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Matrix model of QCD: Edge localized glueballs and phase transitions

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In a matrix model of pure SU(2) Yang-Mills theory, boundaries emerge in the space of $\mathrm{Mat}_3(\mathbb{R})$ and the Hamiltonian requires boundary conditions. We show the existence of edge localized glueball states that can have negative energies. These edge levels can be lifted to positive energies if the gluons acquire a London-like mass. This suggests a new phase of QCD with an incompressible bulk.

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I. INTRODUCTION

Quantum chromodynamics or QCD describing strong interactions is an interacting non-Abelian gauge theory. Non-Abelian gauge theories of high energy physics are based on compact gauge groups. They generally contain a Lie group, like SU(3) or SU(2) that modulo their discrete centers, and are simple. The self-coupling of the gauge field in such a gauge theory is expected to lead to bound states called glueballs. In the presence of matter (say, quarks), these particle excitations interact with hadrons. It is for such reasons that glueballs remain an interesting topic of investigation, despite the fact that they have eluded experimental verification until now.

In a non-Abelian gauge theory, it is impossible to do a global gauge fixing. In particular, Gribov [1] showed that in all such theories, the Coulomb gauge does not fully eliminate the gauge freedom: there are gauge related copies of the connection in this gauge. It was later proved rigorously by Singer [2] and by Narasimhan and Ramadas [3] that there exists, in fact, no condition to eliminate the gauge ambiguities, with the gauge bundle on the configuration space being twisted.

Narasimhan and Ramadas, in their work on SU(2), reduced the considerations to a family of connections parametrized by 3×3 real matrices. The essential topological complexities of exact pure Yang-Mills theory are already captured by this model. Here too, the appropriate $\mathrm{SU}(2)/\mathbb{Z}_2 = \mathrm{SO}(3)$ bundle is twisted, with the twist being inherited from the full pure Yang-Mills theory.

The work of Narasimhan and Ramadas can be extended to SU(3) and other non-Abelian groups. That is because it is based on Maurer-Cartan forms that have a certain universal character. Recently, in [4,5], a matrix model of the $\mathrm{SU}(N)$ Yang-Mills theory was proposed that successfully captures the nontrivial nature of the gauge bundle. There, the Hamiltonian formalism for these matrices as configuration spaces was deduced from the full Yang-Mills theory. The matrix model is constructed by compactifying the spatial $\mathbb{R}^3$ to S^3. The Maurer-Cartan form of $\mathrm{SU}(N)$ is pulled back on the S^3 to obtain a particular subspace of the space of all gauge fields. In this subspace, the gauge fields are $3 \times (N^2 - 1)$ real matrices and the result is the $(0+1)$-dimensional matrix model of $\mathrm{SU}(N)$.

In [4,5], the Hamiltonian formalism for these matrices as configuration spaces was deduced. The colorless eigenstates of the Hamiltonian are interpreted as "glueballs," and it is shown that the glueball spectrum for the SU(2) gauge group has a mass gap. The presence of this gap is often regarded as a signal for confinement. There, the QCD θ-angle is also discussed, and the Dirac operator is constructed. In a numerical study [6], the authors obtained the estimates for glueball masses in the SU(3) matrix model and found an excellent agreement with those obtained from lattice QCD simulations [7], despite the numerics being far simpler and less time consuming in the matrix model. This indicates that the matrix model might emerge as an efficient tool for QCD computations with fair accuracy. This motivates us to further investigate various other aspects of the matrix model in detail, as they can carry useful implications about the full pure Yang-Mills theory.

In this paper, we study certain "singular" boundaries of the 3×3 matrix model of SU(2) Yang-Mills and the special states localized at these boundaries. Such boundary states exist also for SU(3) and other gauge groups, being a reflection of states localized at degenerate connections in exact QCD.

In the space $\mathrm{Mat}_3(\mathbb{R})$ of 3×3 real matrices, boundaries and stratification emerge as follows. A matrix $M \in \mathrm{Mat}_3(\mathbb{R})$ has the singular value decomposition

$$M = LDR^T, \qquad L, R \in SO(3) \tag{1.1}$$

$$D = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \qquad a_1 \geq a_2 \geq a_3 \geq 0. \tag{1.2}$$

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When all the a_i's are unequal, $a_i \neq a_j$ if $i \neq j$, we get the open and dense stratum. At the boundaries, either a pair of a_i or all a_i are equal.

The boundaries $\partial\mathcal{D}$ of the spatial manifold $\mathcal{D}$ have physical consequences. The Laplace and Dirac operators are subject to boundary conditions at $\partial\mathcal{D}$ for self-adjointness. The latter can induce anomalies [8]. They can also create edge-localized states [9] that are of particular interest for topological insulators as discussed previously [10]. For the Dirac operator, when the boundary conditions are of Atiyah-Patodi-Singer type, they lead to the η-invariant, which has an impact on axial anomaly [11]. Such boundary conditions can also make or break supersymmetry or Becchi-Rouet-Stora-Tyutin invariance [12,13].

These known results are the incentives to study the boundaries of $\mathrm{Mat}_3(\mathbb{R})$. As for spatial manifolds with boundaries, here too the Hamiltonian requires boundary conditions. Considering various boundary conditions, we explore the possibility of "edge" localized glueball states [localized near the boundary associated with $\mathrm{Mat}_3(\mathbb{R})$]. They are expected to be novel glueball states and might imply the existence of new phases of QCD. This work focuses on these aspects. It can be extended to $\mathrm{Mat}_8(\mathbb{R})$, which is appropriate for color SU(3). As shown in [6], the glueball spectrum of the matrix model matches excellently with the physical masses predicted by lattice QCD. Similarly, these edge states might also be present in the full pure Yang-Mills theory.

We here confirm first that there do exist such edge states. The energy of these edge states can be negative. Such states are physical only if the gluons acquire mass. This suggests the possibility of new phases in which the gluons become massive just as the photon acquires a London mass in a superconductor. When matter fields are coupled to the matrix model, similar edge states emerge naturally [14]. Here, we demonstrate that the emergence of such edge states is due to the presence of nontrivial boundary conditions on $\mathrm{Mat}_3(\mathbb{R})$. Further, these "superconducting" phases share features with earlier models of quark-gluon plasma [15,16].

