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First law of entanglement rates from holography

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For a perturbation of the state of a conformal field theory (CFT), the response of the entanglement entropy is governed by the so-called "first law" of entanglement entropy, in which the change in entanglement entropy is proportional to the change in energy. Whether such a first law holds for other types of perturbations, such as a change to the CFT Lagrangian, remains an open question. We use holography to study the evolution in time t of entanglement entropy for a CFT driven by a t-linear source for a conserved $U(1)$ current or marginal scalar operator. We find that although the usual first law of entanglement entropy may be violated, a first law for the rates of change of entanglement entropy and energy still holds. More generally, we prove that this first law for rates holds in holography for any asymptotically $(d+1)$-dimensional anti–de Sitter metric perturbation whose t dependence first appears at order z^d in the Fefferman-Graham expansion about the boundary at $z = 0$.

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I. INTRODUCTION, SUMMARY, AND OUTLOOK

Many-body systems in thermal equilibrium are governed by universal laws, the laws of thermodynamics. Many-body systems perturbed out of thermal equilibrium are also governed by universal laws, the laws of hydrodynamics (for sufficiently small perturbations and at sufficiently late times). What laws, if any, govern many-body systems driven far from equilibrium? This question is of central importance in many branches of physics, from cosmology (the electroweak phase transition, the Kibble-Zurek mechanism, etc.) to condensed matter physics (quantum quenches, thermalization, etc.) to heavy ion collisions (thermalization and isotropization of the quark-gluon plasma), and beyond. Many of these phenomena, such as thermalization, necessarily involve interactions. Few reliable techniques exist for studying interacting systems far from equilibrium; hence the question remains open.

Cardy and Calabrese pioneered the use of entanglement entropy (EE), $S_{\rm EE}$, to characterize far-from-equilibrium systems [1–4]. The EE of a subregion of space at a fixed time, t, is defined as the von Neumann entropy of the reduced density matrix, ρ, obtained by tracing out the states in the rest of space (i.e. the region's complement), $S_{\rm EE} \equiv -{\rm tr}(\rho \ln \rho)$.

Cardy and Calabrese focused on a quantum quench of a coupling in the Hamiltonian to a value that produces a Conformal Field Theory (CFT), and used the powerful techniques of (boundary) CFT in spacetime dimension $d = 2$ to compute $S_{\rm EE}$, for a spatial interval of length ℓ. They showed that after the quench ends, $S_{\rm EE}$ evolves linearly in t, and then saturates at a time proportional to ℓ/c, with c the speed of light. They also provided an intuitive model for $S_{\rm EE}$'s evolution, in terms of maximally entangled Einstein-Podolsky-Rosen (EPR) pairs of particles produced by the quench, which are necessarily massless, due to the CFT's scale invariance, and hence move at speed c. Liu and Suh proposed, based on evidence from the anti–de Sitter/CFT (AdS/CFT) correspondence, also known as holography, that when $d > 2$, Cardy and Calabrese's massless particle model becomes an "entanglement tsunami" in which a quench produces a wave front of entangled excitations that moves inward from the region's boundary [5–7].

Crucially, EE obeys constraints that ultimately come from unitarity, and that can be, and have been, used to constrain far-from-equilibrium evolution in quantum systems. For example, for two density matrices ρ and ρ' in the same Hilbert space, their relative entropy, $S(\rho|\rho') \equiv {\rm tr}(\rho \ln \rho) - {\rm tr}(\rho \ln \rho')$, is non-negative, and indeed provides a measure of the "statistical distance" or distinguishability between them. Positivity of relative entropy, $S(\rho|\rho') \geq 0$, played a key role in proving speed limits on entanglement tsunamis [5–9].

Further constraints can be derived from $S(\rho|\rho') \geq 0$. In particular, if ρ and ρ' are close, so that $\rho' = \rho + \delta\rho$ with $\delta\rho$ small, then expanding $S(\rho|\rho')$ to first order in $\delta\rho$ gives a constraint called the "first law of EE" (FLEE) [10–13]:

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$$\delta S_{\rm EE} = \delta\langle H\rangle, \tag{1}$$

where $\delta S_{\rm EE}$ is the change in EE; H is the modular Hamiltonian, defined via $\rho \equiv e^{-H}$; and $\delta\langle H\rangle$ is the change in H's expectation value. If ρ is a thermal density matrix with temperature T, then the FLEE becomes the usual first law of thermodynamics, $\delta S = \delta E/T$, with entropy S and energy E. If ρ is the reduced density matrix of a spatial subregion, then generically H and $\delta\langle H\rangle$ are complicated nonlocal objects that are difficult to calculate. However, for a spherical subregion in a CFT vacuum, H is a product of two factors, integrated over the sphere's volume. The first factor is δE, or in terms of the stress-energy tensor $T_{\mu\nu}$ ($\mu,\nu = 0,1,\ldots,d-1$, with $x^0 \equiv t$), the change in $\langle T_{tt}\rangle$ in the subregion. The second factor depends only on geometric data, including in particular the sphere's radius, R [13,14]. For perturbations with $\delta\langle T_{tt}\rangle$ constant in space, or for spheres sufficiently small that $\delta\langle T_{tt}\rangle$ can be approximated as a constant, the integral is easily performed, with the result

$$\delta S_{\rm EE} = \frac{\delta E}{T_{\rm ent}}, \tag{2}$$

where the "entanglement temperature," $T_{\rm ent}$, depends on R and d,

$$T_{\rm ent}^{\rm sphere} = \frac{d+1}{2\pi R}, \tag{3}$$

but is independent of any other details of the CFT or of the state ρ. For a "strip" subregion, consisting of two parallel planes separated by a distance ℓ, holographic CFTs also obey Eq. (2), but now with [10]

$$T_{\rm ent}^{\rm strip} = \frac{2(d^2-1)\Gamma\left(\frac{d+1}{2(d-1)}\right)\Gamma\left(\frac{d}{2(d-1)}\right)^2}{\sqrt{\pi}\Gamma\left(\frac{1}{d-1}\right)\Gamma\left(\frac{1}{2(d-1)}\right)^2}\frac{1}{\ell}. \tag{4}$$

The FLEE does not hold for arbitrary deformations. In quantum mechanics, the FLEE holds only for "completely positive trace-preserving" maps, linear maps that are combinations of unitary transformations, partial tracing, and adding subsystems—for a precise definition, see for example Appendix of Ref. [12], and references therein.

In a continuum quantum field theory (QFT), what deformations obey the FLEE? Finding a precise answer appears to be more challenging than in quantum mechanics. In particular, in continuum QFT, ρ generically has an infinite number of eigenvalues, so in what sense can a perturbation of the eigenvalues, $\delta\rho$, be small? Currently the best intuition appears to be that, for compact subregions, the FLEE should hold when $\delta\langle T_{\mu\nu}\rangle$ is small, relative to the scale set by the subregion's size [12].

In this paper we will consider perturbations that go beyond a change of state: we will deform a CFT Hamiltonian by a relevant or marginal operator with a t-dependent source, which drives the CFT far from equilibrium. We will focus on sources linear in t, although our most general results apply to a larger class of sources, characterized most precisely via holography, as we discuss below. For our cases, we will show two things: first, generically the naïve FLEE in Eq. (2) is violated, and second, a relation very similar to Eq. (2) holds for the *rates of change* of EE and energy.

We will restrict to CFTs with holographic duals, mainly because holography is currently the easiest way to compute $S_{\rm EE}$ in interacting QFTs. Computing $S_{\rm EE}$ holographically requires two steps. First, we must solve Einstein's equation for the asymptotically AdS_{d+1} metric, G_{mn} ($m,n = 0,1,\ldots,d$), of the holographically dual spacetime. We will mostly work with a Fefferman-Graham (FG) holographic coordinate z with asymptotic AdS_{d+1} boundary at $z=0$, where the CFT "lives." Second, we must compute the area of the extremal surface that at the asymptotic AdS_{d+1} boundary coincides with the entangling region's boundary in the dual QFT. $S_{\rm EE}$ is then that area divided by $4G_N$, with Newton's constant G_N [15–19].

In holography, a deformation of the CFT Hamiltonian by a relevant or marginal operator corresponds to a change of the bulk metric, $G_{mn} \to G_{mn} + \delta G_{mn}$. Our unperturbed metric G_{mn} will be asymptotically AdS_{d+1} and independent of t and the CFT spatial coordinates, but otherwise arbitrary. Our main examples of G_{mn} will be Poincaré patch AdS_{d+1}, dual to the CFT vacuum, and the AdS_{d+1} black brane, dual to the CFT with nonzero T. Our perturbation δG_{mn} will preserve the asymptotic AdS_{d+1}, but generically depend on t. Our only nontrivial assumption will be that t dependence in δG_{mn} first appears at order z^d in the FG expansion. In other words, the t dependence of δG_{mn} will be arbitrary, except that terms in the FG expansion with powers of z smaller than z^d will be t independent. With that assumption, in Sec. II we will prove a "first law of entanglement rates" (FLOER),

$$\partial_t \delta S_{\rm EE} = \frac{\partial_t \delta E}{T_{\rm ent}}, \tag{5}$$

where $T_{\rm ent}$ depends on the unperturbed G_{mn} and the extremal surface therein. If the unperturbed G_{mn} is the Poincaré patch AdS_{d+1}, then $T_{\rm ent}$ is identical to that in Eq. (3) or (4).

Our proof of Eq. (5) can also be straightforwardly extended to deformations by sources which are position dependent instead of time dependent, provided the corresponding assumptions about G_{mn} and δG_{mn} are satisfied. The resulting FLOER involves rates of change in a spatial coordinate $x^1 \equiv x$, rather than t, i.e. $\partial_x \delta S_{\rm EE} = \partial_x \delta E/T_{\rm ent}$. However, given our motivation to understand far-from-equilibrium evolution, and also for clarity, we will continue to refer only to t-dependent sources, unless stated otherwise.

Equation (5) is our main result. The key assumption underlying Eq. (5), that t dependence in δG_{mn} appears first at order z^d in the FG expansion, characterizes the most general class of perturbations for which our FLOER holds, and turns out to be a relatively mild constraint. Indeed, in Secs. III, IV, and V we discuss various nontrivial examples that illustrate how easily our key assumption can be satisfied with a t-linear source. Our examples also provide our other main result: in many of our examples the FLEE in Eq. (2) is explicitly violated, indicating that the FLOER may be more fundamental than the FLEE, as we discuss below.

In Secs. III and IV, we consider holographic CFTs in $d = 3$ and 4, respectively, each with a conserved current J^μ of a global $U(1)$ symmetry. In each case, in the CFT we introduce a constant external electric field $\mathcal{E}$ in the x direction, that is, we add to the CFT Lagrangian a relevant deformation $\propto t\mathcal{E}J^x$, resulting in a current, $\langle J^x\rangle \neq 0$. We introduce no charge density, $\langle J^t\rangle = 0$, so the current arises exclusively from Schwinger pair production, i.e. production of maximally entangled particle-antiparticle (EPR) pairs. Crucially, in our examples, $\langle J^x\rangle$ is t independent. As a result, the Ward identity $\partial_\mu T^{\mu\nu} = F^{\nu\rho}J_\rho$ implies Joule heating, $\partial_t\langle T_{tt}\rangle = \mathcal{E}\langle J^x\rangle$, that is also t independent. As a convenient shorthand, we will call such states "nonequilibrium steady states" (NESS): nonequilibrium because $\partial_t\langle T_{tt}\rangle \neq 0$, but steady states because $\langle J^x\rangle$ and $\partial_t\langle T_{tt}\rangle$ are t independent.

In holography, $T_{\mu\nu}$ is dual to G_{mn}, and J^μ is dual to a $U(1)$ gauge field, A_m. On the gravity side of the duality, our examples in Secs. III and IV thus both have G_{mn} and A_m, albeit with some essential differences.

