PHYSICAL REVIEW D **96,** 075043 (2017)

Neutrino mass matrices with one texture equality and one vanishing neutrino mass

Jinzhong Han,[1,*] Ruihong Wang,[2] Weijian Wang,[3,†] and Xing-Ning Wei[3]

[1]*School of Physics and Telecommunications Engineering, Zhoukou Normal University, Henan 466001, China*

[2]*College of Information Science & Technology, Hebei Agricultural University, Baoding 071001, China*

[3]*Department of Physics, North China Electric Power University, Baoding 071003, China*

(Received 16 May 2017; revised manuscript received 10 August 2017; published 30 October 2017)

In the light of latest data of neutrino oscillation experiments, we carry out a systematic investigation on the texture structures of Majorana neutrino mass matrix M_ν, which contain one vanishing neutrino mass and an equality between two matrix elements. Among 15 logically possible patterns, it is found that for norm order ($m_3 > m_2 > m_1 = 0$) of neutrino masses only three of them are compatible with recent experimental data at the 3σ level, while for inverted order ($m_2 > m_1 > m_3 = 0$) two patterns is phenomenologically allowed. In the numerical analysis, we perform a scan over the parameter space of all viable patterns to get a large sample of scattering points. We present the implications of each allowed pattern for three mixing angles $(\theta_{12}, \theta_2, \theta_3)$, leptonic CP violation and neutrinoless double-beta decay, predicting strong correlations between oscillation parameters. The theoretical realization of a concrete example is discussed in the framework of Froggatt-Nielsen mechanism.

DOI: 10.1103/PhysRevD.96.075043

I. INTRODUCTION

In spite of the evidences for massive neutrinos and the admixtures of the flavor states [1], the origin of the lepton flavor structure remains an open question. A popular approach to understand the leptonic mixing structure is reducing the number of free parameters by adding Abelian or non-Abelian flavor symmetries, leading to specific texture structures of neutrino mass matrix (M_ν). Some common models include texture-zeros [2–7], hybrid textures [8,9], zero trace [10], vanishing minors [11–13], two traceless submatrices [14], equal elements or cofactors [15], hybrid M_ν^{-1} textures [16], partial $\mu-\tau$ symmetry [17].

One can also reduce the number of free parameters of M_ν by assuming one of the neutrinos to be massless. It is well known that in type-I seesaw mechanism, a vanishing neutrino mass is easily realized by assuming two families of heavy right-handed neutrinos. A similar scenario also appears in the radiative seesaw model [18,19]. In the framework of Affleck-Dine mechanism [20–22], an extremely small neutrino mass($\approx 10^{-10}$ eV) is required to successfully produce the leptogenesis [23,24]. Hence it is imperative to ask if the lepton mass matrix with a specific texture structure and a vanishing mass eigenvalue can survive under the current neutrino oscillation data. Several attempts have been made in this direction. In Refs. [25–27], particular attentions has been paid to the neutrino mass matrix structures with one texture-zero and one massless eigenstate.

In the flavor basis where the charged lepton mass matrix is diagonal, we investigate a specific class of neutrino mass matrices which contains a vanishing neutrino mass and a texture equality between two independent entries. The texture equality was proposed in many literatures. Bear in mind the fact that an equality between "12" and "13" element of M_ν has appeared in the tri-bimaximal form of M_ν

$$M_\nu = \begin{pmatrix} a & b & b \\ b & a-c & b+c \\ b & b+c & a-c \end{pmatrix} \tag{1}$$

which can be diagonalized by tri-bimaximal form of Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix

$$U_{\mathrm{PMNS}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \end{pmatrix} \tag{2}$$

and leads to the three neutrino masses eigenvalues $m_1=a-b, m_2=a+2b, m_3=a-b-2c$. If $|a| \simeq |b| \ll |c|$, we obtain a normal hierarchy for neutrino masses with $m_1 \simeq 0$. Clearly, Eq. (2) is not compatible with the observation of no-zero θ_{13} angle. Hence the texture structure with an equality between "12" and "13" elements of M_ν can be considered as a perturbed generalization of Eq. (1), where the number of free parameters is added but the texture equality still holds. The main motivation of this work is three-fold:

*Corresponding Author.
hanjinzhong@zknu.edu.cn
†wjnwang96@aliyun.com

TABLE I. Fifteen possible texture structures with equality between two nonzero elements where "△" denotes the nonzero and equal elements, while "×" stands for the abitrary and nonzero ones.

$P1$	$P2$	$P3$	$P4$	$P5$
$\begin{pmatrix} \triangle & \triangle & \times \\ \triangle & \times & \times \\ \times & \times & \times \end{pmatrix}$	$\begin{pmatrix} \triangle & \times & \triangle \\ \times & \times & \times \\ \triangle & \times & \times \end{pmatrix}$	$\begin{pmatrix} \times & \triangle & \times \\ \triangle & \triangle & \times \\ \times & \times & \times \end{pmatrix}$	$\begin{pmatrix} \times & \times & \times \\ \times & \triangle & \triangle \\ \times & \triangle & \times \end{pmatrix}$	$\begin{pmatrix} \times & \times & \triangle \\ \times & \times & \times \\ \triangle & \times & \triangle \end{pmatrix}$
$P6$	$P7$	$P8$	$P9$	$P10$
$\begin{pmatrix} \times & \times & \times \\ \times & \times & \triangle \\ \times & \triangle & \triangle \end{pmatrix}$	$\begin{pmatrix} \triangle & \times & \times \\ \times & \times & \triangle \\ \times & \triangle & \times \end{pmatrix}$	$\begin{pmatrix} \times & \times & \triangle \\ \times & \triangle & \times \\ \triangle & \times & \times \end{pmatrix}$	$\begin{pmatrix} \times & \triangle & \times \\ \triangle & \times & \times \\ \times & \times & \triangle \end{pmatrix}$	$\begin{pmatrix} \triangle & \times & \times \\ \times & \triangle & \times \\ \times & \times & \times \end{pmatrix}$
$P11$	$P12$	$P13$	$P14$	$P15$
$\begin{pmatrix} \triangle & \times & \times \\ \times & \times & \times \\ \times & \times & \triangle \end{pmatrix}$	$\begin{pmatrix} \times & \times & \times \\ \times & \triangle & \times \\ \times & \times & \triangle \end{pmatrix}$	$\begin{pmatrix} \times & \triangle & \triangle \\ \triangle & \times & \times \\ \triangle & \times & \times \end{pmatrix}$	$\begin{pmatrix} \times & \triangle & \times \\ \triangle & \times & \triangle \\ \times & \triangle & \times \end{pmatrix}$	$\begin{pmatrix} \times & \times & \triangle \\ \times & \times & \triangle \\ \triangle & \triangle & \times \end{pmatrix}$

(i) From the phenomenological viewpoint, either a vanishing neutrino mass or an equality between two nonzero matrix elements imposes one constraint condition on M_ν and reduces the number of free degrees by two. Thus texture equalities are as predictive as the texture zeros.

(ii) On the experimental side, the absolute scale of lightest neutrino mass is still unknown. Within the ΛCDM framework, an upper bound on the sum of neutrino mass $\sum m_i < 0.23$ eV at 95% confidence level has been reported by Plank Collaboration [28]. Recently, combined with the Planck, $\mathrm{TT}, \mathrm{TE}, \mathrm{EE} + \mathrm{lowP} + \mathrm{BAO} + \mathrm{JLA} + H0 + \mathrm{Lensing}$ data, a much tighter bound $\sum m_i < 0.105$, which is almost access to the lower limit of $\sum m_i$ for inverted order spectrum of neutrino masses, is obtained in holographic dark energy scenario [29]. Since we have stood on the verge to distinguish the mass spectrum of three neutrinos through cosmological observations, the phenomenology of some specific texture structures with a vanishing neutrino mass deserves a detailed survey.

(iii) It is generally believed that the observed neutrino mixing pattern indicates some underlying discrete flavor symmetries. The flavor symmetry realization for all possible patterns is beyond the scope of this work. In the framework of the Froggatt-Nielsen (FN) mechanism [30], we consider an explicit model based on $S_{\mu-\tau} \times Z_8$ symmetry suggested by Ref. [17] to realize one of the viable texture equality. Then we adopt the methodology in Ref. [31] by assuming the broken FN symmetry a discrete Z_n. We will see that a ultralight neutrino mass naturally arises as a good approximation to vanishing neutrino masses. We expect that a phenomenological analysis may help us reveal the underlying flavor structure of lepton mixing.

The rest of the paper is organized as follows: In Sec. II, we describe some useful notations and the framework used to obtain the constraint equations. In Sec. III, the details of the numerical analysis are presented. In Sec. IV, we discuss the theoretical realization for a concrete example. The summary is given in Sec. V.

II. FORMALISM AND IMPORTANT RELATIONS

Assuming neutrinos the Majorana particles as proposed in various seesaw models [32], the neutrino mass matrix M_ν is a symmetric and generally complex matrix with six independent entries. We arrive at $C_6^2 = 15$ logically possible patterns to place an equality between two matrix elements, which are shown in Table I.