The first step in the analysis is the partial wave decomposition of wave functions $\Psi\colon \mathrm{Mat}_3(\mathbb{R}) \to \mathbb{C}$ with regard to the two SO(3)'s appearing in (1.1). The Laplacian for the matrix model then separates as shown by Iwai [17]. We discuss this in Sec. II where we also clarify the meaning of the transformation $M \to L'MR'^T$, L', $R' \in \mathrm{SO}(3)$, which commutes with the Laplacian. In this manner, we arrive at the $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_R$ invariant S-wave sector of glueballs.

The eigenvalue problem is singular at the boundaries where two or more a_i becomes equal. It is of the same kind as the singularity at $r = 0$ of radial eigenvalue problem on $\mathbb{R}^d$ ($d =$ dimension). In the latter, as is known, it appears in the volume form $r^{(d-1)} dr d\Omega_{S_{(d-1)}}$, which becomes zero at $r = 0$. We can transfer the $r^{(d-1)}$ factor to the Hamiltonian. Then the new volume form $dr d\Omega_{S_{(d-1)}}$ is well behaved at the origin, while the transformed radial Laplacian,

$$-\partial_r^2 + \frac{(d-3)(d-1)}{4r^2} + \frac{l(l+d-2)}{r^2}, \qquad (1.3)$$

has acquired the singular potential $\frac{(d-3)(d-1)}{4r^2}$ for $d \neq 1, 3$. For all other values of d, the singularity at $r = 0$ calls for special boundary conditions that can be found using Weyl's "limit point-limit circle" theorems [18]. Notice that for $1 < d < 3$, the potential is attractive, whereas for all other values, it is repulsive. In a similar way, in our glueball problem, a potential with singularities of the form $\prod_{i>j}(a_i^2 - a_j^2)^{-1}$ appears. Fortunately, they are amenable to Weyl's approach. These matters are discussed in Sec. III, where we also bring the eigenvalue problem to a stage that can be treated by variational methods.

Section IV reports on the variational calculation. The singularity at the boundary $\partial\mathrm{Mat}_3(\mathbb{R})$ is of the "limit circle" type so that the self-adjoint extensions are characterized by the phases $e^{i\theta}$. The Dirichlet boundary condition has $e^{i\theta} = -1$, while the Neumann boundary condition has $e^{i\theta} = 1$. For the Robin boundary conditions that are near Dirichlet, just as the spatial boundary, edge localized glueball states exist. For certain choices of the boundary condition, these states have positive energy, while some lead to negative energy. In Sec. V, we give the interpretation of the negative energy states in terms of a new QCD phase with an incompressible bulk, in close analogy to superconductivity on a spatial domain $\mathcal{D}$ [9]. In the latter, there are localized low-lying states at the boundary $\partial\mathcal{D}$, whereas the bulk states are gapped. In the new QCD phase, the gluons are massive. Such masses can be generated when matter fields are coupled to the matrix model [14].

In Sec. VI, we highlight certain observations of Iwai [17].[1] Namely, the Hamiltonian does not have a divergent centrifugal barrier term near the boundaries when the wave functions transform nontrivially under $\mathrm{SO}(3)_L$ or $\mathrm{SO}(3)_R$. This is in striking contrast to the Laplacian on $\mathbb{R}^d$, which *does* have a centrifugal potential for nonzero angular momentum, that is for nonsinglet SO(d) representations. That suggests that edge states exist regardless of $\mathrm{SO}(3)_L$ or $\mathrm{SO}(3)_R$ angular momentum. The QCD potential from angular momentum and color excitations does depend on these excitations and change with $\mathrm{SO}(3)_{L,R}$ representations (although it is finite when the boundary is approached) so that the glueball excitations need not be degenerate.

In a different project [6], the glueball masses in the same matrix model have been estimated using the harmonic oscillator eigenstates. Low lying glueball spectra obtained there are remarkably similar to the ones from lattice QCD.

[1]Their significance was pointed out to us by Sachindeo Vaidya.

II. THE HAMILTONIAN AND ITS PARTIAL WAVE REDUCTION

The origin of the matrix model for QCD comes from the well-known "Gribov ambiguity" [1,2]. We explain how that is so in this section, focusing on the SU(2) gauge group. We consider SU(2) and not SU(3), as our numerical work has been on SU(2). Theoretical considerations on SU(3) can be found in [4,5].

The "Gribov ambiguity" can be summarized in the statement that the gauge bundle in any non-Abelian gauge theory that involves a compact semi-simple Lie group is twisted. Therefore, there is no global gauge fixing condition in any such theory.

The full space of connections on $\mathbb{R}^d$ in any gauge theory is infinite dimensional. Narsimhan and Ramadas [3] proved that the exact gauge theory twist is reflected in the following finite-dimensional submanifold of connections parametrized by matrices:

$$\Omega = \mathrm{Tr}\left[\frac{\tau_a}{2} u^{-1} du\right] M_{ab} \tau_b. \tag{2.1}$$

Here we consider spatial dimension 3 and SU(2) gauge group and τ_a's are Pauli matrices. M is a real 3×3 matrix and u is given by the Skyrme ansatz [19]:

$$u(\vec{x}) = \cos\theta(r) + i\tau_i \hat{x}_i \sin\theta(r), \qquad \theta(0) = \pi,$$
$$\theta(\infty) = 0, \qquad \vec{x} \in \mathbb{R}^3, \qquad r = |\vec{x}|, \tag{2.2}$$

θ being a monotonic function of r.

There are two group actions of interest on M:

(1) The first comes from the color SU(2) transformation

$$\Omega \to g\Omega g^{-1}, \qquad g \in SU(2). \tag{2.3}$$

Since

$$g\tau_b g^{-1} = \tau_c Adg_{cb}, \tag{2.4}$$

where $g \to Adg$ is the 3×3 adjoint representation of SU(2), the transformations

$$M \to MAdg^T \tag{2.5}$$

are SU(2) color transformations. Observables are all color singlets.

Only global color acts on Ω: it is partially "gauge fixed" to eliminate space-time dependent transformations.

(2) Under the transformation

$$u \to us, \qquad s \in SU(2), \tag{2.6}$$

we have

$$u^{-1}du \to s^{-1}(u^{-1}du)s. \tag{2.7}$$

Or since

$$s\tau_b s^{-1} = \tau_c Ads_{cb}, \qquad Ads \in SO(3), \tag{2.8}$$

this gives the transformation

$$M \to AdsM. \tag{2.9}$$

Now $s^{-1}(u^{-1}du)s$ is also achieved by the transformation

$$u \to s^{-1}us, \tag{2.10}$$

and that, as (2.2) shows, is a spatial rotation. Hence, (2.9) corresponds to spatial rotation.