In Sec. III, we consider Einstein-Maxwell theory in AdS_4, which arises for example from the consistent truncation of eleven-dimensional supergravity on $\mathrm{AdS}_4 \times S^7$ down to AdS_4 [20,21]. In that example, the dual CFT is the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory, i.e. the $\mathcal{N} = 6$ supersymmetric (SUSY) Chern-Simons-matter CFT in $d = 3$ [22]. Our NESS are dual to spacetimes with a null $U(1)$ field strength and AdS_4-Vaidya metric [23], describing a horizon that moves towards the asymptotically AdS_4 boundary as t increases.

In contrast, in Sec. IV our A_m has a probe Dirac-Born-Infeld (DBI) action in a fixed asymptotically AdS_5 background. Specifically, we consider asymptotically $\mathrm{AdS}_5 \times S^5$ solutions of type IIB supergravity with a number N_f of probe D7-branes along $\mathrm{AdS}_5 \times S^3$. The type IIB solutions are dual to states of $\mathcal{N} = 4$ $SU(N_c)$ SUSY Yang-Mills (SYM) theory in $d = 4$, at large N_c and large 't Hooft coupling, and the probe D7-branes are dual to a number $N_f \ll N_c$ of $\mathcal{N} = 2$ SUSY hypermultiplets in the fundamental representation of $SU(N_c)$, i.e. flavor fields. When $T \neq 0$, $\langle T^{\mu\nu}\rangle$ receives order N_c^2 and $N_f N_c$ contributions from the $\mathcal{N} = 4$ SYM and flavor fields, respectively. We may thus think of the flavors as probes inside a huge heat bath. Our NESS exist because $\mathcal{E}$ pumps energy into the flavor sector at the same constant rate that the flavors dissipate energy into the heat bath. To obtain δS_{EE} we compute only the linearized (not the full nonlinear) backreaction of A_m onto G_{mn}.

Although in Secs. III and IV we focus on particular "top-down" string/M-theory constructions, in each case our analysis should easily generalize to many other systems of $U(1)$ gauge fields in asymptotically AdS_{d+1} spacetimes, either fully backreacted, as in Sec. III, or with linear backreaction of a probe, as in Sec. IV.

In Sec. V, we consider holographic CFTs in $d = 2, 3, 4$ with a marginal scalar operator $\mathcal{O}$, and add to the CFT Lagrangian a deformation $\propto t\mathcal{O}$. In holography, a marginal $\mathcal{O}$ is dual to a massless scalar field, ϕ. In Sec. V we compute only ϕ's linearized backreaction onto G_{mn}, and only in the asymptotically AdS_{d+1} region, which suffices to establish the FLOER. (The Appendix contains the results of the holographic renormalization [24] of ϕ in $d = 3$ and 4 that we use in Sec. V.) In Sec. V we also follow Ref. [25], and add to the CFT Lagrangian a deformation $\propto x\mathcal{O}$. In that case, a spatial FLOER is satisfied trivially, because in the system of Ref. [25] both δS_{EE} and δE turn out to be x independent.

In our examples symmetries actually require $\mathcal{T}_{mn}$ to depend only on z, and not on t. In Secs. III and IV, $U(1)$ gauge invariance implies that $\mathcal{T}_{mn}$ depends only on A_m's field strength, F_{mn}, which is t independent because our solutions for A_m are linear in t. In Sec. V, the massless scalar ϕ has a shift symmetry $\phi \to \phi + C$ with constant C, which implies that $\mathcal{T}_{mn}$ depends only on derivatives of ϕ, and hence is t independent because our solutions for ϕ are linear in t. Time dependence is instead generated by off-diagonal terms $\mathcal{T}_{tz} = \mathcal{T}_{zt}$ which, via Einstein's equation, force δG_{mn} to depend on both z and t. Indeed, such off-diagonal terms in $\mathcal{T}_{mn}$ indicate $\partial_t\langle T_{tt}\rangle \neq 0$ in the dual QFT [26]; i.e. the system is out of equilibrium. We emphasize, however, that while the symmetries of our examples are *sufficient* to guarantee that δG_{mn} obeys our key assumption, they are not strictly *necessary*.

In terms of the CFT generating functional, in all of our examples we deform the CFT by a source *linear* in t. Such deformations are *not* quenches in any conventional sense: our systems do not necessarily approach equilibrium in the infinite past or future. At best, our deformations could perhaps be interpreted as an endless series of global quenches, one right after another, every moment in t. More succinctly, our systems are *driven* by a source linear in t (not periodic in t, in contrast to Ref. [27]). We emphasize again, however, that our examples are only a subset of a much larger class of t-dependent deformations, as mentioned above.

To summarize, we have identified a law governing a certain class of far-from-equilibrium systems. Specifically, we extended the FLEE in Eq. (2) beyond deformations of the state, to deformations of the Hamiltonian, characterized holographically by δG_{mn} whose t dependence first appears at order z^d in the FG expansion. For such deformations, we have shown that the FLOER of Eq. (5) holds, while the FLEE of the form in Eq. (2) in general does not.

Looking to the future, our results have implications both practical and conceptual. In practical terms, the FLOER may be useful because $\partial_t \delta E$ is often easier to calculate than $\partial_t \delta S_{\rm EE}$. In particular, if we can argue that the FLOER holds, and we know $T_{\rm ent}$, then we can obtain $\partial_t \delta S_{\rm EE}$ by calculating $\partial_t \delta E$, for example via the Ward identity $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho$.

Of the many conceptual questions our results raise, we will highlight only three. First, given that the same $T_{\rm ent}$ appears in our FLOER and in the FLEE of Eq. (2), can the FLOER simply be integrated to obtain the FLEE? In our examples where the FLEE is violated, $\delta S_{\rm EE}$ has a t-independent contribution absent from δE. Apparently, integrating the FLOER produces different integration constants in $\delta S_{\rm EE}$ and δE. We suspect that the difference arises from initial conditions. For instance, imagine "turning on" our t-linear source at $t = 0$. We expect EE and energy to be produced immediately. However, the EE is only sensitive to entanglement across the entangling surface, so in an entanglement tsunami description some of the EPR pairs produced at $t = 0$ will contribute to $S_{\rm EE}$ only after some "lag time" required for one EPR partner to leave the subregion. The lag time should be on the order of the subregion's size, as indeed we find in some of our examples.

Second, when the FLOER holds but the FLEE in the form of Eq. (2) is violated, could the FLEE in the form of Eq. (1) still hold? This is only possible if $\delta\langle H\rangle \neq \delta E / T_{\rm ent}$. The crucial point is that we are not comparing two states in the same Hilbert space. We are changing the CFT Hamiltonian, which changes the Hilbert space, and then comparing states in the old and new Hilbert spaces. In such cases, can $S(\rho|\rho')$ even be *defined*, and if so, do $S(\rho|\rho') \geq 0$ and hence the FLEE in Eq. (1) hold? To our knowledge, these questions remain open. The current state of the art appears to be the proof in Ref. [28], for *t-independent* relevant deformations, that $S(\rho|\rho')$ can be defined, and $S(\rho|\rho') \geq 0$, for states in two different Hilbert spaces only if the two theories have the same UV fixed point.[1] The D3/D7 system with massive flavors actually provides a *time-independent* example where the assumptions of Ref. [28] are satisfied but the FLEE in the form of Eq. (2) fails, as we discuss in Sec. IV. In our *time-dependent* examples we could attempt to test the FLEE in Eq. (1) directly, by calculating $\delta\langle H\rangle$ holographically. However, although much is known about the holographic dual of H [30–34], we know of no practical prescription for computing $\delta\langle H\rangle$ holographically, so we will leave such a test for future research.

Third, can we identify more precisely in field theory terms the class of t-dependent deformations for which the FLOER of Eq. (5) holds while the FLEE of Eq. (2) need not? Moreover, can we extend our results to more general systems, either in QFT or in holography? (For work in this direction, see for example Ref. [35].) We believe that these and many other questions relating to the FLOER deserve further study, in large part because they may eventually reveal universal laws governing far-from-equilibrium systems.

II. GENERAL ANALYSIS

In this paper we consider only asymptotically AdS_{d+1} spacetimes. In this section, we exclusively use FG coordinates, in which the metric takes the form

$$ds^2 = G_{mn} dx^m dx^n = \frac{L^2}{z^2}(dz^2 + g_{\mu\nu}(z, x^\rho) dx^\mu dx^\nu), \qquad (6)$$

where $m, n = 0, 1, \ldots, d$ and $\mu, \nu, \rho = 0, 1, \ldots, d-1$, where $x^0 = t$ is time, and L is the radius of the asymptotic AdS_{d+1}, with boundary at $z = 0$. The FG expansion of $g_{\mu\nu}(z, x^\rho)$ about the boundary is of the form

$$g_{\mu\nu}(z, x^\rho) = g^{(0)}_{\mu\nu}(x^\rho) + z^2 g^{(2)}_{\mu\nu}(x^\rho) + \cdots + z^d g^{(d)}_{\mu\nu}(x^\rho) + z^d \log z^2 h^{(d)}_{\mu\nu}(x^\rho) + \cdots, \qquad (7)$$

where the term $\propto z^d \log z^2$ is present only when d is even. The expectation value of the energy-momentum (density) tensor of the dual field theory, $\langle T_{\mu\nu}(x^\rho)\rangle$, takes the generic form [24]

$$\langle T_{\mu\nu}(x^\rho)\rangle = \frac{dL^{d-1}}{16\pi G_N} g^{(d)}_{\mu\nu}(x^\rho) + X_{\mu\nu}[g^{(N)}_{\kappa\lambda}(x^\rho)], \qquad (8)$$

where $X_{\mu\nu}[g^{(N)}_{\kappa\lambda}(x^\rho)]$ is a function of the $g^{(N)}_{\kappa\lambda}(x^\rho)$ with $N < d$. Via Einstein's equation, the $g^{(N)}_{\mu\nu}(x^\rho)$ with $N < d$ are functions of the leading asymptotic coefficients in the near-boundary FG expansions of matter fields, or in dual QFT terms, functions of sources of operators.

Our key assumption is that the $g^{(N)}_{\mu\nu}(x^\rho)$ with $N < d$ are t independent: $g^{(N)}_{\mu\nu}(x^\rho) = g^{(N)}_{\mu\nu}(\vec{x})$, where $\vec{x}$ are the field theory spatial coordinates. In these cases,

$$\partial_t \langle T_{\mu\nu}(x^\rho)\rangle = \frac{dL^{d-1}}{16\pi G_N} \partial_t g^{(d)}_{\mu\nu}(x^\rho), \qquad (9)$$

so in particular the energy density's rate of change, $\partial_t \langle T_{tt}(x^\rho)\rangle$, is fixed by $g^{(d)}_{tt}(x^\rho)$ alone.