In the flavor basis where M_l is diagonal, the Majorana neutrino mass M_ν is related to the diagonal mass matrix $M^D = \mathrm{diag}(m_1, m_2, m_3)$ though the unitary transformation

$$M_\nu = U_{\mathrm{PMNS}} M^D U_{\mathrm{PMNS}}^T. \tag{3}$$

In following analysis, we consider the Pontecorvo-Maki-Nakagawa-Sakata matrix U_{PMNS} [33,34] parametrized as

$$U_{\mathrm{PMNS}} = UP_\nu = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha}{2}} & 0 \\ 0 & 0 & e^{i\frac{\beta}{2}} \end{pmatrix} \tag{4}$$

where the abbreviations $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$ are used. The (α,β) in P_ν denotes two Majorana CP-violating phases and δ stands for Dirac CP-violating phase. We introduce the factors $1/2$ in P_ν due to the fact that the mixing matrix $U_{\rm PMNS}$ appears twice in Eq. (3).

Equation (3) can be reexpressed as

$$M_\nu = U\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} U^T \tag{5}$$

where $\lambda_1 = m_1, \lambda_2 = m_2 e^{i\alpha}, \lambda_3 = m_3 e^{i\beta}$. In this parametrization, the mass matrix elements are given by

$$\begin{aligned}
(M_\nu)_{11} &= m_1 c_{12}^2 c_{13}^2 + m_2 s_{12}^2 c_{13}^2 e^{i\alpha} + m_3 s_{13}^2 e^{i(\beta-2\delta)} \\
(M_\nu)_{12} &= m_1(-s_{12}c_{12}c_{23}c_{13} - c_{12}^2 s_{23}s_{13}c_{13}e^{i\delta}) \\
&\quad + m_2(s_{12}c_{12}c_{23}c_{13}e^{i\alpha} - s_{12}^2 s_{23}s_{13}c_{13}e^{i(\alpha+\delta)}) \\
&\quad + m_3 s_{23}s_{13}c_{13}e^{i(\alpha-\delta)} \\
(M_\nu)_{13} &= m_1(s_{12}c_{12}s_{23}c_{13} - c_{12}^2 c_{23}s_{13}c_{13}e^{i\delta}) \\
&\quad + m_2(-s_{12}c_{12}s_{23}c_{13}e^{i\alpha} - s_{12}^2 c_{23}s_{13}c_{13}e^{i(\alpha+\delta)}) \\
&\quad + m_3 c_{23}s_{13}c_{13}e^{i(\alpha-\delta)} \\
(M_\nu)_{22} &= m_1(s_{12}c_{23} + c_{12}s_{23}s_{13}e^{i\delta})^2 \\
&\quad + m_2(c_{12}c_{23} - s_{12}^2 s_{23}s_{13}e^{i\delta})^2 e^{i\alpha} + m_3 s_{23}^2 c_{13}^2 e^{i\beta} \\
(M_\nu)_{23} &= m_1(-s_{12}^2 s_{23}c_{23} + s_{12}c_{12}(c_{23}^2 - s_{23}^2)s_{13}e^{i\delta} \\
&\quad + c_{12}^2 s_{23}c_{23}s_{13}^2 e^{i\delta}) \\
&\quad + m_2(-c_{12}^2 s_{23}c_{23}e^{i\alpha} - s_{12}c_{12}(c_{23}^2 - s_{23}^2)s_{13}e^{i(\alpha+\delta)} \\
&\quad + s_{12}^2 s_{23}c_{23}s_{13}^2 e^{i(\alpha+\delta)}) + m_3 s_{23}c_{23}c_{13}^2 e^{i\beta} \\
(M_\nu)_{33} &= m_1(s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta})^2 \\
&\quad + m_2(c_{12}s_{23} + s_{12}c_{23}s_{13}e^{i\delta})^2 e^{i\alpha} + m_3 c_{23}^2 c_{13}^2 e^{i\beta}.
\end{aligned} \tag{6}$$

In terms of the texture equality condition [e.g., $(M_\nu)_{ab} = M_\nu)_{cd}$], we obtain the constraint condition equation

$$\sum_{n=1}^{3} (U_{ai}U_{bi} - U_{ci}U_{di})\lambda_i = 0 \tag{7}$$

which, if the lightest neutrino is massless, leads to

$$\begin{aligned}
\xi \equiv \frac{m_2}{m_3} &= \left|\frac{U_{a3}U_{b3} - U_{c3}U_{d3}}{U_{a2}U_{b2} - U_{c2}U_{d2}}\right| \\
\alpha - \beta &= -arg\left(\frac{U_{a3}U_{b3} - U_{c3}U_{d3}}{U_{a2}U_{b2} - U_{c2}U_{d2}}\right)
\end{aligned} \tag{8}$$

for normal order (NO) spectrum. The three neutrino masses are given by

$$m_1 = 0, \qquad m_2 = \sqrt{\delta m^2}, \qquad m_3 = \frac{m_2}{\xi} \tag{9}$$

Note that only the difference $\alpha - \beta$ is physical.

For invert order (IO) spectrum, we obtain

$$\begin{aligned}
\zeta \equiv \frac{m_2}{m_1} &= \left|\frac{U_{a1}U_{b1} - U_{c1}U_{d1}}{U_{a2}U_{b2} - U_{c2}U_{d2}}\right| \\
\alpha &= -\arg\left(\frac{U_{a1}U_{b1} - U_{c1}U_{d1}}{U_{a2}U_{b2} - U_{c2}U_{d2}}\right)
\end{aligned} \tag{10}$$

and three neutrino masses given by

$$m_3 = 0, \qquad m_1 = \sqrt{\Delta m^2 - \frac{\delta m^2}{2}}, \qquad m_2 = \zeta m_1. \tag{11}$$

It is clear that, the Majorana CP-violating phase β becomes unphysical, since m_1 is zero, and can be dropped out in numerical calculation. Hence the neutrino mass ratios and Majorana CP-violating phases are fully determined in terms of the mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and the Dirac CP violating phase δ. The three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ as well as two independent neutrino mass-squared differences $\delta m^2 = m_2^2 - m_1^2$, $\Delta m^2 = |m_3^2 - \frac{1}{2}(m_2^2 + m_1^2)|$ are precisely measured by many neutrino oscillation experiments. We summarize the latest global-fit results [35] of neutrino oscillation parameters in Table II. One can further define the ratio of neutrino mass-squared difference as

$$R_\nu = \frac{\delta m^2}{\Delta m^2} \tag{12}$$

which, using Eqs. (8) and (10), can be expressed as

$$R_\nu = \frac{2\xi^2}{2 - \xi^2} \tag{13}$$

in case of normal order and

$$R_\nu = \frac{\zeta^2 - 1}{\zeta^2 + 1} \tag{14}$$

in case of inverted order.

The Dirac CP-violation in neutrino oscillation experiments can be described by the Jarlskog rephasing invariant quantity

$$J_{CP} = s_{12}s_{23}s_{13}c_{12}c_{23}c_{13}^2 \sin\delta. \tag{15}$$

On the other hand, the Majorana CP violation can be established if any signal of neutrinoless double beta $(0\nu\beta\beta)$ decay is observed. The rate of $0\nu\beta\beta$ decay is determined by the effective Majorana neutrino mass m_{ee}

$$m_{ee} = |m_1 c_{12}^2 c_{13}^2 + m_2 s_{12}^2 c_{13}^2 e^{i\alpha} + m_3 s_{13}^2 e^{i(\beta-2\delta)}| \tag{16}$$

The next generation $0\nu\beta\beta$ experiments, with the aimed sensitivity of m_{ee} being up to 0.01 eV [36], will open the window to weigh both neutrino masses and lepton number

TABLE II. The latest global-fit results of neutrino oscillation parameters given by Ref. [35], with $\delta m^2 = m_2^2 - m_1^2$, $\Delta m^2 = |m_3^2 - \frac{1}{2}(m_2^2 + m_1^2|)$.

Parameter	Order	1σ range	2σ range	3σ range
$\delta m^2/10^{-5}$ eV^2	NO, IO	7.21–7.54	7.07–7.73	6.93–7.96
$\sin^2\theta_{12}/10^{-1}$	NO, IO	2.81–3.14	2.65–3.34	2.50–3.54
$\lvert\Delta m^2\rvert/10^{-3}$ eV^2	NO	2.495–2.567	2.454–2.606	2.411–2.646
	IO	2.473–2.539	2.430–2.582	2.390–2.624
$\sin^2\theta_{13}/10^{-2}$	NO	2.08–2.22	1.99–2.31	1.90–2.40
	IO	2.07–2.24	1.98–2.33	1.90–2.42
$\sin^2\theta_{23}/10^{-1}$	NO	4.10–4.46	3.95–4.70	3.81–6.15
	IO	4.17–4.48 $\oplus$ 5.67–6.05	3.99–4.83 $\oplus$ 5.33–6.21	3.84–6.36
δ/π	NO	1.18–1.61	1.00–1.90	0–0.17 $\oplus$ 0.76–2
	IO	1.12–1.62	0.92–1.88	0–0.15 $\oplus$ 0.69–2

violation. Besides the $0\nu\beta\beta$ experiments, the upper bound on the sum of neutrino masses $\sum m_i < 0.23$ eV[1] is set from cosmology observation. For the texture structures with a vanishing neutrino mass eigenvalue, this bound is always satisfied and has no impact on the numerical results.

