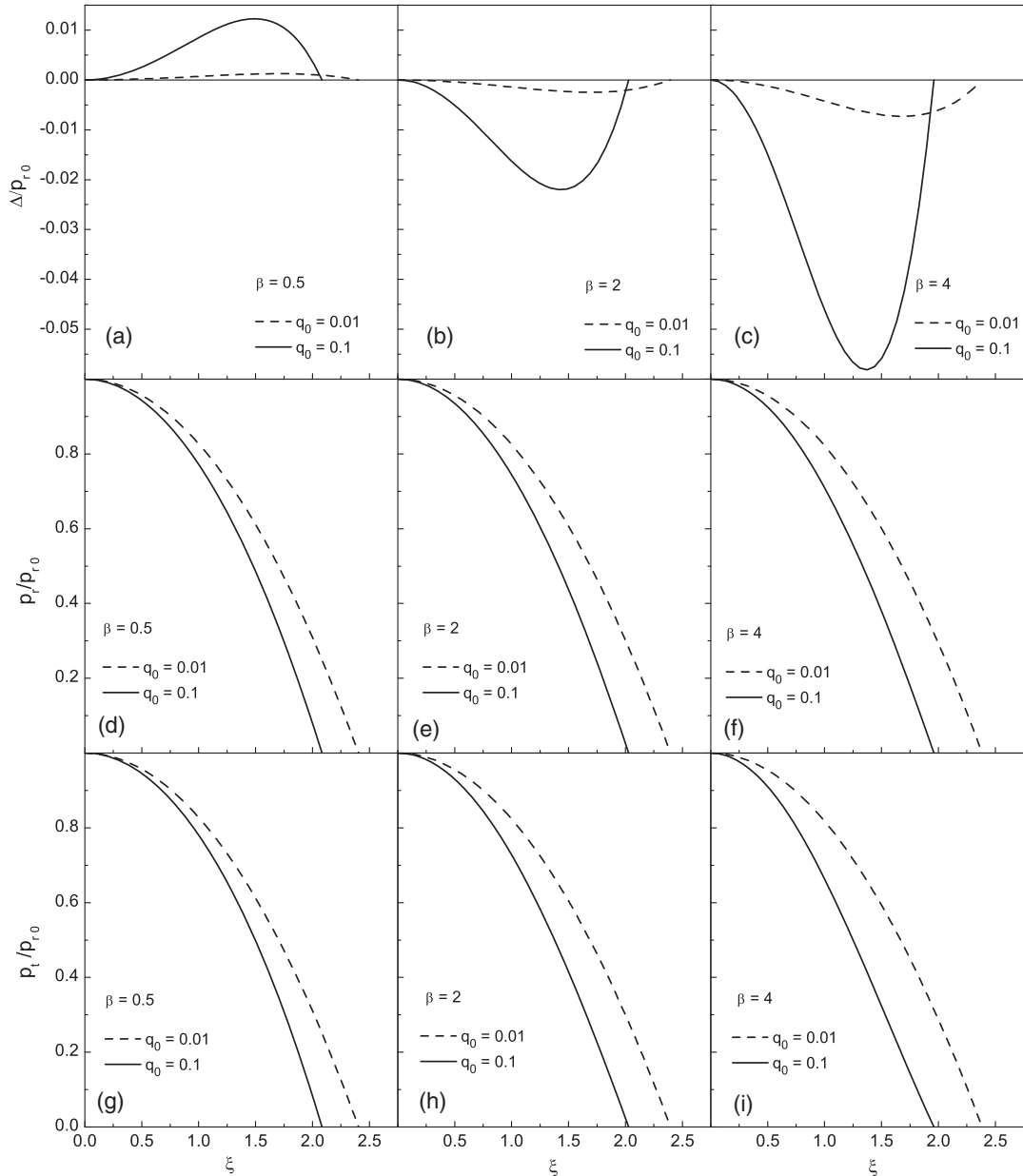
FIG. 1.   The anisotropy parameter Δ (upper row), radial pressure p_r (middle row), and transverse pressure p_t (bottom row) at $n = 0$, normalized to the central radial pressure p_{r0}, as functions of the dimensionless variable ξ for a set of fixed values of β and q_0.

and parameter β, characterizing the impact of the pressure anisotropy. The important peculiarity of our model ansatz (35) is that the anisotropy parameter Δ vanishes both in the center and at the surface of a star. At $\beta < 1$, Δ is positive, and at $\beta > 1$, the opposite is true. The absolute value of Δ increases with q_0.