Our goal is to relate $\partial_t \langle T_{tt}(x^\rho)\rangle$ to $\partial_t S_{\rm EE}$, where in the QFT $S_{\rm EE}$ is the EE between a subregion $\mathcal{A}$ and its

[1] See also Ref. [29] for a discussion of whether $S(\rho|\rho') \geq 0$ holds for a deformation $\propto t\mathcal{O}$, with marginal $\mathcal{O}$, for CFTs on a spatial sphere, holographically dual to gravity in *global* AdS_{d+1}.

complement on a Cauchy surface. To compute S_{EE} holographically, we consider a codimension-two surface $\mathcal{W}$ homologous to $\mathcal{A}$, with $\partial\mathcal{W} = \partial\mathcal{A}$. We describe $\mathcal{W}$'s embedding by a mapping $X^m(\xi)$ from $\mathcal{W}$'s world volume, with coordinates ξ, into the background spacetime. We then define $\mathcal{W}$'s area functional,

$$A[\mathcal{W}] = \int d^{d-1}\xi\sqrt{\gamma}, \tag{10}$$

where γ is the determinant of $\mathcal{W}$'s world volume metric. Extremizing A then gives S_{EE} [17,19],

$$S_{\mathrm{EE}} = \frac{A[\mathcal{W}_{\mathrm{ext}}]}{4G_N}. \tag{11}$$

Imagine we have the solution $X^m_{(0)}$ for $\mathcal{W}^{(0)}_{\mathrm{ext}}$'s embedding in a given background geometry $G^{(0)}_{mn}$, which we assume is asymptotically AdS_{d+1}, but is otherwise arbitrary. If we perturb the metric, $G^{(0)}_{mn} \to G^{(0)}_{mn} + \delta G_{mn}$, which leads to a change in the embedding, $X^m_{(0)} \to X^m_{(0)} + \delta X^m$, then the change in the EE, δS_{EE}, to leading order in δG_{mn} and δX^m, is[2]

$$\delta S_{\mathrm{EE}} = \frac{1}{4G_N}\int_{\mathcal{W}^{(0)}_{\mathrm{ext}}} d^{d-1}\xi\sqrt{\gamma}\left(\theta_m \delta X^m + \frac{1}{2}\Theta^{mn}_{\mathrm{ext}}\delta G_{mn}\right), \tag{12}$$

where θ_m and $\Theta^{mn}_{\mathrm{ext}}$ are variations of A, evaluated on the unperturbed solutions,

$$\theta_m \equiv \frac{1}{\sqrt{\gamma}}\frac{\delta A}{\delta X^m}\bigg|_{X^m_{(0)},G^{(0)}_{mn}}, \qquad \Theta^{mn}_{\mathrm{ext}} = \frac{2}{\sqrt{\gamma}}\frac{\delta A}{\delta G_{mn}}\bigg|_{X^m_{(0)},G^{(0)}_{mn}}. \tag{13}$$

As argued for example in Refs. [36–38], because $\mathcal{W}^{(0)}_{\mathrm{ext}}$ is an extremal surface in the unperturbed geometry $G^{(0)}_{mn}$, by definition $\theta_m = 0$. We therefore find

$$\delta S_{\mathrm{EE}} = \frac{1}{8G_N}\int_{\mathcal{W}^{(0)}_{\mathrm{ext}}} d^{d-1}\xi\sqrt{\gamma}\Theta^{mn}_{\mathrm{ext}}\delta G_{mn}, \tag{14}$$

which generalizes the result of Ref. [37] for δS_{EE} to t-dependent perturbations.

Equation (14) is valid for any holographic spacetime, but for our proof of a FLOER we impose a few restrictions, as follows. First, we assume $G^{(0)}_{mn}$ is asymptotically AdS_{d+1}, and so admits a FG form, and is invariant under translations and rotations in the $\vec{x}$ directions as well as translations in t, so that

$$G^{(0)}_{mn}dx^m dx^n = \frac{L^2}{z^2}(dz^2 + g_{tt}dt^2 + g_{xx}d\vec{x}^2), \tag{15}$$

where g_{tt} and g_{xx} depend only on z. In our examples in the following sections, $G^{(0)}_{mn}$ will be Poincaré patch AdS_{d+1} or an AdS_{d+1} black brane. The assumption that $G^{(0)}_{mn}$ is t independent means the extremal surface $\mathcal{W}^{(0)}_{\mathrm{ext}}$ will actually be a *minimal* surface, $\mathcal{W}^{(0)}_{\mathrm{min}}$, and hence also $\Theta^{mn}_{\mathrm{ext}} \to \Theta^{mn}_{\mathrm{min}}$, the notation that we will use in the following.

We also make three assumptions about the perturbation δG_{mn}. First, we assume δG_{mn} preserves the AdS_{d+1} FG asymptotics, and also preserves translational and rotational symmetry in $\vec{x}$, so that

$$\delta G_{mn}dx^m dx^n = \frac{L^2}{z^2}(\delta g_{tt}dt^2 + \delta g_{xx}d\vec{x}^2), \tag{16}$$

where g_{tt} and g_{xx} depend only on z and t. In particular, as mentioned above, in δg_{tt} and δg_{xx}'s FG expansions we assume that the first t-dependent coefficients are $\delta g^{(d)}_{tt}$ and $\delta g^{(d)}_{xx}$, respectively. All of these assumptions are crucial for our proof of the FLOER, except for translational and rotational symmetry in $\vec{x}$, which we assume only for simplicity of our presentation, but which could be relaxed without spoiling the FLOER. Moreover, our assumptions are relatively mild, being satisfied by an enormous class of holographic spacetimes.

Under these assumptions, plugging the FG expansion of δG_{mn} into Eq. (14) and taking ∂_t of both sides gives us

$$\partial_t\delta S_{\mathrm{EE}} = \frac{L^2}{8G_N}\int_{\mathcal{W}^{(0)}_{\mathrm{min}}} d^{d-1}\xi\sqrt{\gamma}\Theta^{\mu\nu}_{\mathrm{min}}z^{d-2}\partial_t\delta g^{(d)}_{\mu\nu}(t) + \cdots, \tag{17}$$

where $\cdots$ indicates higher powers of (z), which are suppressed for a subregion sufficiently small compared to any other scale. We will henceforth assume that the subregion is sufficiently small to neglect the $\cdots$ terms.

To proceed any further we need an explicit form for $\Theta^{\mu\nu}_{\mathrm{min}}$, for which we must restrict to specific $\mathcal{A}$. We will use two different $\mathcal{A}$'s: a sphere, defined by $|\vec{x}| \le R$, and a strip, defined as two parallel planes separated in $x^1 \equiv x$ by a distance ℓ, and symmetric about $x = 0$.

For the sphere, we employ spherical coordinates, with radial coordinate r. By spherical symmetry we can then parametrize $\mathcal{W}$'s embedding as $r(z)$, so that

$$\sqrt{\gamma} = \left(\frac{L}{z}\right)^{d-1} r^{d-2} g_{xx}^{(d-2)/2}\sqrt{h}\sqrt{1 + g_{xx}r'^2}, \tag{18}$$

where h is the determinant of the metric $h_{\alpha\beta}$ of a unit $(d-2)$ sphere, S^{d-2}. We then find

[2]We may safely assume that under a small perturbation the topology around the entangling wedge does not change, so the homology constraint does not rule out $\mathcal{W}^{(0)}_{\mathrm{min}}$ in the backreacted geometry.

$$\Theta^{mn}_{\min}\partial_m \otimes \partial_n = \left(\frac{z}{L}\right)^2 \left(\frac{(\partial_z + r'\partial_r)^2}{1+g_{xx}r'^2} + \frac{1}{r^2 g_{xx}} h^{\alpha\beta}\partial_\alpha \otimes \partial_\beta\right). \tag{19}$$

For the strip, by translational symmetry in the $\vec{x}$ directions we can parametrize $\mathcal{W}$'s embedding as $x(z)$, so that

$$\sqrt{\gamma} = \left(\frac{L}{z}\right)^{d-1} g_{xx}^{(d-2)/2}\sqrt{1+g_{xx}x'^2}, \tag{20}$$

$$\Theta^{mn}_{\min}\partial_m \otimes \partial_n = \left(\frac{z}{L}\right)^2 \left(\frac{(\partial_z + x'\partial_x)^2}{1+g_{xx}x'^2} + \frac{1}{g_{xx}} \delta^{\alpha\beta}\partial_\alpha \otimes \partial_\beta\right). \tag{21}$$

Since the $\sqrt{\gamma}$ in Eq. (20) depends only on $x'(z)$, and not on $x(z)$, if we plug Eq. (20) into the area functional Eq. (10), then variation with respect to $x'(z)$ gives us a constant of motion, κ. We can then solve algebraically for $x'(z)$ in terms of κ,

$$x'(z) = \pm \frac{1}{\sqrt{\kappa^{d-1}z^{2-2d}g_{xx}^d - g_{xx}}}, \qquad \kappa = \frac{z_*^2}{g_{xx}^*}, \tag{22}$$

where $z_\star$ denotes $\mathcal{W}^{(0)}_{\min}$'s maximal extension in z, fixed by integrating $x'(z)$ from $(z=0)$ to $z_\star$ with the boundary conditions $x(0) = \pm\ell/2$ and by symmetry $x(z_\star) = 0$), and $g_{xx}^* \equiv g_{xx}(z_*)$.

We now plug the $\Theta^{mn}_{\min}$ from Eqs. (19) and (21) into Eq. (17) for $\partial_t S_{\mathrm{EE}}$. Crucially, the $\Theta^{mn}_{\min}$ in Eqs. (19) and (21) depend only on z, so we can trivially perform the integration over all other world volume coordinates ξ. Moreover, in the sum over μ and ν in $\Theta^{\mu\nu}_{\min}\partial_t\delta g^{(d)}_{\mu\nu}(t)$, only the $\vec{x}$ directions contribute, and indeed all contribute equally, due to the rotational symmetry in the $\vec{x}$ directions. Dropping the … terms in Eq. (17), as mentioned above, we thus find, for the sphere and strip, respectively,

$$\partial_t\delta S^{\mathrm{sphere}}_{\mathrm{EE}} = \frac{L^{d-1}}{8G_N}\mathrm{vol}(S^{d-2})\partial_t\delta g^{(d)}_{xx}\int_0^{z_*} dz z r^{d-2} g_{xx}^{\frac{d}{2}-2}\sqrt{1+g_{xx}r'^2}\left(\frac{g_{xx}r'^2}{1+g_{xx}r'^2} + d - 2\right), \tag{23a}$$

$$\partial_t\delta S^{\mathrm{strip}}_{\mathrm{EE}} = \frac{L^{d-1}}{4G_N}\mathrm{vol}(\mathbb{R}^{d-2})\partial_t\delta g^{(d)}_{xx}\int_0^{z_*} dz z g_{xx}^{\frac{d}{2}-2}\frac{(g^*_{xx}z^2/g_{xx}z_*^2)^{d-1} + d - 2}{\sqrt{1-(g^*_{xx}z^2/g_{xx}z_*^2)^{d-1}}}, \tag{23b}$$

where in both cases z_* denotes $\mathcal{W}^{(0)}_{\min}$'s maximal extension in z.

We can write each right-hand side in Eq. (23) in terms of $\partial_t E$, with E the energy inside $\mathcal{A}$, as follows. Translational and rotational symmetry in $\vec{x}$ implies $\langle T_{\mu\nu}\rangle$ is $\vec{x}$ independent, so $\partial_t E$ is simply the volume of $\mathcal{A}$ times $\partial_t\langle T_{tt}\rangle$. From Eq. (9) we have $\partial_t\langle T_{tt}\rangle \propto \partial_t g^{(d)}_{tt}$; however, the right-hand sides of Eq. (23) involve $\partial_t g^{(d)}_{xx}$. To replace $g^{(d)}_{xx}$ with $g^{(d)}_{tt}$, we use the fact that $T_{\mu\nu}$ is traceless, $T^\mu_\mu = 0$, up to a possible Weyl anomaly in even d, and the fact that the Weyl anomaly is t independent for $G^{(0)}_{mn}$ obeying our assumptions, so that $\partial_t T^\mu_\mu = 0$ in any d. As a result, $\partial_t g^{(d)}_{tt} = (d-1)\partial_t g^{(d)}_{xx}$ in any d. Plugging that into Eq. (9) and multiplying by $\mathcal{A}$'s volume we find [for the sphere, the volume of a $(d-1)$ unit ball is $\mathrm{vol}(S^{d-2})/(d-1)$]

$$\partial_t E^{\mathrm{sphere}} = \frac{dL^{d-1}}{16\pi G_N}\mathrm{vol}(S^{d-2})R^{d-1}\partial_t g^{(d)}_{xx}, \tag{24a}$$

$$\partial_t E^{\mathrm{strip}} = \frac{dL^{d-1}}{16\pi G_N}\mathrm{vol}(\mathbb{R}^{d-2})(d-1)\ell\partial_t g^{(d)}_{xx}. \tag{24b}$$

From Eq. (23) we thus identify our FLOER,

$$\partial_t S_{\mathrm{EE}} = \frac{\partial_t E}{T_{\mathrm{ent}}}, \tag{25}$$

with entanglement temperature T_{ent} for the sphere and strip, respectively,

$$(T^{\mathrm{sphere}}_{\mathrm{ent}})^{-1} = \frac{2\pi}{dR^{d-1}}\int_0^{z_\star} dz z r^{d-2} g_{xx}^{\frac{d}{2}-2}\sqrt{1+g_{xx}r'^2}\left(\frac{g_{xx}r'^2}{1+g_{xx}r'^2} + d - 2\right), \tag{26a}$$

$$(T^{\mathrm{strip}}_{\mathrm{ent}})^{-1} = \frac{4\pi}{d(d-1)\ell}\int_0^{z_\star} dz z g_{xx}^{\frac{d}{2}-2}\frac{(g^\star_{xx}z^2/g_{xx}z_\star^2)^{d-1} + d - 2}{\sqrt{1-(g^\star_{xx}z^2/g_{xx}z_\star^2)^{d-1}}}. \tag{26b}$$

If $G^{(0)}_{mn}$ is pure AdS_{d+1}, where $g_{xx}=1$, then $\mathcal{W}^{(0)}_{\rm min}$ for the sphere is given by $r(z)=\sqrt{R^2-z^2}$, for which $z_*=R$, and for the strip, $z_\star=\ell\Gamma(\frac{1}{2(d-1)})/2\sqrt{\pi}\Gamma(\frac{d}{2(d-1)})$ [16]. In these cases $T_{\rm ent}$ takes the same value as in the FLEE, Eqs. (3) and (4), respectively.