III. NUMERICAL ANALYSIS

A. A benchmark point for P12 pattern

First of all, we point out that the requirement of one vanishing neutrino mass is equivalent to the zero determinant of neutrino mass matrix $\mathrm{Det}(M_\nu) = 0$ because

$$\begin{aligned}\mathrm{Det}(M_\nu) &= \mathrm{Det}(U_{\mathrm{PMNS}} M^D U^T_{\mathrm{PMNS}}) \\ &= \mathrm{Det}(U^T_{\mathrm{PMNS}} U_{\mathrm{PMNS}} M^D) \\ &= \mathrm{Det}(U^T_{\mathrm{PMNS}})\mathrm{Det}(U_{\mathrm{PMNS}})\mathrm{Det}(M^D) = 0 \end{aligned} \tag{17}$$

In Ref. [38], the neutrino mass matrices with one texture equality and $\mathrm{Det}(M_\nu) = 0$ has been discussed. The authors claimed that such texture structures are not allowed for $\theta_{13} \neq 0$. However, our analysis demonstrates a different result. Here we take the P12 pattern as an illustration.

For NO spectrum of neutrino masses, we set the benchmark points lying in 3σ range of experimental data as

$$\begin{aligned}(\theta_{12},\theta_{23},\theta_{13}) &= (32.299482°, 44.259012°, 8.240842°) \\ (\delta,\alpha,\beta) &= (165.225824°, -37.807680°, -50.769187°) \\ (m_2, m_3) &= (0.008854\,\mathrm{eV}, 0.051464\,\mathrm{eV}) \end{aligned} \tag{18}$$

then the corresponding neutrino mass matrix is

$$M_\nu \approx \begin{pmatrix} 0.002942 - 0.001901i & -0.001731 + 0.001058i & -0.006257 + 0.004562i \\ -0.001731 + 0.001058i & 0.018449 - 0.021486i & 0.013455 - 0.017614i \\ -0.006257 + 0.004562i & 0.013455 - 0.017614i & 0.018449 - 0.021486i \end{pmatrix} \tag{19}$$

with corresponding $(\delta m^2, \Delta m^2)$ given by

$$\delta m^2 \approx 7.839332 \times 10^{-5}\ \mathrm{eV}^2, \qquad \Delta m^2 \approx 2.609347 \times 10^{-5}\ \mathrm{eV}^2. \tag{20}$$

For IO spectrum of neutrino masses, if benchmark points are taken as following

$$\begin{aligned}(\theta_{12},\theta_{23},\theta_{13}) &= (35.407179°, 39.333087°, 8.400162°) \\ (\delta,\alpha) &= (75.034802°, -75.017322°) \\ (m_1, m_2) &= (0.0502297\ \mathrm{eV}, 0.0509329\ \mathrm{eV}) \end{aligned} \tag{21}$$

we can obtain the neutrino mass matrix

[1]A more robust bound $\sum m_i < 0.17$ is set by latest cosmological data in Ref. [37].

$$M_\nu \approx \begin{pmatrix} 0.036980 - 0.016163i & -0.015747 - 0.020731i & 0.008099 + 0.010964i \\ -0.015747 - 0.020731i & 0.012542 - 0.015988i & -0.013325 + 0.017028i \\ 0.008099 + 0.010964i & -0.013325 + 0.017028i & 0.012542 - 0.015988i \end{pmatrix} \quad (22)$$

with corresponding $(\delta m^2, \Delta m^2)$ given by

$$\delta m^2 \approx 7.113754 \times 10^{-5}\ \mathrm{eV}^2, \\ \Delta m^2 \approx 2.558592 \times 10^{-5}\ \mathrm{eV}^2. \quad (23)$$

It is clearly that, by direct calculation, the P12 pattern of neutrino mass matrix with $M_{\nu 22} = M_{\nu 33}$ and a vanishing neutrino mass (m_1 or $m_3 = 0$) is phenomenologically allowed if appropriate oscillation parameters are chosen.

B. Numerical results and discussion

We have performed a numerical analysis of all fifteen texture structures shown in Table I. For each pattern of M_ν, the a set of random number inputs are generated for the three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and the neutrino mass square differences $(\delta m^2, \Delta m^2)$ in their 3σ range (Table II). Instead, we generate a random input of Dirac CP-violating phase δ in the range of $[0, 2\pi)$. From Eqs. (13) and (14), R_ν is determined by both (θ_{12}, θ_{23}, θ_{13}, δ). This requires the input scattering point of $(\theta_{12}, \theta_{23}, \theta_{13}, \delta)$ is empirically acceptable only when R_ν falls inside the 3σ range $[\delta m^2_{\mathrm{min}}/\Delta m^2_{\mathrm{max}}, \delta m^2_{\mathrm{max}}/\Delta m^2_{\mathrm{min}}]$. From Eqs. (9) and (11), we further get the three absolute scale of neutrino masses $m_{1,2,3}$. Two Majorana CP-violating α and β can be constrained by Eqs. (13), (14) and the allowed range of m_{ee} are subsequently obtained.

Before proceeding, one notes that there exists a so-called $\mu - \tau$ permutation transformation that can relate one texture pattern to another though

$$\tilde{M}_\nu = P_{23} M_\nu P_{23} \quad (24)$$

where

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (25)$$

and the oscillation parameters between M_ν and $\widetilde{M}_\nu$ given by

$$\tilde{\theta}_{12} = \theta_{12}, \qquad \tilde{\theta}_{13} = \theta_{13}, \\ \tilde{\theta}_{23} = \frac{\pi}{2} - \theta_{23}, \qquad \tilde{\delta} = \pi - \delta. \quad (26)$$

One can also prove that two neutrino mass matrices related by $\mu - \tau$ permutation transformation share the same neutrino mass eigenvalues. It is straightforward to verify that such a permutation symmetry exists between

$$P1 \leftrightarrow P2 \qquad P3 \leftrightarrow P5 \qquad P4 \leftrightarrow P6 \\ P7 \leftrightarrow P7 \qquad P8 \leftrightarrow P9 \qquad P10 \leftrightarrow P11 \\ P12 \leftrightarrow P12 \qquad P13 \leftrightarrow P13 \qquad P14 \leftrightarrow P15. \quad (27)$$

Notice that the pattern P7, P12, and P13 transform into themselves under the $\mu - \tau$ permutation. Thus among fifteen texture patterns we studied, only nine of them is independent.

We present the allowed range of oscillation parameters for each viable patterns in Table III and Figs. 1–5 in the Appendix. In the figures, the light blue bands represent the 1σ uncertainty in determination of θ_{12}, θ_{23} and θ_{13} while they plus the beige bands correspond to the 2σ uncertainty. Some interesting observations are summarized as follows

(i) Only the P4, P6, P12, and P13 patterns are phenomenologically allowed by the current experimental data at 3σ confidence level. Among them, the P12 pattern is allowed for both NO and IO spectrum of neutrino masses. The P4 and P6 patterns are allowed for NO spectrum while the P13 pattern is only allowed for IO spectrum of neutrino masses. Furthermore, if the 2σ experimental results are used instead of 3σ data, only P6 pattern for NO spectrum and P12, P13 patterns for IO spectrum can survive the oscillation data.

(ii) The P4 pattern predicts $\theta_{23} > 45°$ while for P6 we get $\theta_{23} < 45°$. In particular, the allowed region of θ_{23} from P12 pattern is tightly located at around 45°. Taking the NO spectrum as an example, we derive the analytical approximate formula up to the second order of $\sin\theta_{13}$

$$\xi = \frac{m_2}{m_3} \approx \frac{1}{\cos\theta_{12}} \left(1 + \tan\theta_{12} \tan 2\theta_{23} \sin\theta_{13} - \frac{1}{4} \tan^2\theta_{12} \tan^2\theta_{23} \sin^2\theta_{13}\right) \quad (28)$$

where, without loss of generality, we have set $\delta = 0$. Clearly, the smallness of mass ratio $\xi \approx \sqrt{\delta m^2/\Delta m^2}$ implies a large cancellation in Eq. (28) which, as shown in Fig. 5, only happens when θ_{23} approaches $\pi/4$ enough. Thus the two highly constrained spaces for θ_{23} besides 45° can be regarded as the two

TABLE III. The various predictions for $P1$–$P15$ patterns. The symbol $\times$ denotes that the corresponding pattern is not phenomenologically allowed under current experimental data.