The radial pressure p_r gradually decreases from its maximum value p_{r0} in the center till it vanishes at the surface of a star. Also, p_r decreases with both increasing β and increasing q_0. Qualitatively the same behavior is demonstrated by the transverse pressure p_t. The important feature of the model ansatz (35) is that not only p_r, but also p_t is positive in the interior of a star and vanishes at its

surface. The positiveness of the radial p_r and transverse p_t pressures in the interior of a star guarantees its mechanical stability. If some of these pressures becomes negative, like the radial pressure p_r in ultrastrong magnetic fields, this leads to the appearance of the corresponding instability; in the case of strong magnetic fields, this is the longitudinal instability developed along the magnetic field direction [22–24,26,27,28,29]. Vanishing of the transverse pressure p_t at the surface of anisotropic fluid stars is also important for their stability [24,25,30,31], although in some studies this is not a required feature [1,11,12].

Figure 2 shows the dimensionless mass M/M^* and dimensionless binding energy E_B/M^* for incompressible
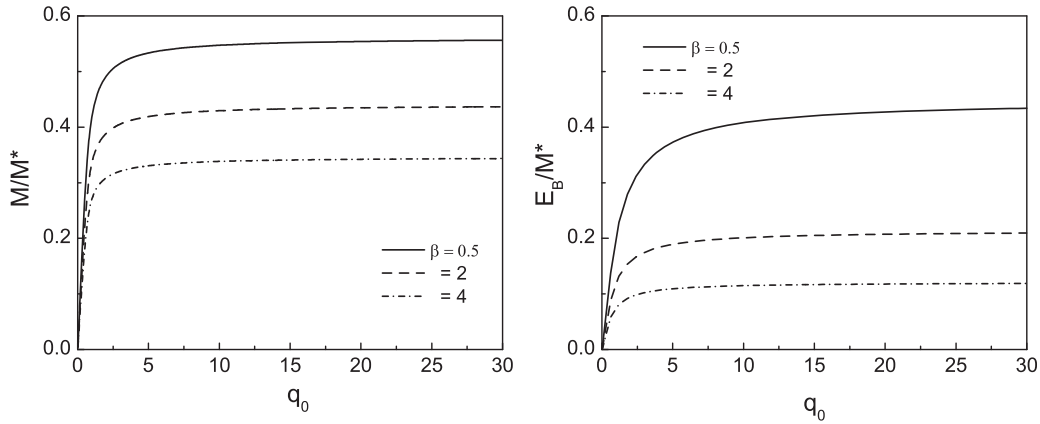
FIG. 2.   The dimensionless mass M/M^* and dimensionless binding energy E_B/M^* at $n = 0$ as functions of the parameter q_0 for the same set of fixed values of the parameter β as in Fig. 1.

anisotropic fluid stars, determined according to Eqs. (27) and (43), respectively, as functions of the parameter q_0 for the set of fixed values of the parameter β. It is seen that both quantities first rapidly increase with q_0 and then gradually approach their asymptotic values, dependent on the given value of β. The mass of a star decreases with increasing the parameter β at the given q_0, and, hence, the pressure anisotropy with $\beta < 1$ ($p_t > p_r$) leads to the increase of the mass of a star compared to that in the isotropic case. The binding energy stays always positive, as required by the stability of a star (although not all stars with $E_B > 0$ are stable, as will be shown in the next section). The gravitational potential energy at $n = 0$ differs only by sign from the binding energy, $\Omega/M^* = -E_B/M^*$, and is always negative.
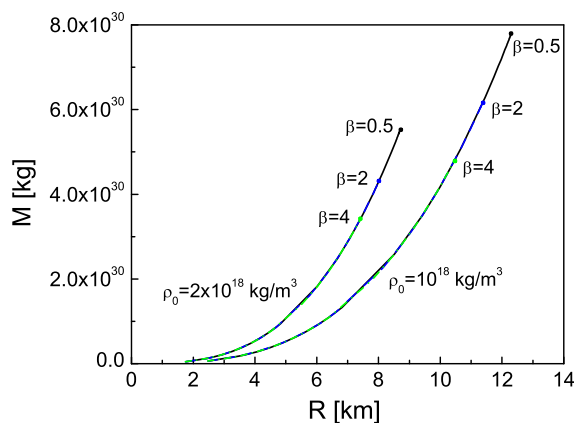
Figure 3 shows the mass-radius relation at $n = 0$ for two values of the central mass density, $\varrho_0 = 10^{18}$ kg/m³ and $\varrho_0 = 2 \times 10^{18}$ kg/m³, obtained by varying the parameter q_0 at fixed values of the parameter β. For the specific central

