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Abnormal isospin violation and $a_0 - f_0$ mixing in the $D_s^+ \to \pi^+\pi^0 a_0(980)(f_0(980))$ reactions

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We have chosen the reactions $D_s^+ \to \pi^+\pi^0 a_0(980)(f_0(980))$ investigating the isospin violating channel $D_s^+ \to \pi^+\pi^0 f_0(980)$. The reaction was chosen because by varying the $\pi^0 a_0(980)(f_0(980))$ invariant mass one goes through the peak of a triangle singularity emerging from $D_s^+ \to \pi^+\bar{K}^* K$, followed by $\bar{K}^* \to \bar{K}\pi^0$ and the further merging of $K\bar{K}$ to produce the $a_0(980)$ or $f_0(980)$. We found that the amount of isospin violation had its peak precisely at the value of the $\pi^0 a_0(980)(f_0(980))$ invariant mass where the singularity has its maximum, stressing the role of the triangle singularities as a factor to enhance the mixing of the $f_0(980)$ and $a_0(980)$ resonances. We calculate absolute rates for the reactions and show that they are within present measurable range. The measurement of these reactions would bring further information into the role of triangle singularities in isospin violation and the $a_0 - f_0$ mixing, in particular, and shed further light into the nature of the low energy scalar mesons.

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I. INTRODUCTION

The issue of the $f_0(980) - a_0(980)$ mixing has attracted much attention in the hadron community due to its potential to learn about the nature of the low lying scalar mesons. First suggested in Ref. [1], it was very early identified as being tied to the mass difference between the charged and neutral kaons [1,2]. Different reactions were suggested to find signals of this mixing in the $pn \to d\eta\pi^0$ [3], the $\gamma p \to p\pi^0\eta$ [4] and the $\pi^- p \to \pi^0\eta n$ [5]. Finally, it was the $J/\psi \to \phi\eta\pi^0$ reaction which showed clearly a mixing. This reaction had been suggested in Ref. [6] and estimates were done there. A more detailed calculation was presented in Ref. [7], using the chiral unitary approach [8,9] to account for the interaction of pseudoscalar mesons that generate the $f_0(980)$ and $a_0(980)$ resonances, and the mechanism for $f_0(980)$ production used in Ref. [10] for the $J/\psi \to \phi\pi\pi$ reaction. In that paper the role of the $K\bar{K}$ loops and of the difference of masses between the K^+ and K^0 was further investigated. A further revision of this issue was done in Ref. [11], where the production model for $J/\psi \to \phi\pi\pi$ and $J/\psi \to \phi\pi\eta$ was improved taking the more complete model of Ref. [12] for $J/\psi \to \phi\pi\pi$. The work of Ref. [11] reproduced very accurately the shape and magnitude of the $a_0(980)$ production in the $J/\psi \to \phi\pi\eta$ reaction [13], together with the $f_0(980)$ production in $J/\psi \to \phi\pi\pi$ with no more free parameters than the one used to regularize the loops in the study of the pseudoscalar-pseudoscalar interaction in Ref. [8]. The new mechanisms used in Ref. [11], accounting for sequential vector and axial-vector meson exchange, were found to be crucial in order to obtain the actual shape and strength (in about a factor of 2) of the mass distributions. Further study of the mixing and suggestion of reactions to observe it was done in Ref. [14].

The concept of an $f_0(980) - a_0(980)$ mixing parameter was accepted when a new reaction came to challenge it. The reaction was the $\eta(1405)$ decay to $\pi^0 f_0(980)$ measured at BESIII [15], which showed an unusually large isospin violation, or equivalently a very large $f_0(980) - a_0(980)$ mixing when compared with the isospin allowed $\eta(1405) \to \pi^0 a_0(980)$. This abnormal mixing found an explanation in Ref. [16] due to the role of a triangle singularity (TS) involving a mechanism in which the $\eta(1405)$ decays to $K^*\bar{K}$, followed by the decay of K^* in $K\pi$ and the merging of $K\bar{K}$ to give the $f_0(980)$ or $a_0(980)$. Further work along these lines was done in Ref. [17] where ambiguities in the size of the $\eta(1405) \to \pi^0 a_0(980)$ in Ref. [16] were solved. More work along these lines followed in Ref. [18], where it was suggested that the $\eta(1405)$ and $\eta(1475)$ are actually the same state.

TSs were introduced by Landau [19] for the decay of an external particle and develop from a mechanism depicted by a Feynman diagram with three intermediate propagators. When the three intermediate particles are simultaneously placed on shell and are collinear in the rest frame of the decaying particle, a singularity can emerge if the process has a classical correspondence, which is known as the Coleman-Norton theorem [20]. A modern and easy formulation of the problem is given in the paper [21].

While finding physical examples was not successful at the origin of the formulation of the TS, the advent of vast experimental information nowadays is providing many examples of TSs, sometimes simulating a resonance, other times providing a mechanism for the production of particular modes in reactions. Suggestions of places to look for triangle singularities were made in Ref. [22]. One of them was the possibility that the COMPASS claimed "$a_1(1420)$" resonance [23] would not be a genuine state but the manifestation of a TS with intermediate states $K^*\bar{K}K$. This hypothesis was made quantitative in Ref. [24]. The suggested mechanism implied the decay of the $a_1(1260)$ into $K^*\bar{K}$, followed by the decay of K^* into $K\pi$ and the further fusion of $K\bar{K}$ into the $f_0(980)$ giving rise to the decay mode $\pi f_0(980)$ observed in the experiment [23]. Further refinements along this line with consideration of the $\rho\pi$ decay of the $a_1(1260)$ resonance were done in Ref. [25], leading to a more accurate determination of the experimental observables and to the same conclusion.

Suggestions that the observed charged charmonium $Z_c(3900)$ [26–29] could be due to a TS were made in Refs. [22,30,31], and similar claims were made regarding other quarkonium [31] and bottomnium [32]. Claims that the narrow pentaquark state found by the LHCb collaboration [33,34] could be due to a triangle singularity were made in Refs. [35,36], but it was shown in Ref. [21] that if the quantum numbers of this state are $3/2^-, 5/2^+$, the $\chi_{c1}p$ that merges to form the final $J/\psi p$ state is at threshold and would be in p or d wave, respectively. Since the TS appears from placing all intermediate states on shell, the signal coming from the suggested mechanism is drastically reduced and the shape is also distorted such that it cannot reproduce the observed signal.

The TS has been discussed in the analysis of some reactions where its consideration can lead to different conclusions than using standard partial wave analysis tools [37–39].

Further examples of TSs have recently been investigated. Some of them show that resonances accepted in the PDG [40] actually correspond to triangle singularities, which produce a peak, although not related to the interaction of quarks or hadrons, but to the structure of the triangle diagram, tied to the masses of the intermediate states. Apart from the case of the $a_1(1420)$ discussed above, the $f_1(1420)$ peak was shown to correspond to the $f_1(1285)$ decay into $\pi a_0(980)$, through a TS, and $K^*\bar{K}$ [41]. The $f_2(1810)$ peak was also shown to come from a TS involving $K^*\bar{K}^*$ production, followed by $K^* \to \pi K$ and $\bar{K}^*K \to a_1(1260)$ [42]. Some other times the TS helps building up a particular decay channel of a resonance generated from the interaction of hadrons. This is the case of the $N(1700)$ which is generated from the ρN interaction with other coupled channels [43,44], but which gets a sizeable $\pi\Delta$ decay channel through the mechanism $N(1700) \to \rho N$ followed by $\rho \to \pi\pi$ and then $\pi N \to \Delta$ [45]. It is also the case of the $N(1875)(3/2^-)$, which emerges from the interaction of $\Delta\pi$ and Σ^*K channels [46], but that builds up the $N(1535)\pi$ and $N\sigma$ decay channels from two TSs [47].