Textures	Spectrum	θ_{23}	δ	Majorana phases	m_{ee} (eV)
$P1$	NO	×	×	×	×
	IO	×	×	×	×
$P2$	NO	×	×	×	×
	IO	×	×	×	×
$P3$	NO	×	×	×	×
	IO	×	×	×	×
$P5$	NO	×	×	×	×
	IO	×	×	×	×
$P4$	NO	49.67°–51.63°	0°–360°	(−8.042°)–7.916°	0.000 98–0.004 11
	IO	×	×	×	×
$P6$	NO	38.16°–40.41°	0°–360°	(−8.042°)–7.985°	0.000 95–0.004 16
	IO	×	×	×	×
$P7$	NO	×	×	×	×
	IO	×	×	×	×
$P8$	NO	×	×	×	×
	IO	×	×	×	×
$P9$	NO	×	×	×	×
	IO	×	×	×	×
$P10$	NO	×	×	×	×
	IO	×	×	×	×
$P11$	NO	×	×	×	×
	IO	×	×	×	×
$P12$	NO	44.17°–44.43° ⊕ 45.59°–45.81°	0°–360°	−179.8°–179.6°	0.000 96–0.004 18
	IO	38.34°–44.92° ⊕ 45.06°–52.88°	0°–83.82° ⊕ 95.75°–263.1° ⊕ 275.8°–360°	(−52.34°)–52.24°	0.034 47–0.050 03
$P13$	NO	×	×	×	×
	IO	38.30°–42.73° ⊕ 46.89°–52.89°	0°–70.32° ⊕ 111.9°–248.6° ⊕ 288.5°–360°	(−179.7°)–(−177.8°) ⊕ 178.1°–180.0°	0.013 72–0.024 53
$P14$	NO	×	×	×	×
	IO	×	×	×	×
$P15$	NO	×	×	×	×
	IO	×	×	×	×

solutions from Eq. (28). One can apply the similar analysis to IO spectrum of neutrino masses, where in leading order of $\sin\theta_{13}$ we have

$$\zeta = \frac{m_2}{m_1} \approx \tan^2\theta_{12}\left(1 + \frac{2\tan 2\theta_{23}\cos\delta\sin\theta_{13}}{\sin\theta_{12}\cos\theta_{12}}\right) \quad (29)$$

One can see that the value of θ_{23} should be located at around $\pi/4$ in order to obtain the mass ratio $\zeta \approx 1$,

(iii) All the viable patterns admit a large Dirac CP violation. We mention that the latest analysis gives a best-fit value of 1.38π for NO spectrum and 1.31π for IO spectrum [35], which strengthens the trend in favor of $\delta \sim 3\pi/2$.

(iv) The viable textures with NO spectrum of neutrino masses predict tiny m_{ee} (at order of 0.001 eV), rendering it very challenging to be detected in future $0\nu\nu\beta$ decay experiments. This is due to the fact that, in the scenario of $m_1 = 0, m_2 = \sqrt{\delta m^2} \ll m_3$, one easily obtains $m_{ee} \approx \sqrt{\Delta m^2} s_{13}^2$ from Eq. (16), with the value of m_{ee} largely suppressed by small s_{13}^2. For IO spectrum of neutrino masses, on the other hand, each viable texture can predict m_{ee} in the order of 0.01 eV, which is promising to be detected in the forthcoming experiments.

IV. THEORETICAL REALIZATION FOR A CONCRETE MODEL

In this section, we present a detailed illustration on how flavor symmetry gives rise to the desired texture structure. The neutrino mass matrices with texture equalities have been realized by using a non-Abelian flavor symmetry e.g., S_3 [9] or A_4 [15]. On the other hand, one notices that as

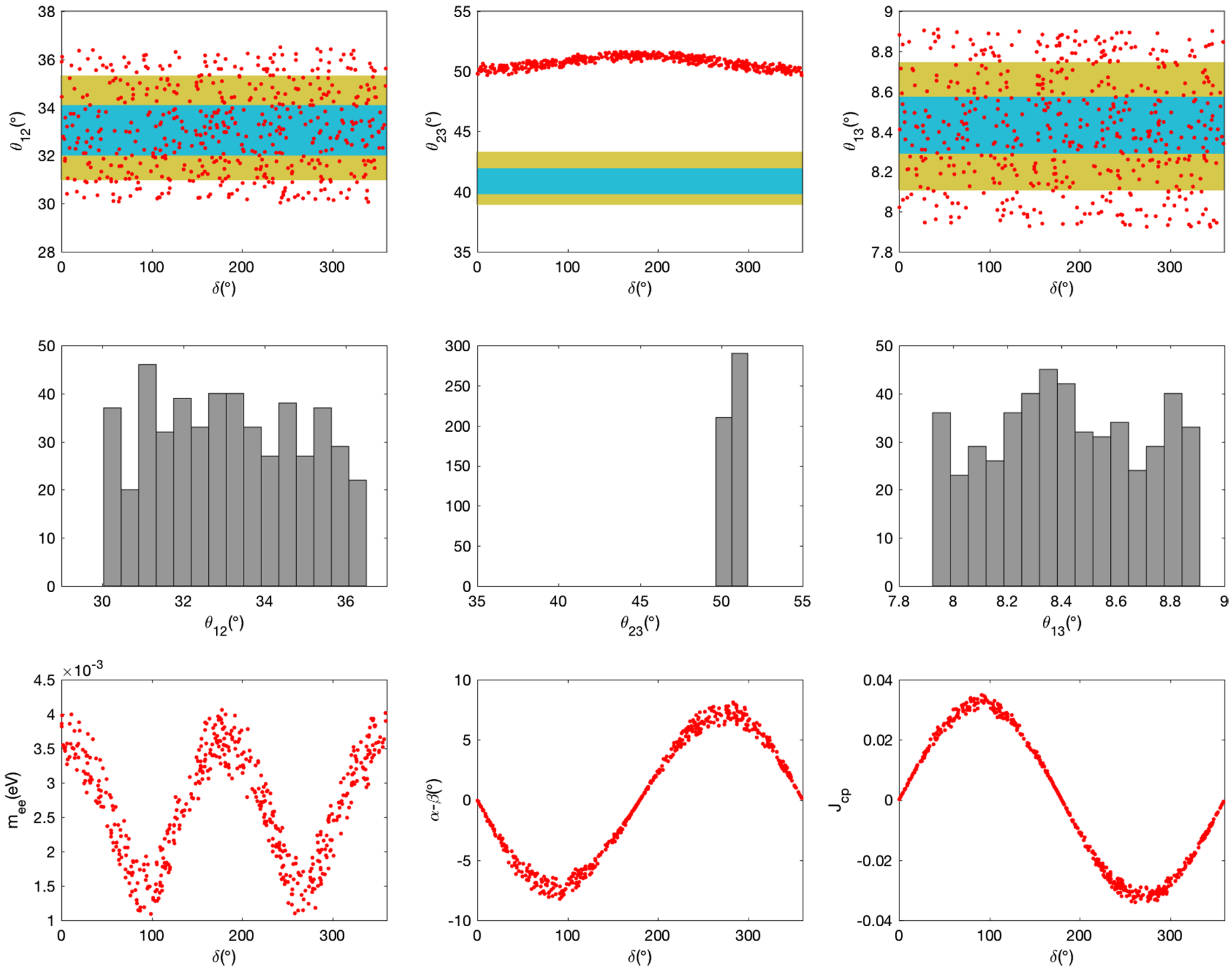

FIG. 1. The correlation plots for NO P4 pattern.

good approximation of vanishing neutrino mass, a ultra-light neutrino masse implies a large mass hierarchy between three generations of neutrinos. To realize hierarchies between neutrino masses, a popular approach is the Froggatt-Nielsen mechanism [30] that was originally developed for quark sector while is also suitable to understand the lepton mass and mixing pattern. In the following, we just take $P12$ pattern as a concrete example, although a complete realization for other patterns is also interesting and necessary.

We extend the particle content of standard model (SM) with three right-handed neutrinos $\nu_{Ri}(i=1,2,3)$, three SM-like doublet scalars $H_i(i=1,2,3)$ responsible for charged lepton masses generation, four additional doublet scalars $\phi_i(i=1,2,3,4)$ responsible for Dirac neutrino mass matrix M_D, two singlet scalars $\sigma_i(i=1,2)$ giving rise to the right-handed neutrino mass matrix M_R. Then the neutrino mass matrix is obtained by the canonical type-I seesaw formula $M_\nu = M_D M_R^{-1} M_D^T$. The general Lagrangian that producing to lepton masses is then given by

$$\mathcal{L} = \left(\frac{\langle\Phi\rangle}{\Lambda}\right)^{Q_{L_i}+Q_{l_{Rj}}} y_{ij}^{(k)} \bar{L}_i H_k l_{Rj} + \left(\frac{\langle\Phi\rangle}{\Lambda}\right)^{Q_{L_i}+Q_{\nu_{Rj}}} h_{ij}^{(k)} \bar{L}_i \tilde{\phi}'_k \nu_{Rj} + \left(\frac{\langle\Phi\rangle}{\Lambda}\right)^{Q_{\nu_{Ri}}+Q_{\nu_{Rj}}} g_{ij}^{(k)} \sigma_k \nu_{Ri} \nu_{Rj} + \text{H.c.}, \tag{30}$$

where $\tilde{\phi}'_k \equiv i\sigma_2 \phi'_k$ and we denote $\epsilon \equiv \langle\Phi\rangle/\Lambda$. The $Q_\alpha(\alpha = L, l_R, \nu_R)$ are interpreted as the FN charges for SM fermion ingredients under which different generations may be charged differently. The flavon Φ obtains the vacuum expectation value (VEV) $\langle\Phi\rangle$ that breaks the FN symmetry. We assign the FN charges for lepton sector as

$$\begin{aligned} \bar{L}_{1,2,3}&: (a+1, a, a) \\ l_{R1,2,3}&: (0, 1, 2) \\ \nu_{R1,2,3}&: (d, c, b). \end{aligned} \tag{31}$$