In brief, the matrix model is constructed by compactifying the spatial $\mathbb{R}^3$ to S^3 of radius R and pulling back the Maurer-Cartan form on SU(N) to obtain a particular subspace of the space of all gauge fields. In this subspace, the gauge fields are $3 \times (N^2 - 1)$ real matrices, yielding a $(0+1)$-dimensional matrix model of SU(N) Yang-Mills theory.

We use [4,5] for the matrix model Hamiltonian and the transformation properties of the states. The Hamiltonian is invariant under color SU(2) and spatial rotations. *The Hamiltonian:* The exact pure Yang-Mills action is

$$S_{\mathrm{QCD}} = -\frac{1}{2g^2}\int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x),$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2.11}$$

It gives the gluon Hamiltonian

$$H_{\mathrm{QCD}} = \frac{1}{2}\int d^3x \mathrm{Tr}\left[g^2 E_i E_i - \frac{1}{g^2} F_{ij}^2\right], \tag{2.12}$$

where the electric field E_i is conjugate to A_i.

The matrix model Hamiltonian follows from (2.12). We introduce

$$E_{i\alpha} = -i\frac{\partial}{\partial M_{i\alpha}} \tag{2.13}$$

as conjugate operators to $M_{i\alpha}$ and write the matrix model Hamiltonian

$$H = -\frac{1}{R}\left[\frac{g^2}{2}\sum_{i,\alpha}\frac{\partial^2}{\partial M_{i\alpha}^2} - V(M)\right],$$
$$V(M) = -\frac{1}{2g^2}\mathrm{Tr}F_{ij}^2. \tag{2.14}$$

R is the radius of the S^3.

The above Hamiltonian only takes into account the classical zero-mode sector of the full field theory. To account for the contribution from the zero-point energy of all the higher, spatially dependent modes in the full quantum field theory, we need to add a constant $C(R)$ to the above Hamiltonian.

$$H = -\frac{1}{R}\left[\frac{g^2}{2}\sum_{i,\alpha}\frac{\partial^2}{\partial M_{i\alpha}^2} - V(M) + C(R)\right]. \tag{2.15}$$

The R dependence comes from the fact that $C(R)$ is the renormalized total zero-point energy (see for example [20]). The numerical values of R and $C(R)$ can be obtained phenomenologically as described in [6].

The curvature F_{ij} is

$$F_{ij} = (d\Omega + \Omega\wedge\Omega)(iX_i, iX_j),$$
$$X_i = \text{angular momentum generators.} \tag{2.16}$$

Since the Skyrme ansatz effectively works on S^3, X_i's replace the spatial translations in F_{ij}. This curvature is computed in [4,5]:

$$F_{ij} = i\epsilon_{ijk}M_{k\alpha}\frac{\tau_\alpha}{2} - i\epsilon_{\alpha\beta\gamma}M_{i\alpha}M_{j\beta}\frac{\tau_\gamma}{2},$$
$$i = 1,2,3, \qquad \alpha = 1,2,3. \tag{2.17}$$

In this way, we get

$$V(M) = \frac{1}{2g^2}[M_{i\alpha}M_{i\alpha} - \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}M_{i\alpha}M_{j\beta}M_{k\gamma}$$
$$+\frac{1}{2}\epsilon_{\alpha_1\beta_1\gamma}\epsilon_{\alpha_2\beta_2\gamma}M_{i\alpha_1}M_{j\beta_1}M_{i\alpha_2}M_{j\beta_2}. \tag{2.18}$$

The scalar product of the functions Ψ for the Hilbert space on which H operates is

$$(\Psi_1, \Psi_2) = \int\prod_{i,\alpha}dM_{i\alpha}\bar{\Psi}_1(M)\Psi_2(M) \tag{2.19}$$

Using the singular value decomposition

$$M = RAS^T, \qquad A \equiv \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$
$$a_1 \geq a_2 \geq a_3 \geq 0, \tag{2.20}$$

we get the simple expression

$$V(M) = \frac{1}{2g^2}[(a_1^2 + a_2^2 + a_3^2) - 6a_1a_2a_2$$
$$+ (a_1^2a_2^2 + a_2^2a_3^2 + a_3^2a_1^2)]. \tag{2.21}$$

The next step for SU(2) or more precisely SO(3), is the separation of variables for $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_R$, which act on the left and right of M respectively. This work has been done by Iwai [17]. As he shows, if $d\Omega_{L,R}$ are the SO(3)-invariant volume forms of $\mathrm{SO}(3)_{L,R}$, then

$$\prod_{i,\alpha}dM_{i\alpha} = \phi(a)\prod_i da_i d\Omega_L d\Omega_R,$$
$$\phi(a) = (a_1^2 - a_2^2)(a_2^2 - a_3^2)(a_1^2 - a_3^2) \geq 0,$$
$$d\Omega_{L,R} = SO(3)_{L,R}\,\text{invariant volume forms.} \tag{2.22}$$

Also, when acting on $\mathrm{SO}(3)_{L,R}$ singlet wave functions, which are our subject of numerical investigations, the Laplacian $-\sum\frac{\partial^2}{\partial M_{i\alpha}^2}$ reduces to

$$-\sum\frac{\partial^2}{\partial M_{i\alpha}^2} \to \Delta = -\left[\frac{\partial^2}{\partial a_1^2} + 2a_1\left(\frac{1}{a_1^2-a_2^2} + \frac{1}{a_1^2-a_3^2}\right)\frac{\partial}{\partial a_1}\right.$$
$$+\frac{\partial^2}{\partial a_1^2} + 2a_2\left(\frac{1}{a_2^2-a_1^2} + \frac{1}{a_2^2-a_3^2}\right)\frac{\partial}{\partial a_2}$$
$$\left.+\frac{\partial^2}{\partial a_3^2} + 2a_3\left(\frac{1}{a_3^2-a_1^2} + \frac{1}{a_3^2-a_2^2}\right)\frac{\partial}{\partial a_3}\right]. \tag{2.23}$$

Since, $d\Omega_{L,R}$ only supply overall factors in the scalar product of singlets, we will ignore them.

