Next-to-leading order correction to the factorization limit of the radiation spectrum

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A next-to-leading order correction to the high-energy factorization limit of radiation spectrum from an ultra-relativistic electron scattering in an external field is evaluated. Generally, it does not express through scattering characteristics, and accounts for smoothness of the crossover between the initial and final electron asymptotes. A few examples of application of this formula are given, including bremsstrahlung in amorphous matter and undulator radiation.

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I. INTRODUCTION

It is well known [1,2] that differential cross-section of bremsstrahlung in the limit of vanishing photon frequency splits into a product of the differential cross-section of elastic scattering and the photon emission probability. The latter is essentially determined by classical electrodynamics: even though the underlying scattering process may be totally quantal, the radiation in the limit $\omega \to 0$ is predominantly generated on large distances during electron rectilinear motion before and after the scattering, and rectilinear motion is always semi-classical.

For ultrarelativistic radiating particles, the mentioned infrared factorization theorem was later superseded by another one [3–6] stating that essentially the same kind of factorization may hold even at $\hbar\omega \sim E$, granted that its actual condition is the smallness of the target extent T compared to the photon formation length[1]

$$l_f(\omega) = \frac{2EE'}{m^2\omega}, \qquad E' = E - \hbar\omega.$$

Ratio $\frac{T}{l_f(\omega)} = \frac{m^2}{2EE'}\omega T$ can be small even at $\hbar\omega \sim E$, provided $E \gg \frac{m^2 T}{\hbar}$. The latter condition is usually well satisfied for ultrarelativistic electrons and microscopic scattering objects (when $T \sim r_B = \hbar^2/me^2 = 137\hbar/m$), but can break down for macroscopic targets. It may be more appropriate, therefore, to speak here formally about the limit $T/E \to 0$ rather than $\omega \to 0$.

Case $\hbar\omega \sim E$ may actually be still treated semiclassically, in spite that the differential radiation probability under substantial photon recoil is influenced by electron spin flips. Spin effects can be incorporated by means of quantum electrodynamics. More importantly, at $E \to \infty$ the electron wavelength $\hbar/E$ becomes short enough for the scattering process to be semiclassical. That definitely must be so in macroscopic external fields, but may hold as well in microscopic ones, provided the final electron is not detected, wherewith interference between different impact parameters disappears [3,7–9]. The transferred momentum then becomes a definite function of the impact parameter, and for each impact parameter there exists a definite radiation emission probability, as in classical electrodynamics.

Away from the infrared limit, shapes of semiclassical radiation can be very diverse, reflecting the diversity of possible electron motions in external fields. To make contact with this complexity, it may be valuable to determine corrections to the infrared limit, as ωT departs from zero. A step toward this goal was made long ago by Low [10], who had shown that the next-to-leading order (NLO) infrared correction to the bremsstrahlung amplitude expresses through the energy derivative of the elastic scattering amplitude at fixed momentum transfer, i.e., is still determined by scattering characteristics. It is easy to see, however, that such a procedure gives a correction of order $\hbar\omega/E$ (involving $\hbar$, and being insensitive to the external field length and strength), i.e., precisely that covered by the modified factorization theorem [3–5]. Furthermore, in the ultrarelativistic QED case, when the elastic scattering amplitude depends on the collision energy just linearly, its nontrivial part depending on the momentum exchange with the target will factor out, anyway, reducing the content of the Low's theorem to that of the modified factorization theorem. Thus, for ultrarelativistic particles such an approach seems to add essentially nothing new,[2] and be rather kinematical than related to the electron dynamics within the target.

In practice, besides that, it often appears that typical radiation emission angles from high-energy electrons are

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[1] Focusing on the ultrarelativistic case, we let $c = 1$.

[2] Historically, however, Low's paper [10] preceded generalized factorization theorem [3–5].

far too small for experimental resolution, so one is content to measurement of the angle-integral radiation spectrum. The task of deriving for it a NLO infrared correction may be not so straightforward, because in the framework of classical particle description, ω dependence enters to the exponent of a plane wave, whereas expanding this exponential to power series and integrating termwise may lead to divergent or improper integrals.