mass density, the masses and radii, corresponding to different fixed q_0 and β, are different. Nevertheless, when q_0 varies at the given β, the current point moves along almost the same curve for this specific central density independently of β, the difference being only in the limiting masses for different β. Note that the maximum mass for incompressible anisotropic fluid stars with a polytropic EoS cannot be reached at finite q_0, but only asymptotically at $q_0 \to \infty$. This is in contrast to the common behavior when a mass-radius curve reaches a maximum value and then decreases (cf. the mass-radius curves, for example, in Refs. [49,50] for strange quark stars). This result does not rely on any numerical procedure. It is just a reflection in the graphical form of the mass-radius relation, given by Eq. (27), with the analytically found solutions of the modified Lane-Emden Eqs. (37) and (38) at $n = 0$, where the quantities $\eta(\xi)$, ξ_R are given by Eqs. (39) and (42).

The recent discovery of two heavy neutron stars PSR J1614-2230 [51] and PSR J0348 + 0432 [52] with the masses $M \sim 2\ M_\odot$ ($M_\odot \approx 1.989 \times 10^{30}$ kg being the solar mass) set the corresponding constraint, to which any reasonable EoS should satisfy. It is seen from Fig. 3 that, despite its simplicity, the polytropic EoS for incompressible anisotropic fluid stars can fulfill this constraint even without reaching the maximum masses.

V. DYNAMICAL STABILITY OF INCOMPRESSIBLE ANISOTROPIC FLUID STARS

Let us consider the stability of spherically symmetric anisotropic stars with respect to radial oscillations, assuming that they do not violate the spherical symmetry. In the spherically symmetric case, allowing for the motions in the radial direction, Einstein Eqs. (3) read

FIG. 3.   The mass-radius relation at $n = 0$ for two values of the central mass density, $\varrho_0 = 10^{18}$ kg/m³ and $\varrho_0 = 2 \times 10^{18}$ kg/m³, with the variable parameter q_0 and fixed values of the parameter β: $\beta = 0.5$ (black curves), $\beta = 2$ (blue curves), and $\beta = 4$ (green curves). The limiting masses for each configuration are shown by full dots.

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi G T_0^0, \qquad (45)$$

$$-e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi G T_1^1, \qquad (46)$$

$$-\frac{1}{2}e^{-\lambda}\left(\nu'' - \frac{1}{2}\nu'\lambda' + \frac{1}{2}\nu'^2 + \frac{\nu' - \lambda'}{r}\right)$$
$$+\frac{1}{2}e^{-\nu}\left(\ddot\lambda + \dot\lambda(\dot\lambda - \dot\nu)\right) = 8\pi G T_2^2 = 8\pi G T_3^3, \quad (47)$$

$$-\frac{e^{-\lambda}}{r}\dot\lambda = 8\pi G T_0^1, \quad (48)$$

where a dot means differentiation with respect to t. The radial component of the equation $T_{i;k}^k = 0$, expressing the vanishing of the divergence of the energy-momentum tensor, can be written as

$$\dot T_1^0 + T_1^{1\prime} + \frac{1}{2}T_1^0(\dot\nu + \dot\lambda) + \frac{\nu'}{2}(T_1^1 - T_0^0) + \frac{2}{r}(T_1^1 - T_2^2) = 0. \quad (49)$$