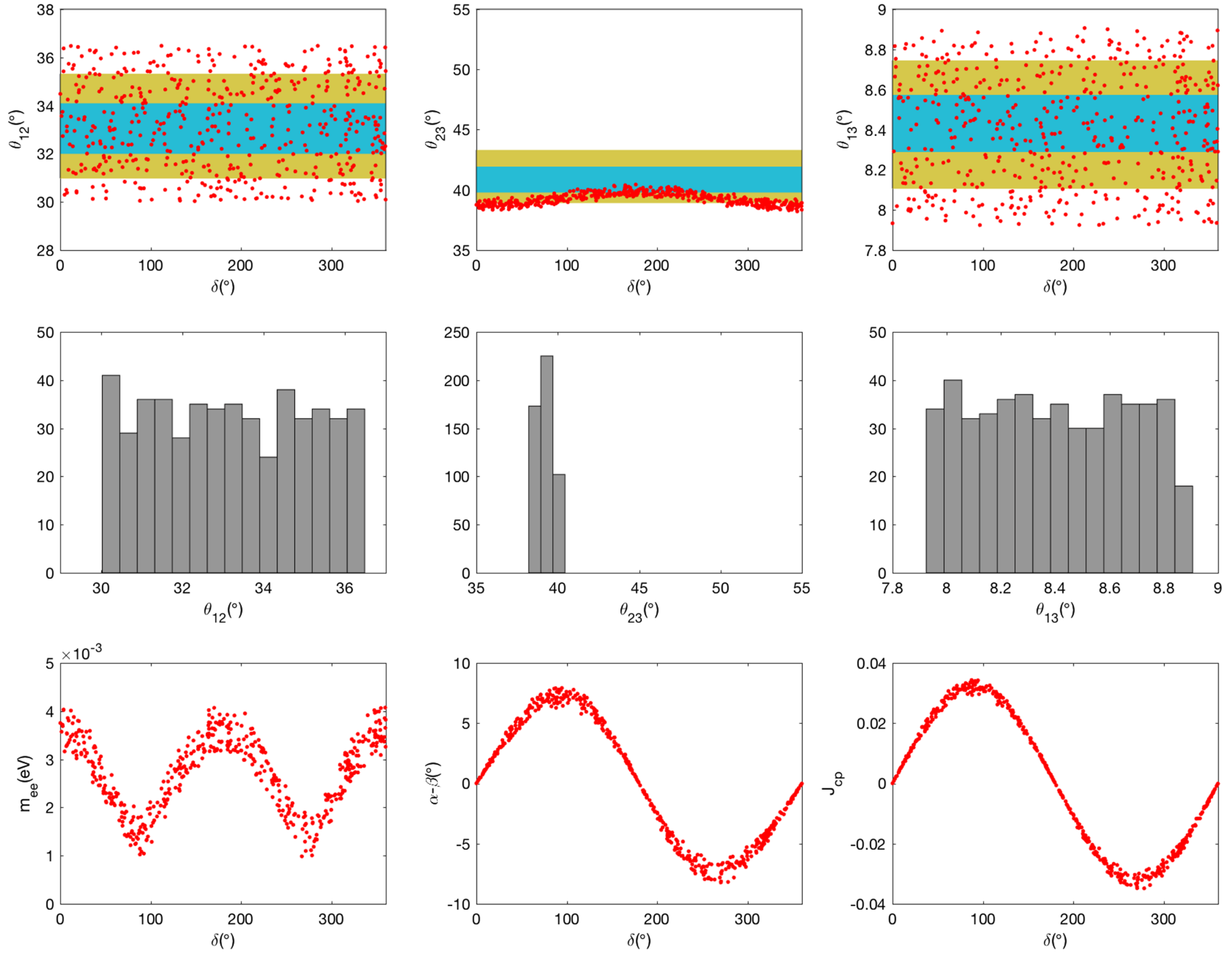

FIG. 2. The correlation plots for NO P6 pattern.

The realistic mass hierarchy between three charged leptons are successfully reproduced by adopting the assignment of FN charges for L_i and l_{RI} with $\epsilon \simeq 0.05$–0.1 [39]. The explicit forms of Dirac neutrino mass matrix M_D and right-handed neutrino mass matrix M_R are

$$M_D = \sum_{k=1}^{3} M_D^{(k)} = \sum_{k=1}^{3} v_k \begin{pmatrix} \epsilon^{a+1} & 0 & 0 \\ 0 & \epsilon^a & 0 \\ 0 & 0 & \epsilon^a \end{pmatrix} \begin{pmatrix} h_{11}^{(k)} & h_{12}^{(k)} & h_{13}^{(k)} \\ h_{21}^{(k)} & h_{22}^{(k)} & h_{23}^{(k)} \\ h_{31}^{(k)} & h_{32}^{(k)} & h_{33}^{(k)} \end{pmatrix} \begin{pmatrix} \epsilon^d & 0 & 0 \\ 0 & \epsilon^c & 0 \\ 0 & 0 & \epsilon^b \end{pmatrix}$$
$$= \sum_{k=1}^{3} v_k \begin{pmatrix} h_{11}^{(k)}\epsilon^{d+a+1} & h_{12}^{(k)}\epsilon^{c+a+1} & h_{13}^{(k)}\epsilon^{b+a+1} \\ h_{21}^{(k)}\epsilon^{d+a} & h_{22}^{(k)}\epsilon^{c+a} & h_{23}^{(k)}\epsilon^{b+a} \\ h_{31}^{(k)}\epsilon^{d+a} & h_{32}^{(k)}\epsilon^{c+a} & h_{33}^{(k)}\epsilon^{b+a} \end{pmatrix} \tag{32}$$

and

$$M_R = \sum_{k=1}^{2} M_R^{(k)} = \sum_{k=1}^{2} v'_k \begin{pmatrix} \epsilon^d & 0 & 0 \\ 0 & \epsilon^c & 0 \\ 0 & 0 & \epsilon^b \end{pmatrix} \begin{pmatrix} g_{11}^{(k)} & g_{12}^{(k)} & g_{13}^{(k)} \\ g_{21}^{(k)} & g_{22}^{(k)} & g_{23}^{(k)} \\ g_{31}^{(k)} & g_{32}^{(k)} & g_{33}^{(k)} \end{pmatrix} \begin{pmatrix} \epsilon^d & 0 & 0 \\ 0 & \epsilon^c & 0 \\ 0 & 0 & \epsilon^b \end{pmatrix} = \sum_{k=1}^{2} v'_k \begin{pmatrix} g_{11}^{(k)}\epsilon^{2d} & g_{12}^{(k)}\epsilon^{d+c} & g_{13}^{(k)}\epsilon^{d+b} \\ g_{21}^{(k)}\epsilon^{d+c} & g_{22}^{(k)}\epsilon^{2c} & g_{23}^{(k)}\epsilon^{c+b} \\ g_{31}^{(k)}\epsilon^{d+b} & g_{32}^{(k)}\epsilon^{c+b} & g_{33}^{(k)}\epsilon^{2b} \end{pmatrix} \tag{33}$$

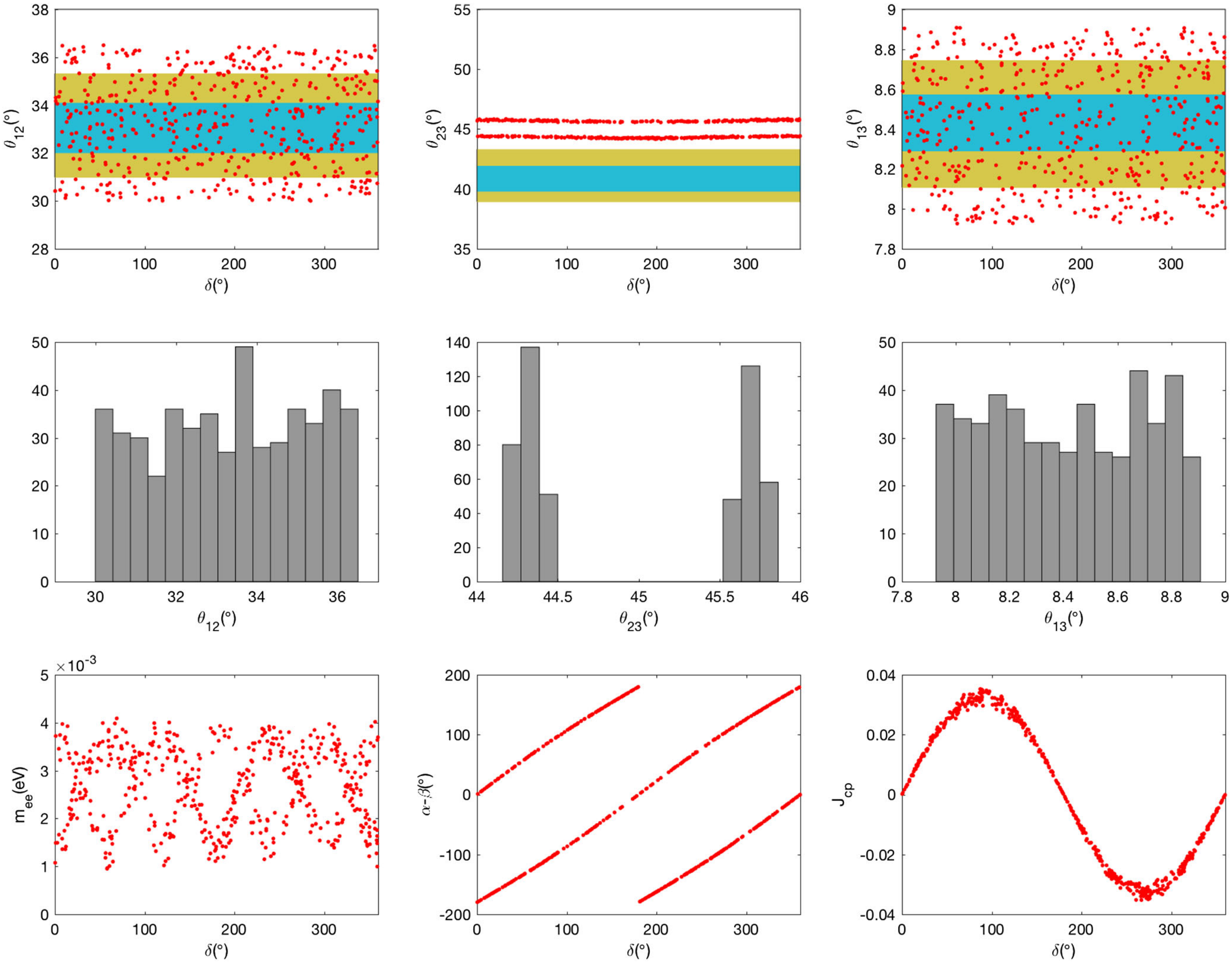

FIG. 3. The correlation plots for NO P12 pattern.