The aim of the present article is to demonstrate that the NLO correction to the angle-integral radiation spectrum has the order $\mathcal{O}[T/l_f(\omega)]$ as $T/l_f \to 0$, and to derive for it a formula valid for arbitrary external field and $\hbar\omega \sim E$, presuming the electron to be ultrarelativistic. It will then be instructive to discuss its physical meaning, sign and magnitude for several physical processes.

II. DOUBLE TIME INTEGRAL REPRESENTATION

As has already been mentioned, the infrared limit of bremsstrahlung emission probability may be inferred from classical electrodynamics. The generic representation for the spectral-angular distribution of classical radiation reads [11,12]

$$\frac{dI}{d\omega d^2n} = e^2 \left| \frac{\omega}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega[t-\boldsymbol{n}\cdot\boldsymbol{r}(t)]} \boldsymbol{n}\times\boldsymbol{v}(t) \right|^2 \tag{1a}$$

$$\equiv e^2 \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega[t-\boldsymbol{n}\cdot\boldsymbol{r}(t)]} \frac{d}{dt} \frac{\boldsymbol{n}\times\boldsymbol{v}(t)}{1-\boldsymbol{n}\cdot\boldsymbol{v}(t)} \right|^2. \tag{1b}$$

At $\omega \to 0$ the exponential factor within the domain of action of the external field may be neglected, whereupon the time integral is trivially taken to give

$$\frac{dI_{\mathrm{BH}}}{d\omega} = \frac{e^2}{(2\pi)^2} \int d^2n \left| \frac{\boldsymbol{n}\times\boldsymbol{v}_f(t)}{1-\boldsymbol{n}\cdot\boldsymbol{v}_f(t)} - \frac{\boldsymbol{n}\times\boldsymbol{v}_i(t)}{1-\boldsymbol{n}\cdot\boldsymbol{v}_i(t)} \right|^2$$
$$= \frac{2e^2}{\pi} \left(\frac{2+\gamma^2 v_{fi}^2}{\gamma v_{fi}\sqrt{1+\gamma^2 v_{fi}^2/4}} \operatorname{arsinh} \frac{\gamma v_{fi}}{2} - 1 \right). \tag{2}$$

It is a function of single variable γv_{fi}, where $\boldsymbol{v}_{fi} = \boldsymbol{v}_f - \boldsymbol{v}_i$ [with $\boldsymbol{v}_i = \boldsymbol{v}(-\infty)$, $\boldsymbol{v}_f = \boldsymbol{v}(+\infty)$] is the electron scattering angle.

At $\hbar\omega \sim E$, quantum electrodynamics gives

$$\frac{dI_{\mathrm{BH}}}{d\omega} = \frac{2e^2}{\pi} \left[\frac{2m(1+\frac{E^2+E'^2}{EE'}\frac{q^2}{4m^2})}{q\sqrt{1+\frac{q^2}{4m^2}}} \operatorname{arsinh}\frac{q}{2m} - 1 \right], \tag{3}$$

where $q \simeq q_\perp$ is the momentum transfer to the target, which is predominantly transverse ($q_z \ll q_\perp$).

The situation becomes more intricate when one aims to derive a next-to-leading order correction to (2). If one attempts to expand the radiation amplitude into power series in ω via expanding the exponential in (1) into Maclaurin series,

$$e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}(t)} = 1 + i\omega[t - \boldsymbol{n}\cdot\boldsymbol{r}(t)] + \mathcal{O}(\omega^2 T^2),$$

the square of the corresponding amplitude would give a real $\mathcal{O}(\omega^2T^2)$ correction, but the angular integral from it will diverge, making such an approach for the radiation spectrum rather ineffectual. Hence, it may be inappropriate to this end to expand the entire phase factor. As will be shown below, in fact, the expansion of $\frac{dI}{d\omega}$ beyond the IR factorization limit begins with a term $\mathcal{O}(\omega T)$.