It is convenient to change the volume form to $\prod da_i$ by changing Δ to

$$\frac{2R}{g^2}\hat{H}_0 = \sqrt{\phi}\Delta\sqrt{\phi} \tag{2.24}$$

and hence H to

$$\hat{H} = \hat{H}_0 + V(M). \tag{2.25}$$

The expression for $\frac{2R}{g^2}\hat{H}_0$ is

$$\frac{2R}{g^2}\hat{H}_0 = -\sum_{i=1}^{3}\frac{\partial^2}{\partial a_i^2} + U(a) \tag{2.26}$$

$$U(a) = \frac{1}{2}\left(\frac{1}{\phi}\frac{\partial^2\phi}{\partial a_i^2}\right)(a) - \frac{1}{4}\left(\frac{1}{\phi}\frac{\partial\phi}{\partial a_i}\right)^2(a), \tag{2.27}$$

In more explicit form,

$$U(a) = \frac{1}{2}\left(\frac{1}{(a_1-a_2)^2}+\frac{1}{(a_1+a_2)^2}+\frac{1}{(a_1-a_3)^2} + \frac{1}{(a_2-a_3)^2}+\frac{1}{(a_1+a_3)^2}+\frac{1}{(a_2+a_3)^2}\right) = \frac{a_1^2+a_2^2}{(a_1^2-a_2^2)^2}+\frac{a_1^2+a_3^2}{(a_1^2-a_3^2)^2}+\frac{a_2^2+a_3^2}{(a_2^2-a_3^2)^2}. \tag{2.28}$$

III. ON BOUNDARY CONDITIONS

The new potential $U(a)$ is singular as $a_i \to a_j$ due to the stratified structure of the matrix orbit space. The bulk corresponds to the orbits of irreducible gauge fields, whereas the boundary contains the orbits of reducible gauge fields. In the pure gauge theory, there are natural boundary conditions for the quantum Hamiltonian. When the gauge theory is coupled to matter fields, some more general conditions can be considered see e.g. [21–28]. Therefore, one needs to consider the possibility of more general boundary conditions to treat these singularities.

The analogous problem for boundary conditions in one variable was treated by Weyl [29], and it goes by the name of "limit point, limit circle" theorem [18,30]. The generalization of Weyl's approach to several variables is due to Harish-Chandra and is described by Knapp [31].[2]

Fortunately, because of certain simplicities, we can treat the domain of self-adjointness of (2.26) without the full machinery in Knapp. The approach we follow is due to [25–28,32].

The general method here for finding all boundary conditions is as follows. Let us consider the asymptotic zero modes Ψ_{azm} of $\hat{H}_0$ that are square integrable in a neighborhood of the singularities of the effective potential, that is

$$\hat{H}_0\Psi_{azm} = \mathcal{O}\left(\frac{1}{\Lambda^2}\right), \tag{3.1}$$

with

$$\int_{\phi(a)\le\Lambda}\prod_i da_i|\Psi_{azm}(a)|^2 < \infty, \tag{3.2}$$

$\Lambda \ll 1$ being an arbitrary small cutoff. In general there is an infinity of such asymptotic zero modes [26]. However, they can be parametrized by a separation of variables in a way similar to what is done in the one-dimensional case (1.3). We can choose a system of coordinates on the surface $\phi(a) = \Lambda$ and one extra *radial* coordinate given by $\phi(a)^{\frac{1}{3}}$. The Hamiltonian $\hat{H}_0$ then splits into two parts: one *radial* term H_ϕ and one *angular* term H_Λ. The asymptotic zero modes can then be split according to the different eigenvalues $\lambda_n \ge 0$ of angular part. Our focus here is on the zero eigenvalue and, hence, on the zero mode of $\hat{H}_0$. In this case there are the two independent zero modes,

$$\Psi_1 = \sqrt{\phi}; \qquad \Psi_2 = \sqrt{\phi}\log\phi. \tag{3.3}$$

Notice the asymptotic zero modes corresponding to the higher modes Δ_Λ vanish at the singular points where $\phi(a)$ vanishes, and they do not require extra parameters to fix the boundary condition. In some sense, the simple structure of the singularity simplifies the analysis of the boundary conditions [26].

We can understand the origin of zero modes very simply before the transformation (2.24): they are just the constant function and $\log\phi$. They correspond to the $\sqrt{r}$ and the $\sqrt{r}\log r$ zero modes of (1.3) for $l = 0$ and $d = 2$.

From the boundary condition it follows that the functions in the domain of the Hamiltonian $\hat{H}_0$ have an asymptotic behavior similar of Ψ_θ, i.e.,

$$\Psi \sim \cos\theta\Psi_1 + \sin\theta\Psi_2 \quad \text{as } \phi(a) \to 0. \tag{3.4}$$

In fact, it is the relative coefficient $\tan\theta$ (which can also be infinite) between Ψ_1 and Ψ_2 is what matters.

Equation (3.4) refers only to the behavior of Ψ as $\phi(a) \to 0$. For a large a_i and/or a large $\phi(a)$, square integrability requires that $\Psi \to 0$. For example, the wave function

$$\Psi = (\Psi_1 + \tan\theta\Psi_2)e^{-\phi(a)\sum_{i=1}^{3}a_i^2}, \tag{3.5}$$

which is globally defined for any a with $\phi(a) > 0$, belongs also to the domain of the Hamiltonian $\hat{H}_0$.

As emphasized, these zero modes appear naturally in the theory, and there is nothing mysterious about them. Working with the volume form $\phi(a)\prod_i da_i$ instead of $\prod_i da_i$, Ψ_1 corresponds to a constant zero mode and, of course, it is a zero mode of the Laplacian. The numerical computation to glueball masses in [6] corresponds to a variational calculation around this mode, which then also brings in nonconstant functions. Therefore, that is the $\theta = 0$ case.

When nothing special happens at the $\phi(a) = 0$ boundary, we only consider Ψ_1. Instead, if we consider both Ψ_1 and Ψ_2 with a nonzero value θ, there can be nontrivial physical effects at the $\phi(a) = 0$ boundary, like the emergence of edge localized states. We demonstrate these in the following section.

IV. EDGE STATES: NUMERICAL RESULTS

In this section, we calculate an upper bound of the ground state energy of

[2]We thank Professor M.S. Narasimhan who helped us in understanding this work and for these references.

$$\mathcal{H} = \hat{H}_0 + V(M), \tag{4.1}$$

[where $\hat{H}_0$ and $V(M)$ are defined in (2.26) and (2.21), respectively] using variational calculations, and we show the existence of edge states as described in the previous section.

For certain choices of θ to be discussed below, energy becomes negative, making the system unstable. Such θ have to be rejected, without further physical inputs. Of course, as we are doing only a variational calculation, we can only numerically check that the energy is positive. That carries the uncertainties of numerical estimates.