Following the spirit of Ref. [17], we consider a cooperation of $\mu - \tau$ permutation symmetry and Z_8 horizontal symmetry i.e $S_{\mu-\tau} \times Z_8$ under which the relevant particle fields transform as

$$\begin{aligned}
&(L_e, L_\mu, L_\tau) \xrightarrow{S} (L_e, L_\mu, L_\tau), \qquad (e_R, \mu_R, \tau_R) \xrightarrow{S} (e_R, \tau_R, \mu_R), \qquad (H_1, H_2, H_3) \xrightarrow{S} (H_1, H_3, H_2)\\
&(\nu_{R1}, \nu_{R2}, \nu_{R3}) \xrightarrow{S} (\nu_{R1}, \nu_{R3}, \nu_{R2}), \qquad (\phi_1, \phi_2, \phi_3, \phi_4) \xrightarrow{S} (\phi_1, \phi_2, \phi_3, -\phi_4), \qquad (\sigma_1, \sigma_2) \xrightarrow{S} (\sigma_1, \sigma_2)\\
&(L_e, L_\mu, L_\tau) \xrightarrow{Z_8} (L_e, -L_\mu, -L_\tau), \qquad (e_R, \mu_R, \tau_R) \xrightarrow{Z_8} (e_R, \tau_R, -\mu_R), \qquad (H_1, H_2, H_3) \xrightarrow{Z_8} (H_1, -H_2, -H_3)\\
&(\nu_{R1}, \nu_{R2}, \nu_{R3}) \xrightarrow{Z_8} (\omega \nu_{R1}, \omega^3 \nu_{R3}, \omega^3 \nu_{R2}), \qquad (\phi_1, \phi_2, \phi_3, \phi_4) \xrightarrow{Z_8} (\omega \phi_1, \omega^3 \phi_2, \omega^7 \phi_3, \omega^3 \phi_4)\\
&(\sigma_1, \sigma_2) \xrightarrow{Z_8} (\omega^6 \sigma_1, \omega^2 \sigma_2). \qquad (34)
\end{aligned}$$

As proposed in Ref. [17], one can further assume a hierarchy in H_i's vacuums ($\langle H_3 \rangle \gg \langle H_1 \rangle, \langle H_2 \rangle$). Then the $S_{\mu-\tau} \times Z_8$-invariant Lagrangian relevant to the charged lepton sector leads to the texture structure of $M_l M_l^\dagger$ given by

$$M_l M_l^\dagger = \begin{pmatrix} A^2 & 0 & 0 \\ 0 & D^2 & DC \\ 0 & DC & C^2 \end{pmatrix} \qquad (35)$$

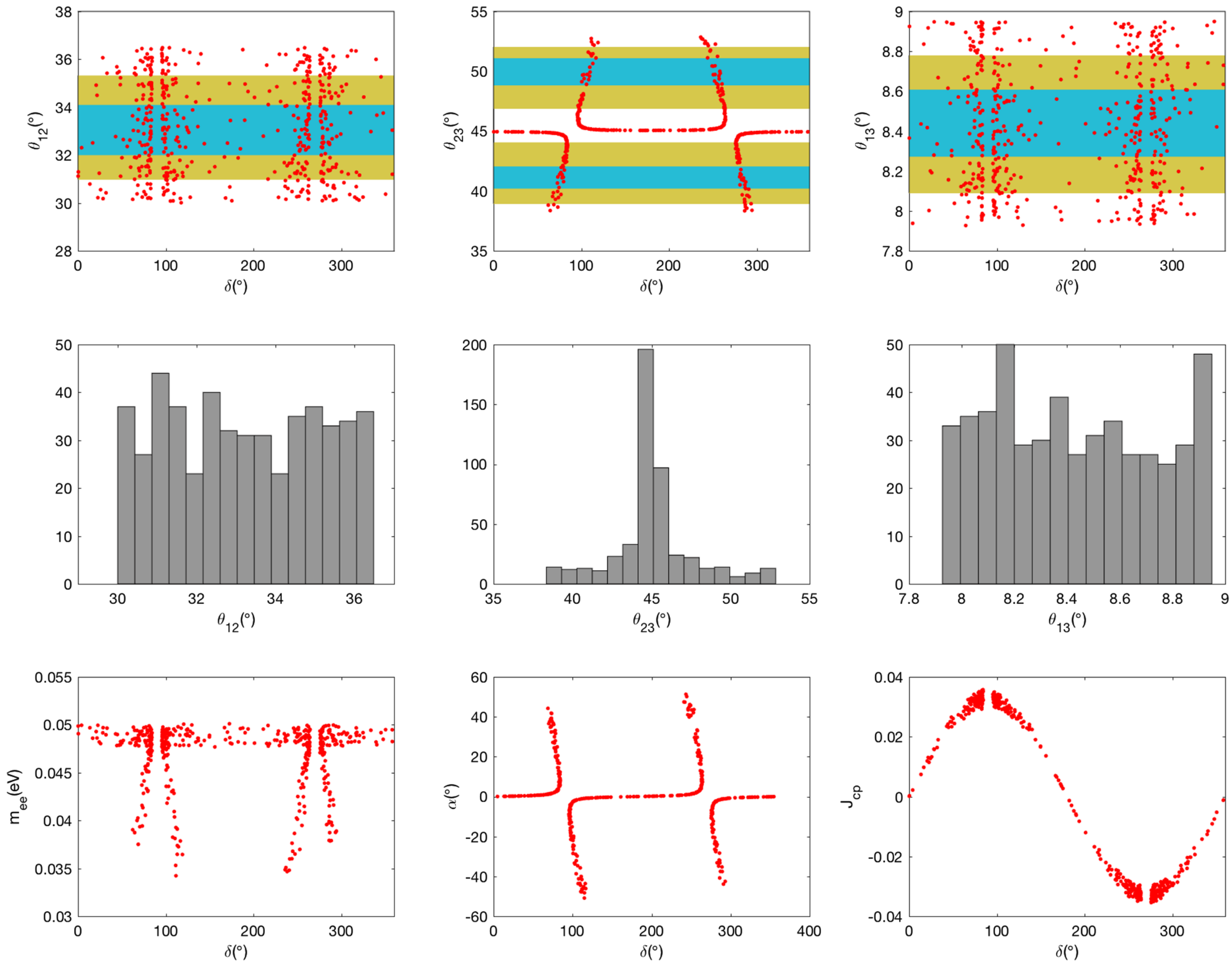

FIG. 4. The correlation plots for IO P12 pattern.

with

$$\frac{B}{C} \sim \frac{m_e}{m_\mu} = 2.8 \times 10^{-4} \qquad \frac{D}{C} \sim \frac{m_\mu}{m_\tau} = 5.9 \times 10^{-2} \tag{36}$$

As required, we construct the model in the flavor basis corrected by an extremely small rotation angle less than 10^{-2}