In the following sections we identify examples in which the *bulk* stress-energy tensor, $\mathcal{T}_{mn}$, produces a perturbation δG_{mn} obeying all of our assumptions, thus leading to a nontrivial FLOER. Moreover, the FLEE in Eq. (2) is typically violated.

III. AdS_4 VAIDYA

In this section we consider solutions of Einstein-Maxwell theory in AdS_4, with bulk action

$$S=\frac{1}{16\pi G_N}\int d^4x\sqrt{-\det(G_{mn})}\left[R+\frac{6}{L^2}-F^2\right], \quad (27)$$

with Ricci scalar R and $U(1)$ field strength F_{mn}. This theory arises for example as a consistent truncation of eleven-dimensional supergravity on S^7 [20,21]. In that case, the dual CFT is the ABJM theory [22], the $\mathcal{N}=6$ SUSY Chern-Simons-matter theory with gauge group $U(N_c)_k\times U(N_c)_{-k}$, in the limits $N_c\to\infty$ and $N_c\gg k^5$, where the Maxwell gauge field is dual to a conserved current J^μ of a $U(1)$ subgroup of the R-symmetry.

A solution of the Einstein-Maxwell theory in AdS_4 that describes a constant external electric field $\mathcal{E}$ in the x direction has the Vaidya metric,

$$ds^2=\frac{L^2}{u^2}[-(1-m(v)u^3)dv^2-2dudv+d\vec{x}^2], \quad (28)$$

with holographic coordinate u, with asymptotic AdS_4 boundary at $u=0$, null time coordinate $v\equiv t-u$, and $m(v)=2\mathcal{E}^2v$ [23]. The metric in Eq. (28) is sourced by a $U(1)$ field strength whose only nonzero components are $F_{xv}=-F_{vx}=\mathcal{E}$, which in the CFT describes an $\mathcal{E}$ that produces $\langle J^x\rangle=\sigma\mathcal{E}$ with conductivity $\sigma=L^2/(4\pi G_N)$ [23]. In the ABJM example, $\sigma=k^{1/2}N_c^{3/2}/(\pi 3\sqrt{2})$ [22]. The bulk stress-energy tensor's only nonzero component is $\mathcal{T}_{vv}=\mathcal{E}^2u^2/L^2$, which via $v=t-u$ produces both diagonal components $\mathcal{T}_{tt}$ and $\mathcal{T}_{uu}$ and off-diagonal components $\mathcal{T}_{tu}=\mathcal{T}_{ut}$, all t independent, as advertised in Sec. I.

The metric in Eq. (28) is well defined only when $m(v)>0$, that is, when $v>0$. In that regime, the metric in Eq. (28) describes a black brane geometry with a horizon moving outward, towards the AdS_4 boundary, in reaction to $\mathcal{E}$ dumping energy into the system at a constant rate $\partial_t\langle T_{tt}\rangle=\mathcal{E}\langle J^x\rangle=\sigma\mathcal{E}^2$. We can write the metric in Eq. (28) in the form $G^{(0)}_{mn}+\delta G_{mn}$, with $G^{(0)}_{mn}$ the metric of pure AdS_4, by switching from v to $t=v+u$:

$$G^{(0)}_{mn}dx^mdx^n=\frac{L^2}{u^2}(du^2-dt^2+d\vec{x}^2), \quad (29a)$$

$$\delta G_{mn}dx^mdx^n=\frac{L^2}{u^2}[2\mathcal{E}^2u^3(t-u)(dt^2-dtdu+du^2)]. \quad (29b)$$

However, just to be clear, $G^{(0)}_{mn}+\delta G_{mn}$ is an *exact* solution of the (full, nonlinear) Einstein equation, not merely a solution to linear order in δG_{mn}.

Crucially, $G^{(0)}_{mn}+\delta G_{mn}$ obeys all the assumptions in Sec. II, and hence will obey a FLOER. However, we will also compute $\delta S_{\rm EE}$ and δE themselves, to show that the FLEE of Eq. (2), $\delta S_{\rm EE}=E/T_{\rm ent}$, is violated.

Equation (14) gives us the $\delta S_{\rm EE}$ induced by $\mathcal{E}$, to leading order in $\mathcal{E}$,

$$\delta S_{\rm EE}=\frac{1}{8G_N}\int_{\mathcal{W}^{(0)}}d^{d-1}\xi\sqrt{\gamma}\Theta^{uu}_{\rm min}\delta G_{uu}, \quad (30)$$

where in this example γ and $\Theta^{mn}_{\rm min}$ are the determinant of the induced metric and the stress-tensor, respectively, of the minimal surface $\mathcal{W}^{(0)}_{\rm min}$ in pure AdS_4. Again, just to be clear, Eq. (30) only captures the leading change in the EE due to $\mathcal{E}$, whereas the metric in Eq. (28) is an exact solution of the Einstein equation. For a spherical subregion, we plug the solution for $\mathcal{W}^{(0)}_{\rm min}$'s embedding, $r(u)=\sqrt{R^2-u^2}$, into Eqs. (18) and (19) for γ and $\Theta^{mn}_{\rm min}$, respectively, and then use δG^{uu} from Eq. (29b), to find from Eq. (30)

$$\delta S_{\rm EE}=\left(\frac{L^2}{4\pi G_N}\mathcal{E}^2\right)2\pi^2R\int_0^R duu(t-u)\left(1-\frac{u^2}{R^2}\right)$$
$$=\mathcal{E}\langle J^x\rangle(\pi R^2)\left(\frac{\pi R}{2}\right)\left(t-\frac{8}{15}R\right),$$

where in the second equality we used $\langle J^x\rangle=\sigma\mathcal{E}$ with $\sigma=L^2/(4\pi G_N)$. Using the Ward identity for the energy density $\partial_t\langle T_{tt}\rangle=\mathcal{E}\langle J^x\rangle$ and the area (πR^2) of a sphere in two spatial dimensions, we identify $\mathcal{E}\langle J^x\rangle(\pi R^2)=\partial_tE$, and from Eq. (3) with $d=3$, we identify $T_{\rm ent}=2/(\pi R)$. We thus find

$$\delta S_{\rm EE}=\frac{\partial_tE}{T_{\rm ent}}\left(t-\frac{8}{15}R\right). \quad (31)$$

The analogous calculation for a strip subregion of width ℓ gives

$$\delta S_{\rm EE}=\frac{\partial_tE}{T_{\rm ent}}\left(t-\frac{8}{5\pi}\ell\right), \quad (32)$$

where $T_{\rm ent}=4\ell/(\pi^2u_*^2)$ with $u_*=\ell\Gamma(1/4)/2\sqrt{\pi}\Gamma(3/4)$ in $d=3$, in agreement with Eq. (4) with $d=3$.

As mentioned above, the metric in Eq. (28) is valid only for $v = t - u > 0$, so Eqs. (31) and (32) are valid only for $t > R$ or $t > u_*$, respectively, so that in both cases $\delta S_{\rm EE} > 0$. Equations (31) and (32) clearly obey the FLOER, $\partial_t \delta S_{\rm EE} = \partial_t E / T_{\rm ent}$, as expected.

To compute δE we switch from the coordinate u in Eq. (29) to the FG coordinate z in Eq. (6), using $1/u^2 = g_{xx}/z^2$. Comparing G_{tt} in the two coordinate systems,

$$G_{tt} = \frac{L^2}{z^2}(-1 + z^3 g_{tt}^{(3)} + \cdots) = \frac{L^2}{u^2}(-1 + u^3(g_{tt}^{(3)} + g_{xx}^{(3)}) + \cdots), \qquad (33)$$

we find $g_{tt}^{(3)} + g_{xx}^{(3)} = 2\mathcal{E}^2 t$. Tracelessness of $T_{\mu\nu}$ gives us $g_{xx}^{(3)} = g_{tt}^{(3)}/2$, so that $g_{tt}^{(3)} = 4\mathcal{E}^2 t/3$. Equation (8) then gives the energy density,

$$\langle T_{tt} \rangle = \frac{3L^2}{16\pi G_N} g_{tt}^{(3)} = \frac{L^2}{4\pi G_N} \mathcal{E}^2 t = \mathcal{E} \langle J^x \rangle t, \qquad (34)$$

so that, unsurprisingly, $\partial_t \langle T_{tt} \rangle = \mathcal{E} \langle J^x \rangle$. As a result, for spherical and strip subregions, $\delta E = \mathcal{E} \langle J^x \rangle (\pi R^2) t$ and $\delta E = \mathcal{E} \langle J^x \rangle (\ell \mathrm{Vol}(\mathbb{R})) t$, respectively, or more simply, $\delta E = t \partial_t E$.

For perturbations of the CFT state, without changes to the CFT Hamiltonian, intuition from QFT [12] and results from holography [10] suggest that for a subregion of fixed size the FLEE of Eq. (2) should hold for sufficiently small δE. Strictly speaking, that criterion does not immediately translate to our case, because we deform the CFT Hamiltonian, by $\mathcal{E}$. Nevertheless, naïvely applying that criterion to our case, we expect the FLEE to hold for t short enough that $\mathcal{E}$ has deposited little energy into the subregion. For example for the sphere we expect the FLEE to hold for t short enough that $\delta E = t \partial_t E \lesssim 1/R$, meaning $t \lesssim (\mathcal{E} \langle J^x \rangle \pi R^3)^{-1}$. We can make that time arbitrarily long by making $\mathcal{E}$ arbitrarily small. In particular, the times for which we expect the FLEE to hold can be made $\gg R$, and hence can easily include the regime $t > R$ where our result for $\delta S_{\rm EE}$ Eq. (31) is valid. However, plugging a time of order R into $\delta S_{\rm EE}$ in Eq. (31), we find that $\delta S_{\rm EE} \neq \delta E / T_{\rm ent}$, due to the term $\propto R$ in Eq. (31). Of course analogous statements apply for $\delta S_{\rm EE}$ of the strip in Eq. (32). In short, in both cases we find that the FLEE of Eq. (2) is violated, as advertised.

Moreover, as mentioned in Sec. I, the "entanglement tsunami" model [5–7] offers a possible explanation for the offending terms, as a difference in initial conditions. As soon as $\mathcal{E}$ is turned on, it pumps energy into the CFT and begins producing massless EPR pairs, doing both at a constant rate and uniformly throughout space. However, the pairs produced at sufficiently early times only contribute to EE after some finite time required to exit the subregion $\mathcal{A}$. As a result, $\delta S_{\rm EE}$ lags behind δE by an amount on the order of $\mathcal{A}$'s size, R or ℓ, as indeed observed in Eqs. (31) and (32). Of course, not all EPR partners are equidistant from $\partial \mathcal{A}$, so the lag is not identically R or ℓ, but is only $\propto R$ or ℓ.