As a variational ansatz for the ground state, we consider

$$\zeta_k = (\Psi_1 + \tan\theta\Psi_2)e^{-k\phi(a)\sum_{i=1}^{3} a_i^2}, \qquad k > 0, \tag{4.2}$$

which has the asymptotic behavior (3.4) and is square integrable in $a_1 \geq a_2 \geq a_3 \geq a_3 \geq 0$. Here, k is the variational parameter.

For fixed values of $\tan\theta$, we numerically compute

$$E(k) = \frac{\int_0^\infty da_1 \int_0^{a_1} da_2 \int_0^{a_2} da_3 \zeta_k^\dagger(\mathcal{H}\zeta_k)}{\int_0^\infty da_1 \int_0^{a_1} da_2 \int_0^{a_2} da_3 \zeta_k^\dagger \zeta_k} \tag{4.3}$$

as a function of the variational parameter k. This provide us by the Rayleigh-Ritz theorem, an upper bound of the ground state energy E of the system:

$$E < E(k_0), \quad \text{where } \left.\frac{dE(k)}{dk}\right|_{k=k_0} = 0. \tag{4.4}$$

We also evaluate

$$E_0(k) = \frac{\int_0^\infty da_1 \int_0^{a_1} da_2 \int_0^{a_2} da_3 \zeta_k^\dagger(\hat{H}_0\zeta_k)}{\int_0^\infty da_1 \int_0^{a_1} da_2 \int_0^{a_2} da_3 \zeta_k^\dagger \zeta_k} \tag{4.5}$$

and show that there exist certain choices of $\tan\theta$ for which $E_0(k_0) < 0$.

Near $\phi(a) = 0$, as both $\zeta_k \to 0$ and $\mathcal{H}\zeta_k \to 0$, the contribution to the integrals in (4.3) is very small from this region. Consequently, we can deform $E(k)$,

$$E(k) = \frac{\int_{2\epsilon}^\infty da_1 \int_{\epsilon}^{a_1-\epsilon} da_2 \int_0^{a_2-\epsilon} da_3 \zeta_k^\dagger(\mathcal{H}\zeta_k)}{\int_{2\epsilon}^\infty da_1 \int_{\epsilon}^{a_1-\epsilon} da_2 \int_0^{a_2-\epsilon} da_3 \zeta_k^\dagger \zeta_k}, \tag{4.6}$$

where ϵ is very small. This is done for the following reason. The integral in the numerator involves $\partial_i\phi(a)$, which does not have the same zeros as $\phi(a)$, and $\partial_i\phi(a)$ can be large even near $\phi(a) = 0$. The integral involves the product of $\phi(a)$ and $\partial_i\phi(a)$'s, and the numerical evaluation might be erroneous because of multiplying a very small number (the zero of the computer) with a very large number.

We checked that as ϵ is reduced, the integrals converge, and that it is enough to consider $\epsilon = 0.01$ for a good estimation of $E(k)$.

For $\tan\theta = 0$, $E_0(k)$ is monotonic and positive, while for $\cot\theta = 0$, $E_0(k)$ is monotonic and negative. Consequently, when we choose $0 < |\theta| \ll \frac{\pi}{2}$, the energy functional $E(k)$ can be positive and have a minima. On the other hand, for $0 \ll |\theta| < \frac{\pi}{2}$, there might be minima, but $E(k)$ might be negative.

In the following, we study the two regimes separately.

A. $0 < |\theta| \ll \frac{\pi}{2}$:

First, we consider $0 < |\theta| \ll \frac{\pi}{2}$ and estimate $E_0(k)$ and $E(k)$ numerically. For various values of θ and g, we have plotted $E(k)$ as a function of k in Fig. 1. The minima of $E(k)$ (as in the plots) give upper bounds of the total energy $E(k_0)$, which are positive such choices of θ. For various values of θ, $E(k_0)$ as a function of the coupling constant g is shown in Fig. 2.

For such choices of θ, due to the exponential factor in the modes ζ_{k_0}, they might localized near the $\phi(a) = 0$ boundary. That can be demonstrated by plotting $|\zeta_{k_0}|^2(a_1, a_2, a_3)$ as a function of a_1, a_2, and a_3 for fixed θ and g. In Fig. 3, we have shown the contour plots of $|\zeta_{k_0}|^2(a_1, a_2, a_3)$ for fixed a_1's in the range $a_2 \in [0, a_1]$ and $a_3 \in [0, a_2]$. The darker regions denote higher values of $|\zeta_{k_0}|^2$.

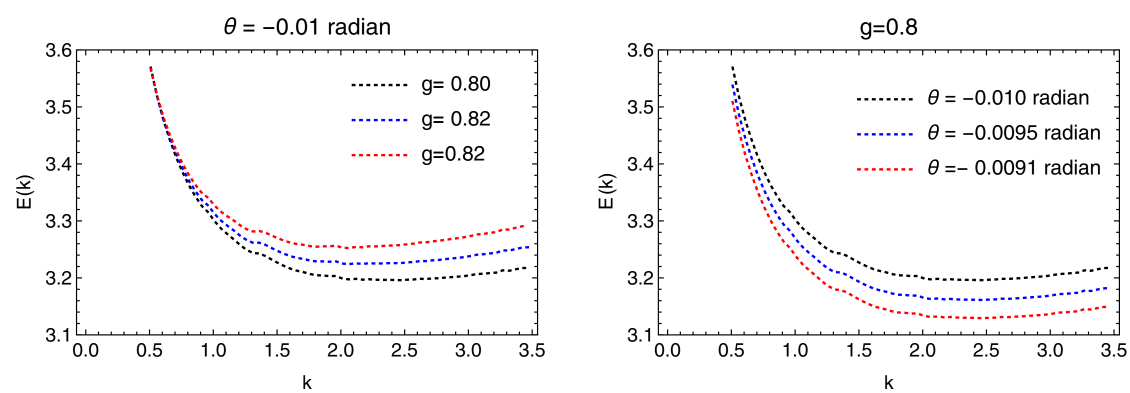

FIG. 1. Plot of $E(k)$ as a function of k for various values of g with a fixed θ (left) and for various values of θ with a fixed g (right).