Now move to the neutrino sector. The $S_{\mu-\tau} \times Z_8$ symmetry imposes $b = c$ and the form of the neutrino mass matrices as follows:

$$M_D^{(1)} = \begin{pmatrix} A_1\epsilon^{d+a+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_D^{(2)} = \begin{pmatrix} 0 & B_2\epsilon^{c+a+1} & -B_2\epsilon^{c+a+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$M_D^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_3\epsilon^{c+a} & D_3\epsilon^{c+a} \\ 0 & D_3\epsilon^{c+a} & C_3\epsilon^{c+a} \end{pmatrix} \qquad M_D^{(4)} = \begin{pmatrix} 0 & B_4\epsilon^{c+a+1} & B_4\epsilon^{c+a+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{37}$$

where

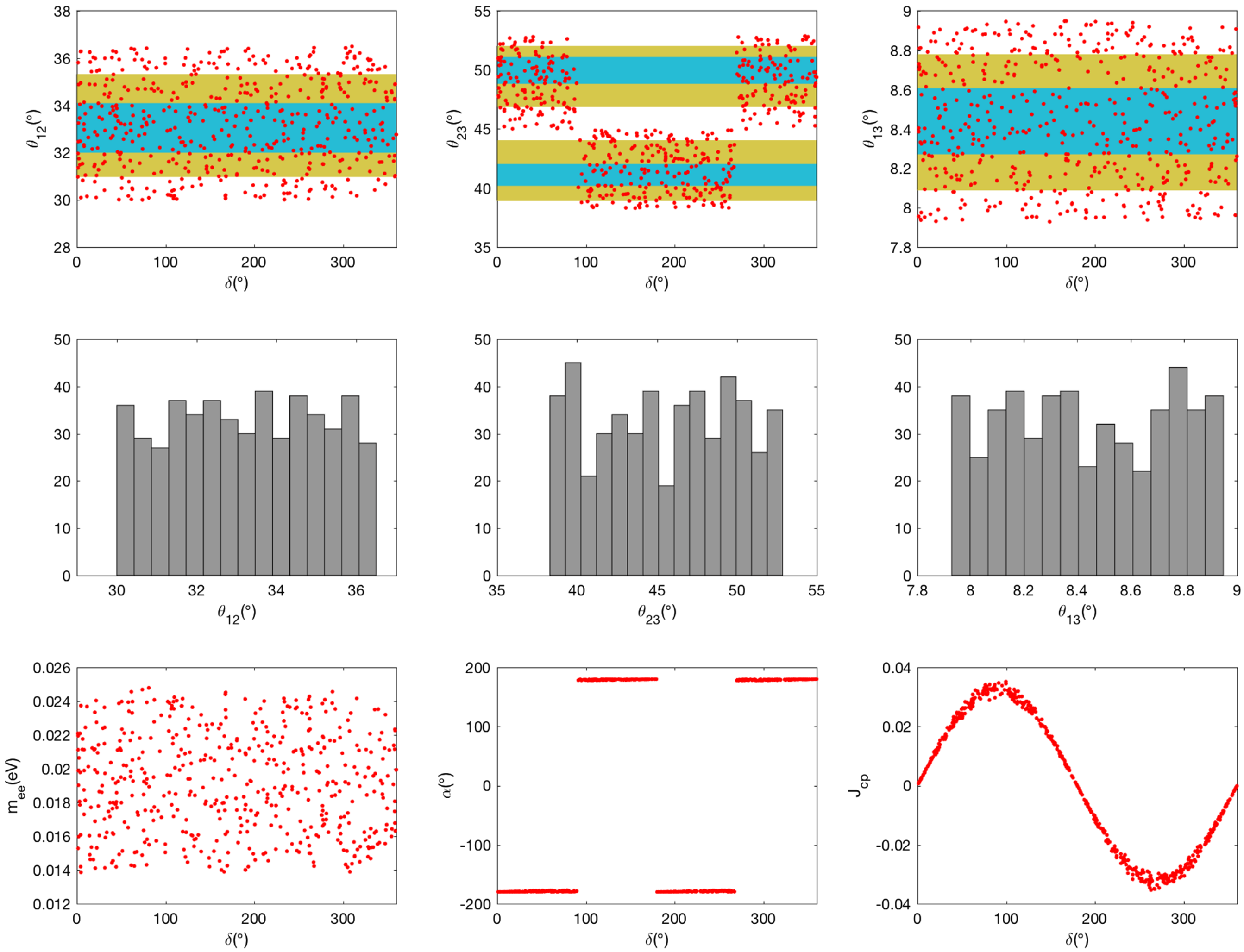

FIG. 5. The correlation plots for IO P13 pattern.

$$A_1 \equiv v_1 h_{11}^{(1)}, \qquad B_2 \equiv v_2 h_{12}^{(2)} = -v_2 h_{13}^{(2)}$$
$$C_3 \equiv v_3 h_{22}^{(3)} = v_3 h_{33}^{(3)} \qquad D_4 \equiv v_4 h_{23}^{(3)} = v_4 h_{32}^{(3)} \tag{38}$$

and

$$M_R^{(1)} = \begin{pmatrix} A_R\epsilon^{2d} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$M_R^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_R\epsilon^{2c} & D_R\epsilon^{2c} \\ 0 & D_R\epsilon^{2c} & C_R\epsilon^{2c} \end{pmatrix} \tag{39}$$

where

$$A_R = v_1' g_{11}^{(1)}, \qquad C_R = v_2' g_{22}^{(2)} = v_2' g_{33}^{(2)},$$
$$D_R = v_2' g_{23}^{(2)} = v_2' g_{32}^{(2)}. \tag{40}$$

Hence the Dirac neutrino mass matrix M_D and the right-handed neutrino mass matrix M_R are given by

$$M_D = \begin{pmatrix} A_D\epsilon^{d+a+1} & B_D'\epsilon^{c+a+1} & -B_D\epsilon^{c+a+1} \\ 0 & C_D\epsilon^{c+a} & D_D\epsilon^{c+a} \\ 0 & D_D\epsilon^{c+a} & C_D\epsilon^{c+a} \end{pmatrix}$$
$$M_R = \begin{pmatrix} A_R\epsilon^{2d} & 0 & 0 \\ 0 & C_R\epsilon^{2c} & D_R\epsilon^{2c} \\ 0 & D_R\epsilon^{2c} & C_R\epsilon^{2c} \end{pmatrix} \tag{41}$$

where we have defined

$$A_D \equiv A_1, \qquad B_D' \equiv B_2 + B_4, \qquad B_D \equiv -B_2 + B_4,$$
$$C_D \equiv C_3, \qquad D_D \equiv D_3. \tag{42}$$

With the help of Eq. (41) and seesaw formula $M_\nu = M_D M_R^{-1} M_D^T$, the effective neutrino mass matrix M_ν is obtained by a direct calculation, leading to the exact form of P_{12} pattern i.e.,

$$M_\nu = \begin{pmatrix} M_{\nu 11} & M_{\nu 12} & M_{\nu 13} \\ M_{\nu 12} & M_{\nu 22} & M_{\nu 23} \\ M_{\nu 13} & M_{\nu 23} & M_{\nu 22} \end{pmatrix} \quad (43)$$

In order to get a ultralight neutrino mass, we adopt the methodology given in Ref. [31]. The basic point is to suppose the broken FN symmetry is a discrete Z_n symmetry instead of continuous $U(1)_{\rm FN}$. One further assumes $n = 2d$ and $c < d$. Then the right-handed neutrino mass matrix M_R in Eq. (41) is

$$M_R = \begin{pmatrix} A_R & 0 & 0 \\ 0 & C_R\epsilon^{2c} & D_R\epsilon^{2c} \\ 0 & D_R\epsilon^{2c} & C_R\epsilon^{2c} \end{pmatrix} \quad (44)$$

We see that In M_R the element A_R is not suppressed by the power of ϵ compared with C_R and D_R, which indicates an extremely large right-handed neutrino masses for N_1 and hence yields an ultralight neutrino mass via type-I seesaw mechanism. The texture hierarchy in the power of ϵ is derived as

$$M_\nu \sim \epsilon^{2a} \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix} \quad (45)$$

Note that the FN charges d and c are completely canceled out when applying seesaw formula and ϵ^{2a} appears as a total factor in the form of M_ν. As first indicated in Ref. [39], the structure of neutrino mass matrices give in Eq. (45) can naturally leads to $\theta_{23} \simeq \frac{\pi}{4}$, which also appears as a remarkable feature of P_{12} pattern (Fig. 3).

V. CONCLUSION

In this paper, we have performed a systematic investigation on the neutrino mass matrix M_ν with and one vanishing neutrino mass ($m_{\rm lightest} = 0$) and one texture equality($M_{\nu ab} = M_{\nu cd}$). Although the zero neutrino mass are strictly set in numerical analysis, our results is also well suited for the scenarios where an ultralight neutrino mass is assumed in the consideration of both experimental and theoretical sides. Using the latest neutrino oscillation and cosmological data, a phenomenological analysis are systematically proposed for all possible patterns. It is found that four out of fifteen possible patterns are compatible with the experimental data at 3σ confidential level. In Figs. 1–5 and Table III, we show the numerical results of viable patterns for normal order and inverted order of neutrino masses. For each viable pattern, allowed regions of neutrino oscillation parameters $(\theta_{12}, \theta_{23}, \theta_{13}, \delta)$, effective Majorana mass m_{ee}, and Jarlskog quantity J_{CP} are presented. We have summarized the main results in the numerical part. These interesting predictions are promising to be explored in the upcoming long-baseline neutrino oscillation experiment, neutrinoless double beta decay experiments, and further cosmological study to the sum of neutrino masses.

Finally, we discussed the flavor symmetry realization of texture structures, where a concrete example has been illustrated in Ref. [17] based on $S_{\mu-\tau} \times Z_8$ symmetry. Inspired by this, we construct the model in the framework of Froggatt-Nielsen (FN) mechanism and subsequently realize the $P12$ pattern that naturally includes an ultralight neutrino mass. The theoretical realization of other patterns deserves further study. Anyway, we except that a cooperation between theoretical study from the flavor symmetry viewpoint and a phenomenology study will help us reveal the underlying structure of massive neutrinos.