The energy-momentum tensor for a spherically symmetric anisotropic star reads

$$T_i^k = (\varepsilon + p_t)u_i u^k - p_t\delta_i^k + (p_r - p_t)s_i s^k, \quad (50)$$

where $u^i = \frac{dx^i}{ds}$ is the fluid four-velocity and s^i is the unit spacelike vector with the properties

$$s^i u_i = 0, s^i s_i = -1. \quad (51)$$

For the motions in the radial direction, the four-vectors u^i and s^i have the structure

$$u^i = \left(\frac{e^{-\frac{\nu}{2}}}{\sqrt{1 - v^2 e^{\lambda-\nu}}}, \frac{v e^{-\frac{\nu}{2}}}{\sqrt{1 - v^2 e^{\lambda-\nu}}}, 0, 0\right), \quad (52)$$

$$s^i = \left(\frac{v e^{\frac{\lambda}{2}-\nu}}{\sqrt{1 - v^2 e^{\lambda-\nu}}}, \frac{e^{-\frac{\lambda}{2}}}{\sqrt{1 - v^2 e^{\lambda-\nu}}}, 0, 0\right), \quad (53)$$

where $v = \frac{dr}{dt}$ is the velocity in the radial direction. In the static limit, Eq. (50) for the energy-momentum tensor goes over to Eq. (2).

Further we will study the small radial oscillations and will represent the unknown quantities as

$$\varepsilon = \varepsilon^0 + \delta\varepsilon, \qquad p_r = p_r^0 + \delta p_r, \qquad p_t = p_t^0 + \delta p_t, \quad (54)$$

$$\nu = \nu^0 + \delta\nu, \qquad \lambda = \lambda^0 + \delta\lambda, \quad (55)$$

where $\delta\varepsilon$, δp_r, δp_t, $\delta\nu$, $\delta\lambda$ are small perturbations with respect to the corresponding values at the state of hydrostatic equilibrium, denoted by the upper index 0. Also, for the small radial oscillations we will consider that $v \ll 1$. In the linear approximation on the small perturbations, for the

nonzero components of the energy-momentum tensor we have

$$T_0^0 = \varepsilon, \qquad T_1^1 = -p_r, \qquad T_2^2 = T_3^3 = -p_t,$$
$$T_0^1 = (\varepsilon^0 + p_r^0)v, \qquad T_1^0 = -e^{\lambda^0-\nu^0}T_0^1, \quad (56)$$

where the quantities ε, p_r, and p_t are given by Eq. (54). Given Eq. (56), Eqs. (45) and (46) can be written in the linearized form

$$\frac{\partial}{\partial r}(re^{-\lambda^0}\delta\lambda) = 8\pi G r^2 \delta\varepsilon, \quad (57)$$

$$\frac{1}{r}e^{-\lambda^0}(\delta\nu' - \nu^{0\prime}\delta\lambda) = \frac{1}{r^2}e^{-\lambda^0}\delta\lambda + 8\pi G\delta p_r. \quad (58)$$

The linearized form of Eq. (48) is

$$-\frac{e^{-\lambda^0}}{r}\dot{\delta\lambda} = 8\pi G(\varepsilon^0 + p_r^0)v. \quad (59)$$

In the linear approximation, Eq. (49) reads

$$-e^{\lambda^0-\nu^0}(\varepsilon^0 + p_r^0)\dot v - \delta p_r' - \frac{1}{2}(\varepsilon^0 + p_r^0)\delta\nu' - \frac{1}{2}(\delta\varepsilon + \delta p_r)\nu^{0\prime}$$
$$+\frac{2}{r}(-\delta p_r + \delta p_t) = 0. \quad (60)$$

Following Ref. [47], it is convenient to introduce a "Lagrange displacement" ψ by the equation $v = \dot\psi$. Then integration of Eq. (59) gives

$$\frac{1}{r}e^{-\lambda_0}\delta\lambda = -8\pi G(\varepsilon^0 + p_r^0)\psi. \quad (61)$$

Note that in the state of hydrostatic equilibrium, as a consequence of Eqs. (4) and (5), the following relationship holds true:

$$\frac{1}{r}e^{-\lambda_0}(\nu^{0\prime} + \lambda^{0\prime}) = 8\pi G(\varepsilon^0 + p_r^0). \quad (62)$$

With account of the last equation, Eq. (61) becomes

$$\delta\lambda = -\psi(\nu^{0\prime} + \lambda^{0\prime}). \quad (63)$$

Taking into account Eq. (61), Eq. (57) reads

$$\delta\varepsilon = -\psi\varepsilon^{0\prime} - \psi p_r^{0\prime} - (\varepsilon^0 + p_r^0)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\psi). \quad (64)$$