Better suited for expansion of the spectrum in powers of ωT (or T/l_f) is representation [8,13–16]

$$\frac{dI}{d\omega} = \omega\frac{e^2}{\pi}\int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} \frac{dt_1}{t_2-t_1}$$
$$\times \left\{ \left(\gamma^{-2} + \frac{E^2+E'^2}{4EE'}[\boldsymbol{v}(t_2)-\boldsymbol{v}(t_1)]^2 \right) \right.$$
$$\times \sin\frac{\omega E}{E'}[t_2 - t_1 - |\boldsymbol{r}(t_2)-\boldsymbol{r}(t_1)|]$$
$$\left. - \gamma^{-2}\sin\frac{\omega E}{E'}(1-v)(t_2-t_1) \right\}, \tag{4}$$

where the trajectory describes a nonradiating particle with the initial conditions of the incoming electron (having energy E), while the block $(1-v)(t_2-t_1) + v(t_2-t_1) - |\boldsymbol{r}(t_2)-\boldsymbol{r}(t_1)|$ entering the argument of the first sine may be expressed through transverse velocity components as

$$v(t_2-t_1) - |\boldsymbol{r}(t_2)-\boldsymbol{r}(t_1)|$$
$$\simeq \frac{1}{2v(t_2-t_1)} \left\{ v^2(t_2-t_1)^2 - \left[\int_{t_1}^{t_2} dt \boldsymbol{v}(t) \right]^2 \right\}$$
$$\simeq \frac{1}{2v} \left\{ \int_{t_1}^{t_2} dt \boldsymbol{v}_\perp^2(t) - \frac{1}{t_2-t_1} \left[\int_{t_1}^{t_2} dt \boldsymbol{v}_\perp(t) \right]^2 \right\}. \tag{5}$$

The latter expression is rotation invariant, but in practice, it may be advantageous to recast it a manifestly rotation invariant form

$$v(t_2-t_1) - |\boldsymbol{r}(t_2)-\boldsymbol{r}(t_1)|$$
$$= \frac{1}{2v} \left\{ \frac{1}{t_2-t_1} \int_{t_1}^{t_2} dt[\boldsymbol{v}_f - \boldsymbol{v}(t)] \cdot \int_{t_1}^{t_2} dt[\boldsymbol{v}(t) - \boldsymbol{v}_i] \right.$$
$$\left. - \int_{t_1}^{t_2} dt[\boldsymbol{v}_f - \boldsymbol{v}(t)]\cdot[\boldsymbol{v}(t)-\boldsymbol{v}_i] \right\} \tag{6}$$

with arbitrary $\boldsymbol{v}_i$, $\boldsymbol{v}_f$. If $\boldsymbol{v}_i$ is chosen to be the initial, and $\boldsymbol{v}_f$ the final electron velocity, the velocity differences appearing in Eq. (6) will vanish correspondingly at $t \to -\infty$ and at $t \to +\infty$, ensuring the convergence of the integrals at large times.

Finally, by virtue of fair straightness of the electron trajectory at high energy, it can as well be reexpressed through the trajectory of an electron with the same impact parameter but energy E', or, more symmetrically, through momentum exchange with the target $\boldsymbol{q}(t) = \int^t dt\boldsymbol{F}(t)$, which is accumulated continuously regardless of whether the photon was emitted or not. In terms of the latter, $\boldsymbol{q}(t) = E\boldsymbol{v}_\perp(t)$, the radiation spectrum expresses as

$$\frac{dI}{d\omega} = \omega\frac{e^2}{\pi\gamma^2}\int_{-\infty}^{\infty}dt_2\int_{-\infty}^{t_2}\frac{dt_1}{t_2-t_1} \times\left\{\left(1+\frac{E^2+E'^2}{4EE'm^2}[\boldsymbol{q}(t_2)-\boldsymbol{q}(t_1)]^2\right) \times\sin\frac{t_2-t_1+\Delta s(t_1,t_2)}{l_f(\omega)} - \sin\frac{t_2-t_1}{l_f(\omega)}\right\}, \quad (7)$$

with

$$\Delta s(t_1,t_2) = \frac{1}{m^2}\left\{\frac{1}{\tau}\int_{t_1}^{t_2}dt[\boldsymbol{q}_f-\boldsymbol{q}(t)]\cdot\int_{t_1}^{t_2}dt[\boldsymbol{q}(t)-\boldsymbol{q}_i] - \int_{t_1}^{t_2}dt[\boldsymbol{q}_f-\boldsymbol{q}(t)]\cdot[\boldsymbol{q}(t)-\boldsymbol{q}_i]\right\}. \quad (8)$$