IV. D3/D7 WITH ELECTRIC FIELD

In this section we study the D3/D7 system [39]. Type IIB supergravity in the near-horizon geometry of $N_c \to \infty$ D3-branes, $\mathrm{AdS}_5 \times S^5$, is dual to $\mathcal{N} = 4$ SYM with gauge group $SU(N_c)$, in the limits $N_c \to \infty$ and 't Hooft coupling $\lambda \to \infty$ [40]. A number N_f of probe D7-branes along $\mathrm{AdS}_5 \times S^3$ is dual to a number $N_f \ll N_c$ of massless $\mathcal{N} = 2$ SUSY hypermultiplets in the fundamental representation of $SU(N_c)$, i.e. flavor fields [39]. The D7-brane world volume $U(N_f)$ gauge fields are dual to conserved $U(N_f)$ flavor symmetry currents.

As mentioned in Sec. I, the probe D7-brane provides a *time-independent* example in which the FLEE of Eq. (1) can hold while that in Eq. (2) is violated. Suppose we give the flavor fields a nonzero $\mathcal{N} = 2$ SUSY-preserving mass, m. The proof of Ref. [28] applies in that case, so if ρ and ρ' are the vacua of the $m = 0$ and $m \neq 0$ theories, then we expect $S(\rho|\rho') \geq 0$ and hence the FLEE of Eq. (1). For the FLEE of Eq. (2), SUSY guarantees $\delta E = 0$. On the other hand, holographic results for $\delta S_{\rm EE}$ of a spherical subregion [41,42] include a term $\propto (mR)^2 \log(\epsilon/R)$, with FG cutoff ϵ. The coefficient of the $\log(\epsilon/R)$ cannot be set to zero by rescaling ϵ, so clearly $\delta S \neq \delta E / T_{\rm ent}$; i.e. the FLEE of Eq. (2) is violated.

To realize out *time-dependent* example, we introduce $T \neq 0$, so that AdS_5 becomes an AdS_5 black brane. The $\mathcal{N} = 4$ SYM and flavor contributions to $\langle T_{\mu\nu} \rangle$ are then order N_c^2 and $N_f N_c \ll N_c^2$, respectively [26], so we may think of the flavors as probes inside an enormous heat bath. We also introduce a constant, external electric field $\mathcal{E}$ in the x direction for the diagonal $U(1) \subset U(N_f)$, producing a current, $\langle J^x \rangle$, of charge carriers in the flavor sector. The charge density vanishes, $\langle J^t \rangle = 0$, so the current comes entirely from Schwinger pair production [43,44]. We consider NESS in which $\langle J^x \rangle$ is t independent because the charge carriers gain energy from $\mathcal{E}$ at the same constant rate $\mathcal{E} \langle J^x \rangle$ that they lose energy to the heat bath [26], as we discuss below.

We use an AdS_5 black brane metric

$$ds^2 = \frac{L^2}{u^2}\left(\frac{du^2}{b(u)} - b(u) dt^2 + d\vec{x}^2\right), \qquad b(u) = 1 - (u/u_h)^4, \qquad (35)$$

with $T = 1/(\pi u_h)$. The D7-branes fill the AdS_5 black brane space and also wrap an equatorial $S^3 \subset S^5$ with radius L.

The only nontrivial contribution to the D7-brane action, S_{D7}, is then the DBI term,

$$S_{\rm D7} = -N_f T_{\rm D7} \int d^8\zeta \sqrt{-\det(\Gamma_{ab} + (2\pi\alpha')F_{ab})}, \tag{36}$$

with D7-brane tension $T_{D7} = (2\pi)^{-7}\alpha'^{-4}g_s^{-1}$, with string length squared α' and coupling g_s; world volume coordinates ζ^a with $a = 0, \ldots, 7$; world volume metric Γ_{ab}; and world volume $U(1)$ field strength $F_{ab} = \partial_a A_b - \partial_b A_a$. To describe $\mathcal{E}$ and $\langle J^x \rangle$ we make the ansatz

$$A_x(t,u) = -\mathcal{E}t + a_x(u), \tag{37}$$

with all of A_a's other components zero. Plugging our ansatz Eq. (37) into S_{D7} in Eq. (36), and trivially performing the integration over the S^3 directions, we find

$$S_{\rm D7} = -N_f T_{\rm D7} L^3 \mathrm{vol}(S^3) \times \int d^5\zeta \frac{L^5}{u^5}\sqrt{1 + (2\pi\alpha')^2 \frac{u^4}{L^4}\left(b(u)a_x'^2(u) - \frac{\mathcal{E}^2}{b(u)}\right)}. \tag{38}$$

For simplicity, we define an "effective tension,"

$$\tilde{T}_{D7} \equiv N_f T_{\rm D7} L^3 \mathrm{vol}(S^3) = \frac{\lambda N_f N_c}{(2\pi)^4}\frac{1}{L^5}, \tag{39}$$

where in the second equality we used $\mathrm{vol}(S^3) = 2\pi^2$, $\lambda \equiv 4\pi g_s N_c$, and $\lambda = L^4/\alpha'^2$ [45].

Crucially, S_{D7} in Eq. (38) depends on $a_x'(u)$ but not on $a_x(u)$; hence we have a first integral of motion, which in the dual CFT is precisely the current: $\frac{\delta S_{D7}}{\delta a_x'} = \langle J^x \rangle$ [45]. We can then solve algebraically for $a_x'(u)$ in terms of $\langle J^x \rangle$,

$$a_x'(u) = \frac{\langle J^x \rangle}{b(u)L}\sqrt{\frac{b(u)/u^4 - (2\pi\alpha')^2\mathcal{E}^2/L^4}{\tilde{T}_{D7}^2(2\pi\alpha')^2 b(u)/u^6 - \langle J^x\rangle^2/L^6}}. \tag{40}$$

To fix $\langle J^x \rangle$ we follow Ref. [45]: we plug the solution for $a_x'(u)$ in Eq. (40) into S_{D7} in Eq. (38) and demand that the result remain real for all $u \in [0, u_h]$, since a nonzero imaginary part of an effective action signals a tachyon [43,44]. We find $\langle J^x \rangle = \sigma\mathcal{E}$, with conductivity

$$\sigma = \frac{N_f N_c T}{4\pi}[1 + \mathcal{E}^2/(\pi\sqrt{\lambda}T^2/2)^2]^{1/4}. \tag{41}$$

To compute the $\delta S_{\rm EE}$ due to $\mathcal{E}$, we must compute the perturbative backreaction of the D7-branes to first order. At first, that looks like a daunting task, since the D7-branes couple not only to the metric but also to the axiodilaton and B-field, and moreover break several symmetries of the background, for example breaking the S^5's $SO(6)$ isometry down to the $SO(4) \times U(1)$ preserved by the equatorial $S^3 \subset S^5$. Fortunately, however, as argued in Refs. [37,46], if the D7-brane world volume fields are independent of the $S^3 \subset S^5$ directions, as in our case, then using an "effective stress-energy tensor," obtained by integrating the AdS_5 part of the D7-brane stress-energy tensor over the S^3, is sufficient for computing $\delta S_{\rm EE}$. In our case, this effective stress-energy tensor is

$$\mathcal{T}^{mn}_{\rm eff} = -\tilde{T}_{D7}\frac{\sqrt{-\det(\Gamma_{mn} + (2\pi\alpha')F_{mn})}}{\sqrt{-\det(G^{(0)}_{mn})}} \times [(\Gamma + (2\pi\alpha')F)^{-1}]^{(mn)}, \tag{42}$$

where $G^{(0)}_{mn}$ is the AdS_5 black brane metric in Eq. (35), Γ_{mn} and F_{mn} are now restricted to the directions in Eq. (35), and (mn) indicates symmetrization over the indices m and n. Splitting $\mathcal{T}^{mn}_{\rm eff}$ into diagonal and off-diagonal parts, $\mathcal{T}^{\rm eff}_{mn} = \mathcal{T}^{\rm diag}_{mn} + \mathcal{T}^{\rm off}_{mn}$, for the $a_x'(u)$ solution in Eq. (40) we find

$$\begin{aligned}\mathcal{T}^{\rm diag}_{mn}dx^m dx^n = -\frac{a_x' L^3}{\langle J^x\rangle u^3}\Bigg[&\left(1 - \frac{\langle J^x\rangle^2}{\tilde{T}_{D7}^2(2\pi\alpha')^2}\frac{u^6}{bL^6}\right)du^2 \\ &- b\frac{b^2 - \langle J^x\rangle^2\mathcal{E}^2u^{10}/(\tilde{T}_{D7}^2L^{10})}{b - (2\pi\alpha')^2\mathcal{E}^2u^4/L^4}dt^2 \\ &+ \frac{\langle J^x\rangle^2u^2}{\tilde{T}_{D7}^2 b a_x'^2 L^2}dx^2 + b(dx^2)^2 + b(dx^3)^2\Bigg],\end{aligned} \tag{43a}$$

$$\mathcal{T}^{\rm off}_{mn}dx^m dx^n = -\mathcal{E}\langle J^x\rangle\frac{u^3}{b(u)L^3}2du dt. \tag{43b}$$

As advertised in Sec. I, $\mathcal{T}^{mn}_{\rm eff}$ is t-independent but has off-diagonal terms $\mathcal{T}^{ut}_{\rm off} = \mathcal{T}^{tu}_{\rm off}$. In fact, $\mathcal{T}^{\rm diag}_{mn}$ and $\mathcal{T}^{\rm off}_{mn}$ turn out to be separately conserved, so if we linearize Einstein's equation in δG_{mn}, and split δG_{mn} into parts sourced by $\mathcal{T}^{\rm diag}_{mn}$ and $\mathcal{T}^{\rm off}_{mn}$, respectively, $\delta G_{mn} = \delta G^{\rm diag}_{mn} + \delta G^{\rm off}_{mn}$ (which are not necessarily diagonal and off-diagonal themselves), then we can solve for $\delta G^{\rm diag}_{mn}$ and $\delta G^{\rm off}_{mn}$ separately.

We have checked explicitly that a t-independent solution for $\delta G^{\rm diag}_{mn}$ exists, whose existence relies crucially on the fact that $\mathcal{T}^{\rm diag}_{mn}$ is invariant under t-reversal. At leading order in $\mathcal{E}$, $\mathcal{T}^{\rm diag}_{mn}$'s backreaction is just a shift of the cosmological constant, as expected: the DBI action in Eq. (36) with trivial world volume fields is a contribution to the cosmological constant $\propto T_{D7}$. The cosmological constant is $\propto 1/L^2$, and roughly speaking L in Planck units is dual to the number of degrees of freedom in the CFT, measured for example in even d by a central charge [47]. In particular, adding a space-filling probe DBI action with trivial world volume fields corresponds to adding degrees of freedom, such as adding flavor fields to $\mathcal{N}=4$ SYM. Such a deformation results in a FLEE of the form in Eq. (2), but with a "chemical potential" term arising from the change in the number of degrees of freedom [48].

On the other hand, $\mathcal{T}^{\rm off}_{mn}$ breaks t-reversal, and hence so does $\delta G^{\rm off}_{mn}$. Indeed, the solution for $\delta G^{\rm off}_{mn}$ is

$$\delta G^{\rm off}_{mn} dx^m dx^n = \frac{16\pi G_N}{3L}\mathcal{E}\langle J^x\rangle t u^2\left(dt^2 + \frac{du^2}{b^2(u)}\right). \quad (44)$$

If $\delta G^{\rm off}_{mn}$ grows too big, then the linearized approximation breaks down; hence the linearized solution in Eq. (44) is valid only for sufficiently small $\mathcal{E}\langle J^x\rangle t$.

Strictly speaking, in this example $G_{mn} = G^{(0)}_{mn} + \delta G_{mn}$ does not obey all the assumptions in Sec. II. For instance, as mentioned above $\delta G^{\rm diag}_{mn}$ is asymptotically AdS_5, but shifts L, something we did not account for in Sec. II. However, a key step in Sec. II was taking ∂_t of $\delta S_{\rm EE}$, so in fact we only need the t-dependent part of δG_{mn} to obey our assumptions. In this example, all of δG_{mn}'s t dependence is in $\delta G^{\rm off}_{mn}$. Indeed, $G^{(0)}_{mn} + G^{\rm off}_{mn}$ obeys all the assumptions in Sec. II, and hence this example must obey a FLOER.