In view of the condition of hydrostatic equilibrium (7), the last equation can be written as

$$\delta\varepsilon = -\psi\varepsilon^{0\prime} - (\varepsilon^0 + p_r^0)\frac{e^{\frac{\nu_0}{2}}}{r^2}\frac{\partial}{\partial r}(r^2 e^{\frac{-\nu_0}{2}}\psi) - \frac{2\psi}{r}(p_t^0 - p_r^0). \quad (65)$$

Next, considering Eq. (58), given Eqs. (61) and (62), we get

$$(\varepsilon^0 + p_r^0)\delta\nu' = \left(\delta p_r - (\varepsilon^0 + p_r^0)\left(\nu^{0\prime} + \frac{1}{r}\right)\psi\right)(\nu^{0\prime} + \lambda^{0\prime}).$$

$$(66)$$

Now let us assume that all perturbations contain the dependence on time only through the exponential factor $e^{i\omega t}$, where ω is the frequency of radial oscillations. Then, taking Eq. (66) into account, Eq. (60) can be written as

$$\omega^2 e^{\lambda^0 - \nu^0}(\varepsilon^0 + p_r^0)\psi = \delta p_r' + \left(\nu^{0\prime} + \frac{\lambda^{0\prime}}{2}\right)\delta p_r$$
$$+ \frac{\nu^{0\prime}}{2}\delta\varepsilon - \frac{1}{2}(\varepsilon^0 + p_r^0)\left(\nu^{0\prime} + \frac{1}{r}\right)$$
$$\times (\nu^{0\prime} + \lambda^{0\prime})\psi + \frac{2}{r}(\delta p_r - \delta p_t),$$

$$(67)$$

where all variations now stand for the amplitudes of the corresponding quantities with the time dependence given by the exponent $e^{i\omega t}$. Note that the variation $\delta\varepsilon$ is expressed through the Lagrange displacement ψ by Eq. (64). In order to relate the perturbation of the radial pressure δp_r to ψ, we will assume, following Ref. [47], the conservation of the total baryon number. The corresponding continuity equation in general relativity reads

$$\frac{\partial}{\partial x^k}(n_b u^k) + n_b u^k \frac{\partial \ln \sqrt{-g}}{\partial x^k} = 0, \qquad (68)$$

where n_b is the baryon number density, g is the determinant of the metric tensor, $g \equiv \det ||g_{ik}|| = -\exp(\nu + \lambda)r^4 \sin^2\theta$. Taking into account Eq. (52), the last equation in the linear approximation on the small perturbations takes the form

$$e^{-\frac{\nu^0}{2}}\dot{\delta n_b} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(n_b^0 e^{-\frac{\nu^0}{2}}vr^2\right) + \frac{1}{2}n_b^0 e^{-\frac{\nu^0}{2}}\dot{\delta\lambda}$$
$$+ \frac{1}{2}n_b^0 e^{-\frac{\nu^0}{2}}v(\nu^{0\prime} + \lambda^{0\prime}) = 0, \qquad (69)$$

where $n_b^0(r)$ is the baryon number density in the state of hydrostatic equilibrium. Since $v = \dot\psi$, this equation can be integrated and one gets

$$\delta n_b + \frac{e^{\frac{\nu^0}{2}}}{r^2}\frac{\partial}{\partial r}(n_b^0 e^{-\frac{\nu^0}{2}}r^2\psi) + \frac{1}{2}n_b^0(\delta\lambda + \psi(\nu^{0\prime} + \lambda^{0\prime})) = 0.$$

$$(70)$$

In view of Eq. (63), the last term in Eq. (70) vanishes and one obtains

$$\delta n_b = -\frac{e^{\frac{\nu^0}{2}}}{r^2}\frac{\partial}{\partial r}(n_b^0 e^{-\frac{\nu^0}{2}}r^2\psi). \qquad (71)$$