In some other cases a TS has been shown to solve some known puzzle, like the enhancement in the $\gamma p \to K^+\Lambda(1405)$ cross section around $\sqrt{s} = 2110$ MeV [48] which was discussed from the TS perspective in Ref. [49], and the $\pi N(1535)$ contribution to the $\gamma p \to \pi^0\eta p$ reaction [50], which was discussed from that perspective in Ref. [51].

Finally, based on known hadron dynamics, it has become relatively easy to make predictions of peaks that should show up in some reactions, which are solely tied to TSs. In this line we can quote the $B^- \to K^-\pi^-D_{s0}^+$ and $B^- \to K^-\pi^-D_{s1}^+$ reactions [52], where one finds this type of nonresonant peak at 2850 MeV in the invariant mass of $\pi^-D_{s0}^+$ pair and at 3000 MeV in the invariant mass of $\pi^-D_{s1}^+$ pair respectively [52], or the $B_c \to B_s\pi\pi$ reaction, which develops a peak at 5777 MeV in the invariant mass of $B_s^0\pi^+$ [53].

Coming back to the $f_0(980) - a_0(980)$ mixing, the works done on the subject have shown that the different K^+ and K^0 masses are responsible for this mixing. In this sense, mechanisms that proceed via a triangle singularity with $K\bar{K}$ in the intermediate states of the triangle diagram should stress this mixing and make the isospin violating process more efficient. This is because the triangle singularity emerges from having the particles on shell, and this is where the differences of masses play a more relevant role. In this sense, the $\eta(1405) \to \pi^0 f_0(980)$ reaction is a good example. However, the reaction occurs at a fixed energy, 1405 MeV. The purpose of the present work is to suggest a reaction where we can change the initial energy to show the isospin violation as a function of the energy, and see that it peaks at the energy where the TS develops. We have found such a case in the $D_s^+ \to \pi^+\pi^0 a_0(980)(f_0(980))$ reactions which we discuss here. The D_s^+ state decays to $\pi^+ s\bar{s}$ and the $s\bar{s}$ quarks hadronize to $\bar{K}^*K$; the $\bar{K}^*$ decays to $\bar{K}\pi^0$ and the $K\bar{K}$ merge to produce the $a_0(980)$ or $f_0(980)$. Since the $s\bar{s}$ system is in $I = 0$ the $\pi^0 a_0(980)$ mode is the isospin allowed channel, while the $\pi^0 f_0(980)$ mode is the isospin forbidden one. We see that both decay modes are enhanced around a $\pi^0 a_0(980)$ or $\pi^0 f_0(980)$ invariant mass of 1420 MeV, but the isospin forbidden channel is more enhanced than the isospin allowed one. Also we can evaluate absolute rates and show that they are well within the present measurable range. Since the evaluations are based on the notion that the $f_0(980)$ and $a_0(980)$ resonances are generated from the interaction of coupled channels of pseudoscalar mesons, the rates obtained are tied to this picture and an eventual agreement of the future experiment with the predictions done here would further support this picture for which there is already much phenomenological support [54–56].

II. FORMALISM

A. The $D_s^+ \to \pi^+K^0\bar{K}^{*0}$ reaction

If we look at the D_s^+ Cabibbo favored and color favored decay process at the quark level, we have the diagram given in Fig. 1(a), corresponding to external emission in the

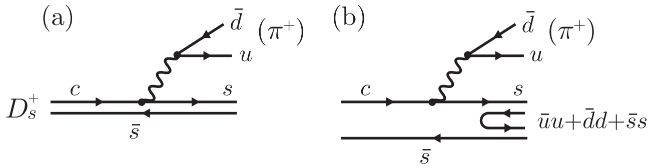

FIG. 1. (a) Diagrammatic representation of $D_s \to \pi^+ \bar{s}s$. (b) Hadronization process through $\bar{q}q$ creation with vacuum quantum number.

classification of Refs. [57,58]. The process is isospin selective because $s\bar{s}$ has $I = 0$. The $s\bar{s}$ can hadronize with strong interaction leading to two mesons in $I = 0$ incorporating a $\bar{q}q$ pair with the quantum numbers of the vacuum. In order to see the meson content of $s\bar{s}(u\bar{u} + d\bar{d} + s\bar{s})$, we use the arguments of Refs. [59,60] with the $q\bar{q}$ matrix in terms of mesons and we find

$$s\bar{s}(\bar{u}u + \bar{d}d + \bar{s}s) = K^+K^- + K^0\bar{K}^0 + \cdots, \quad (1)$$

where the points $\cdots$ indicate terms in η, η' which play no role in the reaction that we study. The decomposition in Eq. (1) has to do with flavor alone, and what it tells us is that we get the $K\bar{K}$ combination in $I = 0$ [(K^+, K^0) and $(\bar{K}^0, -K^-)$ are the isospin doublets in our notation]. However, we can get equally $K\bar{K}^*$ and this is the channel that we pick up to study our process. Hence we look at the decay

$$D_s^+ \to \pi^+(K^+K^{*-} + K^0\bar{K}^{*0}). \quad (2)$$

The reason to choose this channel is that we have the rate for $D_s^+ \to \pi^+K^{*-}K^+ \to \pi^+\pi^0K^-K^+$ decay [61]. Since K^{*-} has a branching fraction twice as big for $\pi^-\bar{K}^0$ than for π^0K^-, the rate for $D_s^+ \to \pi^+K^{*-}K^+$ is three times bigger than the one for $\pi^-\pi^0K^-K^+$ and thus

$$\mathrm{BR}(D_s^+ \to \pi^+K^{*-}K^+) = 3 \times (6.37 \pm 0.21 \pm 0.56) \times 10^{-2}, \quad (3)$$

quite a large rate.

In the reaction of Eq. (2) angular momentum is conserved and since we have a vector meson ($J^P = 1^-$) and pseudoscalar meson (0^-) in the final state we need a p-wave. It is easy to see that nonrelativistically the right coupling is $\vec{\epsilon}^*_{\bar{K}^*} \cdot \vec{p}_{\pi^+}$. Indeed, the $W^+\pi^+$ vertex goes as $(\partial_\mu\pi^+)W^{+\mu}$ [62,63] and the csW as $\gamma^\nu(1-\gamma_5)W_\nu$ [57,64]. The $\gamma^i\gamma_5$ matrix is proportional to σ^i at the quark level which is needed to pass from a pseudoscalar to a vector and we are left with the $\partial_i\pi^+$ component. Hence, we take

$$t_{D_s^+\to\pi^+K^+K^{*-}} = C\vec{\epsilon}_{K^{*-}} \cdot \vec{p}_{\pi^+}, \quad (4)$$

and we take C as constant since there is not much phase space for this reaction. When evaluating the triangle diagram we work in the $\bar{K}^*K$ system at rest where the $\bar{K}^*$ has a small three momentum. This is also the case in the $\pi^+K^0\bar{K}^{*0}$ reaction and we neglect the $\epsilon^0(\bar{K}^*)$ component in Eq. (4), but evaluate $\vec{p}_{\pi^+}$ in the $K\bar{K}^*$ rest frame.