However, to dispel any doubt, we have calculated $\partial_t S_{\rm EE}$ following the steps in Sec. II, adapted to the coordinate u of Eq. (35), with the results

$$\partial_t\delta S^{\rm sphere}_{\rm EE} = \mathcal{E}\langle J^x\rangle\left(\frac{4}{3}\pi R^3\right)\frac{2\pi}{R^3}\int_0^{u_\star} du \frac{r^2 u}{\sqrt{b^3(u)(1+b(u)r'^2)}}, \quad (45a)$$

$$\partial_t\delta S^{\rm strip}_{\rm EE} = \mathcal{E}\langle J^x\rangle(\ell\,{\rm vol}(\mathbb{R}^2))\frac{4\pi}{3\ell}\int_0^{u_\star} du\, u\sqrt{\frac{1-(u/u_\star)^6}{b^3(u)}}, \quad (45b)$$

where for the sphere $r(u)$ is the solution for the minimal surface's embedding, and for both the sphere and strip $u_\star$ is the minimal surface's maximal extension in u, in the unperturbed AdS_5 black brane geometry of Eq. (35). For the strip, $u_\star$ is related to the width ℓ by

$$\ell = 2\int_0^{u_\star} du \frac{(u/u_\star)^3}{\sqrt{b(u)(1-(u/u_\star)^6)}}. \quad (46)$$

Identifying $\partial_t\delta E = \mathcal{E}\langle J^x\rangle(\frac{4}{3}\pi R^3)$ or $\partial_t\delta E = \mathcal{E}\langle J^x\rangle(\ell\,{\rm vol}(\mathbb{R}^2))$ for the sphere and strip, respectively, we thus find

$$\partial_t\delta S^{\rm sphere}_{\rm EE} = \partial_t\delta E\frac{2\pi}{R^3}\int_0^{u_\star} du\frac{r^2 u}{\sqrt{b^3(u)(1+b(u)r'^2)}}, \quad (47a)$$

$$\partial_t\delta S^{\rm strip}_{\rm EE} = \partial_t\delta E\frac{4\pi}{3\ell}\int_0^{u_\star} du\, u\sqrt{\frac{1-(u/u_\star)^6}{b^3(u)}}. \quad (47b)$$

A straightforward calculation confirms that for the AdS_5 black brane the integrals in Eqs. (47a) and (47b) reproduce $T_{\rm ent}$ from Eqs. (26a) and (26b), respectively. We have thus explicitly shown that this example obeys the FLOER.

As mentioned in Sec. I, the FLOER may be useful because $\partial_t\delta E$ is often easier to calculate than $\partial_t\delta S_{\rm EE}$. Indeed, for probe branes we can calculate $\partial_t\delta E$ in the probe limit, without computing backreaction, following Refs. [26,49]. The probe flavor's order $N_f N_c$ contribution to the energy density, $\delta\langle T_{tt}\rangle$, is given holographically by the energy density on the D7-brane, $\mathcal{T}^t{}_t$, integrated over the $S^3\subset S^5$ and u directions,

$$\delta\langle T_{tt}\rangle = -\int_0^{u_h} du\sqrt{-\det(\Gamma_{ab})}\mathcal{T}^t{}_t. \quad (48)$$

Taking ∂_t of Eq. (48) and using $\nabla_c(\sqrt{-\det(\Gamma_{ab})}\mathcal{T}^c{}_t) = 0$), from conservation of $\mathcal{T}_{ab}$, we find

$$\partial_t\delta\langle T_{tt}\rangle = \int_0^{u_h} du\,\partial_u\sqrt{-\det(\Gamma_{ab})}\mathcal{T}^u{}_t = \left[\sqrt{-\det(\Gamma_{ab})}\mathcal{T}^u{}_t\right]_0^{u_h}. \quad (49)$$

From the $\mathcal{T}^u_t$ in Eq. (43), we find that the rate of energy density gain at the boundary, dual to the energy density pumped into the probe sector by $\mathcal{E}$, and energy density loss at the horizon, dual to the energy density that the probe sector dumps into the heat bath, are equal:

$$\sqrt{-\det(\Gamma_{ab})}\mathcal{T}^u{}_t|_{u=0} = \sqrt{-\det(\Gamma_{ab})}\mathcal{T}^u{}_t|_{u=u_h} = -\mathcal{E}\langle J^x\rangle. \quad (50)$$

The *total* rate of change of energy density in Eq. (49) thus vanishes, $\partial_t\delta\langle T_{tt}\rangle = 0$, producing a NESS, as advertised. Presumably, the $\partial_t\delta E$ that appears in the FLOER in Eq. (47) comes from the energy injected into the subregion by $\mathcal{E}$, i.e. from the $u=0$ contribution to $\partial_t\delta\langle T_{tt}\rangle$ in Eq. (49). In short, we can calculate $\partial_t\delta E$ directly in the probe limit, avoiding any backreaction, simply by evaluating $\sqrt{-\det(\Gamma_{ab})}\mathcal{T}^u{}_t$ at $u=0$.

In general, when $\mathcal{E}\neq 0$ a probe brane's induced metric Γ_{ab} has a horizon distinct from that of the background metric [45,50,51]. A temperature can be associated with the world volume horizon [52–59], which in general is distinct from the background temperature T, clearly indicating that the system is out of equilibrium. The world volume horizon may represent the EE of the Schwinger pairs produced by $\mathcal{E}$ [60]. However, whether any meaningful notion of entropy can be associated to the world volume horizon is unclear. An obvious guess is a Bekenstein-Hawking entropy, the horizon's area over $4G_N$. However, the DBI action does not describe gravitational degrees of freedom, and Γ_{ab} is not necessarily a solution of Einstein's equation, so although we can compute the area of the world volume horizon, what should play the role of $4G_N$? The open string coupling [56]? In any case, the world volume horizon did not appear to play any special role in our calculation of EE, and in particular, our result for the EE does not appear to be proportional to the area of the world volume horizon.

Although we focused on the D3/D7 system, the analysis in this section should straightforwardly generalize to many other systems involving a space-filling probe DBI action with $\mathcal{E} \neq 0$ in an asymptotically AdS_{d+1} spacetime.

V. MASSLESS SCALARS

In this section we study holographic CFTs deformed by marginal scalar operators $\mathcal{O}$ with a source linear in time t or in a spatial direction x. Explicit examples of such CFTs are $\mathcal{N} = 4$ SYM in $d = 4$, which has three exactly marginal scalar operators [61,62], and ABJM theory, where the Chern-Simons level, or equivalently the 't Hooft coupling, is exactly marginal.

A marginal scalar operator $\mathcal{O}$ is holographically dual to a massless scalar field ϕ, whose stress-energy tensor $\mathcal{T}_{mn}$ depends only on derivatives of ϕ, due to invariance under constant shifts of ϕ. A linear source for $\mathcal{O}$ produces a $\mathcal{T}_{mn}$ that depends only on the holographic radial coordinate, but may have nontrivial off-diagonal components, producing a δG_{mn} that may depend on t or x, but obeys the assumptions in Sec. II; hence the FLOER will be obeyed.

However, we compute δS_{EE} and δE separately for $d = 2$, 3, 4, and show that in all cases the FLEE of Eq. (2) is violated. More specifically, we solve Einstein's equation for δG_{mn} near the asymptotic AdS_{d+1} boundary, obtaining explicit expressions for only a subset of δG_{mn}'s FG coefficients, while any remaining FG coefficients could in principle be fixed by imposing regularity of δG_{mn} in the bulk. These asymptotic solutions for δG_{mn} suffice to establish violation of the FLEE of Eq. (2).

A. Linear time dependence

In this subsection we consider $(d+1)$-dimensional Einstein-Hilbert gravity coupled to a massless scalar field ϕ, with bulk action

$$S = \int d^{d+1}x\sqrt{-\det(G_{mn})} \times \left[\frac{1}{16\pi G_N}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{2}(\partial\phi)^2\right]. \tag{51}$$

As in Sec. II, we consider $G_{mn} = G^{(0)}_{mn} + \delta G_{mn}$ in FG form, with $G^{(0)}_{mn}$ the AdS_{d+1} metric,

$$G^{(0)}_{mn}dx^m dx^n = \frac{L^2}{z^2}(dz^2 - dt^2 + d\vec{x}^2). \tag{52}$$

A solution for ϕ admits the FG expansion,

$$\phi = \phi_0 + \cdots z^d\phi_d + \cdots, \tag{53}$$

where the coefficients ϕ_0, ϕ_d, etc. generically depend on t and $\vec{x}$. The coefficient ϕ_0 is dual to the source for $\mathcal{O}$, so we introduce $\phi_0 = -ct$ with constant $c > 0$ of dimension $[t]^{-1}$. The remaining coefficients in ϕ's and δG_{mn}'s FG expansions can then depend only on t, although their explicit solutions depend on d, so in the following we consider $d = 2$, 3, 4 in turn.

For each of $d = 2$, 3, 4, we compute δS_{EE} and δE for a sphere or strip subregion. More specifically, to compute δS_{EE} we use Eq. (14), whose inputs are $\sqrt{\gamma}$, $\Theta^{mn}_{\mathrm{min}}$, and δG_{mn}. We plug the solution for $\mathcal{W}^{(0)}_{\mathrm{min}}$'s embedding, for example $r(z) = \sqrt{R^2 - z^2}$ for the sphere, into Eqs. (18) and (19) to obtain γ and $\Theta^{mn}_{\mathrm{min}}$, respectively. As mentioned above, we solve for δG_{mn} only near the asymptotic AdS_{d+1} boundary, and then extract δE via holographic renormalization [24]. The details of the holographic renormalization appear in the Appendix, where we also check several Ward identities. [In each case, the $\partial_t\langle T_{tt}\rangle$ from holographic renormalization reproduces Eq. (9).] For each of $d = 2$, 3, 4, we find that the FLOER is obeyed, as expected, while the FLEE of Eq. (2) is violated.

a. Boundary dimension $d = 2$: The holographic renormalization for a massless scalar in AdS_3 appears in Ref. [24]. Plugging a Minkowski metric at the AdS_3 boundary and $\phi_0 = -ct$ into the results of Ref. [24] yields

$$G_{xx} = \frac{L^2}{z^2}\left[1 + z^2 g^{(2)}_{xx} - z^2\log(z^2/L^2)2\pi\frac{G_N}{L}c^2 + \cdots\right], \tag{54a}$$

$$\langle T_{xx}\rangle = \frac{L}{8\pi G_N}g^{(2)}_{xx} + c^2\left(\frac{1}{4} - \eta\right), \tag{54b}$$

$$\langle T^{\mu}{}_{\mu}\rangle = \frac{1}{2}c^2, \tag{54c}$$

where the term $\propto\eta$ in $\langle T_{xx}\rangle$ is scheme dependent, and comes from the finite counterterm

$$S_{CT} = \eta\int d^2x\sqrt{-\det(\tilde{g}_{\mu\nu})}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{55}$$

added to the bulk action S in Eq. (51) (with $d = 2$) at a regulating cutoff surface $z = \epsilon$, with induced metric $\tilde{g}_{\mu\nu}$. Plugging $\langle T_{xx}\rangle$ from Eq. (54b) into $\langle T^{\mu}{}_{\mu}\rangle = \frac{1}{2}c^2$ from Eq. (54c) then gives $\langle T_{tt}\rangle = \langle T_{xx}\rangle - \frac{1}{2}c^2$.

The bulk stress-energy tensor $\mathcal{T}_{mn}$ is quadratic in $\partial_m\phi$ and hence $\propto c^2$. We treat $\mathcal{T}_{mn}$ as a perturbation, and so linearize Einstein's equation, producing δG_{mn} of order c^2. The change in energy inside the sphere $|x| < R$,

$$\delta E = (2R)\delta\langle T_{tt}\rangle = (2R)\frac{L}{8\pi G_N}\delta g^{(2)}_{xx} - (2R)c^2\left(\frac{1}{4}+\eta\right), \tag{56}$$

is then $\propto c^2$, and in particular, $\delta g^{(2)}_{xx} \propto c^2$. As mentioned above, we compute $\delta S_{\rm EE}$ from Eq. (14), with the result

$$\delta S_{\rm EE} = \frac{\delta E}{T_{\rm ent}} + \frac{\pi}{9}c^2R^2(-6\log(2R/L) + 8 + 12\eta), \tag{57}$$

where $T_{\rm ent} = 3/(2\pi R)$, as in Eq. (3) with $d = 2$. The $\delta S_{\rm EE}$ in Eq. (57) has some scheme dependence, via the term $\propto \eta$, and in particular, a shift of η produces a shift of the argument of the logarithm in the term $\propto -R^2\log(2R/L)$. Crucially, however, the choice of η is part of the definition of the QFT, so η cannot depend on the size R of some arbitrarily chosen subregion, and so η cannot affect the *coefficient* of the term $\propto -R^2\log(2R/L)$. As a result, the latter coefficient is scheme independent and hence physically meaningful.