III. FACTORIZATION LIMIT AND NLO CALCULATION

The Bethe-Heitler limit results from Eqs. (4), (6) if the electron trajectory is replaced by its counterpart corresponding to an instantaneous momentum transfer equal to q_{fi} (say, at time $t=0$):

$$\frac{dI_{\mathrm{BH}}}{d\omega} = \omega\frac{e^2}{\pi\gamma^2}\int_0^{\infty}dt_2\int_{-\infty}^{0}\frac{dt_1}{t_2-t_1} \times\left\{\left(1+\frac{E^2+E'^2}{4EE'm^2}q_{fi}^2\right)\sin\frac{\tau-\frac{t_1t_2}{\tau}\frac{q_{fi}^2}{m^2}}{l_f(\omega)} - \sin\frac{\tau}{l_f(\omega)}\right\}. \quad (9)$$

Passage here to variables $\tau = t_2 - t_1$, $w = t_2/\tau$ [17] reduces it to a single algebraic integral:

$$\frac{dI_{\mathrm{BH}}}{d\omega} = \omega\frac{e^2}{\pi\gamma^2}\int_0^1 dw\int_0^{\infty}d\tau \times\left\{\left(1+\frac{E^2+E'^2}{4EE'm^2}q_{fi}^2\right)\sin\frac{[1+\frac{q_{fi}^2}{m^2}w(1-w)]\tau}{l_f(\omega)} - \sin\frac{\tau}{l_f(\omega)}\right\}$$
$$= \frac{2e^2}{\pi}\frac{E'}{E}\left\{\int_0^1 dw\frac{1+\frac{E^2+E'^2}{4EE'm^2}q_{fi}^2}{1+\frac{q_{fi}^2}{m^2}w(1-w)} - 1\right\},$$

which is taken to give (3). To evaluate a correction to it, one has to subtract (9) from (4), and investigate the difference in the limit $T/l_f(\omega)\to 0$.

To this end, it is convenient to select a finite domain $0 < t < T$, containing all the deflecting fields, so that outside of it one may let $\boldsymbol{q}(t\le 0) = \boldsymbol{q}_i$, $\boldsymbol{q}(t\ge T) = \boldsymbol{q}_f$. In the double time integral, it is then possible to replace the lower limit for t_2 by 0, and split the t_1 integral as $\int_{-\infty}^{t_2}dt_1\ldots = \int_{-\infty}^{0}dt_1\ldots + \int_0^{t_2}dt_1\ldots$. That gives

$$\frac{dI}{d\omega} - \frac{dI_{\mathrm{BH}}}{d\omega} = I_1 + I_2$$

with

$$I_1 = \omega\frac{e^2}{\pi\gamma^2}\int_0^{\infty}dt_2\int_{-\infty}^{0}\frac{dt_1}{\tau} \times\left\{\left(1+\frac{E^2+E'^2}{4EE'm^2}[\boldsymbol{q}(t_2)-\boldsymbol{q}_i]^2\right)\sin\frac{\tau+\Delta s(t_1,t_2)}{l_f(\omega)} - \left(1+\frac{E^2+E'^2}{4EE'm^2}q_{fi}^2\right)\sin\frac{\tau-\frac{t_1t_2}{\tau m^2}q_{fi}^2}{l_f(\omega)}\right\}, \quad (10)$$

[the last term stemming from Eq. (9)] and

$$I_2 = \omega\frac{e^2}{\pi\gamma^2}\int_0^{\infty}dt_2\int_0^{t_2}\frac{dt_1}{\tau} \times\left\{\left(1+\frac{E^2+E'^2}{4EE'm^2}[\boldsymbol{q}(t_2)-\boldsymbol{q}(t_1)]^2\right)\sin\frac{\tau+\Delta s(t_1,t_2)}{l_f(\omega)} - \sin\frac{\tau}{l_f(\omega)}\right\}. \quad (11)$$