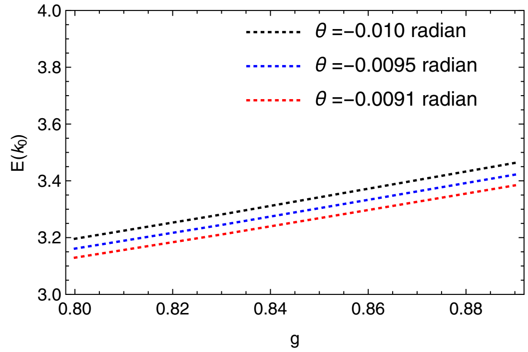

FIG. 2. Plot of $E(k_0)$ as a function of g.

From the figures, we can see that when a_1 is small, $a_1 \geq a_2 \geq a_3 \geq 0$ is the region close to the vertex of the wedgelike region spanned by a_1, a_2, and a_3. We see that $|\zeta_{k_0}|^2$ is significantly larger in this region. For a larger a_1, $|\zeta_{k_0}|^2$ is localized near the $a_2 = a_3$ boundary (the diagonal lines in the boxes), the $a_1 = a_2$ boundary (the right-hand limits of the boxes), and near the $a_1 = a_2 = a_3$ boundary (the top-right corners of the boxes), while in the interior region, where the a_i's are distinctly different, $|\zeta_{k_0}|^2$ is significantly damped. Thus, we conclude that ζ_{k_0} describes states localized near the $\phi(a) = 0$ boundary.

As $|\theta|$ decreases, the value of k_0 decreases, which can be seen from Fig. 1. As a result, the exponential factor decays slowly for smaller a $|\theta|$, and $|\zeta_{k_0}|^2$ spreads more into the bulk. This is consistent with the fact that there is no edge state at $\theta = 0$.

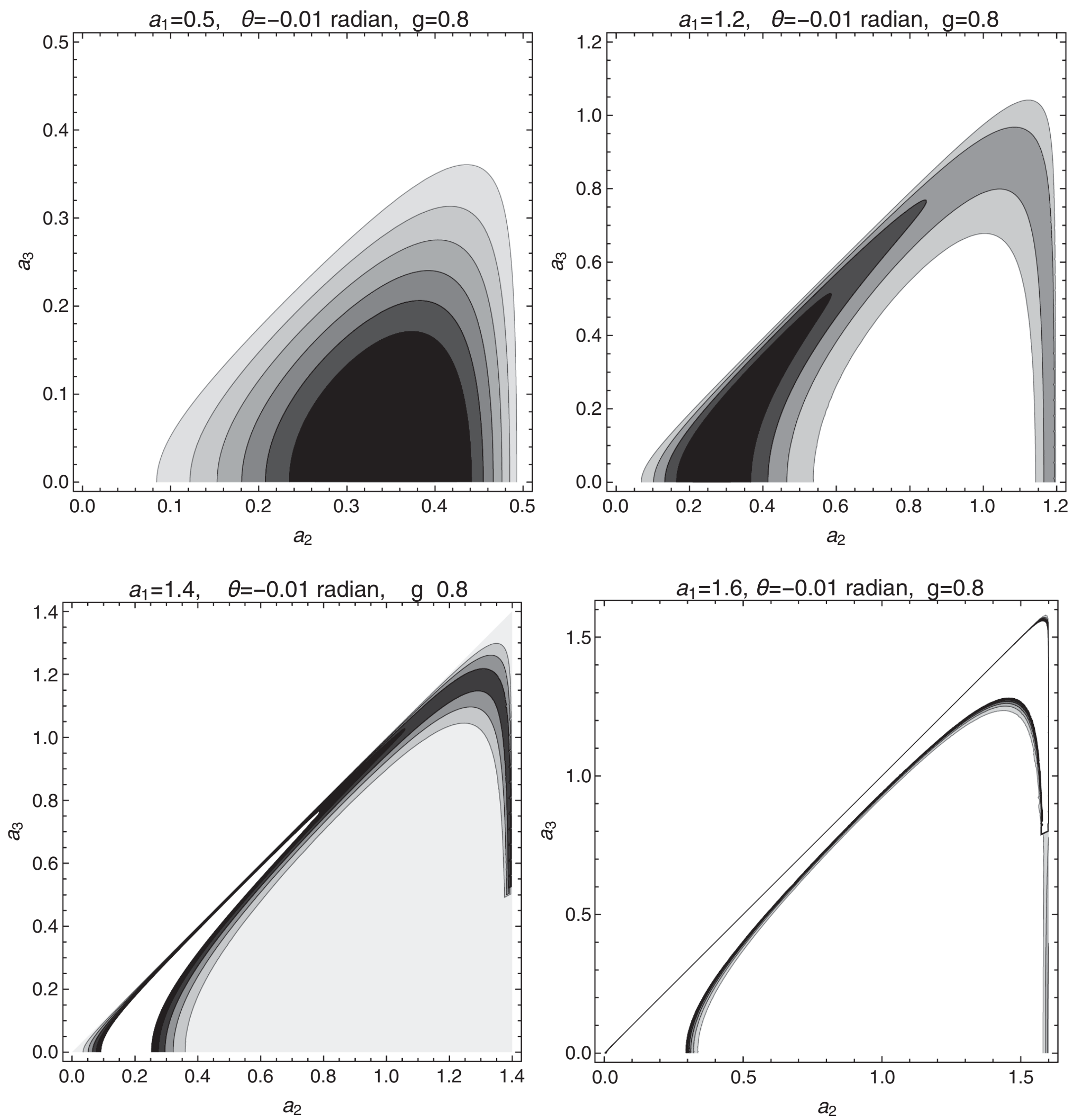

FIG. 3. Plot of $|\zeta_{k_0}|^2$ as a function of a_2 and a_3 for fixed values of a_1 and $\theta = -0.01$ radian. Here, we used $g = 0.8$ and $k_0 \approx 2.35$, which is obtained from the minima in Fig. 1 with $\theta = -0.01$ radian.