As discussed in Sec. III, we naïvely expect the FLEE of Eq. (2) to hold at sufficiently early times such that $t\partial_t E \lesssim 1/R$. The diffeomorphism Ward identity $\nabla^\mu T_{\mu\nu} = \mathcal{O}\partial_\nu\phi_0$ implies $\partial_t\langle T_{tt}\rangle = c\langle\mathcal{O}\rangle$, and hence $\partial_t E = (2R)c\langle\mathcal{O}\rangle$. We thus expect the FLEE to hold for $t \lesssim (c\langle\mathcal{O}\rangle 2R^2)^{-1}$, which can be made arbitrarily long by making c arbitrarily small, and can hence include the regime $t \simeq R$. As argued above, δE's only dependence on c is $\delta E \propto c^2$, and when $t \simeq R$ dimensional analysis requires $\delta E \propto c^2R$ or $c^2R\log R$. In that case, in the $\delta S_{\rm EE}$ in Eq. (57) the terms $\propto \delta E/T_{\rm ent}$ and $\propto c^2R^2$ are of the same order, so the FLEE of Eq. (2) is clearly violated, as advertised.

b. Boundary dimension $d = 3$: The details of the holographic renormalization for a massless scalar in asymptotically AdS_4 spacetimes appear in the appendix. In particular, plugging a Minkowski metric at the AdS_4 boundary and $\phi_0 = -ct$ into Eqs. (A1) and (A3b) yields

$$G_{xx} = \frac{L^2}{z^2}\left[1 + z^2 2\pi\frac{G_N}{L^2}c^2 + z^3 g^{(3)}_{xx} + \cdots\right], \tag{58a}$$

$$\langle T_{\mu\nu}\rangle = \frac{3L^2}{16\pi G_N}g^{(3)}_{\mu\nu}, \tag{58b}$$

and $T^\mu{}_\mu = 0$, as expected in $d = 3$. As in the $d = 2$ case above, a linearized perturbation δG_{mn} is $\propto c^2$, so the change in the energy inside the sphere $|\vec{x}| < R$ is

$$\delta E = (\pi R^2)\delta\langle T_{tt}\rangle = (\pi R^2)\frac{3L^2}{16\pi G_N}\delta g^{(3)}_{tt}, \tag{59}$$

where $\delta g^{(3)}_{tt} \propto c^2$. The δE for the strip is identical, but with $(\pi R^2) \to \ell\mathrm{Vol}(\mathbb{R})$. As mentioned above, from Eq. (14) we compute $\delta S_{\rm EE}$ for the sphere,

$$\delta S_{\rm EE} = \frac{\delta E}{T_{\rm ent}} + \frac{2\pi}{3}c^2R^2, \tag{60}$$

where $T_{\rm ent} = 2/(\pi R)$ as in Eq. (3) with $d = 3$, and for the strip,

$$\delta S_{\rm EE} = \frac{\delta E}{T_{\rm ent}} + \frac{\pi^{5/2}}{3\sqrt{2}\Gamma(3/4)^2}c^2 z_*\mathrm{vol}(\mathbb{R}), \tag{61}$$

where $T_{\rm ent} = 4\ell/(\pi^2 z_*^2)$ with $z_* = \ell\Gamma(1/4)/2\sqrt{\pi}\Gamma(3/4)$, as in Eq. (4) with $d = 3$. Via essentially the same arguments as those below Eq. (57), for sufficiently small c we can enter a regime where naïvely we expect the FLEE of Eq. (2) to hold, but the two terms in Eq. (60) or (61) are of the same order. The FLEE of Eq. (2) is then clearly violated, as advertised.

c. Boundary dimension $d = 4$: The details of the holographic renormalization for a massless scalar in asymptotically AdS_5 spacetimes appear in the appendix. In particular, plugging a Minkowski metric at the AdS_5 boundary and $\phi_0 = -ct$ into Eqs. (A5) and (A8b) yields

$$G_{xx} = \frac{L^2}{z^2}\left[1 + z^2\frac{2\pi}{3}\frac{G_N}{L^3}c^2\left(1 - z^2\log(z^2/L^2)2\pi\frac{G_N}{L^3}c^2\right) + z^4 g^{(4)}_{xx} + \cdots\right], \tag{62a}$$

$$\langle T_{xx}\rangle = \frac{L^3}{4\pi G_N}g^{(4)}_{xx} - \frac{5\pi}{18}\frac{G_N}{L^3}c^4, \tag{62b}$$

$$\langle T^\mu{}_\mu\rangle = \frac{2\pi}{3}\frac{G_N}{L^3}c^4. \tag{62c}$$

Plugging $\langle T_{xx}\rangle$ from Eq. (62b) into $\langle T^\mu{}_\mu\rangle$ from Eq. (62c) then gives $\langle T_{tt}\rangle = 3\langle T_{xx}\rangle - \langle T^\mu{}_\mu\rangle$. As in the $d = 2$ case above, a linearized perturbation δG_{mn} is $\propto c^2$, so the change in the energy inside the sphere $|\vec{x}| < R$ is

$$\delta E = \frac{4}{3}\pi R^3\delta\langle T_{tt}\rangle = \frac{L^3}{G_N}R^3\delta g^{(4)}_{xx}, \tag{63}$$

where $\delta g^{(4)}_{xx} \propto c^2$. As mentioned above, from Eq. (14) we compute $\delta S_{\rm EE}$ for the sphere,

$$\delta S_{\rm EE} = \frac{\delta E}{T_{\rm ent}} - \frac{\pi^2}{9}c^2R^2(5 - 6\log 2 + 6\log(\epsilon/R)), \tag{64}$$

with UV cutoff $z = \epsilon$. The $\delta S_{\rm EE}$ in Eq. (64) has some scheme dependence, via the term $\propto c^2R^2\log(\epsilon/R)$, such that rescaling ϵ shifts the terms $\propto c^2R^2$. However, the *coefficient* of the term $\propto c^2R^2\log(\epsilon/R)$ is invariant under rescalings of ϵ, i.e. is scheme independent, and hence is physically meaningful.

Via essentially the same arguments as those below Eq. (57), for sufficiently small c we can enter a regime where naïvely we expect the FLEE of Eq. (2) to hold, but all terms in Eq. (64) are of order c^2R^2. The FLEE of Eq. (2) is then clearly violated, as advertised.

B. Linear spatial dependence

In this subsection we consider the model of Ref. [25], containing a $U(1)$ gauge field A_m and massless scalars ϕ_I with $I = 1, \ldots, d-1$ in asymptotically AdS_{d+1} spacetime. We consider the solutions of Ref. [25] describing charged black branes with ϕ_I linear in a spatial direction x, dual to CFT states with nonzero chemical potential, μ, and x-linear sources for a set of exactly marginal scalar operators $\mathcal{O}_I$. The main result of Ref. [25] was that the x-linear sources break translational symmetry in the CFT and hence produce the effects of momentum relaxation, such as a Drude peak in the $U(1)$ conductivity. We are instead interested in the x-linear sources as perturbations of the CFT at nonzero μ. In $d = 3$ we will show that the FLEE in Eq. (2) is violated, while both δS_{EE} and δE are independent of t and $\vec{x}$, and hence will trivially obey a FLOER involving any CFT coordinate. Previous calculations of EE in the model of Ref. [25] appear for example in Ref. [63].

The model of Ref. [25] has bulk action

$$S = \int d^{d+1}x\sqrt{-\det(G_{mn})} \times \left[\frac{1}{16\pi G_N}\left(R + \frac{d(d-1)}{L^2} - F^2\right) - \frac{1}{2}\sum_{I=1}^{d-1}(\partial\phi_I)^2\right], \tag{65}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$. We consider the solutions of Ref. [25] describing a static, charged black brane with scalar hair linear in x,

$$G_{mn}dx^m dx^n = \frac{L^2}{u^2}\left(\frac{du^2}{f(u)} - f(u)dt^2 + d\vec{x}^2\right), \tag{66a}$$

$$A_t = \mu\left[1 - \left(\frac{u}{u_h}\right)^{d-2}\right], \tag{66b}$$

$$\phi_I = \vec{\alpha}_I \cdot \vec{x}, \tag{66c}$$

with horizon at $u = u_h$ and all other components of A_m vanishing. The constant vector $\vec{\alpha}_I$ in Eq. (66c) has components $(\alpha_I)_i$ with $i = 1, \ldots, d-1$ defined such that

$$\sum_{I=1}^{d-1}(\alpha_I)_i(\alpha_I)_j = \alpha^2\delta_{ij}, \tag{67}$$

with constant α^2. The blackening function $f(u)$ appearing in the metric in Eq. (66a) is

$$f(u) = 1 - M\left(\frac{u}{L^2}\right)^d + \left(\frac{u_h\mu}{\beta}\right)^2\left(\frac{u}{u_h}\right)^{2(d-1)} - \frac{8\pi G_N}{(d-2)}\alpha^2 u^2, \tag{68a}$$

$$M \equiv \left[1 + \left(\frac{u_h\mu}{\beta}\right)^2 - \frac{8\pi G_N}{(d-2)}\alpha^2 u_h^2\right]\left(\frac{L^2}{u_h}\right)^d, \tag{68b}$$

$$\beta^2 \equiv \frac{d-1}{d-2}\frac{L^2}{2}. \tag{68c}$$

When $\alpha^2 = 0$, this solution reduces to the AdS_{d+1}-Reissner-Nordström charged black brane.

We henceforth specialize to $d = 3$, the case for which the holographic renormalization of this model was performed in Ref. [25]. When $d = 3$, the asymptotic change of coordinates,

$$u = z - z^3 2\pi G_N\alpha^2 - z^4\frac{M}{6L^6} + \mathcal{O}(z^5), \tag{69}$$

brings the metric in Eq. (66a) into the asymptotic FG form:

$$G_{mn}dx^m dx^n = \frac{L^2}{z^2}(dz^2 + g_{tt}dt^2 + g_{xx}d\vec{x}^2), \tag{70a}$$

$$g_{tt} = -1 + z^2 4\pi G_N\alpha^2 + z^3\frac{2M}{3L^6} + \mathcal{O}(z^4), \tag{70b}$$

$$g_{xx} = 1 + z^2 4\pi G_N\alpha^2 + z^3\frac{M}{3L^6} + \mathcal{O}(z^4). \tag{70c}$$

The holographic renormalization in Ref. [25] then gives for the energy density

$$\langle T_{tt}\rangle = \frac{3L^2}{8\pi G_N}g_{xx}^{(3)} = \frac{M}{8\pi G_N L^4}. \tag{71}$$

For sufficiently small α^2, we may treat the terms $\propto \alpha^2$ in Eq. (68) as perturbations, and write $G_{mn} = G_{mn}^{(0)} + \delta G_{mn}$, with $G_{mn}^{(0)}$ the AdS_4-Reissner-Nordström metric, and δG_{mn} of order α^2. In particular,

$$\delta g_{xx} = z^2 4\pi G_N\alpha^2 - z^3\frac{8\pi G_N\alpha^2}{3u_h} + \mathcal{O}(z^4). \tag{72}$$

Using Eq. (71) we thus find that the change in energy inside a spherical subregion comes from the change in the order z^3 term in the FG asymptotics,

$$\delta E = (\pi R^2)\frac{3L^2}{8\pi G_N}\delta g_{xx}^{(3)} = -(\pi R^2)\frac{L^2\alpha^2}{u_h}, \tag{73}$$

and the change in energy inside a strip subregion is identical, but with $(\pi R^2) \to \ell\,\mathrm{Vol}(\mathbb{R})$.