ACKNOWLEDGMENTS

The work of Jinzhong Han is supported by National Natural Science Foundation of China (NSFC) under Grant No. U1404113, the Aid Project for the Mainstay Young Teachers in Henan Provincial Institutions of Higher Education of China under Grant No. 2016GGJS-135. The work of Weijian Wang is supported by National Natural Science Foundation of China under Grant No. 11505062, Special Fund of Theoretical Physics under Grant No. 11447117 and Fundamental Research Funds for the Central Universities (2016MS133).

[1] Q. R. Ahmad *et al.* (SNO Collaboration), Phys. Rev. Lett. **89**, 011301 (2002); K. Eguchi *et al.* (KamLAND Collaboration), Phys. Rev. Lett. **90**, 021802 (2003); M. H. Ahn *et al.* (K2K Collaboration), Phys. Rev. Lett. **90**, 041801 (2003); F. P. An *et al.* (DAYA-BAY Xollaboration), Phys. Rev. Lett. **108**, 171803 (2012); J. K. Ahn *et al.* (RENO Collaboration), Phys. Rev. D **108**, 191802 (2012).

[2] P. H. Frampton, S. L. Glashow, and D. Marfatia, Phys. Lett. B **536**, 79 (2002); H. Fritzsch, Z.-z. Xing, and S. Zhou, J. High Energy Phys. 09 (2011) 083.

[3] W. Grimus, A. S. Joshipura, L. Lavoura, and M. Tanimoto, Eur. Phys. J. C **36**, 227 (2004).

[4] Z. Z. Xing, Phys. Lett. B **530**, 159 (2002); A. Merle and W. Rodejohann, Phys. Rev. D **73**, 073012 (2006); S. Dev,

S. Kumar, S. Verma, and S. Gupta, Phys. Rev. D **76**, 013002 (2007); S. Dev, S. Kumar, S. Verma, and S. Gupta, Nucl. Phys. **B784**, 103 (2007); S. Dev and S. Kumar, Mod. Phys. Lett. A **22**, 1401 (2007); S. Kumar, Phys. Rev. D **84**, 077301 (2011); P. O. Ludl, S. Morisi, and E. Peinado, Nucl. Phys. **B857**, 411 (2012); W. Grimus and P. O. Ludl, arXiv:1208.4515; D. Meloni and G. Blankenburg, Nucl. Phys. **B867**, 749 (2013).

[5] G. C. Branco, D. Emmannuel-Costa, R. González Felipe, and H. Serôdio, Phys. Lett. B **670**, 340 (2009).

[6] M. Randhawa, G. Ahuja, and M. Gupta, Phys. Lett. B **643**, 175 (2006); G. Ahuja, S. Kumar, M. Randhawa, M. Gupta, and S. Dev, Phys. Rev. D **76**, 013006 (2007); G. Ahuja, M. Gupta, M. Randhawa, and R. Verma, Phys. Rev. D **79**, 093006 (2009); J. A. Acosta, A. Aranda, M. A. Buen-Abad, and A. D. Rojas, Phys. Lett. **18B**, 1413 (2013); S. Sharma, P. Fakay, G. Ahuja, and M. Gupta, arXiv:1402.0628; S. Sharma, P. Fakay, G. Ahuja, and M. Gupta, J. Phys. G **40**, 055003 (2013); S. Dev, L. Singh, and D. Raj, Eur. Phys. J. C **75**, 394 (2015).

[7] M. Gupta and G. Ahuja, Int. J. Mod. Phys. A **23**, 1270033 (2012).

[8] S. Kaneko, H. Sawanaka, and M. Tanimoto, J. High Energy Phys. 08 (2005) 073; S. Dev, S. Verma, and S. Gupta, Phys. Lett. B **687**, 53 (2010); S. Goswami, S. Khan, and A. Watanable, Phys. Lett. B **687**, 53 (2010); W. Grimus and P. O. Ludl, J. Phys. G **40**, 055003 (2013).

[9] J.-Y. Liu and S. Zhou, Phys. Rev. D **87**, 093010 (2013).

[10] X.-G. He and A. Zee, Phys. Rev. D **68**, 037302 (2003).

[11] L. Lavoura, Phys. Lett. B **609**, 317 (2005); E. I. Lashin and N. Chamoun, Phys. Rev. D **78**, 073002 (2008); E. I. Lashin and N. Chamoun, Phys. Rev. D **80**, 093004 (2009).

[12] S. Dev, S. Gupta, and R. R. Gautam, Mod. Phys. Lett. A **26**, 501 (2011); S. Dev, S. Gupta, R. R. Gautam, and L. Singh, Phys. Lett. B **706**, 168 (2011); T. Araki, J. Heeck, and J. Kubo, J. High Energy Phys. 07 (2012) 083; S. Verma, Nucl. Phys. **B854**, 340 (2012); S. Dev, R. R. Gautam, and L. Singh, Phys. Rev. D **89**, 013006 (2014); S. Dev, R. R. Gautam, and L. Singh, Phys. Rev. D **89**, 013006 (2014); W. Wang, Phys. Lett. B **733**, 320 (2014); W. Wang, Phys. Rev. D **90**, 033014 (2014); W. Wang, S. Y. Guo, and Z. G. Wang, Mod. Phys. Lett. A **31**, 1650080 (2016).

[13] S. Dev, S. Verma, S. Gupta, and R. R. Gautam, Phys. Rev. D **81**, 053010 (2010); J. Liao, D. Marfatia, and K. Whisnant, J. High Energy Phys. 09 (2014) 013.

[14] H. A. Alhendi, E. I. Lashin, and A. A. Mudlej, Phys. Rev. D **77**, 013009 (2008).

[15] S. Dev, R. R. Gautam, and L. Singh, Phys. Rev. D **87**, 073011 (2013).

[16] W. Wang, Eur. Phys. J. C **73**, 2551 (2013); S. Dev, R. R. Gautam, and L. Singh, Phys. Rev. D **88**, 033008 (2013).

[17] E. I. Lashin, N. Chamoun, C. Hamzaoui, and S. Nasri, Phys. Rev. D **89**, 093004 (2014).

[18] W. Wang and Z. L. Han, Phys. Rev. D **92**, 095001 (2015).

[19] W. Wang, R. Wang, Z. L. Han, and J. Z. Han, arXiv:1705.00414.

[20] I. Affleck and M. Dine, Nucl. Phys. **B249**, 361 (1985).

[21] H. Murayama and T. Yanagida, Phys. Lett. B **322**, 349 (1994).

[22] M. Dine, L. Randall, and S. D. Thomas, Nucl. Phys. **B458**, 291 (1996).

[23] T. Asaka, M. Fujii, K. Hamaguchi, and T. Yanagida, Phys. Rev. D **62**, 123514 (2000).

[24] M. Fujii, K. Hamaguchi, and T. Yanagida, Phys. Rev. D **64**, 123526 (2001).

[25] S. Antusch, S. F. King, C. Luhn, and M. Spinrath, Nucl. Phys. **B856**, 328 (2012).

[26] W. Rodejohann, M. Tanimoto, and A. Watanabe, Phys. Lett. B **710**, 636 (2012).

[27] R. R. Gautam, M. Singh, and M. Gupta, Phys. Rev. D **92**, 013006 (2015).

[28] P. A. R. Ade *et al.* (Planck Collaboration), Astron. Astrophys. **571**, A16 (2014).

[29] S. Wang, Y. F. Wang, D. M. Xia, and X. Zhang, Phys. Rev. D **94**, 083519 (2016).

[30] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. **B174**, 227 (1979).

[31] M. Fujii, K. Hamaguchi, and T. Yanagida, Phys. Rev. D **63**, 123513 (2001).

[32] H. Fritzsch, M. Gell-Mann, and P. Minkowski, Phys. Lett. **59B**, 256 (1975); P. Minkowski, Phys. Lett. **67B**, 421 (1977); T. Yanagida, in *Proceedings of Workshop on Unified Theory and the Baryon Number of the Universe*, edited by O. Sawada and A. Sugamoto (KEK, Tsukuba, 1979), p. 95; M. Gell-Mann, P. Ramond, and Slansky, in *Supergravity*, edited by P. van. Nieuwenhuizen and D. Z. Freeman (North-Holland, Amsterdam, 1979), p. 315; R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980); J. Schechter and J. W. F. Valle, Phys. Rev. D **22**, 2227 (1980); J. Schechter and J. W. F. Valle, Phys. Rev. D **25**, 774 (1982).

[33] B. Pontecorvo, Zh. Eksp. Teor. Fiz. **33**, 549 (1957); Z. Maki, M. Nakagawa, and N. Sakata, Prog. Theor. Phys. **28**, 870 (1962).

[34] C. Patrignani *et al.* (Particle Data Group), Chin. Phys. C **40**, 100001 (2016).

[35] F. Capozzi, E. Di Valentino, E. Lisi, A. Marrone, A. Melchiorri, and A. Palazzo, Phys. Rev. D **95**, 096014 (2017).

[36] S. M. Bilenky and C. Giunti, Mod. Phys. Lett. A **27**, 1230015 (2012).

[37] F. Couchot, S. Henrot-Versillé, O. Perdereau, S. Plaszczynski, B. Rouillé D'orfeuil, M. Spinelli, and M. Tristram, arXiv:1703.10829.

[38] B. C. Chauhan, J. Pulido, and M. Picariello, Phys. Rev. D **73**, 053003 (2006).

[39] J. Sato and T. Yanagida, Phys. Lett. B **493**, 356 (2000).