Since we need the constant C in the evaluation of the triangle diagram we proceed to its evaluation by using Eqs. (3) and (4). Summing over the polarization of the K^{*-}, we have for the $D_s^+ \to \pi^+K^{*-}K^+$ reaction,

$$\frac{d\Gamma_{D_s^+\to\pi^+K^+K^{*-}}}{dM_{\mathrm{inv}}(K^+K^{*-})} = \frac{1}{(2\pi)^3}\frac{p_{\pi^+}\tilde{p}_{K^{*-}}}{4m^2_{D_s^+}}C^2p'^2_{\pi^+}, \quad (5)$$

where p_{π^+} is the π^+ momentum in the D_s^+ rest frame, $\tilde{p}_{K^{*-}}$ the one of the K^{*-} in the K^+K^{*-} rest frame and p'_{π^+} the π^+ momentum in the latter frame. These momenta are given by

$$p_{\pi^+} = \frac{\lambda^{1/2}(m^2_{D_s^+}, m^2_{\pi^+}, M^2_{\mathrm{inv}}(K^+K^{*-}))}{2m_{D_s^+}}, \quad (6)$$

$$\tilde{p}_{K^{*-}} = \frac{\lambda^{1/2}(M^2_{\mathrm{inv}}(K^+K^{*-}), m^2_{K^+}, m^2_{K^{*-}})}{2M_{\mathrm{inv}}(K^+K^{*-})}, \quad (7)$$

$$p'_{\pi^+} = \frac{\lambda^{1/2}(m^2_{D_s^+}, m^2_{\pi^+}, M^2_{\mathrm{inv}}(K^+K^{*-}))}{2M_{\mathrm{inv}}(K^+K^{*-})}, \quad (8)$$

where $\lambda(x,y,z)$ is the Källen function defined by $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$.

B. The $D_s^+ \to \pi^+\pi^0a_0(980)$ $(f_0(980))$ reactions

In order to produce the $a_0(980)$ or $f_0(980)$, we look at the decay products $\pi^0\eta$ and $\pi^+\pi^-$ of the $a_0(980)$ and $f_0(980)$ respectively. The mechanism to produce the $a_0(980)$ is depicted in Fig. 2. The mechanism of Fig. 2 involves a triangle diagram. The $\bar{K}^*$ decays to $\pi^0\bar{K}$ and then the remaining K and this $\bar{K}$ fuse to give the $a_0(980)$ or $f_0(980)$. The sum of the two diagrams is constructive for $\pi^0\pi^0\eta$ production via π^0a_0 and destructive for $\pi^0\pi^+\pi^-$ production via π^0f_0. In the case the K^+ and K^0 masses are equal, we would have the s-wave $K^+K^- \to \pi^0\eta$ and $K^0\bar{K}^0 \to \pi^0\eta$ amplitudes opposite, but the $K^+K^- \to \pi^0\eta$ and $K^0\bar{K}^0 \to \pi^+\pi^-$ equal. Taking account of the fact that the vertex $\bar{K}^{*0} \to \pi^0\bar{K}^0$ has the opposite sign to

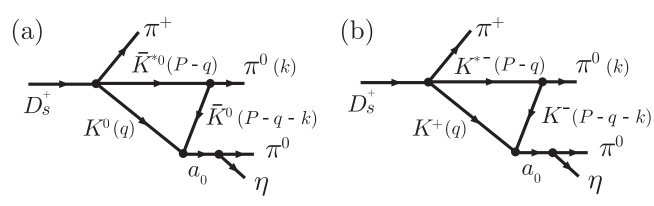

FIG. 2. Triangle mechanism which produces $\pi^+\pi^0a_0(980)$. The $\pi^+\pi^0f_0(980)$ channel could be seen replacing $\pi^0\eta$ by $\pi^+\pi^-$ at the end. The momenta of the particles are given in the brackets.

$K^{*-} \to \pi^0 K^-$, the sum of diagrams in Fig. 2 for $\pi^0 f_0$ production (assuming also equal $\bar{K}^{*0}$ and K^{*-} masses equal) would cancel and we would have exact $I = 0$ ($\pi^0 a_0(980)$) production, corresponding to the original $\bar{s}s$ state, and no $I = 1$ ($\pi^0 f_0(980)$) production. When the K masses are allowed to have their physical values we get two sources of isospin symmetry breaking, from the $K\bar{K} \to \pi^0\eta(\pi^+\pi^-)$ amplitudes, when they are evaluated with the actual K masses, and from the loop function of Fig. 2, which is different for the two diagrams thanks to the different K masses (also $\bar{K}^*$). The interesting thing is that we can now tune the invariant mass of $\pi^0 a_0$ ($\pi^0 f_0$) by changing the energy of the emitted π^+, and for a certain value of this invariant mass, we get a triangle singularity that enhances the production of both $\pi^0 a_0$ and $\pi^0 f_0$ modes. The TS places the $K\bar{K}^*\bar{K}$ on shell in the loop integration when the momenta of the $\bar{K}^*$ and π^0 from the $\bar{K}^*$ decay have the same direction. Since the different masses of the charged and neutral kaon cause the $\pi^0 f_0(980)$ production, the on-shell contribution is the most sensitive to these differences and we expect that the TS will enhance the $\pi^0 f_0$ production versus the $\pi^0 a_0$ one.

We proceed now to the evaluation of the diagram of Fig. 2. Apart from the vertex of Eq. (4), we need the $\bar{K}^* \to \pi\bar{K}$ vertices that are obtained from the ordinary Lagrangian,

$$\mathcal{L}_{VPP} = -ig\langle[P, \partial_\mu P]V^\mu\rangle; \qquad g = \frac{m_V}{2f_\pi}, \tag{9}$$

where P and V are the ordinary pseudoscalar and vector meson SU(3) matrices [43], m_V the vector mass ($m_V \sim 800$ MeV) and f_π the pion decay constant $f_\pi = 93$ MeV. This produces a vertex

$$t_{\bar{K}^{*0}\to\pi^0\bar{K}^0} = \frac{g}{\sqrt{2}}(p_{\bar{K}^0} - p_{\pi^0})^\mu \epsilon_{\bar{K}^{*0}\mu}, \tag{10}$$

and opposite sign for $K^{*-} \to \pi^0 K^-$.

With the former ingredients, the amplitude for the diagram of Fig. 2(a) is given by

$$\begin{aligned} t = {} & \frac{1}{\sqrt{2}} g C t_{K^0\bar{K}^0,\pi^0\eta} \sum_{\text{pol}} i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_{K^+}^2 + i\epsilon} \\ & \times \frac{1}{(P-q)^2 - m_{\bar{K}^{*0}}^2 + i\epsilon} \frac{1}{(P-q-k)^2 - m_{K^-}^2 + i\epsilon} \\ & \times [\vec{\epsilon}_{\bar{K}^{*0}} \cdot (2\vec{k} + \vec{q})][\vec{\epsilon}_{\bar{K}^{*0}} \cdot \vec{p}_{\pi^+}]. \end{aligned} \tag{11}$$

Summing upon the polarizations of the intermediate vector meson and taking $\vec{P} = 0$, Eq. (11) reads

$$t = \frac{1}{\sqrt{2}} g C t_{K^0\bar{K}^0,\pi^0\eta} i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_{K^+}^2 + i\epsilon} \frac{1}{(P-q)^2 - m_{\bar{K}^{*0}}^2 + i\epsilon} \frac{1}{(P-q-k)^2 - m_{K^-}^2 + i\epsilon} \vec{p}_{\pi^+} \cdot (2\vec{k} + \vec{q}). \tag{12}$$