In contrast, $\delta S_{\rm EE}$ depends on both $\delta g_{xx}^{(2)}$ and $\delta g_{xx}^{(3)}$. Indeed, applying the results of Sec. II, we find for spherical and strip subregions, respectively,

$$\delta S_{\rm EE}^{\rm sphere} = \frac{\mathrm{vol}(S^1)}{8G_N}\int_0^{z_\star} dz \left(\frac{L}{z}\right)^2 r g_{xx}^{-1/2}\sqrt{1+g_{xx}r'^2}\left(\frac{g_{xx}r'^2}{1+g_{xx}r'^2}+1\right)(\delta g_{xx}^{(2)}z^2+\delta g_{xx}^{(3)}z^3\ldots), \tag{74a}$$

$$\delta S_{\rm EE}^{\rm strip} = \frac{\mathrm{vol}(\mathbb{R}^1)}{4G_N}\int_0^{z_\star} dz \left(\frac{L}{z}\right)^2 g_{xx}^{-1/2}\frac{(g_{xx}^\star z^2/g_{xx}z_\star^2)^2+1}{\sqrt{1-(g_{xx}^\star z^2/g_{xx}z_\star^2)^2}}(\delta g_{xx}^{(2)}z^2+\delta g_{xx}^{(3)}z^3\ldots). \tag{74b}$$

In the $\delta S_{\rm EE}$ in Eq. (74), a contribution $\propto \delta E$ can only possibly come from the terms involving $\delta g_{xx}^{(3)}$, so the terms involving $\delta g_{xx}^{(2)}$ represent violations of the FLEE of Eq. (2). On the other hand, both δE and $\delta S_{\rm EE}$ are independent of t and $\vec{x}$, so a FLOER involving any CFT coordinate is trivially obeyed, as advertised.

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APPENDIX: HOLOGRAPHIC RENORMALIZATION OF MASSLESS SCALAR

In this appendix we present results for the holographic renormalization of a massless scalar field ϕ coupled to an asymptotically AdS_4 or AdS_5 metric G_{mn}. A massless scalar field ϕ and metric G_{mn} are dual to a marginal scalar operator $\mathcal{O}$ and the stress-energy tensor $T_{\mu\nu}$, respectively. For a scalar of any mass coupled to gravity, a convenient form of Einstein's equations appears in Ref. [24]. For a massless scalar, we solve the Einstein's equations in Ref. [24] asymptotically,[3] and then compute $\langle\mathcal{O}\rangle$, $\langle T_{\mu\nu}\rangle$, the diffeomorphism, and the Weyl Ward identities in terms of the coefficients of ϕ and G_{mn}'s asymptotic expansions in Eqs. (53) and (7), respectively. We use the results for $\langle T_{\mu\nu}\rangle$ and $T^\mu{}_\mu$ in Sec. V A to compute the change in energy inside a CFT subregion due to a t-linear source for $\mathcal{O}$.

In contrast to the body of the paper, in this appendix we choose units with $L\equiv 1$, and we use notation $\sqrt{-G}\equiv\sqrt{-\det(G_{mn})}$, and similarly for other metrics.

d. Boundary dimension $d=3$: For a massless scalar field ϕ coupled to an asymptotically AdS_4 metric G_{mn}, we find

$$\phi_2 = \frac{1}{2}\nabla^2\phi_0, \tag{A1a}$$

$$g_{\mu\nu}^{(2)} = -R_{\mu\nu}[g^{(0)}]+\frac{1}{4}R[g^{(0)}]+8\pi G_N\partial_\mu\phi_0\partial_\nu\phi_0 - 2\pi G_N g_{\mu\nu}^{(0)}(\partial\phi_0)^2, \tag{A1b}$$

$$\mathrm{Tr}g^{(2)} = -\frac{1}{4}R[g^{(0)}]+2\pi G_N(\partial\phi_0)^2, \tag{A1c}$$

$$\mathrm{Tr}g^{(3)} = 0, \tag{A1d}$$

$$\nabla^\nu g_{\mu\nu}^{(2)} = \partial_\mu \mathrm{Tr}g^{(2)} + 16\pi G_N\phi_2\partial_\mu\phi_0, \tag{A1e}$$

$$\nabla^\nu g_{\mu\nu}^{(3)} = 16\pi G_N\phi_3\partial_\mu\phi_0, \tag{A1f}$$

where ∇^μ is with respect to $g^{(0)}$, indices are raised and lowered with $g^{(0)}$, and $\mathrm{Tr}g^{(N)}\equiv g_{\mu\nu}^{(0)}g^{(N)\mu\nu}$. The renormalized action is

$$S_{\rm ren} = \lim_{\epsilon\to 0}\left\{\frac{1}{16\pi G_N}\left[\int d^4x\sqrt{-G}(R+6)+2\int_{z=\epsilon}d^3x\sqrt{-\tilde{g}}K[\tilde{g}]\right]-\frac{1}{2}\int d^4x\sqrt{-G}G^{mn}\partial_m\phi\partial_n\phi + \frac{1}{16\pi G_N}\int_{z=\epsilon}d^3x\sqrt{-\tilde{g}}(4+R[\tilde{g}]-8\pi G_N\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)\right\}, \tag{A2}$$

[3] In our conventions the Riemann tensor has the opposite sign compared to that in Ref. [24].

where $\tilde{g}_{\mu\nu}$ is the induced metric on a regulating cutoff surface $z=\epsilon$, with extrinsic curvature $K[\tilde{g}]$ and Ricci scalar $R[\tilde{g}]$. The final line of Eq. (A2) consists of counterterms at $z=\epsilon$. Varying $S_{\rm ren}$ in Eq. (A2) with respect to sources, we obtain the one point functions

$$\langle\mathcal{O}\rangle = 3\phi_3, \tag{A3a}$$

$$\langle T_{\mu\nu}\rangle = \frac{3}{16\pi G_N} g^{(3)}_{\mu\nu}, \tag{A3b}$$

although the values of ϕ_3 and $g^{(3)}_{\mu\nu}$ cannot be fixed by our near-boundary analysis alone. Equations (A1d) and (A1f) yield the diffeomorphism and Weyl Ward identities, respectively,

$$\nabla^\mu\langle T_{\mu\nu}\rangle = \langle\mathcal{O}\rangle\partial_\nu\phi_0, \tag{A4a}$$

$$\langle T^\mu{}_\mu\rangle = 0. \tag{A4b}$$

e. Boundary dimension $d=4$: For a massless scalar field ϕ coupled to an asymptotically AdS_5 metric G_{mn}, we find, with the same conventions as in Eq. (A1),

$$\phi_2 = \frac{1}{4}\nabla^2\phi_0, \tag{A5a}$$

$$\psi_4 = -\frac{1}{32}(\nabla^2)^2\phi_0 + \frac{1}{8}\frac{1}{\sqrt{-g^{(0)}}}\partial_\mu\left(\sqrt{-g^{(0)}}g^{(2)\mu\nu}\partial_\nu\phi_0\right) - \frac{1}{16}\partial_\mu \mathrm{Tr}g^{(2)}g^{(0)\mu\nu}\partial_\nu\phi_0 - \frac{1}{16}\mathrm{Tr}g^{(2)}\nabla^2\phi_0, \tag{A5b}$$

$$g^{(2)}_{\mu\nu} = -\frac{1}{2}R_{\mu\nu}[g^{(0)}] + \frac{1}{12}g^{(0)}_{\mu\nu}R[g^{(0)}] + 4\pi G_N\partial_\mu\phi_0\partial_\nu\phi_0 - \frac{2\pi G_N}{3}g^{(0)}_{\mu\nu}(\partial\phi_0)^2, \tag{A5c}$$

$$\begin{aligned} h^{(4)}_{\mu\nu} = &+\frac{1}{4}R_{\mu\nu}[g^{(2)}] + \frac{1}{2}g^{(2)}_{\mu\lambda}g^{(2)\lambda}_{\nu} - \frac{1}{8}g^{(0)}_{\mu\nu}\mathrm{Tr}[(g^{(2)})^2] - \frac{1}{4}\pi G_N g^{(0)}_{\mu\nu}(\nabla^2\phi_0)^2 \\ &-\frac{1}{2}\pi G_N(\partial_\mu\phi_0\partial_\nu\nabla^2\phi_0 + \partial_\mu\nabla^2\phi_0\partial_\nu\phi_0), \end{aligned} \tag{A5d}$$

$$\mathrm{Tr}h^{(4)} = 0, \tag{A5e}$$

$$\nabla^\nu h^{(4)}_{\mu\nu} = 4\pi G_N\psi_4\partial_\mu\nabla^2\phi_0, \tag{A5f}$$

$$\mathrm{Tr}g^{(4)} = \frac{1}{4}\mathrm{Tr}[(g^{(2)})^2] - \frac{1}{2}\pi G_N(\nabla^2\phi_0)^2, \tag{A5g}$$

$$\nabla^\nu g^{(4)}_{\mu\nu} = -\frac{1}{4}\partial_\mu\mathrm{Tr}[(g^{(2)})^2] + 16\pi G_N\phi_4\partial_\mu\phi_0 - \frac{1}{2}\pi G_N(\nabla^2\phi_0)\partial_\mu(\nabla^2\phi_0), \tag{A5h}$$

where ψ_4 is the coefficient of the $z^4\log z^2$ term in ϕ's asymptotic expansion. The renormalized action is

$$\begin{aligned} S_{\rm ren} = \lim_{\epsilon\to 0}\Bigg\{&\frac{1}{16\pi G_N}\left[\int d^5x\sqrt{-G}(R+12) + 2\int_{z=\epsilon}d^4x\sqrt{-\tilde{g}}K[\tilde{g}]\right] - \frac{1}{2}\int d^5x\sqrt{-G}G^{mn}\partial_m\phi\partial_n\phi \\ &+\frac{1}{16\pi G_N}\int_{z=\epsilon}d^4x\sqrt{-\tilde{g}}\left(6 + \frac{1}{2}R[\tilde{g}] + 4\pi G_N\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + a_{(4)}\epsilon^2\log\epsilon\right)\Bigg\}, \end{aligned} \tag{A6}$$

where the final line consists of counterterms at $z=\epsilon$, and

$$a_{(4)} \equiv \frac{1}{\epsilon^2}\left(-\frac{1}{4}R^{\mu\nu}[\tilde{g}]R_{\mu\nu}[\tilde{g}] + \frac{1}{12}R[\tilde{g}]^2 + 4\pi G_N R^{\mu\nu}[\tilde{g}]\partial_\mu\phi\partial_\nu\phi - \frac{136}{9}\pi^2G_N^2(\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\phi)^2 - \pi G_N(\nabla^2_{\tilde{g}}\phi)^2\right), \tag{A7}$$

where $R_{\mu\nu}[\tilde{g}]$ is the Ricci tensor of $\tilde{g}_{\mu\nu}$, and $\nabla^2_{\tilde{g}}$ is with respect to $\tilde{g}_{\mu\nu}$. Varying $S_{\rm ren}$ in Eq. (A6) with respect to sources, we obtain the one point functions

$$\langle\mathcal{O}\rangle = 4\phi_4 + 6\psi_4 + \phi_2\mathrm{Tr}g^{(2)}, \tag{A8a}$$

$$\langle T_{\mu\nu}\rangle = \frac{1}{8\pi G_N}\left(2g^{(4)}_{\mu\nu} + 3h^{(4)}_{\mu\nu} - g^{(2)}_{\mu\lambda}g^{(2)\lambda}_{\nu} + \frac{1}{2}g^{(2)}_{\mu\nu}\mathrm{Tr}g^{(2)} + \frac{1}{2}g^{(0)}_{\mu\nu}\mathrm{Tr}[(g^{(2)})^2] - \frac{1}{4}g^{(0)}_{\mu\nu}[\mathrm{Tr}g^{(2)}]^2 - g^{(0)}_{\mu\nu}\mathrm{Tr}g^{(4)}\right). \tag{A8b}$$

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