In I_1, according to Eq. (6), the nonlinear part of the phase of the first sine can be written

$$\Delta s = \frac{1}{m^2}\left\{\frac{1}{\tau}\left(-t_1\boldsymbol{q}_{fi}+\int_{-\infty}^{\infty}dtt\frac{d\boldsymbol{q}}{dt}\right)\cdot\left(t_2\boldsymbol{q}_{fi}-\int_{-\infty}^{\infty}dtt\frac{d\boldsymbol{q}}{dt}\right) - \int_{-\infty}^{\infty}dt[\boldsymbol{q}_f-\boldsymbol{q}(t)]\cdot[\boldsymbol{q}(t)-\boldsymbol{q}_i]\right\}$$
$$= -\frac{t_1t_2}{\tau m^2}q_{fi}^2 + \mathcal{O}(Tq_{fi}^2/m^2), \quad (12)$$

which thus appears to be close to that in the second sine. Being devided by $l_f(\omega)$, the first term in (12) can still be nonvanishing as $T/l_f\to 0$, because both typical contributing times expand proportionally to $l_f(\omega)$, but $\frac{T}{l_f}\frac{q_{fi}^2}{m^2}$ does vanish in this limit. Therefore, there is complete cancellation between the last two lines in Eq. (10) when $t_2 > T$. In the difference of those terms, concentrated at $t_2 < T$, it is justified to neglect $\Delta s/l_f$ in the phase of the first sine, wherewith this sine factors out:

$$I_1 \simeq \omega \frac{e^2}{\pi} \frac{E^2 + E'^2}{4E^3 E'} \int_0^\infty dt_2 \int_{-\infty}^0 \frac{dt_1}{\tau} \times \{[\boldsymbol{q}(t_2) - \boldsymbol{q}_i]^2 - q_{fi}^2\} \sin \frac{\tau}{l_f(\omega)}. \tag{13}$$

Now passage to variable $\varphi = \tau / l_f(\omega)$,

$$I_1 \simeq \omega \frac{e^2}{\pi} \frac{E^2 + E'^2}{4E^3 E'} \int_0^\infty dt_2 \{[\boldsymbol{q}(t_2) - \boldsymbol{q}_i]^2 - q_{fi}^2\} \times \int_{t_2/l_f(\omega)}^\infty \frac{d\varphi}{\varphi} \sin \varphi, \tag{14}$$

proves that it tends to

$$I_1 \underset{T/l_f(\omega)\to 0}{\longrightarrow} \omega \frac{e^2}{2} \frac{E^2 + E'^2}{4E^3 E'} \times \int_0^\infty dt_2 \{[\boldsymbol{q}(t_2) - \boldsymbol{q}_i]^2 - [\boldsymbol{q}_f - \boldsymbol{q}_i]^2\}. \tag{15}$$

(Note that in deriving this limit we did not expand the sine in powers of ω, thus avoiding spurious divergences.)

In I_2, the leading contribution comes from $\int_T^\infty dt_2 \times \int_0^T dt_1 \ldots$, since $\omega \int_0^T dt_2 \int_0^{t_2} dt_1 \ldots = \mathcal{O}(T^2/l_f^2)$ (being of a higher order of smallness), because there, for finite integration limits and $T/l_f \to 0$, the sine in the integrand can be linearized, bringing an extra T/l_f factor. In the band $0 < t_1 < T$, $t_2 > T$ (a region symmetrical to that making the leading contribution to I_1), we can linearize the phase by omitting term $\frac{t_1 t_2}{l_f \tau} \frac{q_{fi}^2}{m^2}$ (now because for finite t_1, $\frac{t_1 t_2}{l_f \tau} \frac{q_{fi}^2}{m^2} \underset{l_f\to\infty}{\longrightarrow} 0$), and in the limit $l_f \to \infty$ get