Since in the integral of Eq. (12) the only vector not integrated is $\vec{k}$, we use $\int d^3q f(\vec{q}, \vec{k}) q_j = k_j \int d^3q f(\vec{q}, \vec{k})(\vec{q} \cdot \vec{k})/\vec{k}^2$ with $f(\vec{q}, \vec{k})$ being the remaining terms in Eq. (12), and then we can write Eq. (12) as

$$t = \frac{1}{\sqrt{2}} g C t_{K\bar{K}^0,\pi^0\eta} t_T(\vec{p}_{\pi^+} \cdot \vec{k}), \tag{13}$$

where t_T is given by

$$t_T = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_{K^0}^2 + i\epsilon} \frac{1}{(P-q)^2 - m_{\bar{K}^{*0}}^2 + i\epsilon} \cdot \frac{1}{(P-q-k)^2 - m_{\bar{K}^0}^2 + i\epsilon} \left(2 + \frac{\vec{q} \cdot \vec{k}}{\vec{k}^2}\right). \tag{14}$$

Performing analytically the q^0 integration in Eq. (14), we obtain [21,65]

$$\begin{aligned} t_T = {} & \int \frac{d^3q}{(2\pi)^3} \frac{1}{8\omega_{K^0}\omega_{\bar{K}^{*0}}\omega_{\bar{K}^0}} \frac{1}{k^0 - \omega_{\bar{K}^0} - \omega_{\bar{K}^{*0}} + i\Gamma_{\bar{K}^{*0}}/2} \frac{1}{M_{\text{inv}}(\pi^0 a_0) + \omega_{K^0} + \omega_{\bar{K}^0} - k^0} \\ & \cdot \frac{2M_{\text{inv}}(\pi^0 a_0)\omega_{K^0} + 2k^0\omega_{\bar{K}^0} - 2(\omega_{K^0} + \omega_{\bar{K}^0})(\omega_{K^0} + \omega_{\bar{K}^{*0}} + \omega_{\bar{K}^0})}{[M_{\text{inv}}(\pi^0 a_0) - \omega_{K^0} - \omega_{\bar{K}^0} - k^0 + i\epsilon][M_{\text{inv}}(\pi^0 a_0) - \omega_{\bar{K}^{*0}} - \omega_{K^0} + i\Gamma_{\bar{K}^{*0}}/2]} \left(2 + \frac{\vec{q} \cdot \vec{k}}{\vec{k}^2}\right), \end{aligned} \tag{15}$$

where $\omega_{K^0} = \sqrt{\vec{q}^2 + m_{K^0}^2}$, $\omega_{\bar{K}^0} = \sqrt{(\vec{q} + \vec{k})^2 + m_{\bar{K}^0}^2}$, $\omega_{\bar{K}^{*0}} = \sqrt{\vec{q}^2 + m_{\bar{K}^{*0}}^2}$, $k^0 = \frac{M_{\text{inv}}^2(\pi^0 a_0) + m_{\pi^0}^2 - M_{\text{inv}}^2(\pi^0\eta)}{2M_{\text{inv}}(\pi^0 a_0)}$, and $k = \frac{1}{2M_{\text{inv}}(\pi^0 a_0)} \lambda^{1/2}(M_{\text{inv}}^2(\pi^0 a_0), m_{\pi^0}^2, M_{\text{inv}}^2(\pi^0\eta))$. In Eq. (13), there is information on $\sqrt{s} = P^0$, $M_{\text{inv}}(\pi^0\eta)$ and $\cos\theta$ with θ the angle between $\vec{p}_{\pi^+}$ and $\vec{k}$, but in the integral over the phase space of $|t|^2$, $1/2 \int d\cos\theta \cos^2\theta = 1/3$, and we can define a t_{eff} such that

$$|t_{\rm eff}|^2 = \frac{1}{3}\vec{p}^{\prime 2}_{\pi^+}\vec{k}^2\left|\frac{1}{\sqrt{2}}Cgt_T t_{K^0,\overline{K^0},\pi^0\eta}\right|^2, \qquad (16)$$

and then, summing the two diagrams of Fig. 2, in analogy to Ref. [66] we find

$$\frac{d^2\Gamma}{dM_{\rm inv}(\pi^0 a_0)dM_{\rm inv}(\pi^0\eta)} = \frac{1}{(2\pi)^5}\frac{p_{\pi^+}k\tilde{p}_\eta}{4m^2_{D_s^+}}|t'_{\rm eff}|^2, \qquad (17)$$

where $\tilde{p}_\eta$ is the η momentum in the $\pi^0\eta$ center-of-mass frame, and

$$|t'_{\rm eff}|^2 = \frac{1}{6}C^2g^2p'^2_{\pi^+}k^2|t_T(K^0\bar{K}^0\bar{K}^{*0})t_{K^0\bar{K}^0,\pi^0\eta} - t_T(K^+K^-K^{*-})t_{K^+K^-,\pi^0\eta}|^2. \qquad (18)$$

For the case of $f_0(980)$ production, we use the same Eq. (18) substituting $\pi^0\eta$ in T matrices by $\pi^+\pi^-$. We can see there that since $t_{K^0\bar{K}^0,\pi^0\eta} = -t_{K^+K^-,\pi^0\eta}$ and $t_{K^0\bar{K}^0,\pi^+\pi^-} = t_{K^+K^-,\pi^+\pi^-}$ in the strict isospin limit, the two terms in Eq. (18) add for the case of the a_0 production and subtract in the case of the f_0 production. In the strict isospin limit, the two terms cancel for the f_0 production, as it should be.

The integrand of Eq. (15) is regularized including the factor $\theta(q_{\rm max} - |\vec{q}^*|)$, where $\vec{q}^*$ is the momentum of the K in the rest frame of a_0 (f_0) [see Eq. (22) of Ref. [21]], with $q_{\rm max} = 600$ MeV as it is needed in the chiral unitary approach that reproduces the $f_0(980)$ and $a_0(980)$ (see Refs. [59,67]).

III. RESULTS

In Fig. 3, we show the results of t_T as a function of $\sqrt{s} \equiv M_{\rm inv}(\pi^0 a_0)$ taking for $M_{\rm inv}(\pi^0\eta)$ [or $M_{\rm inv}(\pi^+\pi^-)$] the value of 980 MeV. We can see that the amplitude has a shape similar to a Breit-Wigner with ${\rm Re}(t_T)$ and ${\rm Im}(t_T)$ interchanged ($t_T \sim -it_{BW}$). Yet the origin of this structure does not come from any particular interaction, but solely from the analytical structure of the loop function. We can see that $|t_T|$ has a peak around 1420 MeV and its origin is the triangle singularity developed by the amplitude. Indeed,

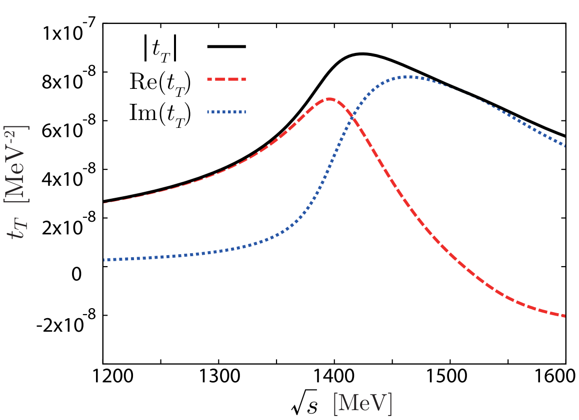

FIG. 3. ${\rm Re}(t_T)$, ${\rm Im}(t_T)$ and $|t_T|$ of Eq. (15).

according to Ref. [21] the diagrams of Fig. 2 develop a singularity where in the d^4q integration the $K^0\bar{K}^{*0}$ are placed on shell simultaneously, as well as the $K^0\bar{K}^0$, and the angle between the $\bar{K}^{*0}$ and the π^0 coming from its decay is 0. Analytically this is given by Eq. (18) of Ref. [21] and $q_{\rm on} = q_{a-}$. One can see that this occurs at about 1420 MeV (one must choose the mass of a_0 slightly above $m_K + m_{\bar{K}}$ to have the relationship fulfilled). However, the actual singularity (a sharp peak) becomes a broad bump, as seen in Fig. 3, when we consider explicitly the width of the $\bar{K}^*$ in the integral of t_T, $\omega_{\bar{K}^*} \to \omega_{\bar{K}^*} - i\Gamma_{\bar{K}^*}/2$ in Eq. (15).