Let the EoS of the system have the following general structure: $n_b = n_b(\varepsilon, p_r)$. Then it follows from Eqs. (65) and (71) that in the linear approximation

$$\delta p_r = -p_r^{0\prime}\psi - \gamma p_r^0 \frac{e^{\frac{\nu^0}{2}}}{r^2}\frac{\partial}{\partial r}(e^{-\frac{\nu^0}{2}}r^2\psi) - \frac{2}{r}\frac{\partial p_r}{\partial\varepsilon}(p_t^0 - p_r^0)\psi,$$

$$(72)$$

where the adiabatic coefficient γ is determined by

$$\gamma = \frac{1}{p_r\frac{\partial n_b}{\partial p_r}}\left(n_b - (\varepsilon + p_r)\frac{\partial n_b}{\partial\varepsilon}\right) \qquad (73)$$

and, analogously to Ref. [47], is considered to be a constant for the matter inside a star. For the polytropic EoS (14), $\frac{\partial p_r}{\partial\varepsilon} = \gamma - 1$. Substituting expressions (65) and (72) for $\delta\varepsilon$ and δp_r into Eq. (67), and using the field Eq. (6) and equation of hydrostatic equilibrium (7), in the case of a polytropic EoS, one gets

$$\omega^2 e^{\lambda^0 - \nu^0}(\varepsilon^0 + p_r^0)\psi = \frac{2\psi}{r}p_r^{0\prime} - \frac{2\psi}{r}\left(\gamma\left(\nu^{0\prime} + \frac{\lambda^{0\prime}}{2} + \frac{2}{r}\right) + \frac{2}{r}\right)(p_t^0 - p_r^0)$$
$$+ 8\pi G e^{\lambda^0}p_t^0(\varepsilon^0 + p_r^0)\psi - \gamma\frac{d}{dr}\left(\frac{2}{r}(p_t^0 - p_r^0)\psi\right) - \frac{\psi}{\varepsilon^0 + p_r^0}\left(p_r^{0\prime} - \frac{2}{r}(p_t^0 - p_r^0)\right)^2$$
$$- \gamma e^{-(\nu^0 + \frac{\lambda^0}{2})}\frac{d}{dr}\left(e^{\frac{3\nu^0 + \lambda^0}{2}}\frac{p_r^0}{r^2}\frac{d}{dr}(r^2 e^{-\frac{\nu^0}{2}}\psi)\right) - \frac{2}{r}\left(\gamma p_r^0 \frac{e^{\frac{\nu^0}{2}}}{r^2}\frac{d}{dr}(r^2 e^{-\frac{\nu^0}{2}}\psi) + \delta p_t\right). \qquad (74)$$

Solutions of Eq. (74) for the frequencies of radial oscillations should be sought under the boundary conditions

$$\psi(r = 0) = 0, \qquad \delta p_r(r = R) = 0. \qquad (75)$$

In order to get the variational basis for finding the frequencies ω, let us multiply both parts of Eq. (74) on $r^2\psi \exp(\frac{\nu^0 + \lambda^0}{2})$ and integrate over the range of r. We will write the corresponding equation already for incompressible fluid stars ($n = 0$), when the polytropic exponent $\gamma \to \infty$. Omitting the upper indices zero as no longer necessary, one gets

$$\omega^2 \int_0^R e^{\frac{3\lambda-\nu}{2}}(\varepsilon+p_r)r^2\psi^2 dr = \gamma \int_0^R e^{\frac{\lambda+3\nu}{2}}\frac{p_r}{r^2}\left(\frac{d}{dr}(r^2 e^{-\frac{\nu}{2}}\psi)\right)^2 dr - \gamma \int_0^R e^{\frac{\lambda+\nu}{2}}r^2\psi\frac{d}{dr}\left(\frac{2}{r}(p_t-p_r)\psi\right)dr$$

$$- 2\gamma \int_0^R e^{\frac{\lambda+\nu}{2}}r\psi^2\left(\nu'+\frac{\lambda'}{2}+\frac{2}{r}\right)(p_t-p_r) - 2\gamma \int_0^R e^{\frac{\lambda}{2}+\nu}\psi\frac{p_r}{r}\frac{d}{dr}(r^2 e^{-\frac{\nu}{2}}\psi)dr. \quad (76)$$

In the variational Eq. (76), the Lagrange displacement ψ should be chosen such that ω^2 is minimized. If all frequencies of radial oscillations are real, a spherically symmetric anisotropic star is dynamically stable; if some frequency appears to be imaginary, the configuration is unstable. A sufficient condition for the occurrence of the dynamical instability is vanishing of the right-hand side of Eq. (76) for some trial form of the Lagrange displacement ψ satisfying the boundary conditions.