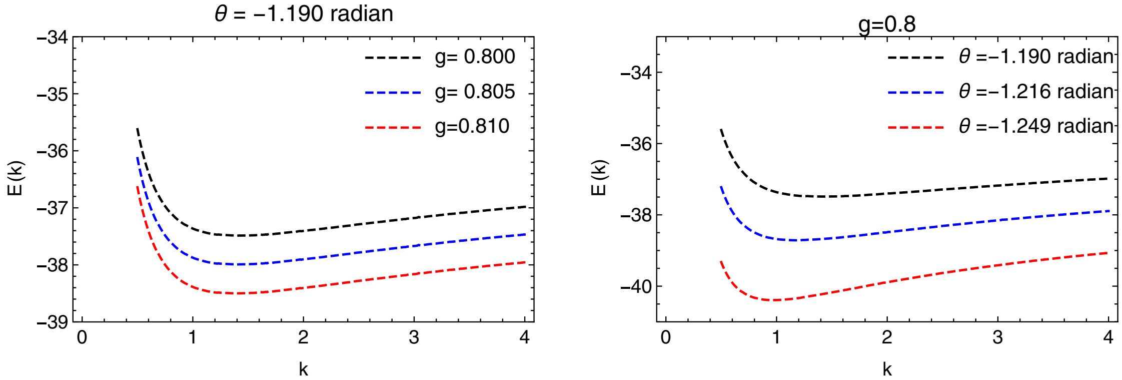

FIG. 4. Plot of $E(k)$ as a function of k for various values of g with a fixed θ (left) and for various values of θ with a fixed g (right).

B. $0 \ll |\theta| < \frac{\pi}{2}$:

For a large value of $|\theta|$, we indeed find that $E(k)$ is negative. For various values of θ and g, we have plotted $E(k)$ as a function of k in Fig. 4.

Again, the minima gives the upper bound of the total energy $E(k_0)$. We have plotted $E(k_0)$ as a function of the coupling constant g for various values of θ in Fig. 5.

Again, if we plot $|\zeta_{k_0}|^2$ as a function of a_1, a_2, and a_3 (as in Fig. 3) for a fixed θ and g, we will find that the modes are localized near the $\phi = 0$ boundary, so these are edge states but with a negative $\langle H_0 + V(M)\rangle$.

As $|\theta|$ increases towards $\frac{\pi}{2}$, k_0 decreases (as can be seen from Fig. 4). Consequently, the modes spread more into the bulk.

When $0 > \langle H_0 + V(M)\rangle \geq -C(R)$, the total energy of these edge states are positive, and such edge states are physical.

On the other hand, when $\langle H_0 + V(M)\rangle < -C(R)$, the total energy is negative. As glueballs are bosons, if such negative energy states exist, there is no Pauli exclusion principle to prevent states with an arbitrary number of such edge localized glueballs with negative energies. Therefore, the energy will be unbounded from below, and the vacuum will be unstable. Therefore, these edge states should be considered *unphysical*.

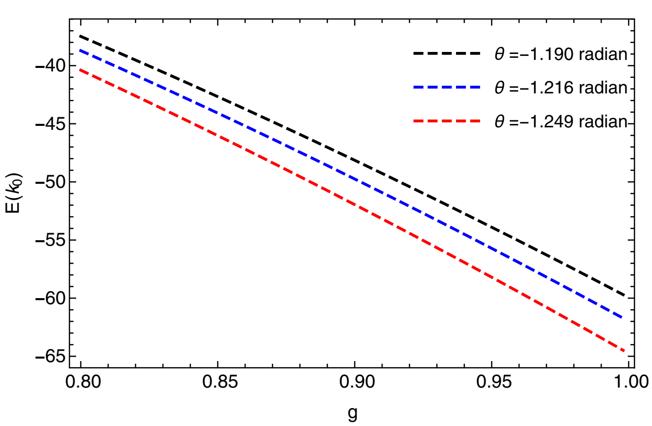

FIG. 5. Plot of $E(k_0)$ as a function of g.

V. NEGATIVE ENERGY EDGE STATES AND INDICATIONS OF PHASE TRANSITION

In the previous section, we have shown that the total energy of the edge states in the matrix model can be negative [i.e., $\langle H_0 + V(M)\rangle < -C(R)$]. In a pure gauge theory, these negative energy states are unphysical. However, as we argue below on the inclusion of matter fields, these states can have positive energy and, therefore, can exist.

A similar situation has been treated by Asorey *et al.* [9,10] for spin-zero (and one) fields on a spatial disk $\mathcal{D}$ with boundary $\partial\mathcal{D}$. If $\hat{n}$ is the outward-drawn unit, normal at $\partial\mathcal{D}$, and ∂_n denotes $\hat{n}\cdot\vec{\nabla}$ at $\partial\mathcal{D}$, the scalar Laplacian $\Delta = -\sum_i \frac{\partial^2}{\partial x_i^2}$ is (essentially self-adjoint for the Robin boundary conditions)

$$(\Psi + i\partial_n\Psi)|_{\partial\mathcal{D}} = e^{i\theta}(\Psi - i\partial_n\Psi)|_{\partial\mathcal{D}}, \qquad e^{i\tilde{\theta}} \in U(1). \tag{5.1}$$

When $e^{i\tilde{\theta}} = 1$, (5.1) gives the Neumann boundary condition $\partial_n\Psi|_{\partial\mathcal{D}} = 0$, while if $e^{i\tilde{\theta}} = -1$, it gives the Dirichlet boundary condition $\Psi|_{\partial\mathcal{D}} = 0$. Near the Dirichlet point, there are Robin boundary conditions

$$\Psi|_{\partial\mathcal{D}} = \lambda\partial_n\Psi|_{\partial\mathcal{D}}. \tag{5.2}$$

It is an important result of [9] that the Laplacian $-\sum_i \frac{\partial^2}{\partial x_i^2}$ has edge-localized negative energy states if $\lambda > 0$. Hence, the free Laplacian $-\sum_i \frac{\partial^2}{\partial x_i^2}$ cannot be second quantized.

However, it was proved [10] that $-\sum_i \frac{\partial^2}{\partial x_i^2}$ has a lower bound:

$$-\sum_i \frac{\partial^2}{\partial x_i^2} \geq -m_0^2(\lambda). \tag{5.3}$$

Hence,

$$-\sum_i \frac{\partial^2}{\partial x_i^2} + m_0^2(\lambda) \geq 0, \tag{5.4}$$

and the Lagrangian density

$$\mathcal{L} = -\phi^* \left(\frac{\partial^2}{\partial x_0^2} - \sum_i \frac{\partial^2}{\partial x_i^2} - m_0^2(\lambda) \right) \phi \tag{5.5}$$

allows a consistent quantization.

In addition, (5.5) is the Lagrangian for a superconductor. (For the latter, ϕ should be a vector field, but that is not important here). Further the field ϕ in the ground state decays as it enters $\mathcal{D}$ due to Meissner effect:

$$\phi = \phi_0 e^{-(r_0 - r) m_0^2(\lambda)} \tag{5.6}$$

near $\partial\mathcal{D}$, with r decreasing away from the boundary. From (5.6), we get

$$(\phi - m_0^2(\lambda) \partial_r \phi)|_{\partial\mathcal{D}} = 0. \tag{5.7}$$

This is (5.2) for $\lambda = m_0^2(\lambda)$.