In Fig. 4, we show the results of Eq. (17) for $[d^2\Gamma_{D_s^+\to\pi^+\pi^0\pi^0\eta}/dM_{\rm inv}(\pi^0 a_0)dM_{\rm inv}(\pi^0\eta))]/\Gamma_{D_s^+}$ or

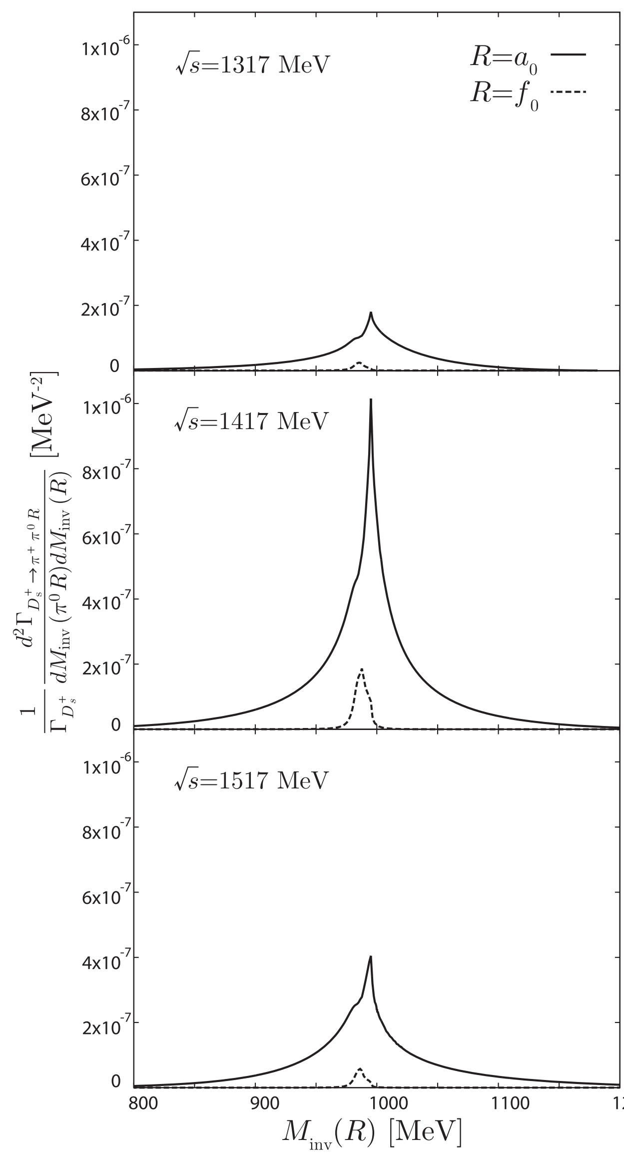

FIG. 4. $[d^2\Gamma_{D_s^+\to\pi^+\pi^0\pi^0\eta}/dM_{\rm inv}(\pi^0 a_0)dM_{\rm inv}(\pi^0\eta))]/\Gamma_{D_s^+}$ and $[d^2\Gamma_{D_s^+\to\pi^+\pi^0\pi^+\pi^-}/dM_{\rm inv}(\pi^0 f_0)dM_{\rm inv}(\pi^+\pi^-))]/\Gamma_{D_s^+}$ as functions of $M_{\rm inv}(\pi^0\eta)$ or $M_{\rm inv}(\pi^+\pi^-)$ for fixed value of $M_{\rm inv}(\pi^0 a_0)$ or $M_{\rm inv}(\pi^0 f_0)$ as 1317, 1417, and 1517 MeV, respectively. $M_{\rm inv}(R)$ for $R = a_0$ (f_0) means $M_{\rm inv}(\pi^0\eta)$ ($M_{\rm inv}(\pi^+\pi^-)$).

$[d^2\Gamma_{D_s^+\to\pi^+\pi^0\pi^+\pi^-}/dM_{\rm inv}(\pi^0 f_0)dM_{\rm inv}(\pi^+\pi^-))]/\Gamma_{D_s^+}$ as a function of $M_{\rm inv}(\pi^0\eta)$ or $M_{\rm inv}(\pi^+\pi^-)$ with a fixed value of $M_{\rm inv}(\pi^0 a_0)$ or $M_{\rm inv}(\pi^0 f_0)$ at 1317, 1417 and 1517 MeV. For this we have used Eqs. (3) and (5) to determine C^2. What we see in the figure is that we get two peaks, corresponding to the typical $\pi^0\eta$ mass distribution of the $a_0(980)$ and the $\pi^+\pi^-$ mass distribution of the $f_0(980)$. The a_0 peaks around 995 MeV and the $f_0(980)$ around 985 MeV. We also observe a larger strength for $\pi^0 a_0$ production (isospin allowed mode) than for the $\pi^0 f_0$ production (isospin suppressed mode). However, the amount of the $\pi^0 f_0$ production is sizable. The strength of the two distributions at the respective peaks for $M_{\rm inv}(\pi^0 a_0)$ $(M_{\rm inv}(\pi^0 f_0))$ at $\sqrt{s} = 1417$ MeV is about 16% for $\pi^0 f_0$ versus $\pi^0 a_0$, a sizable isospin violation. We also see that when we change $M_{\rm inv}(\pi^0 a_0)$ $(M_{\rm inv}(\pi^0 f_0))$ by 100 MeV up and down from this middle energy the strength of both distributions is sizably decreased. The maximum strength corresponds to $M_{\rm inv}(\pi^0 a_0)$ $(M_{\rm inv}(\pi^0 f_0)) \sim 1420$ MeV where the peak of the singularity of the triangle diagram appears. We also observe that the relative weight of the peaks $\pi^0 f_0$ and $\pi^0 a_0$ is decreased by about a factor of 2, indicating that the maximum of the isospin violation appears at the $M_{\rm inv}(\pi^0 a_0)$ $(M_{\rm inv}(\pi^0 f_0))$ where we have the peak of the triangle singularity. It should be noted that, although one could interpret this as a $a_0 - f_0$ mixing we have deliberately avoided this perspective and independently have calculated the rate for $\pi^0 f_0$ and $\pi^0 a_0$ production. The isospin violation ($\pi^0 f_0$ production) is possible because the $t_{K\bar{K},\pi^0\eta}$ and $t_{K\bar{K},\pi^+\pi^-}$ amplitudes already contain isospin symmetry breaking terms as soon as the chiral unitary approach is implemented with different masses of the kaons. The second reason is the loop of the triangle diagram that also induces isospin violation from the different masses of the kaons and the K^*. We have checked that the most important source for this isospin breaking comes from the triangle singularity.