$$\begin{aligned} I_2 &\simeq \omega \frac{e^2}{\pi} \frac{E^2 + E'^2}{4E^3 E'} \int_0^T dt_1 [\boldsymbol{q}_f - \boldsymbol{q}(t_1)]^2 \int_T^\infty \frac{dt_2}{\tau} \sin \frac{\tau}{l_f(\omega)} \\ &= \omega \frac{e^2}{\pi} \frac{E^2 + E'^2}{4E^3 E'} \int_0^T dt_1 [\boldsymbol{q}_f - \boldsymbol{q}(t_1)]^2 \int_{(T-t_1)/l_f(\omega)}^\infty \frac{d\varphi}{\varphi} \sin \varphi \\ &\underset{T/l_f(\omega)\to 0}{\longrightarrow} \omega \frac{e^2}{2} \frac{E^2 + E'^2}{4E^3 E'} \int_0^T dt_1 [\boldsymbol{q}_f - \boldsymbol{q}(t_1)]^2 \end{aligned} \tag{16}$$

(again, avoiding troublesome expansion of $\sin \varphi$ to power series).

Combining (15) and (16), on account of identity

$$(\boldsymbol{q} - \boldsymbol{q}_i)^2 + (\boldsymbol{q}_f - \boldsymbol{q})^2 - (\boldsymbol{q}_f - \boldsymbol{q}_i)^2 = -2(\boldsymbol{q}_f - \boldsymbol{q}) \cdot (\boldsymbol{q} - \boldsymbol{q}_i \Big),$$

we are led to

$$\frac{dI}{d\omega} \underset{T/l_f(\omega)\to 0}{\simeq} \frac{dI_{\mathrm{BH}}}{d\omega} \left(\frac{q_{fi}}{m}, \frac{\hbar\omega}{E} \right) + C_1 \omega + \mathcal{O}[T^2/l_f^2(\omega)] \tag{17}$$

with

$$C_1 = -e^2 \frac{E^2 + E'^2}{4E^3 E'} \int_{-\infty}^\infty dt [\boldsymbol{q}(t) - \boldsymbol{q}_i] \cdot [\boldsymbol{q}_f - \boldsymbol{q}(t)] \tag{18a}$$

$$= -e^2 \frac{E^2 + E'^2}{4E^3 E'} \int_{-\infty}^\infty dt \int_{-\infty}^t dt' \boldsymbol{F}_\perp(t') \cdot \int_t^\infty dt'' \boldsymbol{F}_\perp(t''). \tag{18b}$$

Here the lower and upper integration limits were replaced by infinity, presuming the integrand to vanish rapidly enough at $t < 0$ and $t > T$. Let us stress that expression (18), being quadratic in q (or $F_\perp$), nonetheless implies no restrictions on ratio q/m, i.e., on the dipole or nondipole character of the radiation. Evidently, integral (18) takes into account the smoothness of the crossover between the rectilinear asymptotes of the electron trajectory.

For $\hbar\omega \ll E$, Eq. (18) reduces to

$$C_1 = -\frac{e^2}{2} \int_{-\infty}^\infty dt [\boldsymbol{v}(t) - \boldsymbol{v}_i] \cdot [\boldsymbol{v}_f - \boldsymbol{v}(t)], \tag{19}$$

i.e., for a given trajectory it does not depend on the electron Lorentz-factor; that can be attributed to the radiophysical character of the radiation process in this limit (cf. [17]).

To precisely understand the physical meaning of the obtained result, note that structure (18), (19) coincides with that of the second term in the right-hand side (rhs) of Eq. (6). Since the meaning of its left-hand side is clear enough (being a time delay due to the trajectory curvature), it remains to figure out the meaning of the first term in the rhs at $t_1 \to -\infty$, $t_2 \to \infty$. To this end, integrating by parts as in Eq. (12), rewrite it as

$$\begin{aligned} &\int_{t_1}^{t_2} dt [\boldsymbol{v}_f - \boldsymbol{v}(t)] \cdot \int_{t_1}^{t_2} dt [\boldsymbol{v}(t) - \boldsymbol{v}_i] \\ &\underset{\substack{t_1\to-\infty \\ t_2\to\infty}}{\simeq} - \left(t_1 \boldsymbol{v}_{fi} - \int_{-\infty}^\infty dt t \frac{d\boldsymbol{v}}{dt} \right) \cdot \left(t_2 \boldsymbol{v}_{fi} - \int_{-\infty}^\infty dt t \frac{d\boldsymbol{v}}{dt} \right). \end{aligned} \tag{20}$$