Let us introduce, following Ref. [37], the auxiliary function $\chi = e^{-\frac{\nu}{2}}\psi$. After changing the integration variable in Eq. (76) according to Eq. (22), substituting $p_r = q_0\varrho_0\theta$ and $\varepsilon = \varrho_0$, and using Eq. (44) for the anisotropy parameter $\Delta = p_t - p_r$ and expressions for the metric functions at $n = 0$

$$e^{-\lambda} = 1 - \frac{2q_0\eta(\xi)}{\xi} = 1 - \frac{2q_0\xi^2}{3}, \quad e^\nu = \frac{1-\frac{2GM}{R}}{(1+\beta q_0\theta)^{\frac{2}{\beta}}}, \quad (77)$$

Eq. (76) takes the form

$$\frac{\omega^2}{\omega_0^2}\frac{1}{1-\frac{2GM}{R}}\int_0^{\xi_R}\frac{(1+q_0\theta)\xi^2\chi^2}{(1-\frac{2q_0\xi^2}{3})^{\frac{3}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{1}{\beta}}}$$

$$= \gamma \int_0^{\xi_R}\frac{\theta(\frac{d}{d\xi}(\xi^2\chi))^2}{\xi^2(1-\frac{2q_0\xi^2}{3})^{\frac{1}{2}}(1+\beta q_0\theta)^{\frac{3}{\beta}}}d\xi$$

$$- \frac{\gamma(1-\beta)}{2}\int_0^{\xi_R}\frac{\xi^2\chi}{(1-\frac{2q_0\xi^2}{3})^{\frac{1}{2}}}\frac{d}{d\xi}\left(\frac{\nu'\chi\theta}{(1+\beta q_0\theta)^{\frac{1}{\beta}}}\right)\frac{d\xi}{(1+\beta q_0\theta)^{\frac{2}{\beta}}}$$

$$- \frac{\gamma(1-\beta)}{2}\int_0^{\xi_R}\frac{\xi^2\chi^2\nu'\theta(\nu'+\frac{\lambda'}{2}+\frac{2}{\xi})}{(1-\frac{2q_0\xi^2}{3})^{\frac{1}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{3}{\beta}}}$$

$$- 2\gamma \int_0^{\xi_R}\frac{\chi\theta\frac{d}{d\xi}(\xi^2\chi)}{\xi(1-\frac{2q_0\xi^2}{3})^{\frac{1}{2}}(1+\beta q_0\theta)^{\frac{3}{\beta}}}d\xi, \quad (78)$$

TABLE I. The critical values of the parameter q_0 for the appearance of the dynamical instability of an incompressible anisotropic fluid star at different values of the parameter β and two types of the trial functions used in the calculations.

	q_{0c} evaluated with the trial function	
β	$\chi_1 = e^{-\frac{\nu}{2}}\xi$	$\chi_2 = \sqrt{\xi}$
0.1	1.391	-
0.3	1.796	-
0.5	2.526	-
0.7	4.210	-
0.9	11.646	-

where $\omega_0^2 = 4\pi\varrho_0 G$. Let us use the trial functions of the form

$$\chi_1 = e^{-\frac{\nu}{2}}\xi, \qquad \chi_2 = \sqrt{\xi}. \quad (79)$$

Then for each given β we will try to find q_{0c} such that the right-hand side of Eq. (78) vanishes, and, hence, the dynamical instability for an incompressible anisotropic fluid star occurs at $q_0 > q_{0c}$.

The results of these calculations are presented in Table I. The most important conclusion is that there are solutions for q_{0c} in the case of the trial function χ_1 at $\beta < 1$, i.e., for $\Delta = p_t - p_r > 0$ (and there are no solutions at $\beta > 1$). This means that the local pressure anisotropy with $p_t > p_r$ can affect the dynamical stability of spherically symmetric incompressible fluid stars, the result which is in contrast to the conclusion for incompressible isotropic fluid stars with the polytropic EoS (13) in Ref. [37], which are stable against radial oscillations.