We should also note that we have not explicitly used the $f_0(980)$ and $a_0(980)$ resonances in the approach. They are dynamically generated by the $\pi^0\eta$, $\pi\pi$, $K\bar{K}$, $\eta\eta$ channels [8,59,67–70] and they are implicitly contained in the $t_{K\bar{K},\pi^0\eta}$ and $t_{K\bar{K},\pi^+\pi^-}$ amplitudes. Note that the apparent width of the $f_0(980)$ distribution is about 10 MeV, much narrower than the f_0 natural width of about 30–50 MeV, because as discussed in Refs. [2,17], the width of the isospin violating distribution is of the order of the magnitude of the difference of the K^+ and K^0 masses. This was seen clearly in the experiment in the $\eta(1405) \to \pi^0 f_0(980)$ [15] and one should not take this width as a measure of the $f_0(980)$ width, which should be looked at in isospin allowed processes.

As we have seen, the amount of isospin violating $\pi^0 f_0$ production is a function of $M_{\rm inv}(\pi^0 f_0)$ and hence, as already discussed in Ref. [17], the concept of a universal

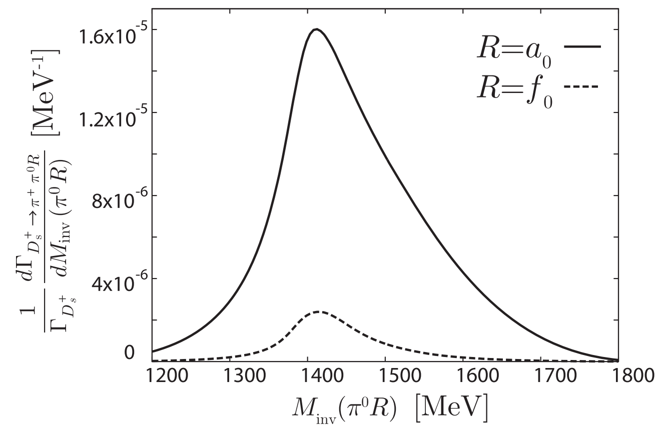

FIG. 5. $d\Gamma/dM_{\rm inv}(\pi^0 a_0)$ and $d\Gamma/dM_{\rm inv}(\pi^0 f_0)$ integrated over the respective a_0 and f_0 mass distributions (see the text). Only the $\pi^+\pi^-$ mode of f_0 and $\pi^0\eta$ mode of a_0 are considered here.

$a_0 - f_0$ mixing parameter is not an appropriate one. It is better to talk in terms of a_0 isospin allowed and f_0 isospin forbidden production, or vice versa, which depend on the particular experiment, and even for the same experiment on the particular part of the phase space chosen, as we have seen here.

In order to give a perspective of the amount of isospin violation as a function of $M_{\rm inv}(\pi^0 a_0)$ $(M_{\rm inv}(\pi^0 f_0))$, we apply the following criteria. The $f_0(980)$ production has a narrow range and we integrate its strength between $M_{\rm inv}(\pi^+\pi^-) \in [970\ {\rm MeV}, 1000\ {\rm MeV}]$. The $\pi^0\eta$ mass distribution around the $a_0(980)$ has the typical cusp form [71,72] and has a broad distribution. Yet it is customary experimentally not to associate the whole strength with the $a_0(980)$ but subtract a smooth background (note that the amplitudes of the chiral unitary approach are for $K\bar{K} \to \pi^0\eta$ and contain background and pole contributions simultaneously). In Ref. [67] a smooth background was constructed adjusting a phase space distribution to the sides of the $\pi^0\eta$ distribution, such that the apparent width of the a_0 is about 70–80 MeV, in the middle of 50–100 MeV of the PDG [40]. Then, the strength of the "a_0" was about one third of the strength integrated from $M_{\rm inv}(\pi^0\eta) \in [700\ {\rm MeV}, 1200\ {\rm MeV}]$ (see Fig. 3 of Ref. [67]). Then, in Fig. 5 we plot the strength of the integrated mass distributions of $\pi^0\eta$ and $\pi^+\pi^-$ with this criterion. We can see in Fig. 5 that both the $\pi^0 a_0$ and $\pi^0 f_0$ strength peak around $M_{\rm inv}(\pi^0 a_0) \sim 1420$ MeV as a consequence of the TS.

In Fig. 6, we plot the ratio of $d\Gamma/dM_{\rm inv}(\pi^0 f_0)$ and $d\Gamma/dM_{\rm inv}(\pi^0 a_0)$. We see in Fig. 6 that the ratio of f_0 to a_0 production is strongly dependent on the $\pi^0 R$ $(R = f_0, a_0)$ invariant mass. By going 100 MeV above and below the peak, the ratio decreases by about a factor of 2 and keeps decreasing as we go further away from the peak. As we can see, the TS has acted as a magnifier for the isospin violating $\pi^0 f_0$ production process, as we had anticipated.

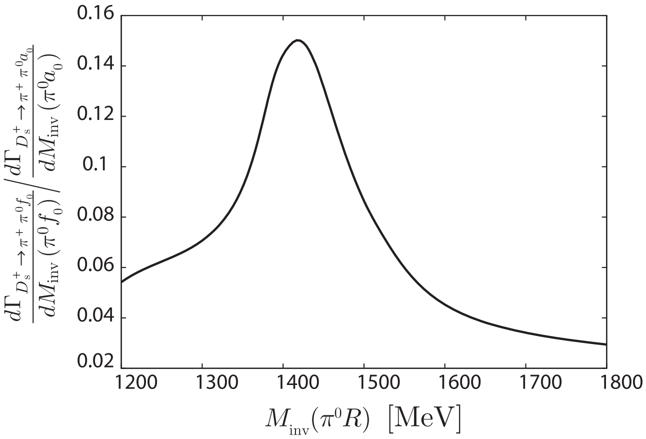

FIG. 6. Ratio of $d\Gamma/dM_{\rm inv}(\pi^0 a_0)$ and $d\Gamma/dM_{\rm inv}(\pi^0 f_0)$ as a function of $M_{\rm inv}(\pi^0 R)$ $(R = f_0, a_0)$.

Finally, we give numbers for the integrated rates of $\pi^+\pi^0 f_0$ and $\pi^+\pi^0 a_0$ production by integrating $d\Gamma/dM_{\rm inv}(\pi^0 R)$ in the range of invariant masses of Fig. 5. We find the numbers

$$\begin{aligned} &{\rm BR}(D_s^+ \to \pi^+\pi^0\text{“}f_0\text{”}) = (3.28 \pm 0.31) \times 10^{-4}, \\ &{\rm BR}(D_s^+ \to \pi^+\pi^0\text{“}a_0\text{”}) = 3.28 \pm 0.31) \times 10^{-3}, \end{aligned} \tag{19}$$

which are within present measurable range. Note that the numbers of Eq. (19) are not for the full f_0 and a_0 production. Indeed, in the PDG [40] we have $\Gamma_{K\bar{K}}/\Gamma_{\pi^0\eta} = 0.183$. Also the f_0 decays into $\pi^+\pi^-$ and $\pi^0\pi^0$ and $\Gamma_{\pi^+\pi^-} = 2\Gamma_{\pi^0\pi^0}$. Hence, to correct for that we must divide the “a_0” production by 0.85 and multiply the “f_0” production by 3/2. With this, the numbers of Eq. (19) become

$$\begin{aligned} &{\rm BR}(D_s^+ \to \pi^+\pi^0 f_0) = (4.91 \pm 0.46) \times 10^{-4}, \\ &{\rm BR}(D_s^+ \to \pi^+\pi^0 a_0) = (3.85 \pm 0.36) \times 10^{-3}. \end{aligned} \tag{20}$$

The errors in Eqs. (19) and (20) come solely from the experimental errors in the evaluation of C via Eq. (3) summing the errors in quadrature, but they can easily be double of it accepting similar theoretical errors from different sources, as done in other examples [59,67,73].