Choosing the zero time such that $\int_{-\infty}^\infty dt t \frac{d\boldsymbol{v}}{dt} = 0$ (which is always possible, e.g., when the motion is planar), integrals $\int_{t_1}^{t_2} dt [\boldsymbol{v}_f - \boldsymbol{v}(t)]$ and $\int_{t_1}^{t_2} dt [\boldsymbol{v}(t) - \boldsymbol{v}_i]$ (representing the particle transverse coordinates with respect to $\boldsymbol{v}_f$ or $\boldsymbol{v}_i$) at $t_1 \to -\infty$, $t_2 \to \infty$ are proportional correspondingly to t_1 and t_2, i.e., both the initial and the final trajectory asymptotes issue from the origin. Correspondingly, in that limit the first term in the rhs of Eq. (6) tends to the time delay $v(t_2 - t_1) - |\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)|$ for a trajectory having the shape of an angle along the initial and final electron asymptotes. The second term thus represents a difference between the time delay for the actual trajectory and for its angle-shaped approximation. If it is impossible to adjust the time origin such that $\int_{-\infty}^\infty dt t \frac{d\boldsymbol{v}}{dt} = 0$, it suffices to demand that $\boldsymbol{v}_{fi} \cdot \int_{-\infty}^\infty dt t \frac{d\boldsymbol{v}}{dt} = 0$.

Generally, for monotonous electron deflection, $C_1 \leq 0$ (the particle "cuts the corner"), whereas for an oscillatory electron motion within the target, $C_1 \geq 0$. As a cross-check, note that for classical radiation at double scattering through angles χ_1 and χ_2 with a time separation t_{21}, Eq. (19) gives

$$C_1 = -\frac{e^2}{2}\chi_1 \cdot \chi_2 t_{21},$$

which coincides with the result obtained in [17]. After an initial decline, however, the spectrum will start rising, due to resolution of smaller parts of the electron trajectory.

In case of a monotonous electron deflection, as in a magnet of length T, C_1 grows with T cubically, wherefore at low ω it may compete with the "volume" (synchrotron-like) contribution, which is proportional to T.

In case of undulator radiation, when $\boldsymbol{F}_\perp(t) = \boldsymbol{F}_0 \cos\frac{2\pi t}{T_1}$ within the interval $0 < t < NT_1$, with $N \gg 1$ being the number of oscillation periods, from (18b) we get

$$\frac{C_1}{NT_1} \underset{N\to\infty}{\simeq} e^2 \frac{E^2 + E'^2}{8E^3 E'} \left(\frac{F_0 T_1}{2\pi}\right)^2. \tag{21}$$

Finally, in an amorphous medium modeled by action of a delta-correlated (Langevin) force, averaging of (18b) gives zero:

$$\left\langle \int_{-\infty}^{t} dt' \boldsymbol{F}_\perp(t') \cdot \int_{t}^{\infty} dt'' \boldsymbol{F}_\perp(t'') \right\rangle \propto \int_{-\infty}^{t} dt' \int_{t}^{\infty} dt'' \delta(t' - t'') = 0. \tag{22}$$

IV. SUMMARY

The distinctions of our result (18) from the Low theorem [10] are that it applies to: (i) the angle-integral radiation spectrum, which is a more inclusive quantity than the radiation amplitude considered in [10]; (ii) ultrarelativistic electrons and small-angle photon emission, allowing for $\hbar\omega \sim E$, whereas the small parameter is $T/l_f(\omega)$. From the physical point of view, it is essential that the NLO expansion for $dI/d\omega$ begins with $\mathcal{O}(T/l_f)$, since in the double time integral representation (4) it originates from the region where only one of the contributing times is large: $|t_1|$ for I_1 and t_2 for I_2.

The obtained correction, by virtue of its simplicity, can be used for accurate connection of the infrared (or generalized factorization) limit of the bremsstrahlung spectrum with its behavior at higher ω, probing the interior of the target. At its application, it is worth minding that the correction is insensitive to nondipole radiation effects. For instance, relation (21), well known for dipole undulators, must hold as well for wigglers.

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