It is seen also that the choice of the trial function does matter: the use of χ_1 allows us to find the critical value q_{0c} for $\beta < 1$ while the right-hand side of Eq. (78) does not vanish for the trial function χ_2 at any β. Figure 4 shows the behavior of these trial functions at $\beta = 0.5$, $q_0 = 3$. While the derivative $\chi_2'(\xi)$ is always positive, the derivative $\chi_1'(\xi)$ changes sign in the interval $[0, \xi_R]$, and, hence, the sub-integral functions containing $\chi'(\xi)$ contribute qualitatively differently to the respective integrals for $\chi_1(\xi)$ and $\chi_2(\xi)$.

If dynamical instability occurs at $\beta < 1$, the question naturally arises: is the mass of an incompressible anisotropic fluid star still compatible with the two-solar-mass constraint at the moment of the appearance of dynamical instability at the critical value q_{0c}? Table II shows the values of the mass of an incompressible anisotropic fluid
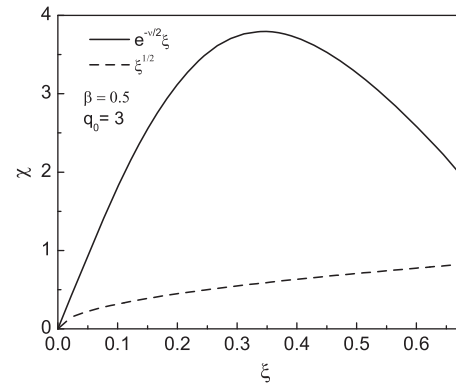

FIG. 4. The trial functions $\chi_1 = e^{-\frac{\nu}{2}}\xi$ and $\chi_2 = \sqrt{\xi}$ at $\beta = 0.5$, $q_0 = 3$.

TABLE II. The mass M_c of an incompressible anisotropic fluid star at the critical value q_{0c} for the appearance of dynamical instability in the case of the trial function $\chi_1(\xi)$.

ϱ_0 (kg/m^3)	M_c evaluated at the critical value q_{0c} (kg)				
	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
10^{18}	6.999×10^{30}	7.030×10^{30}	7.064×10^{30}	7.097×10^{30}	7.122×10^{30}
2×10^{18}	4.949×10^{30}	4.971×10^{30}	4.995×10^{30}	5.018×10^{30}	5.036×10^{30}

star at the critical value q_{0c} in the case of the trial function $\chi_1(\xi)$. It is seen that for both values of the central mass density ϱ_0, used in the calculations, the two-solar-mass constraint is still satisfied at the moment of the appearance of dynamical instability.

In summary we have considered, within the framework of general relativity, spherically symmetric relativistic stars with a polytropic equation of state which possess local pressure anisotropy. The generalized Lane-Emden equations have been derived for the case of the arbitrary anisotropy parameter $\Delta = p_t - p_r$. In this research, in order to study the effects of the pressure anisotropy, we follow a phenomenological approach, in which the anisotropy parameter is set with the help of some phenomenological ansatz. The conclusions obtained within this approach are of a rather general character and independent of the details of a specific physical mechanism. Specifically, the generalized Lane-Emden equations are applied to the special ansatz (35) for the anisotropy parameter Δ in the form of the differential relation between Δ and the metric function ν. The analytical solutions of the obtained equations have been found for incompressible fluid stars and then used to obtain their mass-radius relation, gravitational energy, and binding energy. It has

been clarified that the pressure anisotropy with $p_t > p_r$ leads to an increase of the mass of a star compared to that in the isotropic case, and this factor can be helpful in explaining the observational data of heavy compact stars with the mass $M \sim 2\,M_\odot$. Also, following the Chandrasekhar variational approach [47], the dynamical stability of incompressible anisotropic fluid stars with a polytropic EoS against radial oscillations has been studied. It has been shown that local pressure anisotropy with $p_t > p_r$ can make the incompressible fluid stars unstable with respect to radial oscillations, in contrast to incompressible isotropic fluid stars with a polytropic EoS which are dynamically stable [37]. As shown in the model calculations of the given work, if dynamical instability occurs, the mass of an incompressible anisotropic fluid star at the moment of the appearance of instability is still compatible with the two-solar-mass constraint.

Note that in the interior of magnetars—strongly magnetized neutron or quark stars—magnetic fields of about 10^{18} G, or even larger, can potentially occur. Such strong magnetic fields can produce substantial pressure anisotropy with $p_t > p_r$ [22–34], and, hence, can cause the dynamical instability of a magnetized stellar object.

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