There is another point worth making. In Fig. 3, we see that both ${\rm Re}(t_T)$ and ${\rm Im}(t_T)$ have a peak. This is a bit different from other cases, where only one of these parts of t_T have a peak, but not the two [45,47,51]. The present case resembles more the one of Ref. [52], where one peak was associated to a threshold and the other one to a triangle singularity. In the present case, the peak of ${\rm Re}(t_T)$ appears because of the $\bar{K}^{*0}K^0$ threshold, while the one of ${\rm Im}(t_T)$ comes from the triangle singularity. Yet, by looking at Fig. 3 and the mass distribution of Fig. 5, it is clear that around the peak of the distributions most of the strength comes from the triangle singularity.

IV. FURTHER CONSIDERATIONS

In the first place, we make some considerations about the dynamics that we use. In the diagram of Fig. 1(a), we show the weak process at the quark level. The weak Lagrangian contracting the W propagator is of the type $\gamma^\mu(1-\gamma_5)\gamma_\mu(1-\gamma_5)$ [57,58] acting on quarks, which must be converted in hadronic matrix elements using implicitly or explicitly wave functions of hadrons in terms of quarks. In Fig. 1(b), the $s\bar{s}$ quarks hadronize to produce two mesons. The evaluation of these matrix elements combining the weak Lagrangians with QCD for the strong interaction part is quite involved and uncertain [57,58]. Approximations are required, like the factorization assumption using light cone sum rules [74]. Other times amplitudes for different processes are related in terms of a few SU(3) irreducible amplitudes, which are not calculated [75]. The perturbative QCD approach has also been used in Ref. [76]. In general, one must rely on some form factor which is obtained from experimental data and discrepancies between different methods are of 2 orders of magnitude for many reactions [57,58]. Two recent examples on the dispersion of the theoretical results can be seen in Refs. [53,77] in the study of the $B_c^+ \to B^+\bar{K}^{*0}$ and $B^0 \to J/\psi\gamma$ reactions, respectively. In view of this, since we want to be rather quantitative in the predictions, we take the input needed for the $D_s^+ \to \pi^+K^{*-}K^+$ amplitude from experiment. The branching ratio for this reaction is very large, around 19% [see Eq. (3)], which guarantees a sizable strength for related reactions, like the one we study. The form of the amplitude is easily given by the spin and angular structure, as shown in Eq. (4), and the experiment provides us with the unknown coefficient C. From there on, the step to the process of Fig. 2 through a triangle diagram is straightforward and well determined. The amplitudes $K\bar{K} \to \pi^0\eta(\pi^+\pi^-)$ are given with the unitary extension of the chiral Lagrangians, which provide the effective theory of QCD at low energies [62,63]. The vertex $K^* \to K\pi$ is standard, provides the right K^* decay width and is derived within the extension of the chiral Lagrangians to incorporate vector mesons [78].

On the other hand, the triangle diagram of Fig. 2 is bound to have a relevance since according to the Coleman Norton theorem [20] it can occur at the classical level once we prove that the condition $q_{\rm on} = q_{a-}$ of Ref. [21] is fulfilled, which is the case here.

We stress that the procedure followed is neatly nonperturbative. Although the mechanism of Fig. 2 looks like selecting a one loop term from a perturbative expansion, this is not the case. The mechanism has to be interpreted as having $D_s^+ \to \pi^+\pi^0K\bar{K}$ followed by the final state interaction of $K\bar{K}$, which is intrinsically nonperturbative and is done summing the Bethe-Salpeter series of terms with a kernel for $K\bar{K}$ (and coupled channels) provided by the chiral Lagrangians, the essence of the chiral unitary

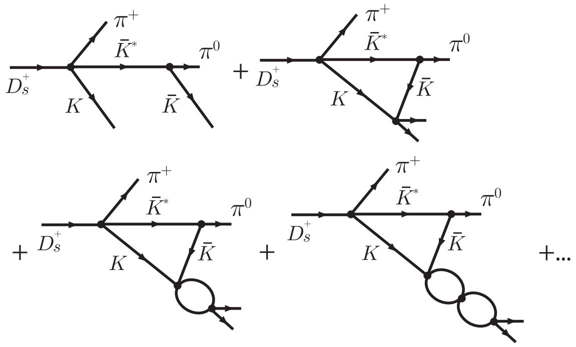

FIG. 7. Interpretation of the diagrams of Fig. 2 as a final state interaction of $K\bar{K}$ after the $D_s^+ \to \pi^+\pi^0 K\bar{K}$ decay.

approach. This is depicted in Fig. 7. Note that in Fig. 7, if we put a $\pi^0\eta$ in the final state of $K\bar{K}$ rescattering we have the nonperturbative series (the tree level will not contribute, since we do not have $K\bar{K}$ in the final state),

$$\begin{aligned} t &= \tilde{t}_T V_{K\bar{K},\pi^0\eta} + \tilde{t}_T V_{K\bar{K},MM} G_{MM} V_{MM,\pi^0\eta} \\ &\quad + \tilde{t}_T V_{K\bar{K},MM} G_{MM} V_{MM,M'M'} G_{M'M'} V_{M'M',\pi^0\eta} + \cdots \\ &= \tilde{t}_T t_{K\bar{K},\pi^0\eta}, \end{aligned} \tag{21}$$

where $\tilde{t}_T$ is given by Eq. (11) removing $t_{K^0\bar{K}^0,\pi^0\eta}$, V_{ij} and G are the kernel and the two meson loop function, respectively, MM, $M'M'$ are intermediate states of the coupled channels (we sum over them) and we have used the Bethe-Salpeter equation,

$$\begin{aligned} t_{MM,M'M'} &= V_{MM,M'M'} \\ &\quad + V_{MM,M''M''} G_{M''M''} V_{M''M'',M'M'} \\ &\quad + V_{MM,M''M''} G_{M''M''} V_{M''M'',M'''M'''} \\ &\quad \cdot G_{M'''M'''} V_{M'''M''',M'M'} + \cdots \\ &= V_{MM,M'M'} + V_{MM,M''M''} G_{M''M''} t_{M''M'',M'M'}. \end{aligned} \tag{22}$$

In Eq. (22), the kernel V (potential) is provided by the chiral Lagrangians [62,63], which are given by local, contact, terms. Yet, it has been shown in Ref. [79] that this theory can be cast into an equivalent one, the local hidden gauge approach [78], where the source of interaction is the t-channel exchange of vector mesons (ρ, ω, ϕ in this case). The chiral Lagrangian is obtained neglecting the q^2/M_V^2 terms, where q is the momentum transfer in $MM \to M'M'$, something justified at the low energies involved in the rescattering process.

All the former arguments indicate that we can evaluate accurately the strength of the process proceeding through the mechanism of Fig. 2. The next question that one can ask is whether there are no other mechanisms that can produce the same final state and that do not require this triangle mechanism, proceeding at tree level or through loops with two propagators instead. We rule out the tree level since the $f_0(980)$, $a_0(980)$ are generated dynamically from the interaction of pseudoscalar mesons in the chiral unitary approach.