Thus, the negative energy levels signify the transition to the superconducting phase.

The wave functions Ψ vanishing at $\partial\mathcal{D}$ have nonnegative energies even without $m_0^2(\lambda)$:

$$\left(\Psi, -\sum_i \frac{\partial^2}{\partial x_i^2} \Psi \right) = \left(\sum_i \frac{\partial}{\partial x_i} \Psi, \sum_i \frac{\partial}{\partial x_i} \Psi \right) \geq 0 \quad \text{for } \Psi|_{\partial\mathcal{D}} = 0. \tag{5.8}$$

Hence, with the addition of $m_0^2(\lambda)$, the edge states get lifted to positive energies (which can be adjusted to be low lying), while bulk states get gapped, with bulk energies $> |m_0(\lambda)|$.

The possibility of "superconducting" phases has been considered in the quark-gluon plasma phase of QCD [15,16]. Such color superconductivity is expected to be the ground state when the temperatures are low and the baryon chemical potential is high. When massless quarks are coupled to the pure Yang-Mills theory, indeed color-flavor locked phase or 2SC (when one quark does not participate in the condensation) phases can emerge [33–35]. The superconducting phases emerge when the global symmetries $\mathrm{SU}(3)_F$ and $U(1)_B$ are broken. In the quark-gluon plasma phase, the symmetry group is $\frac{SU(3)_C \times SU(3)_F \times U(1)_B}{Z_3 \times Z_3}$. In the superconducting phase, the pairing of two quarks of the same helicity is dominant, and the presence of this diquark condensate spontaneously breaks the symmetry to $\frac{SU(3) \times Z_3}{Z_3 \times Z_3}$. This spontaneous breaking of the flavor symmetry and $U(1)_B$ naturally leads to a phase of massive gluons.

In the matrix model too, we can consider gluons coupled to the flavor symmetry breaking diquark condensate. In that case, the matrix model is constructed by pulling back the Maurer-Cartan one forms of $\frac{SU(3) \times Z_3}{Z_3 \times Z_3}$ instead of $\frac{SU(3)_C \times SU(3)_F \times U(1)_B}{Z_3 \times Z_3}$. In this matrix model, the gluons are massive. The mass term lifts the edge levels to positive energies, and at the same time, it creates a gap in the bulk levels, making them incompressible. Thus, in such a massive gluon phase, the aforementioned edge states do exist.

VI. EDGE STATES: ANGULAR MOMENTUM AND COLOR

The Schrödinger Hamiltonian on $\mathbb{R}^3$, on the separation of variables, acquires the centrifugal term $\frac{l(l+1)}{r^2}$. This term eliminates the boundary condition ambiguities at $r = 0$ from all except the S-wave.

However, it is a surprising result of Iwai (Secs. III. 3, V. 2, V. 3 in [17])[3] that the induced potential in the Hamiltonian $H = \Delta + V(M)$ [see (2.23)] for color or angular momentum states is *finite* as $\phi(a) \to 0$. That means that edge states are also present with angular momentum and color excitations.

Their energies will depend on angular momentum and color, because the induced potential depends on them. It will be interesting to study this energy dependence on angular momentum and color.

VII. DISCUSSIONS

In a matrix model of SU(2) Yang-Mills theory, the Hamiltonian requires boundary conditions on the boundaries of $\mathrm{Mat}_3(\mathbb{R})$. We have shown that for certain choices of these boundary conditions, there are glueball states localized near the boundaries. The energy of these edge states can be negative, in which case they can only be physical, if a London-like mass term is added to lift the total energies to a positive value. In the presence of matter, such a mass term can indeed be generated, and there, these edge states comprised of massive gluons constitute a superconducting phase of QCD.

In this matrix model, one can construct colored states of the Hamiltonian as well. However, as shown in [4,5], all observables are color singlet functions of M. Thus, the colored states naturally decouple from the color singlet theory.

Also, the colored states are mixed, while the colorless ones are pure [4,5]. That is why it is not possible to evolve

[3]This point was emphasized to us by Sachindeo Vaidya.

from a colorless state to the tensor product of colored states by unitary time evolution.

Under $M_{ia} \to -M_{ia}$, the singular values are invariant: $a_i \to a_i$. Consequently, the ground state obtained by the variational computation has even parity, as expected. Here, we used the zero modes of H_0 to construct the variational ansatz. For a better approximation, we can, in principle, include nonzero modes as well in the variational ansatz. However, there will be additional computational complexity owing to many nonvanishing terms.

Although, we have demonstrated the presence of these edge states in a SU(2) Yang-Mills theory, the analysis and the conclusions can be readily extended to $N > 2$. However, for a large N, the singular value decomposition becomes difficult, because under color, the gauge fields transforms as $M \to M(Ad(h))^T$, $h \in \mathrm{SU}(N)$.

To study the large N limit, we should start with the observation: our matrix model is very similar to a three-matrix model describing N-coincident D-branes coupled to a Ramond-Ramond 4-form field [36]. In particular, the potential of our matrix model

$$V(M) = \frac{1}{2Rg^2} Tr\left(M_i M_i + i\epsilon_{ijk} M_i [M_j, M_k] - \frac{1}{2}[M_i, M_j]^2\right), \qquad M_i \equiv M_{ia} T_a, \tag{7.1}$$

(T_a's are generators of $\mathrm{SU}(N)$ in the fundamental representation) has extrema describing $N \times N$ fuzzy sphere algebras (similar to [36]). Here, the difference between $N = 2$ and $N > 2$ appears: for $N = 2$, only the nontrivial extremum is described by the fuzzy sphere algebra in a two-dimensional irreducible representation, while for $N > 2$, the algebra can be N-dimensional irreducible or any possible reducible representation of SU(2).

The vacua corresponding to the irreducible and the reducible representations are degenerate, and transitions between them occur by quantum tunneling (as discussed in [37]), which we intend to study in the future.

One can also obtain the quantum states of these fuzzy spheres (both reducible and irreducible) by a Gelfand-Naimark-Segal construction and compute the von Neumann entropy associated with the fuzzy spheres (as in [38]) to study the evolution of the system.

The setting in the previous paragraph is perfect to extend the study of the our matrix model to a large N limit and discuss its possible equivalence to the Calogero model in the light of [39]. This will be the future direction of our investigation.

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