With this aim in mind and still looking for processes that benefit from the color counting rule and the Cabibbo favored transitions, we find the candidate that could compete with the mechanism that we have considered. This would be the process with exactly the same topology as in Fig. 1, but where a ρ^+ is produced from the W instead of a π^+. Upon hadronization of the $s\bar{s}$ quark pair, we get $K\bar{K}$ components in $I=0$ which couple directly to the $f_0(980)$. Note that in this case the $f_0(980)$ is produced in an isospin conserving reaction and hence could, in principle, be more important than the isospin suppressed mode that we have studied before. We can make an estimate of the branching fraction for the $D_s^+ \to \rho^+ f_0(980)$ production by comparing it to the $B_s^0 \to J/\psi f_0(980); f_0 \to \pi^+\pi^-$ of $(1.19 \pm 0.23) \times 10^{-4}$. For reasons of angular momentum conservation, both reactions go in p-wave and the width goes as p^3. Since $p_\rho = 447$ MeV$/c$ and $p_{J/\psi} = 1601$ MeV$/c$ in these reactions, one could expect a small branching fraction for $D_s^+ \to \rho^+ f_0(980)$, of the order of 2.4×10^{-6}, which could explain why this mode is not reported in PDG [40]. A rate of this order of magnitude is much smaller than the results obtained with the triangle mechanism in Eq. (20) of $(4.91 \pm 0.46) \times 10^{-4}$.

There is, however, another argument that we must invoke concerning the $D_s^+ \to \rho^+ f_0$ and the triangle mechanism. By placing the three particles in the triangle loop in shell, which gives rise to the peak of the singularity, we obtained an invariant mass of $\pi^0 f_0$ of about 1420 MeV. Taking this value and playing with the phase space, we see that the invariant mass of the π^+ from the $D_s^+ \to \pi^+\bar{K}^* K$ vertex, and the π^0 from the $\bar{K}^*$ decay ranges within [290 MeV, 986 MeV]. The $\pi^+\pi^0$ coming from the decay of the ρ^+ have a range of invariant mass given by the width of the ρ around the ρ mass. In an experiment where $\pi^+\pi^0$ and f_0 are looked for in the final state, the $\pi^+\pi^0$ mass distribution can be measured and the ρf_0 part of it, assuming that it is much bigger than our estimation, can easily be separated since it concentrates in a small region of the invariant mass allowed for the triangle diagram. Eliminating that part of the invariant mass spectrum, there is still a large fraction of it from where the effect of the triangle singularity enhancing the isospin forbidden f_0 production can be seen clearly.

In principle, other processes than the one we consider could produce the same final state. Note that the resonances decay into stable particles which are those finally detected, in our case $\pi^+\pi^-$ for the $f_0(980)$ and $\pi^0\eta$ for the $a_0(980)$. Hence, the $D_s^+ \to \pi^+\pi^0 a_0(980)$ will actually be detected in the $\pi^+\pi^0\pi^0\eta$ mode. Our calculations consider these final states explicitly. Yet, there might be other processes that also have $\pi^+\pi^0\pi^0\eta$ in the final state and could contribute in the region of the resonances. If the resonances are very narrow, the amount of contribution from other nonresonant processes below the peak is negligible. The f_0 and a_0 are sufficiently narrow to qualify for this case. In other cases, these contributions do not have resonant shape and add a background to the resonant contribution that an experimental analysis can easily separate. One can consider potential sources of such a background to suggest how to disentangle them. One example was given before with the $D_s^+ \to \rho^+ f_0$. Here we consider another case: $D_s^+ \to \pi^+ \eta'(958)$. Since the η' has an $s\bar{s}$ component, it can be created via the mechanism of Fig. 1(a). Next the η' can decay to $\pi^0\pi^0\eta$, with a branching fraction of about 23% [40] and then we have $\pi^+\pi^0\pi^0\eta$ in the final state, like the $D_s^+ \to \pi^+\pi^0 a_0(980) \to \pi^+\pi^0\pi^0\eta$. However, since $(p_{\pi^0} + p_{\pi^0} + p_\eta)^2 = m_{\eta'}^2$, the maximum invariant mass of a $\pi^0\eta$ is $m_{\eta'} - m_{\pi^0} \equiv 823$ MeV and one does not get any contribution from this process in the $a_0(980)$ region.

V. CONCLUSIONS

The abnormal isospin violation observed in the $\eta(1405) \to \pi^0 f_0(980)$ reaction [15] and its interpretation as the consequence of a triangle singularity in Refs. [16,17] prompted us to dig into the problem looking for a reaction where the energy to produce $\pi^0 f_0(980)$ could be changed at will. This would allow us to see if, indeed, the TS has a clear effect enhancing the isospin violation close to the peak of the singularity. We found such a reaction in $D_s \to \pi^+\pi^0 a_0(980)(f_0(980))$, where the freedom to change the energy of the π^+ allows one to change the invariant mass of the $\pi^+\pi^0 a_0(980)(f_0(980))$ system and investigate the amount of isospin breaking as a function of this invariant mass. The reaction allows one to get a range of $\pi^+\pi^0 a_0(980)(f_0(980))$ invariant masses that passes through 1420 MeV, the energy where the triangle mechanism $D_s^+ \to \bar{K}^* K$, followed by $\bar{K}^* \to \bar{K}\pi^0$ and the further merging of $K\bar{K}$ to produce the $a_0(980)$ or $f_0(980)$ has a triangle singularity. We could see that, indeed, the isospin violating process of $D_s \to \pi^+\pi^0 f_0(980)$ was enhanced versus the isospin allowed $D_s \to \pi^+\pi^0 a_0(980)$ as one passed thought the TS peak. This is due to the fact that the isospin violating reaction was made possible by the different masses of K^+ and K^0, and these differences are stressed by the triangle singularity that places the intermediate particles (and here the two kaons) on shell, where the difference of the masses matters most.

It is curious that a weak reaction that violates isospin in the weak vertex is chosen to investigate isospin violation due to strong interactions. However, due to Cabibbo selectivity, color enhancement and the topology of the weak processes, these weak reactions offer very good filters of isospin in some cases [60]. This was the case in the reaction chosen. Indeed, the Cabibbo favored, color favored mode of D_s decay is $D_s^+ \to \pi^+ s\bar{s}$, and the $s\bar{s}$ system has $I = 0$. After hadronization with $\bar{q}q$ pairs, the emerging $K\bar{K}^*$ state will be in $I = 0$. In our picture, in which the $f_0(980)$ and $a_0(980)$ resonances are dynamically generated by the interaction of pseudoscalar mesons, it is this interaction in the final state which is responsible for the isospin violation. Since we prove that the isospin mixing depends on the reaction and for reactions like the present one, depends on the region of the phase space chosen, we deliberately chose not to talk about $f_0(980) - a_0(980)$ mixing, and the mixing parameter, because it is not a universal magnitude. It is better to talk in terms of independent $f_0(980)$ or $a_0(980)$ production and then investigate the amount of isospin violation. There is mixing of the two resonances but this in encoded in the $K\bar{K}, \pi^0\eta$ and $K\bar{K}, \pi^+\pi^-$ amplitudes and the loop functions of the triangle mechanism, and thus is very much dependent on the reaction and regions of phase space. From our perspective the results obtained have an extra value. While the enhancement due to the triangle singularity could be reached in different ways, the strength obtained and its energy dependence is very much tied to the nature of the $f_0(980)$ and $a_0(980)$ resonances, which we have assumed as dynamically generated from the interaction of pseudoscalar mesons. The rates obtained are within present measurable range and we can only encourage experimental teams to carry out this reaction, which undoubtedly will bring further light into the issue of $f_0(980) - a_0(980)$ mixing and the nature of the low mass scalar mesons.

ACKNOWLEDGMENTS

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